Primer for the Matlab Function:
`lplq.m`

Alessandro Buccini and Lothar Reichel

1 Introduction

This primer describes one MATLAB function for the iterative solution of a ill-posed inverse problem of the form

$$A\mathbf{x} = \mathbf{b}^\delta, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$ is a severely ill-conditioned matrix and the right-hand side $\mathbf{b}^\delta$ is contaminated by noise. The function `lplq.m` iteratively solves the minimization problem

$$\mathbf{x}^* = \arg\min \frac{1}{p} \left\|A\mathbf{x} - \mathbf{b}^\delta\right\|_p^p + \frac{\mu}{q} \left\|L\mathbf{x}\right\|_q^q,$$

where $\mu > 0$, $\|z\|_p^p = \sum_{i=1}^n |x_i|^p$, and $L \in \mathbb{R}^{s\times n}$. The iterative method implemented by this function is described in [1], where several properties are discussed.

2 Files and Installation

The software is available from the publisher ETNA (Electronic Transactions on Numerical Analysis) and will be stored as a compressed file entitled `lplq.zip` accompanying paper [1]. Installation details are discussed in this primer as well as in the `README.md` file. The files in `lplq.zip` are listed in Table 1. All files should be extracted and placed in the same directory before use. The code has been developed and tested using MATLAB version R2021b by MathWorks. No other MathWorks products or toolboxes are required to run the codes. To use the provided demos, the MATLAB package “Image Processing Toolbox” by MathWorks is required for visualizing the results.

Table 1: *Files and directories in the `lplq` package.*

File	Description
`codePrimer.pdf`	This document.
`License.md`	Markdown file containing license for package use.
`README.md`	Markdown file with installation instructions and details similar to this primer document.
`lplq.m`	The $\ell^p - \ell^q$ algorithm for linear discrete ill-posed problems described in [1].
`l23Reg.m`	The $\ell^2 - \ell^{2/3}$ shrinkage algorithm for linear discrete ill-posed problems described in [2]. Used in the demo `Test23.m` to showcase the performances of the $\ell^p - \ell^q$ algorithm.
`Test.m`	A demo to showcase the use of the $\ell^p - \ell^q$ algorithm on six examples. Each examples shows the usage of a different selection rule for the regularization parameter.
`Test23.m`	A comparison between $\ell^p - \ell^q$ method and the $\ell^2 - \ell^{2/3}$ shrinkage algorithm.
`@Aclass`	Directory containing a class that implements the blurring operator of black-and-white images.
`@Aclass3`	Directory containing a class that implements the blurring operator of color images.
`@TVclass`	Directory containing a class that implements the Total Variation operator for black-and-white images.
`@TVclass3`	Directory containing a class that implements the Total Variation operator for color images.
`Cameraman_dataset.mat`	Dataset for the examples in `Test.m`.
`Peppers_dataset.mat`	Dataset for the examples in `Test.m`.

Continued on next page

Table 1: *Files and directories in the `lplq` package.* (Continued)

File	Description
`Satellite_dataset.mat`	Dataset for the examples in `Test.m`.
`Saturn_dataset.mat`	Dataset for the examples in `Test.m`.
`Tomography_dataset.mat`	Dataset for the examples in `Test.m`.
`Tree_dataset.mat`	Dataset for the examples in `Test.m`.

3 Input and Output Parameters for Functions

The input parameter syntax for the functions as well as their outputs is displayed in Table 2. A more detailed description of each input parameter is provided in Table 3 and a description of the outputs is given in Table 4.

Table 2: *Syntax for `lplq` and `l23Reg` functions.*

File	Syntax
`lplq.m`	`[x,outInfo] = lplq(A,b,options)`
	`default_options = lplq('defaults')`
`l23Reg.m`	`[x,RRE] = l23Reg(A,b,mu,maxit,tol,x_true)`

Table 3: *Input parameters for `lplq` and `l23Reg` functions.*

Function	Parameter	Description
`lplq.m`	`A`	Severely ill-conditioned $m \times n$ matrix of the system (1). It can be a matrix or an object for which the operations `A*x` and `A'*x` are defined.
	`b`	Column vector containing the data.
	`options`	Structure containg the options for the algorithm; see [1, Table 4.1] for more details. This argument is optional. To obtain a structure containing the default values for `options` use `default_options = lplq('defaults')`.

Continued on next page

Table 3: *Input parameters for `lplq` and `l23Reg` functions.* (Continued)

Function	Parameter	Description
`l23Reg.m`	`A`	Severely ill-conditioned $m \times n$ matrix of the system (1). It can be a matrix or an object for which the operations `A*x`, `A'*x`, and `normest`, are defined.
	`b`	Column vector containing the data.
	`mu`	Value of the regularization parameter
	`maxit`	Maximum number of iterations.
	`tol`	Tolerance for the stopping criterion.
	`x_true`	Exact solution of the problem (optional).

Table 4: *Outputs from `lplq` and `l23Reg` functions.*

Function	Parameter	Description
`lplq.m`	`x`	Approximate solution of the problem (1).
	`outInfo`	Structure containing some information on the iterations; see [1, Table 4.2] for more details.
`l23Reg.m`	`x`	Approximate solution of the problem (1).
	`RRE`	Vector containing the Relative Restoration Error at each iteration. This vector is populated only if `x_true` is given and non-empty.

4 Demos and Further Resources

Two demos for the method are included in the `lplq.zip` package. The demo `Test23.m` compares the $\ell^p - \ell^q$ algorithm and the $\ell^2 - \ell^{2/3}$ shrinkage method, while the demo `Test.m` showcases each selection rule for the regularization parameter in the $\ell^p - \ell^q$ algorithm described in [1, Section 3]. The two files reproduce the examples in [1, Section 5].

References

[1] A. Buccini and L. Reichel, *Software for limited memory restarted ℓ^p-ℓ^q minimization methods using generalized Krylov subspaces*, Electron. Trans. Numer. Anal., Submitted (2024), pp. 1–24.

[2] Y. Zhang and W. Ye, *$L_{2/3}$ regularization: Convergence of iterative thresholding algorithm*, J. Vis. Commun. Image Represen., 33 (2015), pp. 350–357.