Electronic Transactions on Numerical Analysis.
Volume 8, pp. 154-168, 1999.

ISSN 1068-9613.

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WHITNEY ELEMENTS ON PYRAMIDS*

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Abstract. Conforming finite elements in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$ can be regarded as discrete differential forms (Whitney–forms). The construction of such forms is based on an interpolation idea, which boils down to a simple extension of the differential form to the interior of elements. This flexible approach can accommodate elements of more complicated shapes than merely tetrahedra and bricks. The pyramid serves as an example for the successful application of the construction: New Whitney forms are derived for it and they display all desirable properties of conforming finite elements.

Key words. Whitney elements, edge elements, pyramidal element.

AMS subject classifications. 65N30 41A10 58A15.

1. Introduction. The true meaning of differential operators like div and $\mathbf{curl}$ is only revealed when they are looked at from the perspective of differential forms. In many cases, the calculus of differential forms is a very natural and powerful tool to express the partial differential equations arising from mathematical modeling of physical phenomena. This holds true, in particular, in electromagnetism and thermodynamics [3, 13, 32]. For the sake of numerical simulation the model equations have to be cast in a discrete form in which the interesting quantities are determined by only a finite number of degrees of freedom.

Hence, it highly desirable to have discrete differential forms at one's disposal that inherit essential properties of their continuous counterparts. Provided that discrete differential forms are available, the first order equations of the physical model can be directly mapped to systems of equations. Disguised as a finite volume scheme this is the gist of the Finite Integration Technique in electromagnetism [24, 34, 35].

When the Galerkin approach for the discretization of the weak form of the model equations is chosen, which underlies the finite element method [10, 11], it has been realized that discrete differential forms supply excellent choices for finite element approximation spaces [8]. They immediately supply conforming finite elements, for instance, in $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$. In particular in the field of computational electromagnetism this insight has boosted the popularity of so-called edge elements [1, 6, 23, 25, 31, 33]. They are representatives of discrete 1-forms and the natural discrete space of electric and magnetic fields.

Discrete differential forms are built upon triangulations of the domain of interest. By a triangulation we mean a partition of Ω into closed non-overlapping convex polyhedrons (elements) such that every vertex is a vertex of all adjacent elements [15]. For simplicial triangulations Whitney had introduced discrete differential forms in 1957 [36]. Inside each element they are linear polynomials, but a generalization to higher polynomial degrees is possible [20, 21, 30]. Independently, several authors [26, 27, 29] devised vector valued finite elements that can be regarded as special cases of discrete differential forms. In a sense, the perspective of differential forms brings about valuable unification.

We will adopt the term "Whitney-forms" for all discrete differential forms of lowest degree. Their generic feature is a special choice of degrees of freedom. Generally speaking, discrete l-forms, $l \in \mathbb{N}$, are fixed by the values of their integrals over l-faces of the elements. Consequently, in the case of 1-forms in three dimensions (edge elements), the degrees of freedom are provided by path integrals along the edges of the mesh. Correspondingly, the fluxes through faces of elements uniquely describe a discrete Whitney-2-form. Up to now, Whitney

* Received March 12, 1999 . Accepted for publication November 20,1999. Recommended by O. Widlund.
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forms have been constructed for various shapes of elements, for simplices, hypercubes and prisms [18, 22, 26].

The big advantage of finite element schemes is their enormous flexibility in terms of meshes facilitating the resolution of complex geometries and local mesh refinement. In principle, tetrahedral meshes can handle all situations. However, on behalf of efficient implementation, often a combination of both tetrahedral and hexahedral elements should be preferred [5]. Then, if so-called hanging nodes destroying the integrity of the mesh, are to be avoided, the mesh has to be padded with pyramids [4].

This paper pursues a systematic approach to construct Whitney forms by means of a generalized concept of interpolation. Great attention will be payed to finding a whole sequence of discrete differential forms of order 1 through 3 so that the central exact sequence property of differential forms [14] is preserved on the discrete level.

The plan of the paper is as follows: In the next section we briefly discuss differential forms and desirable properties of their discrete counterparts. In the third section we introduce the setting and disclose why a naive attempt to cope with pyramids fails. In section 4 we will elaborate on the idea that the construction of Whitney-forms can be viewed as an interpolation of a special kind. We first confine ourselves to tetrahedral meshes. In the fifth section we will apply this idea to pyramids and present the finite elements obtained thus. Finally, the sixth section is devoted to a straightforward verification that the new elements meet all the requirements and possess reasonable approximation properties.

2. Discrete differential forms. There is huge body of literature on the calculus of differential forms. For an exposition we refer to [14]. In $\mathbb{R}^n$ an l-form, $0 \le l \le n$, is a mapping of $\mathbb{R}^n$ into the $\binom{n}{l}$-dimensional vector space of alternating l-multilinear forms on $\mathbb{R}^n$. After a basis of $\mathbb{R}^n$ has been chosen, there is a canonical way to identify differential forms with vectorfields, their "vector proxies". The usual identification in $\mathbb{R}^3$ is depicted in table 2.1. Using

Differential form	Related function u/vectorfield $\mathbf{u}$
$\mathbf{x} \mapsto \omega(\mathbf{x})$	$\omega(\mathbf{x}) = u(\mathbf{x})$
$\mathbf{x} \mapsto \{\mathbf{v} \mapsto \omega(\mathbf{x})(\mathbf{v})\}$	$\omega(\mathbf{x})(\mathbf{v}) = \langle \mathbf{u}(\mathbf{x}), \mathbf{v} \rangle$
$\mathbf{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2) \mapsto \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2)\}$	$\omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2) = \langle \mathbf{u}(\mathbf{x}), \mathbf{v}_1 \times \mathbf{v}_2 \rangle$
$\mathbf{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \mapsto \omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\}$	$\omega(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = u(\mathbf{x}) \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

TABLE 2.1
Relationship between differential forms and vectorfields in 3D

this identification the exterior derivative d of differential forms spawns the familiar differential operators of vector analysis (see table 2.2). The appropriate transformation of differential forms under a smooth change Φ of variables is described by the pullback operator, whose meaning for the vector proxies is listed in table 2.2. A crucial feature of the pullback is that it commutes both with integration and the exterior derivative.

Given a triangulation $\mathcal{T}_h$ (in the sense of [15]) of some domain $\Omega \in \mathbb{R}^n$, we choose some polytope as a reference element for each type of element occurring in $\mathcal{T}_h$. We demand that for each element we can find a smooth, regular, maybe affine, mapping onto a suitable reference element. On the reference element we define spaces of discrete differential forms and degrees of freedom. Following the concept of affine equivalent finite elements [15, 21], global spaces and degrees of freedom can be declared via transformations, which are provided by the pullback of differential forms (given in table 2.2).

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Forms	d	pullback	patching condition
0-form	$\mathbf{grad}$	$\mathfrak{F}^0(u)(\widehat{\mathbf{x}}) = u(\Phi^{-1}(\hat{\mathbf{x}}))$	C^0-continuity
1-form	$\mathbf{curl}$	$\mathfrak{F}^1(\mathbf{u})(\widehat{\mathbf{x}}) = D\Phi^T(\Phi^{-1}(\hat{\mathbf{x}}))\mathbf{u}(\Phi^{-1}(\hat{\mathbf{x}}))$	tangential continuity
2-form	div	$\mathfrak{F}^2(\mathbf{u})(\widehat{\mathbf{x}}) = \det(D\Phi)D\Phi(\Phi^{-1}(\hat{\mathbf{x}}))\mathbf{u}(\Phi^{-1}(\hat{\mathbf{x}}))$	normal continuity
3–form	0	$\mathfrak{F}^3(u)(\widehat{\mathbf{x}}) = \det D\Phi(\Phi^{-1}(\hat{\mathbf{x}}))u(\Phi^{-1}(\hat{\mathbf{x}}))$	—

TABLE 2.2

Meaning of exterior derivative, continuity of traces, pullback for vector proxies of differential forms of different order in three dimensions

We aim at conforming finite element spaces. Consequently, the traces of discrete differential forms onto any interelement boundary (a $(n-1)$–face) have to be unique and they have to be fixed by the degrees of freedom associated with that face. This makes the vector proxies fulfill the patching condition from table 2.2 and guarantees that they provide finite elements conforming in $H^1(\Omega)$, $\boldsymbol{H}(\mathbf{curl};\Omega)$, and $\boldsymbol{H}(\mathrm{div};\Omega)$, respectively.

In addition an "*exact sequence property*" must hold for the spaces of discrete differential forms if Ω is contractible: The exterior derivative of a discrete l-form is to yield a valid discrete $l+1$-form. In addition, any discrete $l+1$-form with vanishing exterior derivative should have a representation as the exterior derivative of some discrete l-form.

Finally, the discrete differential forms have to possess approximation properties, in order to be useful for Galerkin discretizations. It is a standard insight in finite elements that satisfactory approximation properties are directly linked to the fact that all polynomials of a certain degree are contained in the spaces on the reference elements [11]. In the case of Whitney-forms that provide only first order schemes, we have to make sure that all constant forms can be represented.

3. Construction by transformation. As consequence of affine equivalence, the construction of the local finite element spaces can be entirely carried out on a *reference element*. By transformation the scheme is then fixed for any other element. Whitney-forms for the cube Q are well known [26]. We recall the local spaces $\mathcal{W}^l(Q)$ of vector proxies for Whitney-l-forms, $l \in \mathbb{N}$:

- 0-forms: $\mathcal{W}^0(Q) = \mathcal{Q}_{1,1,1}(Q)$;
- 1-forms: $\mathcal{W}^1(Q) = \mathcal{Q}_{1,1,0}(Q) \times \mathcal{Q}_{1,0,1}(Q) \times \mathcal{Q}_{0,1,1}(Q)$;
- 2-forms: $\mathcal{W}^2(Q) = \mathcal{Q}_{0,0,1}(Q) \times \mathcal{Q}_{0,1,0}(Q) \times \mathcal{Q}_{1,0,0}(Q)$;
- 3-forms: $\mathcal{W}^3(Q) = \mathcal{Q}_{0,0,0}(Q)$.

Here, $\mathcal{Q}_{k_1,k_2,k_3}$ denotes the spaces of 3-variate tensor-product polynomials with degree $\leq k_j$ in the independent variable x_j, $j = 1, 2, 3$.

It is tempting to treat the pyramid P as a degenerate cube. For instance, the transformation

$$\Phi : \begin{cases} Q & \mapsto & P \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & \begin{pmatrix} (1-z)x \\ (1-z)y \\ z \end{pmatrix} \end{cases} \quad \Longleftrightarrow \quad \Phi^{-1} : \begin{cases} P & \mapsto & Q \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & \begin{pmatrix} x/(1-z) \\ y/(1-z) \\ z \end{pmatrix} \end{cases}$$

"collapses" the cube into a pyramid. In particular the unit cube $Q =]0;1[^3$ will be mapped onto the pyramid P with vertices $\mathbf{a}_1 = (0,0,0)$, $\mathbf{a}_2 = (1,0,0)$, $\mathbf{a}_3 = (0,1,0)$, $\mathbf{a}_4 = (1,1,0)$, $\mathbf{a}_5 = (0,0,1)$ (see figure 3.1). We are going to use this very pyramid P as reference pyramid.

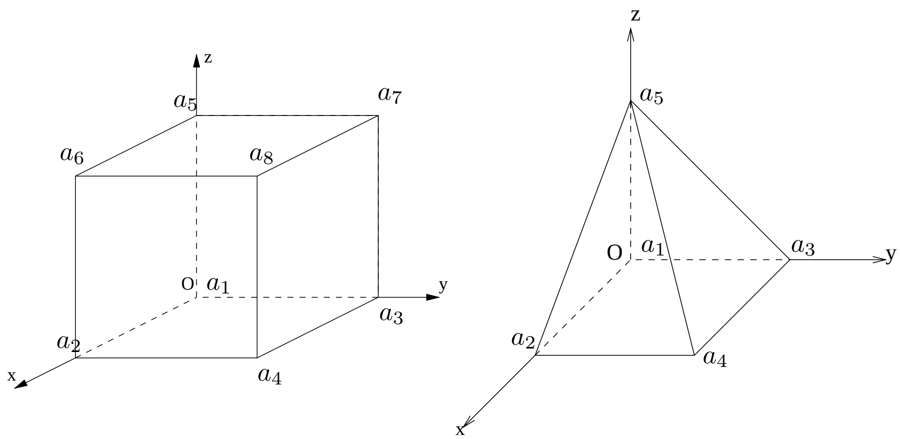

FIG. 3.1. *Rectangular and pyramidal reference element*

Now, we can use the transformation rule from table 2.2 for a function u on T that corresponds to a 0-form :

$$\mathfrak{F}^0(u)(\hat{\mathbf{x}}) := u(\Phi^{-1}(\hat{\mathbf{x}}))\ , \quad \hat{\mathbf{x}} \in P\ . \tag{3.1}$$

Pick linear functions $\beta_1, \dots, \beta_5 \in \mathcal{Q}_{1,1,1}(Q)$ such that $\beta_i(\mathbf{q}_j) = \delta_{ij}$, $i = 1, \dots, 4$, $j = 1, \dots, 8$, and $\beta_5(\mathbf{q}_j) = 0$ for $j = 1, 2, 3, 4$, $\beta_5(\mathbf{q}_j) = 1$ for $j = 5, 6, 7, 8$. The numbering of the vertices $\mathbf{q}_i$, $i = 1, \dots, 8$ of the cube is given in Figure 3.1. Note that $\beta_5 \equiv 1$ on the top plane of the cube. Thus it is a promising candidate for a function that the mapping $\mathfrak{F}^0$ will take to a Whitney-0-form basis function associated with vertex #5 of the pyramid. In detail the images of these functions under the transformation read

(3.2)

$$\begin{array}{rclcrcl}
\beta_1 & = & (1-x)(1-y)(1-z) & & \pi_1 := \mathfrak{F}^0\beta_1 & = & \dfrac{(1-z-x)(1-z-y)}{1-z} \\
\beta_2 & = & x(1-y)(1-z) & & \pi_2 := \mathfrak{F}^0\beta_2 & = & \dfrac{x(1-z-y)}{1-z} \\
\beta_3 & = & (1-x)y(1-z) & \Rightarrow & \pi_3 := \mathfrak{F}^0\beta_3 & = & \dfrac{(1-z-x)y}{1-z} \\
\beta_4 & = & xy(1-z) & & \pi_4 := \mathfrak{F}^0\beta_4 & = & \dfrac{xy}{1-z} \\
\beta_5 & = & z & & \pi_5 := \mathfrak{F}^0\beta_5 & = & z.
\end{array}$$

We refer to Figure 3.1 for the coordinate directions. Straightforward computations establish a few facts about the transformed functions:

LEMMA 3.1. *The functions* $\pi_1, \dots, \pi_5$ *from* (3.2) *fulfill:*

(i) $\pi_i(\mathbf{a}_j) = \delta_{ij}$, $i, j = 1, \dots, 5$.
(ii) *The restrictions of* $\pi_1, \dots, \pi_5$ *to the square bottom plane of the pyramid are bilinear in* x, y*, their restrictions to the triangular faces are linear.*
(iii) *Any linear function on the pyramid can be represented as a linear combination of* $\pi_1, \dots, \pi_5$.
(iv) *The* π_i, $i = 1, \dots, 5$*, form a non-negative partition of unity.*

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We conclude that $\{\pi_1, \dots, \pi_5\}$ is a valid nodal basis for the local space of Whitney-0-forms on the pyramid P. Here, "nodal" means that they form a set dual to the set of degrees of freedom. We stress that the second property ensures that the local space on pyramids fits those on tetrahedra and hexahedra; if the degrees of freedoms, that is, the function values at the vertices of a commmon face of two elements, coincide, then overall continuity of the finite element function across this face is guaranteed. This is the well-known compatibility condition for 0-form and H^1-conformity, respectively.

At first glance, the same procedure should succeed for the other forms, too, now using the appropriate transformations $\mathfrak{F}^l$ for vector proxies of l-forms given in table 2.2. For standard Whitney-1-forms on the cube Q the nodal basis function associated with edge #7 (that is $[a_4, a_8]$ in figure 3.1) and its image under the mapping $\mathfrak{F}^1$ read

$$\boldsymbol{\beta}_7(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ xy \end{pmatrix} \quad \Longrightarrow \quad (\mathfrak{F}^1\boldsymbol{\beta}_7)(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{xy}{(z-1)^2} \end{pmatrix} .$$

We know that the compatibility condition for 1-forms boils down to the continuity of the tangential components across interelement faces. For the triangular face spanned by the vertices $\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$, which has a normal vector $\mathbf{n} = \frac{1}{2}\sqrt{2}(0,1,1)^T$, we find

$$(\mathfrak{F}^1\boldsymbol{\beta}_7) \times \mathbf{n} = -\frac{1}{2}\sqrt{2}\frac{t_1}{1-t_2} \cdot \mathbf{e}_1 \ , \tag{3.3}$$

where t_1 and t_2 are the local coordinates of the face chosen such that $(x,y,z)^T = (t_1, 1-t_2, t_2)^T$ and $\mathbf{e}_1 = (1,0,0)^T$ is the t_1-coordinate direction. The tangential components of edge element vectorfields on a face of a tetrahedron are linear with respect to any local Cartesian coordinate system. Obviously, the expression from (3.3) is not linear. The bottom line is that the mapped 1-forms cannot be matched with conventional edge elements on tetrahedra. The same holds true for 2-forms. This demonstrates the failure of the mapping approach and calls for a different construction on a pyramid.

4. Interpolation on simplices. Sloppily speaking a differential form of order l can be regarded as a mapping assigning to each smooth oriented manifold of dimension l a real number, the value of its integral [16]. Vice versa, once all these integrals are known, the form is uniquely determined. This view permits us to tackle the construction of discrete differential forms as an interpolation problem: Given the values of the integrals over only a finite number of convex manifolds (the vertices, edges or faces of the mesh), find a simple way to express integrals over general mani–folds through these values. Of course, this task of interpolation has many solutions. To obtain practical finite elements, we strive to come up with a procedure as simple as possible.

In fact, all we need to specify is a way to evaluate the integrals over simplices. Write $[\mathbf{x}_1, \dots, \mathbf{x}_{l+1}]$ for the convex span of $\mathbf{x}_1, \dots, \mathbf{x}_{l+1} \in \mathbb{R}^3$. Orientation is induced by the ordering of the vertices. If the integrals of a smooth l-form ω over all such simplices are known, we get from the definition of a differential form [14]

$$\omega(\mathbf{x})(\mathbf{v}_1, \dots, \mathbf{v}_l) = l! \lim_{t\to 0} \frac{1}{t^l} \int\limits_{[\mathbf{x}_1,\dots,\mathbf{x}_{l+1}]} \omega \ , \tag{4.1}$$

where $\mathbf{x}_1 = \mathbf{x}$, $\mathbf{x}_{i+1} = \mathbf{x} + t\mathbf{v}_i$, for $i = 1, \dots, l$, $\mathbf{v}_i \in \mathbb{R}^3$. Recall that an l-form evaluated at a point yields an alternating l-linear form on $\mathbb{R}^3$.

We first illustrate the idea of interpolation in the case of a tetrahedral mesh, where no complications are encountered. Let T be a non-degenerate tetrahedron with vertices $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$. We will use the term l-face, $l = 0, 1, 2$, to refer to a vertex ($l = 0$), an edge ($l = 1$) or a face ($l = 2$).

For 0-forms the degrees of freedom are just the values of the associated continuous function ϕ at the vertices of T. The simplest way to extend these values is linear interpolation

$$\phi(\mathbf{x}) = \sum_{i=1}^{4} \phi(\mathbf{a}_i)\lambda_i(\mathbf{x}) \ , \tag{4.2}$$

where λ_i is the barycentric coordinate function of the tetrahedron associated with vertex $\mathbf{a}_i$. Note that, equivalently, we could have introduced the λ_i as the canonical basis functions for Whitney-0-forms. Now, our goal is to find the counterparts of linear interpolation for forms of higher order l, $l = 1, 2, 3$.

For discrete l-forms the degrees of freedoms are the integrals

$$\int\limits_{[\mathbf{a}_{j_1},\dots,\mathbf{a}_{j_{l+1}}]} \omega \ ,$$

where $1 \leq j_1 \leq \dots \leq j_{l+1} \leq 4$. We point out that the order of the vertices fixes an orientation of the face, which, in turn, affects the sign of the integral. We can read (4.2) as follows: An interior point of the simplex is represented as a weighted sum of its vertices. The weights, in this case values of the barycentric coordinates, tell us, how to interpolate the integrals of the 0-form. Thus, the essential idea is to represent any l-simplex inside T by a "weighted sum" of its l-faces.

In the case of 1-forms consider the arbitrary 1–simplex $[\mathbf{x}, \mathbf{y}]$, an oriented line, with $\mathbf{x}, \mathbf{y} \in T$, $\mathbf{x} = \sum_i \lambda_i(\mathbf{x})\mathbf{a}_i$, $\mathbf{y} = \sum_i \lambda_i(\mathbf{y})\mathbf{a}_i$. Then

$$\begin{aligned}
[\mathbf{x}, \mathbf{y}] &= \{t\mathbf{x} + (1-t)\mathbf{y} \ ; \ 0 \leq t \leq 1\} \\
&= \left\{ \sum_i [t\lambda_i(\mathbf{x}) + (1-t)\lambda_i(\mathbf{y})]\mathbf{a}_i \ ; \ 0 \leq t \leq 1 \right\} \\
&= \left\{ \sum_i [t \sum_j \lambda_j(\mathbf{y})\lambda_i(\mathbf{x}) + (1-t) \sum_j \lambda_j(\mathbf{x})\lambda_i(\mathbf{y})]\mathbf{a}_i \ ; \ 0 \leq t \leq 1 \right\} \\
&= \left\{ \sum_i \sum_j \lambda_i(\mathbf{x})\lambda_j(\mathbf{y})[t\mathbf{a}_i + (1-t)\mathbf{a}_j] \ ; \ 0 \leq t \leq 1 \right\} \ .
\end{aligned}$$

Hence, taking into account orientation, we will represent

$$\int\limits_{[\mathbf{x},\mathbf{y}]} \omega := \sum_i \sum_j \lambda_i(\mathbf{x})\lambda_j(\mathbf{y}) \int\limits_{[\mathbf{a}_i,\mathbf{a}_j]} \omega = \sum_{i<j} [\lambda_i(\mathbf{x})\lambda_j(\mathbf{y}) - \lambda_i(\mathbf{y})\lambda_j(\mathbf{x})] \int\limits_{[\mathbf{a}_i,\mathbf{a}_j]} \omega \ . \tag{4.3}$$

Plugging this formula into (4.1) and using that the exterior derivative of a 0-form is the gradient, we get

$$\begin{aligned}
\omega(\mathbf{x})(\mathbf{v}) &= \lim_{t\to 0} \frac{1}{t} \sum_{i<j} [\lambda_i(\mathbf{x})(\lambda_j(\mathbf{x}+t\mathbf{v}) - \lambda_j(\mathbf{x})) - \lambda_j(\mathbf{x})(\lambda_i(\mathbf{x}+t\mathbf{v}) - \lambda_i(\mathbf{x}))] \int\limits_{[\mathbf{a}_i,\mathbf{a}_j]} \omega \\
&= \sum_{i<j} [\lambda_i(\mathbf{x}) d\lambda_j(\mathbf{x})(\mathbf{v}) - \lambda_j(\mathbf{x}) d\lambda_i(\mathbf{x})(\mathbf{v})] \int\limits_{[\mathbf{a}_i,\mathbf{a}_j]} \omega \ . \qquad (4.4)
\end{aligned}$$

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Now, take into account that the vectorfield $\mathbf{u}$ belonging to ω is defined by $\omega(\mathbf{x})(\mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$, $\mathbf{v} \in \mathbb{R}^3$, where $\langle \cdot, \cdot \rangle$ stands for the Euclidean inner product (cf. Table 2.1). It is evident from (4.4) that for the vector proxy we get

$$\mathbf{u}(\mathbf{x}) = \sum_{i<j} (\mathbf{grad}\,\lambda_i(\mathbf{x}) \cdot \lambda_j(\mathbf{x}) - \mathbf{grad}\,\lambda_j(\mathbf{x}) \cdot \lambda_i(\mathbf{x})) \int\limits_{[\mathbf{a}_i, \mathbf{a}_j]} \omega \ .$$

It is just the standard edge element basis functions [7]

$$\boldsymbol{\beta}_{ij} := \mathbf{grad}\,\lambda_i \cdot \lambda_j - \mathbf{grad}\,\lambda_j \cdot \lambda_i \tag{4.5}$$

that have emerged, weighted with the values of the degrees of freedom. From (4.3) we infer that for $1 \le i \ne j \le 4, 1 \le k \ne l \le 4$

$$\int\limits_{[\mathbf{a}_i, \mathbf{a}_j]} \langle \boldsymbol{\beta}_{kl}, \mathbf{t} \rangle \, d\Gamma = \begin{cases} \pm 1 & \text{if } \{i,j\} = \{k,l\} \\ 0 & \text{else }, \end{cases}$$

as expected for basis functions.

Discrete 2-forms can be constructed in a similar fashion. In this case plane triangles $[\mathbf{x}, \mathbf{y}, \mathbf{z}]$, $\mathbf{x}, \mathbf{y}, \mathbf{z} \in T$, in the interior of the tetrahedron have to be represented as "combinations" of faces. Using barycentric coordinates, we can write

$$\begin{aligned} [\mathbf{x}, \mathbf{y}, \mathbf{z}] &= \{t_1\mathbf{x} + t_2\mathbf{y} + t_3\mathbf{z};\ 0 \le t_i \le 1,\ t_1 + t_2 + t_3 = 1\} \\ &= \left\{ \sum_{i,j,k=1}^{4} \lambda_i(\mathbf{x})\lambda_j(\mathbf{y})\lambda_k(\mathbf{z})(t_1\mathbf{a}_i + t_2\mathbf{a}_j + t_3\mathbf{a}_k);\ \begin{array}{l} 0 \le t_i \le 1,\ i = 1,2,3 \\ t_1 + t_2 + t_3 = 1 \end{array} \right\}. \end{aligned}$$

This suggests the formula

$$\int\limits_{[\mathbf{x},\mathbf{y},\mathbf{z}]} \omega = \sum_{i<j<k} \left(\sum_{\pi \in \mathrm{Perm}\{i,j,k\}} \mathrm{sgn}(\pi) \lambda_{\pi(i)}(\mathbf{x}) \lambda_{\pi(j)}(\mathbf{y}) \lambda_{\pi(k)}(\mathbf{z}) \int\limits_{[\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k]} \omega \right) . \tag{4.6}$$

Using (4.1), after tedious computations we arrive at a representation for the vector proxy of ω,

$$\mathbf{u}(\mathbf{x}) = \sum_{i<j<k} \boldsymbol{\beta}_{ijk} \int\limits_{[\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k]} \langle \mathbf{u}, \mathbf{n} \rangle \, d\Gamma \ ,$$

with the basis functions for Whitney-2-forms [7]

$$\boldsymbol{\beta}_{ijk} := 2(\lambda_i\,\mathbf{grad}\,\lambda_j \times \mathbf{grad}\,\lambda_k + \lambda_j\,\mathbf{grad}\,\lambda_k \times \mathbf{grad}\,\lambda_i + \lambda_k\,\mathbf{grad}\,\lambda_i \times \mathbf{grad}\,\lambda_j) \ . \tag{4.7}$$

Again, the canonical basis functions for lowest order face elements have emerged from the construction.

5. Interpolation for the pyramid.

What foils a straightforward application of the interpolation idea to a pyramid is both the apparent lack of natural barycentric coordinates and the fact that certain convex spans of vertices do not occur as edges or faces, respectively. The first difficulty is easily overcome by resorting to the functions $\pi_1, \ldots, \pi_5$ from (3.2), which provide a basis for Whitney-0-forms on the pyramid; From lemma 3.1(iv) we get

$\mathbf{x} = \sum_i \pi_i(\mathbf{x})\mathbf{a}_i$ for any $\mathbf{x} \in P$. Hence, the π_i, $i = 1, \dots, 5$ are a full replacement for the barycentric coordinates. Thus, we can simply state formulas (4.3) and (4.6) with λ_i replaced by π_i and summation ranging between 1 and 5.

Then we face the second problem, since the edges $[\mathbf{a}_2, \mathbf{a}_3]$, $[\mathbf{a}_1, \mathbf{a}_4]$ and the faces $[\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_5]$, $[\mathbf{a}_1, \mathbf{a}_4, \mathbf{a}_5]$, $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4]$, $[\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4]$ occur in the formula, but no degrees of freedom are specified on them. The idea is to *express each integral over an non-existent edge or face by a weighted sum of degrees of freedom* observing the following rule: *Expressions for integrals over edges contained in a face of P may only be based on degrees of freedoms associated with that face.* This rule is necessary to get compatibility across faces, because only degrees of freedom belonging to a face may contribute to the tangential/normal trace of the interpolant onto that face.

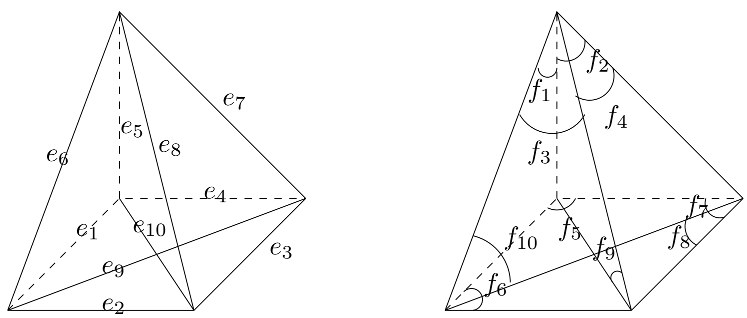

FIG. 5.1. *Numbering of the "edges"* $e_1 = [\mathbf{a}_1, \mathbf{a}_2]$, $e_2 = [\mathbf{a}_2, \mathbf{a}_4]$, $e_3 = [\mathbf{a}_3, \mathbf{a}_4]$, $e_4 = [\mathbf{a}_1, \mathbf{a}_3]$, $e_5 = [\mathbf{a}_1, \mathbf{a}_5]$, $e_6 = [\mathbf{a}_2, \mathbf{a}_5]$, $e_7 = [\mathbf{a}_3, \mathbf{a}_5]$, $e_8 = [\mathbf{a}_4, \mathbf{a}_5]$, $e_9 = [\mathbf{a}_2, \mathbf{a}_3]$, $e_{10} = [\mathbf{a}_1, \mathbf{a}_4]$ *and "faces"* $f_1 = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5]$, $f_2 = [\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5]$, $f_3 = [\mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_5]$, $f_4 = [\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5]$, $f_5 = [\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_2]$, $f_6 = [\mathbf{a}_1, \mathbf{a}_4, \mathbf{a}_2]$, $f_7 = [\mathbf{a}_1, \mathbf{a}_4, \mathbf{a}_3]$, $f_8 = [\mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_3]$, $f_9 = [\mathbf{a}_1, \mathbf{a}_5, \mathbf{a}_4]$, $f_{10} = [\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_5]$

Recall that we confine ourselves to constructing Whitney-forms on the reference pyramid only; any pyramid $\widetilde{P}$ of the actual mesh can be mapped onto P by a smooth transformation $\Phi : \widetilde{P} \to P$. Then the Whitney-forms on $\widetilde{P}$ arise from those on P by the pullback transformations specified in table 2.2.

Let us denote like in Figure 5.1 the edges and the faces and let the basis of the reference pyramid be $f_b = [\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_2]$.

Keeping in mind the above rule, it is clear how to choose the weights for the non-existent edges of the pyramid. They are all contained in the bottom square. Hence,

$$\begin{aligned}\int_{e_9} \omega &= \nu_1 \int_{e_1} \omega + \nu_2 \int_{e_2} \omega + \nu_3 \int_{e_3} \omega + \nu_4 \int_{e_4} \omega \\ \int_{e_{10}} \omega &= \mu_1 \int_{e_1} \omega + \mu_2 \int_{e_2} \omega + \mu_3 \int_{e_3} \omega + \mu_4 \int_{e_4} \omega \,.\end{aligned}$$

In addition, the discrete 1-form when restricted to the bottom square must agree with the trace onto a face of a discrete 1-form on a cube. In other words, we can just take the cue from discrete 1-forms on a square to fix the weights ν_i and μ_i uniquely. Expressions for Whitney-1-forms on a square are well known and evaluation of their integrals along the diagonal yields $\mu_i = \frac{1}{2}$, $i = 1, 2, 3, 4$, $\nu_i = \frac{1}{2}$, $i = 2, 4$, $\nu_i = -\frac{1}{2}$, $i = 1, 3$. Then we crank up the machine of section 4. Using a definition of $\boldsymbol{\vartheta}_{ij} = \mathbf{grad}\, \pi_i \cdot \pi_j - \mathbf{grad}\, \pi_j \cdot \pi_i$ similar to that of (4.5), we end up with the following expressions for the canonical basis functions $\boldsymbol{\gamma}_i$, $i = 1, \dots, 8$, for

Whitney-1-forms on pyramids:

$$\begin{array}{rclrcl}
\boldsymbol{\gamma}_1 &=& \boldsymbol{\vartheta}_{12} + \frac{1}{2}\boldsymbol{\vartheta}_{14} - \frac{1}{2}\boldsymbol{\vartheta}_{23}, & \boldsymbol{\gamma}_5 &=& \boldsymbol{\vartheta}_{15};\\
\boldsymbol{\gamma}_2 &=& \frac{1}{2}\boldsymbol{\vartheta}_{23} + \boldsymbol{\vartheta}_{24} + \frac{1}{2}\boldsymbol{\vartheta}_{14}, & \boldsymbol{\gamma}_6 &=& \boldsymbol{\vartheta}_{25};\\
\boldsymbol{\gamma}_3 &=& \frac{1}{2}\boldsymbol{\vartheta}_{14} + \boldsymbol{\vartheta}_{34} - \frac{1}{2}\boldsymbol{\vartheta}_{23}, & \boldsymbol{\gamma}_7 &=& \boldsymbol{\vartheta}_{35};\\
\boldsymbol{\gamma}_4 &=& \frac{1}{2}\boldsymbol{\vartheta}_{14} + \boldsymbol{\vartheta}_{13} + \frac{1}{2}\boldsymbol{\vartheta}_{23}, & \boldsymbol{\gamma}_8 &=& \boldsymbol{\vartheta}_{45}.
\end{array}$$

Computing the gradients we get the related vectorfields ($(x,y,z)^T \in P$):

$$\boldsymbol{\gamma}_1 = \begin{pmatrix} 1-z-y \\ 0 \\ x - \dfrac{xy}{1-z} \end{pmatrix}, \quad \boldsymbol{\gamma}_2 = \begin{pmatrix} 0 \\ x \\ \dfrac{xy}{1-z} \end{pmatrix}, \quad \boldsymbol{\gamma}_3 = \begin{pmatrix} y \\ 0 \\ \dfrac{xy}{1-z} \end{pmatrix}, \quad \boldsymbol{\gamma}_4 = \begin{pmatrix} 0 \\ 1-z-x \\ y - \dfrac{xy}{1-z} \end{pmatrix},$$

$$\boldsymbol{\gamma}_5 = \begin{pmatrix} z - \dfrac{yz}{1-z} \\ z - \dfrac{xz}{1-z} \\ 1-x-y+\dfrac{xy}{1-z} - \dfrac{xyz}{(1-z)^2} \end{pmatrix}, \qquad \boldsymbol{\gamma}_6 = \begin{pmatrix} -z + \dfrac{yz}{1-z} \\ \dfrac{xz}{1-z} \\ x + \dfrac{xy}{1-z} - \dfrac{xyz}{(1-z)^2} \end{pmatrix},$$

$$\boldsymbol{\gamma}_7 = \begin{pmatrix} \dfrac{yz}{1-z} \\ -z + \dfrac{xz}{1-z} \\ y + \dfrac{xy}{1-z} - \dfrac{xyz}{(1-z)^2} \end{pmatrix}, \qquad \boldsymbol{\gamma}_8 = \begin{pmatrix} -\dfrac{yz}{1-z} \\ -\dfrac{xz}{1-z} \\ \dfrac{xy}{1-z} - \dfrac{xyz}{(1-z)^2} \end{pmatrix}.$$

The problem for 2-forms is more delicate. It boils down to determining the ten weights η_i, κ_i, $i = 1, \dots, 5$, in

$$\begin{array}{rcl}
\int\limits_{f_9} \omega &=& \eta_b \int\limits_{f_b} \omega + \eta_1 \int\limits_{f_1} \omega + \eta_2 \int\limits_{f_2} \omega + \eta_3 \int\limits_{f_3} \omega + \eta_4 \int\limits_{f_4} \omega \\
\int\limits_{f_{10}} \omega &=& \kappa_b \int\limits_{f_b} \omega + \kappa_1 \int\limits_{f_1} \omega + \kappa_2 \int\limits_{f_2} \omega + \kappa_3 \int\limits_{f_3} \omega + \kappa_4 \int\limits_{f_4} \omega \, .
\end{array} \tag{5.1}$$

Three different considerations guide to search for the weights:

Firstly, we point out that we need not worry about the weights of the four triangles contained in the bottom square. Parallel to the above reasoning they can be fixed by examining discrete 2-forms on the square, which are just constants. This implies

$$\int\limits_{f_5} \omega = \int\limits_{f_6} \omega = \int\limits_{f_7} \omega = \int\limits_{f_8} = \frac{1}{2} \int\limits_{f_b} \omega \, .$$

Secondly, as we emphasized in section 2, on behalf of basic approximation properties, the constant forms have to be contained in the space of discrete 2-forms on the reference element. Accordingly, the weights η_i and κ_i (cf. (5.1)) for the interior faces have to be chosen such that (5.1) is satisfied for $\omega \equiv \mathrm{const.}$. Switching to vector proxies, we have to ensure that

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the equations hold for the three constant vector fields $(1,0,0)^T$, $(0,1,0)^T$, and $(0,0,1)^T$. Straightforward calculation of the integrals yields respectively

$$\begin{array}{rcrcrcrcrcl} 0\cdot\eta_b & + & 0\cdot\eta_1 & - & 1\cdot\eta_2 & + & 1\cdot\eta_3 & + & 0\cdot\eta_4 & = & -1; \\ 0\cdot\eta_b & - & 1\cdot\eta_1 & + & 0\cdot\eta_2 & + & 0\cdot\eta_3 & + & 1\cdot\eta_4 & = & 1; \\ -1\cdot\eta_b & + & 0\cdot\eta_1 & + & 0\cdot\eta_2 & + & 1\cdot\eta_3 & + & 1\cdot\eta_4 & = & 0. \end{array} \tag{5.2}$$

The same linear system of equations can be obtained for the weights κ_i. Still, (5.2) is an underdetermined linear system. Thus, we have to employ a third consideration to get additional conditions. They are provided by the "exact sequence property" of section 2 in conjunction with Stokes' theorem:

The space of discrete 3-forms on the pyramid will be of dimension 1. In other words, discrete 3-forms have to be constant. Consequently, all discrete 2-forms must have constant exterior derivatives. Writing T for the tetrahedron $[\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_4,\mathbf{a}_5]$ contained in the pyramid, we get

$$\int_T d\omega_h = \frac{\mathrm{vol}(T)}{\mathrm{vol}(P)}\int_P d\omega_h = \frac{1}{2}\int_P d\omega_h$$

for any discrete 2-form ω_h. By Stokes' theorem applied to both P and T

$$\begin{aligned} \frac{1}{2}\left(\int_{f_b}\omega_h + \int_{f_1}\omega_h + \int_{f_2}\omega_h + \int_{f_3}\omega_h + \int_{f_4}\omega_h\right) &= \frac{1}{2}\int_P d\omega_h = \\ &= \int_{f_1}\omega_h + \int_{f_3}\omega_h + \int_{f_6}\omega_h + \int_{f_9}\omega_h\ , \end{aligned}$$

and the last integral can be replaced by its representation from (5.1). When we plug in "test forms" into the resulting equations, we get more linear equations for the weights. Appropriate test forms are provided by the, hitherto unknown, basis forms $\boldsymbol{\zeta}_i$, satisfying $\int_{f_i}\boldsymbol{\zeta}_j = \delta_{ij}$, $i,j = b,1,2,3,4$. This gives the five linear conditions

$$\begin{array}{rcrcrcrcrcll} 1\cdot\eta_b & + & 0\cdot\eta_1 & + & 0\cdot\eta_2 & + & 0\cdot\eta_3 & + & 0\cdot\eta_4 & = & 0 & [\text{for } \boldsymbol{\zeta}_b]; \\ 0\cdot\eta_b & + & 1\cdot\eta_1 & + & 0\cdot\eta_2 & + & 0\cdot\eta_3 & + & 0\cdot\eta_4 & = & -\frac{1}{2} & [\text{for } \boldsymbol{\zeta}_1]; \\ 0\cdot\eta_b & + & 0\cdot\eta_1 & + & 1\cdot\eta_2 & + & 0\cdot\eta_3 & + & 0\cdot\eta_4 & = & \frac{1}{2} & [\text{for } \boldsymbol{\zeta}_2]; \\ 0\cdot\eta_b & + & 0\cdot\eta_1 & + & 0\cdot\eta_2 & + & 1\cdot\eta_3 & + & 0\cdot\eta_4 & = & -\frac{1}{2} & [\text{for } \boldsymbol{\zeta}_3]; \\ 0\cdot\eta_b & + & 0\cdot\eta_1 & + & 0\cdot\eta_2 & + & 0\cdot\eta_3 & + & 1\cdot\eta_4 & = & \frac{1}{2} & [\text{for } \boldsymbol{\zeta}_4]\ . \end{array} \tag{5.3}$$

These equations fix the weights η_i, $i = b,1,2,3,4$ and they are compatible with (5.2). The total system is overdetermined, but has the solution $\eta_1 = \eta_3 = -\frac{1}{2}$, $\eta_2 = \eta_4 = \frac{1}{2}$, $\eta_b = 0$. A similar reasoning gives us the other weights κ_i, such that we obtain the following expressions for the canonical basis functions $\boldsymbol{\zeta}_i$, $i = 1,2,3,4,b$ for Whitney–2–forms on pyramids:

$$\boldsymbol{\zeta}_1 = \boldsymbol{\tau}_1 - \frac{1}{2}\boldsymbol{\tau}_{10} + \frac{1}{2}\boldsymbol{\tau}_9, \qquad \boldsymbol{\zeta}_2 = \boldsymbol{\tau}_2 - \frac{1}{2}\boldsymbol{\tau}_{10} - \frac{1}{2}\boldsymbol{\tau}_9,$$

$$\boldsymbol{\zeta}_3 = \boldsymbol{\tau}_3 + \frac{1}{2}\boldsymbol{\tau}_9 + \frac{1}{2}\boldsymbol{\tau}_{10}, \qquad \boldsymbol{\zeta}_4 = \boldsymbol{\tau}_4 + \frac{1}{2}\boldsymbol{\tau}_{10} - \frac{1}{2}\boldsymbol{\tau}_9,$$

$$\boldsymbol{\zeta}_b = -\frac{1}{2}\boldsymbol{\tau}_5 - \frac{1}{2}\boldsymbol{\tau}_6 - \frac{1}{2}\boldsymbol{\tau}_7 - \frac{1}{2}\boldsymbol{\tau}_8,$$

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where τ_l corresponds to f_l, $l = 1, \dots, 10$ and they are given, for the face $f_l = [a_i, a_j, a_k]$, by formula (4.7) deduced in the previous section, but this time with π playing the role of λ:

$$\boldsymbol{\tau}_l = 2(\pi_i \,\mathbf{grad}\, \pi_j \times \mathbf{grad}\, \pi_k + \pi_j \,\mathbf{grad}\, \pi_k \times \mathbf{grad}\, \pi_i + \pi_k \,\mathbf{grad}\, \pi_i \times \mathbf{grad}\, \pi_j)$$

After computations we get the related vectorfields ($(x, y, z)^T \in P$):

$$\boldsymbol{\zeta}_1 = \begin{pmatrix} -\dfrac{xz}{1-z} \\ y - 2 + \dfrac{y}{1-z} \\ z \end{pmatrix}, \qquad \boldsymbol{\zeta}_2 = \begin{pmatrix} x - 2 + \dfrac{x}{1-z} \\ -\dfrac{yz}{1-z} \\ z \end{pmatrix},$$

$$\boldsymbol{\zeta}_3 = \begin{pmatrix} x + \dfrac{x}{1-z} \\ -\dfrac{yz}{1-z} \\ z \end{pmatrix}, \ \boldsymbol{\zeta}_4 = \begin{pmatrix} -\dfrac{xz}{1-z} \\ y + \dfrac{y}{1-z} \\ z \end{pmatrix}, \ \boldsymbol{\zeta}_5 = \begin{pmatrix} x \\ y \\ z - 1 \end{pmatrix}.$$

As mentioned above, the discrete 3-forms on P are just constants. So we have finally found a complete sequence of spaces of Whitney-forms on P:

$$\begin{aligned} W^0 &= \text{ span } \{\pi_1, \dots, \pi_5\}; \\ W^1 &= \text{ span } \{\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_8\}; \\ W^2 &= \text{ span } \{\boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_5\}; \\ W^3 &= \text{ span } \{1\}. \end{aligned}$$

6. Properties. In the course of the construction in the previous section we took great pains to ensure that interpolation remained local on faces of the pyramid. In addition the weights were chosen to match the two-dimensional Whitney-forms on the faces. Evidently, these two conditions make the patching condition hold for the new Whitney-forms, when used on a mesh containing pyramids, tetrahedra, and bricks.

One aspect of the exact sequence property is readily confirmed: By straightforward computations we get

$$\begin{aligned} \mathbf{grad}\, \pi_1 &= -\gamma_1 - \gamma_4 - \gamma_5; \\ \mathbf{grad}\, \pi_2 &= \gamma_1 - \gamma_2 - \gamma_6; \\ \mathbf{grad}\, \pi_3 &= \gamma_4 - \gamma_3 - \gamma_7; \\ \mathbf{grad}\, \pi_4 &= \gamma_2 + \gamma_3 - \gamma_8; \\ \mathbf{grad}\, \pi_5 &= \gamma_5 + \gamma_6 + \gamma_7 + \gamma_8; \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{\gamma}_1 &= -\boldsymbol{\zeta}_5 + \boldsymbol{\zeta}_1; \\ \mathbf{curl}\, \boldsymbol{\gamma}_2 &= -\boldsymbol{\zeta}_5 + \boldsymbol{\zeta}_3; \\ \mathbf{curl}\, \boldsymbol{\gamma}_3 &= \boldsymbol{\zeta}_5 - \boldsymbol{\zeta}_4; \\ \mathbf{curl}\, \boldsymbol{\gamma}_4 &= \boldsymbol{\zeta}_5 - \boldsymbol{\zeta}_2; \\ \mathbf{curl}\, \boldsymbol{\gamma}_5 &= -\boldsymbol{\zeta}_1 + \boldsymbol{\zeta}_2; \\ \mathbf{curl}\, \boldsymbol{\gamma}_6 &= -\boldsymbol{\zeta}_3 + \boldsymbol{\zeta}_1; \\ \mathbf{curl}\, \boldsymbol{\gamma}_7 &= \boldsymbol{\zeta}_4 - \boldsymbol{\zeta}_2; \\ \mathbf{curl}\, \boldsymbol{\gamma}_8 &= \boldsymbol{\zeta}_3 - \boldsymbol{\zeta}_4 \ . \end{aligned} \tag{6.2}$$

We remark that the weights in the above sums are clear from Stokes' theorem; they have to agree with the entries of the vertex-edge and edge-face incidence matrices for a single pyramid (cf. [9]).

To prove the second assertion of the exact sequence property, we have to rely on an auxiliary result, the so-called "commuting diagram property". We denote by $\mathcal{DF}^l$ the space of continuous l-forms and write $\mathcal{I}^l$ the (local) interpolation operator from $\mathcal{DF}^l$ onto W^l. It maps an l-form ω onto that discrete l-form that has the same integrals over l-faces of P as ω.

THEOREM 6.1 (Commuting Diagram Property). *The diagram*

$$\begin{array}{ccccccc}
\mathcal{DF}^0 & \xrightarrow{d} & \mathcal{DF}^1 & \xrightarrow{d} & \mathcal{DF}^2 & \xrightarrow{d} & \mathcal{DF}^3 \\
\mathcal{I}^0\big\downarrow & & \mathcal{I}^1\big\downarrow & & \mathcal{I}^2\big\downarrow & & \mathcal{I}^3\big\downarrow \\
W^0 & \xrightarrow{d} & W^1 & \xrightarrow{d} & W^2 & \xrightarrow{d} & W^3
\end{array}$$

commutes.

Proof. We have to show that $d(\mathcal{I}^l\varphi) = \mathcal{I}^{l+1}d\varphi$, which is equivalent to $\mathcal{I}^{l+1}d(\varphi - \mathcal{I}^l\varphi) = 0$, that is $\xi(d(\varphi - \mathcal{I}^l\varphi)) = 0$, for all degrees of freedom ξ. Hence, it is sufficient to prove that, if τ is an l–form, which makes all the degrees of freedom vanish, then $\xi(d\tau) = 0$, for all degrees of freedom. But this is obvious by Stokes' theorem. For instance, for 1–forms:

$$\int_f d\tau = \sum_{e\in f} \int_e \tau = 0,$$

where e are edges belonging to the face f. □

We remark that the commuting diagram property is a key device in the theory of mixed finite elements [12]. Also note that from theorem 6.1 we learn that all constants are contained in W^1, since all linear functions belong to W^0.

THEOREM 6.2 (Existence of discrete potentials). *One has*

$$W^1 \cap \mathit{ker}(d) = d\,W^0, \quad W^2 \cap \mathit{ker}(d) = d\,W^1.$$

Proof. Take ω in W^1 such that $d\omega = 0$, so there is a continuous 0–form φ such that $\omega = d\varphi$. Pick $a := \mathcal{I}\varphi \in W^0$ and use the commuting diagram property in order to obtain $d\,a = \mathcal{I}d\varphi = \mathcal{I}\omega = \omega$. The second assertion can be established in the same way. Until now, we referred only to the local properties, but we can follow the approach of the proof [21, Thm. 18] to conclude the global existence of the discrete potentials for contractible domains. □

When discrete differential forms are used in a finite element framework, L^2-inner products of basis functions and their exterior derivatives have to be evaluated in order to get the entries of stiffness and mass matrices and load vectors. For second order variational problems, which typically occur in electromagnetism, those are obtained through integrals of the form $\int_{\widetilde{P}} \langle \alpha(\mathbf{x})b_i, b_j\rangle \, d\mathbf{x}$ and $\int_{\widetilde{P}} \langle \alpha(\mathbf{x})db_i, db_j\rangle \, d\mathbf{x}$ for every element $\widetilde{P}$ of the finite element mesh. Here b_i stands for some nodal basis function of the global space of discrete l-forms, $l = 0, 1, 2, 3$. The coefficient function $\alpha(\mathbf{x})$ is to be bounded. First, note that by theorem 6.1 db_i agrees with a linear combination of basis functions in the space of discrete $l+1$-forms. Secondly, the pullbacks of table 2.2 take the integrals to the reference pyramid and preserve basis functions. Eventually, all we need to evaluate are integrals of the forms

$$\int_P \alpha(\mathbf{x})\pi_j\pi_k \, d\mathbf{x} \quad , \quad \int_P \left\langle A(\mathbf{x})\boldsymbol{\gamma}_j, \boldsymbol{\gamma}_k\right\rangle \, d\mathbf{x} \quad , \quad \int_P \left\langle A(\mathbf{x})\boldsymbol{\zeta}_j, \boldsymbol{\zeta}_k\right\rangle \, d\mathbf{x} \; .$$

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where $\alpha : P \to \mathbb{R}, A : P \to \mathbb{R}^{3,3}$ are bounded functions. However, as some of the basis functions on P are rational polynomial with a pole for $z = 1$, the evaluation of the integrals might run into difficulties. At second glance, this is not true, as a straightforward computation confirms that $\pi_j \in L^2(P)$, $\boldsymbol{\gamma}_j \in \boldsymbol{L}^2(P)$, and $\boldsymbol{\zeta}_j \in \boldsymbol{L}^2(P)$. It turns out that the critical monomials in z just cancel. This is illustrated by the following example:

$$\int_P \pi_5 \left\langle \boldsymbol{\gamma}_1^{(3)}, \boldsymbol{\gamma}_2^{(3)} \right\rangle d\mathbf{x} = \tag{6.3}$$
$$= \int_0^1 \frac{z}{1-z} \int_0^{1-z} y \int_0^{1-z} x^2 dx dy dz - \int_0^1 \frac{z}{(1-z)^2} \int_0^{1-z} y^2 \int_0^{1-z} x^2 dx dy dz$$
$$= \frac{1}{6} \int_0^1 z(1-z)^4 dz - \frac{1}{9} \int_0^1 z(1-z)^4 dz = \frac{1}{1620} = 0.00062$$

We point out that integrals of the form (6.3) occur whenever the coefficient functions $\alpha(\mathbf{x})$, $A(\mathbf{x})$ are replaced by their (component-wise) interpolant in the space of Whitney 0-forms. Hence, the values of elementary integrals like (6.3) may be computed in advance and stored in a table.

We start discussing the approximation properties of Whitney forms on pyramids by noting that the local spaces on the reference pyramid contain all constants. In the case of 0-forms even all affine - linear functions belong to $W^0(P)$. As a consequence, since $W^l(P) \subset L^2(P)$, $l = 0, 1, 2$, we conclude from the Bramble-Hilbert-lemma (see, e.g., [15]) and continuity properties of the interpolation operators [2, 17] that there exist constants $c_0, c_1, c_2 > 0$ such that

$$\begin{aligned} \left\|u - I^0 u\right\|_{L^2(P)} &\le c_0 \, |u|_{H^2(P)}, \quad \forall u \in H^2(P); \\ \left\|u - I^1 u\right\|_{\boldsymbol{L}^2(P)} &\le c_1 \, \left(|\mathbf{u}|_{\boldsymbol{H}^1(P)} + \|\mathbf{curl}\,\mathbf{u}\|_{\boldsymbol{H}^1(P)}\right), \forall \mathbf{u} \in \boldsymbol{H}^1(\mathbf{curl}; P); \\ \left\|u - I^2 u\right\|_{\boldsymbol{L}^2(P)} &\le c_2 \, |\mathbf{u}|_{\boldsymbol{H}^1(P)}, \quad \forall \mathbf{u} \in \boldsymbol{H}^1(P) \, . \end{aligned} \tag{6.4}$$

The next step involves classical affine equivalence techniques [15]. They are based on the assumption of *shape-regularity* of the mesh. This condition carries the customary geometric meaning that the ratio of the radii of the largest inscribed ball and smallest circumscribed ball is bounded by the same constant for all elements of the mesh. In particular, for any pyramid $\widetilde{P}$ we can find a diffeomorphism $\Phi : \widetilde{P} \to P$ such that with $h :=$ diam $\widetilde{P}$, $|\det \Phi| \le k_1 h^{-3}$, $\|D\Phi\|_{L^\infty(\widetilde{P})} \le k_2 h^{-1}$, $\left\|D\Phi^{-1}\right\|_{L^\infty(P)} \le k_3 h$ uniformly with respect to all pyramids of the mesh.

Then we use the appropriate pullback of $u/\mathbf{u}$ on both sides of the estimates (6.4). Lengthy computations, whose details are given in [26, 28], yield

$$\begin{aligned} \left\|u - \widetilde{I}^0 u\right\|_{L^2(\widetilde{P})} &\le C_0 h^2 \, |u|_{H^2(\widetilde{P})}, \quad \forall u \in H^2(\widetilde{P}); \\ \left\|u - \widetilde{I}^1 u\right\|_{\boldsymbol{L}^2(\widetilde{P})} &\le C_1 h \, \left(|\mathbf{u}|_{\boldsymbol{H}^1(\widetilde{P})} + \|\mathbf{curl}\,\mathbf{u}\|_{\boldsymbol{H}^1(\widetilde{P})}\right), \forall \mathbf{u} \in \boldsymbol{H}^1(\mathbf{curl}; \widetilde{P}); \\ \left\|u - \widetilde{I}^2 u\right\|_{\boldsymbol{L}^2(\widetilde{P})} &\le C_2 h \, |\mathbf{u}|_{\boldsymbol{H}^1(\widetilde{P})}, \quad \forall \mathbf{u} \in \boldsymbol{H}^1(\widetilde{P}) \, . \end{aligned} \tag{6.5}$$

The constants C_0, C_1, C_2 only depend on k_1, k_2, k_3 and the constants in (6.4). They are hence independent on $\widetilde{P}$, and the inequalities (6.5) can be converted into global approximation estimates on the entire mesh. In sum, the pyramidal Whitney-forms perfectly match the approximation properties of their tetrahedral and hexahedral counterparts [12, 19].

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