

FAST AND ACCURATE COMPUTATION OF THE ZEROS OF GEGENBAUER–SOBOLEV ORTHOGONAL POLYNOMIALS*

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Abstract. Gegenbauer–Sobolev polynomials are related to the classical Gegenbauer polynomials, and they satisfy orthogonality relations with respect to a Sobolev inner product. In this paper, we focus on the study of the location and computation of the zeros of Gegenbauer–Sobolev polynomials. In particular, some interlacing properties between the zeros of the Gegenbauer–Sobolev polynomials and those of the Gegenbauer polynomials are shown. Moreover, two variants of a Newton-like method, known as the Ehrlich–Aberth method, are applied to compute the zeros of the considered polynomials. The Ehrlich–Aberth method is iterative, requiring the evaluation of the polynomial and its derivative at all the approximations of the zeros. This is accomplished by exploiting a relation that links the sequence of Gegenbauer–Sobolev polynomials with the sequence of Gegenbauer polynomials. Furthermore, in order to ensure fast convergence, the proposed algorithms exploit the derived interlacing properties to construct the initial guesses of the zeros required by the Ehrlich–Aberth method. The algorithms are shown to be accurate and very fast, with a computational complexity of the order of $\mathcal{O}(n^2)$ for a Gegenbauer–Sobolev polynomial of degree n .

Key words. Sobolev orthogonal polynomials, Gegenbauer polynomials, zeros of polynomials

AMS subject classifications. 33C47, 65D32, 65F15

1. Introduction. Sobolev orthogonal polynomials have been extensively studied in the last decades with respect to their analytic properties, such as their asymptotic behavior and the location of their zeros; see, e.g., [7, 18, 19, 20, 24, 25, 26, 27, 29, 31].

Some algorithms to compute the zeros of Sobolev orthogonal polynomials have been proposed [12, 13]. The coefficients of their recurrence relation, constructed either by the modified Chebyshev or by the Stieltjes algorithm [12, 13], are embedded into an upper Hessenberg matrix of order n , whose eigenvalues are the zeros of the considered Sobolev orthogonal polynomial of degree n . This approach has two drawbacks: the computational complexity of constructing the Hessenberg matrix is $\mathcal{O}(n^3)$, and its eigenvalues are very ill-conditioned [19, 22, 32]. As a consequence, stable methods to compute the eigenvalues of the Hessenberg matrix, such as the QR algorithm [10], still yield inaccurate results [19, 22, 32]. A balancing procedure to transform the above Hessenberg matrix into a similar one with well-conditioned eigenvalues has been proposed in [19, 20, 22]. This balancing procedure yields accurately computed eigenvalues, but the overall complexity of the modified algorithm is still $\mathcal{O}(n^3)$.

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The aim of the present paper is to propose two fast and accurate $\mathcal{O}(n^2)$ -algorithms for computing all the zeros of a class of Sobolev orthogonal polynomials of degree n , known as Gegenbauer–Sobolev polynomials [24]. The proposed algorithms are based on a variant of the Ehrlich–Aberth method. This method simultaneously computes all the zeros of a polynomial, and its efficiency strongly depends on a good choice of the initial approximation of the zeros as well as on a fast and accurate way to evaluate the polynomial and its derivative at the approximations of the zeros [1, 5, 9]. In this work, good initial approximations of the zeros are obtained by exploiting the interlacing properties satisfied by the zeros of the Gegenbauer–Sobolev polynomials. Moreover, in the first algorithm, the evaluation of the polynomial and its derivative at a point exploits a recurrence relation that links the sequence of monic Gegenbauer–Sobolev polynomials and the sequence of monic Gegenbauer polynomials [24]. In the second algorithm, the first substep is replaced by the computation of the null space of a rectangular matrix. Notice that the key point is that Gegenbauer polynomials are symmetrically self-coherent according to H. G. Meijer [28].

The manuscript is organized as follows. The main features of the Gegenbauer polynomials and Gegenbauer–Sobolev polynomials are described in Section 2. The considered numerical methods are proposed in Sections 3 and 4. Numerical tests are presented in Section 5, followed by the conclusions.

2. Gegenbauer polynomials and Gegenbauer–Sobolev polynomials. In this section, we introduce the main properties of monic Gegenbauer polynomials as well as those of monic Gegenbauer–Sobolev polynomials. Furthermore, the formulas linking the sequence of monic Gegenbauer polynomials with the sequence of monic Gegenbauer–Sobolev polynomials are recalled.

Let

$$\rho^{(\alpha)}(x) = (1 - x^2)^{\alpha - \frac{1}{2}}, \quad \alpha > -1/2,$$

and let $\{\mathcal{C}_j^{(\alpha)}(x)\}_{j \geq 0}$ be the sequence of monic Gegenbauer polynomials [30], with $\mathcal{C}_j^{(\alpha)}(x)$ of degree j , orthogonal with respect to the inner product

$$(f, g) = \int_{-1}^1 f(x)g(x)\rho^{(\alpha)}(x)dx,$$

i.e., if $i \neq j$, then

$$\int_{-1}^1 \mathcal{C}_i^{(\alpha)}(x)\mathcal{C}_j^{(\alpha)}(x)\rho^{(\alpha)}(x)dx = 0.$$

Such polynomials satisfy the three-term recurrence relation

$$(2.1) \quad \begin{cases} \mathcal{C}_0^{(\alpha)}(x) = 1, \\ \mathcal{C}_1^{(\alpha)}(x) = x, \\ \mathcal{C}_{j+1}^{(\alpha)}(x) = x\mathcal{C}_j^{(\alpha)}(x) - \gamma_j^{(\alpha)}\mathcal{C}_{j-1}^{(\alpha)}(x), \end{cases} \quad j \geq 1,$$

with

$$\gamma_j^{(\alpha)} = \frac{1}{4} \frac{j}{j + \alpha} \frac{j + 2\alpha - 1}{j + \alpha - 1}.$$

Let $\lambda > 0$, and let $\{\mathcal{Q}_j^{(\alpha)}(x)\}_{j \geq 0}$ be the sequence of monic Gegenbauer–Sobolev polynomials [24], with $\mathcal{Q}_j^{(\alpha)}(x)$ of degree j , orthogonal with respect to the Sobolev inner product

$$(f, g)_S = \int_{-1}^1 f(x)g(x)\rho^{(\alpha)}(x)dx + \lambda \int_{-1}^1 f'(x)g'(x)\rho^{(\alpha)}(x)dx,$$

i.e., if $i \neq j$, then

$$\begin{aligned} &(\mathcal{Q}_i^{(\alpha)}, \mathcal{Q}_j^{(\alpha)})_S \\ &= \int_{-1}^1 \mathcal{Q}_i^{(\alpha)}(x)\mathcal{Q}_j^{(\alpha)}(x)\rho^{(\alpha)}(x)dx + \lambda \int_{-1}^1 \mathcal{Q}_i^{(\alpha)'}(x)\mathcal{Q}_j^{(\alpha)'}(x)\rho^{(\alpha)}(x)dx = 0. \end{aligned}$$

Let $x_{n,1} < x_{n,2} < \dots < x_{n,n}$ and $x_{n,1}^{(\alpha)} < x_{n,2}^{(\alpha)} < \dots < x_{n,n}^{(\alpha)}$ be the zeros of $\mathcal{Q}_n^{(\alpha)}(x)$ and $\mathcal{C}_n^{(\alpha)}(x)$, respectively. They are real and simple [24, 30]. Furthermore, for $\alpha \geq 1/2$, the $\{x_{n,j}\}_{j=1}^n$ are all contained in the interval $[-1, 1]$, and, for $-1/2 < \alpha < 1/2$, there is at most one pair of zeros, symmetric with respect to the origin, outside the interval $[-1, 1]$ [24].

The following recurrence relation between the monic Gegenbauer polynomials and the monic Gegenbauer–Sobolev polynomials holds [24]:

$$(2.2) \quad \begin{cases} \mathcal{C}_0^{(\alpha)}(x) = \mathcal{Q}_0^{(\alpha)}(x) = 1, \\ \mathcal{C}_1^{(\alpha)}(x) = \mathcal{Q}_1^{(\alpha)}(x) = x, \\ \mathcal{C}_j^{(\alpha)}(x) - \xi_{j-2}^{(\alpha)}\mathcal{C}_{j-2}^{(\alpha)}(x) = \mathcal{Q}_j^{(\alpha)}(x) - d_{j-2}(\lambda)\mathcal{Q}_{j-2}^{(\alpha)}(x), & j \geq 2, \quad \alpha > -1/2, \\ \xi_0^{(\alpha)} = 0, \\ \xi_j^{(\alpha)} = \frac{1}{4} \frac{j+2}{j+\alpha+1} \frac{j+1}{j+\alpha}, & j \geq 1, \quad \alpha > -1/2, \\ d_0(\lambda) = 0, \\ d_1(\lambda) = \frac{3}{2(\alpha+2)(\alpha+1)(1+2(\alpha+1)\lambda)}, \\ d_j(\lambda) = \frac{\xi_j^{(\alpha)}\gamma_j^{(\alpha)}\gamma_{j-1}^{(\alpha)}}{\gamma_{j-1}^{(\alpha)}(\lambda j^2 + \gamma_j^{(\alpha)}) + \xi_{j-2}^{(\alpha)}(\xi_{j-2}^{(\alpha)} - d_{j-2}(\lambda))}, & j \geq 2, \quad \alpha > -1/2. \end{cases}$$

Hence, given the sequence $\mathcal{C}_j^{(\alpha)}(x)$, $j = 0, 1, \dots, n$, by (2.1), the sequence of Gegenbauer–Sobolev polynomials $\mathcal{Q}_j^{(\alpha)}(x)$, $j = 0, 1, \dots, n$, can be retrieved from (2.2). Moreover, observe that, by (2.2), the sequence of Gegenbauer–Sobolev polynomials of even degree and that of odd degree can be computed independently.

The sequence $\{\mathcal{C}_j^{(\alpha)'}(x)\}_{j=0}^n$ is obtained by considering the derivative of (2.1),

$$(2.3) \quad \begin{cases} \mathcal{C}_0^{(\alpha)'}(x) = 0, \\ \mathcal{C}_1^{(\alpha)'}(x) = 1, \\ \mathcal{C}_{j+1}^{(\alpha)'}(x) = x\mathcal{C}_j^{(\alpha)'}(x) + \mathcal{C}_j^{(\alpha)}(x) - \gamma_j^{(\alpha)}\mathcal{C}_{j-1}^{(\alpha)'}(x), & j \geq 1. \end{cases}$$

Therefore, given $\mathcal{C}_j^{(\alpha)'}(x)$, $j = 0, 1, \dots, n$, the sequence $\mathcal{Q}_j^{(\alpha)'}(x)$, $j = 0, 1, \dots, n$, can be computed by differentiating (2.2):

$$(2.4) \quad \begin{cases} \mathcal{C}_0^{(\alpha)'}(x) = \mathcal{Q}_0^{(\alpha)'}(x) = 0, \\ \mathcal{C}_1^{(\alpha)'}(x) = \mathcal{Q}_1^{(\alpha)'}(1) = 1, \\ \mathcal{C}_j^{(\alpha)'}(x) - \xi_{j-2}^{(\alpha)} \mathcal{C}_{j-2}^{(\alpha)'}(x) = \mathcal{Q}_j^{(\alpha)'}(x) - d_{j-2}(\lambda) \mathcal{Q}_{j-2}^{(\alpha)'}(x), \quad j \geq 0. \end{cases}$$

The following interlacing property is a straightforward consequence of Rolle's theorem:

$$(2.5) \quad x_{n,1}^{(\alpha)} < x_{n-1,1}^{(\alpha+1)} < x_{n,2}^{(\alpha)} < x_{n-1,2}^{(\alpha+1)} < \dots < x_{n-1,n-2}^{(\alpha+1)} < x_{n,n-1}^{(\alpha)} < x_{n-1,n-1}^{(\alpha+1)} < x_{n,n}^{(\alpha)},$$

because [6, p. 905]

$$\mathcal{C}_n^{(\alpha)'}(x) = n \mathcal{C}_{n-1}^{(\alpha+1)}(x), \quad n \geq 1.$$

Observe that, if $\bar{x}_{n,k}$ and $\bar{w}_{n,k}$, $k = 1, \dots, n$, are, respectively, the nodes and the weights of the n -point Gauss–Lobatto quadrature rule with respect to the Gegenbauer weight $\rho^{(\alpha)}(x)$ with the additional points $\{-1, 1\}$, then [23]

$$\begin{aligned} \bar{x}_{n,1} &= -1, \\ \bar{x}_{n,k+1} &= x_{n-2,k}^{(\alpha+1)}, \quad k = 1, \dots, n-2, \\ \bar{x}_{n,n} &= 1, \\ \bar{w}_{n,k} &> 0, \quad k = 1, \dots, n. \end{aligned}$$

The interlacing properties satisfied by the zeros of $\mathcal{Q}_n^{(\alpha)}(x)$, for $\alpha = 1/2$, have been studied in [19]. The following theorem holds:

THEOREM 2.1. *Let $\alpha \geq 1/2$ and $n \in \mathbb{N}$, $n \geq 2$, n even. Then*

$$(2.6) \quad \begin{aligned} -1 &= \bar{x}_{n+1,1} \\ &= \bar{x}_{n,1} < x_{n,1} < x_{n,1}^{(\alpha)} < \bar{x}_{n+1,2} < \bar{x}_{n,2} < x_{n,2} < x_{n,2}^{(\alpha)} < \dots \\ &< \bar{x}_{n+1, \frac{n}{2}} < \bar{x}_{n, \frac{n}{2}} < x_{n, \frac{n}{2}}^{(\alpha)} < x_{n, \frac{n}{2}}, \\ \bar{x}_{n+1, \frac{n}{2}+1} &= 0, \end{aligned}$$

$$(2.7) \quad \begin{aligned} x_{n,m}^{(\alpha)} &< x_{n,m} < \bar{x}_{n,m} < \bar{x}_{n+1,m+1} < \dots \\ &< x_{n,n-1}^{(\alpha)} < x_{n,n-1} < \bar{x}_{n,n-1} < \bar{x}_{n+1,n} < x_{n,n}^{(\alpha)} < x_{n,n} \\ &< \bar{x}_{n,n} = \bar{x}_{n+1,n+1} = 1, \end{aligned}$$

with $m = \frac{n}{2} + 1$.

Proof. The case $\alpha = \frac{1}{2}$ was proved in [19]. Let us prove (2.6) for $\alpha > \frac{1}{2}$. Property (2.7) can be proved by symmetry. Note that the inequalities

$$-1 < x_{n,1} < x_{n,1}^{(\alpha)} < x_{n,2} < x_{n,2}^{(\alpha)} < \dots < x_{n, \frac{n}{2}} < x_{n, \frac{n}{2}}^{(\alpha)}$$

are proved in [24].

Let us consider the inequalities

$$\begin{aligned} -1 < x_{n,1}^{(\alpha)} < \bar{x}_{n+1,2} < \bar{x}_{n,2} < x_{n,2}^{(\alpha)} < \cdots < \bar{x}_{n+1,\frac{n}{2}} < \bar{x}_{n,\frac{n}{2}} < x_{n,\frac{n}{2}}^{(\alpha)}, \\ & \updownarrow \\ -1 < x_{n,1}^{(\alpha)} < x_{n-1,1}^{(\alpha+1)} < x_{n-2,1}^{(\alpha+1)} < x_{n,2}^{(\alpha)} < \cdots < x_{n-1,\frac{n}{2}-1}^{(\alpha+1)} < x_{n-2,\frac{n}{2}-1}^{(\alpha+1)} < x_{n,\frac{n}{2}}^{(\alpha)}. \end{aligned}$$

The following was shown in [8, Theorem 1]:

$$x_{n-2,j}^{(\alpha+1)} < x_{n,j+1}^{(\alpha)}, \quad j = 1, \dots, \frac{n}{2} - 1.$$

The inequalities

$$x_{n-1,j}^{(\alpha+1)} < x_{n-2,j}^{(\alpha+1)}, \quad j = 1, \dots, \frac{n}{2} - 1,$$

hold since $x_{n-1,j}^{(\alpha+1)}$ and $x_{n-2,j}^{(\alpha+1)}$ are, respectively, zeros of the Gegenbauer polynomials $C_{n-1}^{(\alpha+1)}(x)$ and $C_{n-2}^{(\alpha+1)}(x)$ and these zeros satisfy the interlacing property [30]. The inequalities

$$x_{n,1}^{(\alpha)} < x_{n-1,1}^{(\alpha+1)}, \quad j = 1, \dots, \frac{n}{2} - 1,$$

follow from (2.5).

Let us prove that

$$\bar{x}_{n,j} = x_{n-2,j-1}^{(\alpha+1)} < x_{n,j}, \quad j = 2, \dots, \frac{n}{2}.$$

For this purpose, we define the polynomials of degree $n - 4$

$$\omega_i(x) = \prod_{\substack{j=1 \\ j \neq i, n-i-1}}^{n-2} (x - x_{n-2,j}^{(\alpha+1)}), \quad i = 1, \dots, n - 2.$$

By [24, Lemma 5.2 and Corollary 5.3], there exist a unique polynomial $\hat{\omega}_i(x)$ of degree $n - 4$ and a unique constant $\hat{c}(\omega_i) > 0$ such that

$$\mathcal{F}\hat{\omega}_i(x) = (1 - x^2)\omega_i(x) + (1 - x^2)^2\hat{c}(\omega_i)$$

and

$$\begin{aligned} (2.8) \quad 0 &= (Q_n^{(\alpha)}, \hat{\omega}_i)_S \\ &= \int_{-1}^1 \omega_i(x) Q_n^{(\alpha)}(x) \rho^{(\alpha)}(x) dx + \hat{c}(\omega_i) \int_{-1}^1 (1 - x^2)^2 Q_n^{(\alpha)}(x) \rho^{(\alpha)}(x) dx, \end{aligned}$$

with

$$(2.9) \quad \int_{-1}^1 (1 - x^2)^2 Q_n^{(\alpha)}(x) \rho^{(\alpha)}(x) dx > 0,$$

and $\mathcal{F}$ is a linear operator defined on the linear space of real polynomials

$$\mathcal{F} = (1 - x^2)\mathcal{I} + \lambda(2\alpha - 1)\mathcal{D} - \lambda(1 - x^2)^2\mathcal{D}^2,$$

where $\mathcal{I}$ and $\mathcal{D}$ are the identity and derivative operators, respectively.

Hence, by (2.8) and (2.9),

$$(2.10) \quad \int_{-1}^1 \omega_i(x) Q_n^{(\alpha)}(x) \rho^{(\alpha)}(x) dx < 0.$$

Applying the n -point Gauss–Lobatto–Gegenbauer quadrature rule with weights $\bar{w}_{n,j} > 0$, $j = 1, \dots, n$, and nodes

$$\begin{aligned} \bar{x}_{n,1} &= -1, \\ \bar{x}_{n,j+1} &= x_{n-2,j}^{(\alpha+1)}, \quad j = 1, \dots, n-2, \\ \bar{x}_{n,n} &= 1, \end{aligned}$$

to (2.10), we obtain

$$\begin{aligned} \int_{-1}^1 \omega_i(x) Q_n^{(\alpha)}(x) \rho^{(\alpha)}(x) dx &= \sum_{j=1}^n \bar{w}_{n,j} \omega_i(\bar{x}_{n,j}) Q_n^{(\alpha)}(\bar{x}_{n,j}) \\ &= \bar{w}_{n,1} \omega_i(\bar{x}_{n,1}) Q_n^{(\alpha)}(\bar{x}_{n,1}) + \bar{w}_{n,n} \omega_i(\bar{x}_{n,n}) Q_n^{(\alpha)}(\bar{x}_{n,n}) \\ &\quad + \sum_{j=1}^{n-2} \bar{w}_{n,j+1} \omega_i(\bar{x}_{n,j+1}) Q_n^{(\alpha)}(\bar{x}_{n,j+1}) \\ &= \bar{w}_{n,1} \omega_i(-1) Q_n^{(\alpha)}(-1) + \bar{w}_{n,n} \omega_i(1) Q_n^{(\alpha)}(1) \\ &\quad + \sum_{j=1}^{n-2} \bar{w}_{n,j+1} \omega_i(x_{n-2,j}^{(\alpha+1)}) Q_n^{(\alpha)}(x_{n-2,j}^{(\alpha+1)}) \\ &= \bar{w}_{n,1} \omega_i(-1) Q_n^{(\alpha)}(-1) + \bar{w}_{n,n} \omega_i(1) Q_n^{(\alpha)}(1) \\ &\quad + \bar{w}_{n,i+1} \omega_i(x_{n-2,i}^{(\alpha+1)}) Q_n^{(\alpha)}(x_{n-2,i}^{(\alpha+1)}) \\ &\quad + \bar{w}_{n,n-i} \omega_i(x_{n-2,n-i-1}^{(\alpha+1)}) Q_n^{(\alpha)}(x_{n-2,n-i-1}^{(\alpha+1)}). \end{aligned}$$

Observe that $\omega_i(-1) = \omega_i(1) > 0$ and $Q_n^{(\alpha)}(-1) = Q_n^{(\alpha)}(1) > 0$. Thus,

$$\bar{w}_{n,1} \omega_i(-1) Q_n^{(\alpha)}(-1) = \bar{w}_{n,n} \omega_i(1) Q_n^{(\alpha)}(1) > 0$$

since ω_i and $Q_n^{(\alpha)}$ are both monic polynomials of even degree with all their zeros in $(-1, 1)$. Therefore,

$$\omega_i(x_{n-2,i}^{(\alpha+1)}) Q_n^{(\alpha)}(x_{n-2,i}^{(\alpha+1)}) = \omega_i(x_{n-2,n-i-1}^{(\alpha+1)}) Q_n^{(\alpha)}(x_{n-2,n-i-1}^{(\alpha+1)}) < 0.$$

Since $\text{sign}(\omega_i(x_{n-2,i}^{(\alpha+1)})) = (-1)^{i-1}$, we conclude that $\text{sign}(Q_n^{(\alpha)}(x_{n-2,i}^{(\alpha+1)})) = (-1)^i$. $\square$

REMARK 2.2. For $-\frac{1}{2} \leq \alpha < \frac{1}{2}$, as proved in [24], only the following inequalities hold:

$$\begin{aligned} x_{n,1} &< x_{n,1}^{(\alpha)} < x_{n,2} < x_{n,2}^{(\alpha)} < \dots < x_{n,\frac{n}{2}}^{(\alpha)} < x_{n,\frac{n}{2}}, \\ x_{n,m}^{(\alpha)} &< x_{n,m} < \dots < x_{n,n-1}^{(\alpha)} < x_{n,n-1} < x_{n,n}^{(\alpha)} < x_{n,n}. \end{aligned}$$

REMARK 2.3. Extensive numerical tests show that the results of Theorem 2.1 also hold when n is odd, although no analytical proof is available.

3. The Ehrlich–Aberth method. The Ehrlich–Aberth method simultaneously computes all the zeros of a polynomial, starting from initial approximations of them. It converges cubically, or even faster if all the zeros are simple [5].

Starting from a vector $\mathbf{z}^{(0)} = [z_{n,1}^{(0)}, \dots, z_{n,n}^{(0)}]^T$, which consists of initial approximations of the zeros of $\mathcal{Q}_n^{(\alpha)}(x)$, at the $(i+1)$ st iteration, $i = 0, 1, \dots$, these approximations are updated as follows:

$$(3.1) \quad z_{n,j}^{(i+1)} = z_{n,j}^{(i)} - \frac{\frac{\mathcal{Q}_n^{(\alpha)}(z_{n,j}^{(i)})}{\mathcal{Q}_n^{(\alpha)'}(z_{n,j}^{(i)})}}{1 - \frac{\mathcal{Q}_n^{(\alpha)}(z_{n,j}^{(i)})}{\mathcal{Q}_n^{(\alpha)'}(z_{n,j}^{(i)})} \sum_{\substack{k=1 \\ k \neq j}}^n \frac{1}{z_{n,j}^{(i)} - z_{n,k}^{(i)}}}, \quad \begin{cases} j = 1, \dots, n, \\ i = 0, 1, \dots \end{cases}$$

Similar to Newton's method, at each step it is required to compute the polynomial and its derivative at the current approximations of the zeros. At the $(i+1)$ st iteration, $i \geq 0$, the computation of $\mathcal{Q}_n^{(\alpha)}(z_{n,j}^{(i)})$ and $\mathcal{Q}_n^{(\alpha)'}(z_{n,j}^{(i)})$, $j = 1, \dots, n$, can be performed simultaneously on parallel computers.

Given, respectively, $\mathcal{C}_j^{(\alpha)}(x)$ and $\mathcal{C}_j^{(\alpha)'}(x)$, $j = 0, 1, \dots, n$, by (2.1) and (2.3), the polynomials $\mathcal{Q}_n^{(\alpha)}(x)$ and $\mathcal{Q}_n^{(\alpha)'}(x)$ can be recursively computed by (2.2) and (2.4) in an accurate manner [21] with $\mathcal{O}(n)$ floating-point operations. Therefore, $\mathcal{O}(n^2)$ floating-point operations are required to update all the roots of $\mathcal{Q}_n^{(\alpha)}(x)$.

A good set of initial approximations $\mathbf{z}^{(0)}$ of the zeros of $\mathcal{Q}_n^{(\alpha)}(x)$ is needed to ensure fast convergence of the Ehrlich–Aberth method. From the interlacing properties reported in Theorem 2.1, the following initial approximations of the zeros of $\mathcal{Q}_n^{(\alpha)}(x)$ can be considered:

$$(3.2) \quad z_{n,j}^{(0)} = \frac{x_{n,j}^{(\alpha)} + \bar{x}_{n,j}}{2}, \quad \Leftrightarrow \quad \begin{cases} z_{n,1}^{(0)} = \frac{x_{n,1}^{(\alpha)} - 1}{2}, \\ z_{n,j}^{(0)} = \frac{x_{n,j}^{(\alpha)} + x_{n-2,j-1}^{(\alpha+1)}}{2}, \quad j = 2, \dots, n-1, \\ z_{n,n}^{(0)} = \frac{x_{n,n}^{(\alpha)} + 1}{2}. \end{cases}$$

for $j = 1, \dots, n$,

Although the interlacing properties of Theorem 2.1 do not hold in general for $-\frac{1}{2} < \alpha < \frac{1}{2}$, the Ehrlich–Aberth method with the proposed initial guesses (3.2) shows similar behavior in this case as well.

To compute the initial approximations $z_{n,j}^{(0)}$, $j = 1, \dots, n$, in (3.2), the zeros of the Gegenbauer polynomials $\mathcal{C}_n^{(\alpha)}$ and $\mathcal{C}_{n-2}^{(\alpha)}$ are needed. They are computed as the eigenvalues of symmetric tridiagonal matrices of orders n and $n-2$, respectively, requiring a computational complexity of $\mathcal{O}(n^2)$ [16]. Alternative and faster algorithms, based on asymptotic formulas, can also be considered for computing the initial approximations [14, 15]. As reported in Section 5, on average, fewer than 4 iterations are required by this method to converge. Therefore, the overall computational complexity is of order $\mathcal{O}(n^2)$.

4. Numerical algorithms. Evaluating $\mathcal{C}_n^{(\alpha)}(x)$ and $\mathcal{C}_n^{(\alpha)'}(x)$, respectively, by the forward recurrence approaches (2.1) and (2.3) is numerically stable for $|x| \leq 1$ [11], requiring $7n$ floating-point operations. Moreover, the evaluation of $\mathcal{Q}_n^{(\alpha)}(x)$ and $\mathcal{Q}_n^{(\alpha)'}(x)$ by (2.2) and (2.4), respectively, is backward stable, requiring $8n$ floating-point operations. Therefore, each

iteration of the Ehrlich–Aberth method, denoted in the sequel by EA_1 , has computational complexity $\mathcal{O}(n^2)$.

Here, we also propose another algorithm to evaluate $Q_n^{(\alpha)}(x)$ and $Q_n^{(\alpha)'}(x)$ in a backward-stable fashion. The $(i + 1)$ st iteration of the Ehrlich–Aberth method, $i \geq 1$, can be divided into two steps:

The first step consists of computing $C_n^{(\alpha)}(x)$ and $C_n^{(\alpha)'}(x)$. Let

$$C_n(x) = \begin{bmatrix} -x & 1 & & & \\ \gamma_1^{(\alpha)} & -x & 1 & & \\ & \gamma_2^{(\alpha)} & \ddots & \ddots & \\ & & \ddots & -x & 1 \\ & & & \gamma_{n-1}^{(\alpha)} & -x & 1 \end{bmatrix} \in \mathbb{R}^{n \times (n+1)}, \quad \mathbf{c}_n(x) = \begin{bmatrix} C_0^{(\alpha)}(x) \\ C_1^{(\alpha)}(x) \\ C_2^{(\alpha)}(x) \\ \vdots \\ C_n^{(\alpha)}(x) \end{bmatrix} \in \mathbb{R}^{n+1}.$$

Then, (2.1) can be written in matrix form as

$$(4.1) \quad C_n(x)\mathbf{c}_n(x) = \mathbf{0}_{n+1},$$

i.e., the vector $\mathbf{c}_n(x)$ spans the right null space of $C_n(x)$. This can be computed, as described below, by applying n Givens rotations to the right of $C_n(x)$ in order to make it lower triangular. In order to simplify the notation, we denote $L_n^{(0)} = C_n(x)$. Then, we construct the matrices

$$L_n^{(i)} \equiv L_n^{(i-1)}G_i^T, \quad i = 1, \dots, n,$$

where G_i are the Givens rotations

$$G_i = \begin{bmatrix} I_{i-1} & & & \\ & c_i & s_i & \\ & -s_i & c_i & \\ & & & I_{n-i} \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)},$$

with c_i and s_i such that

$$\begin{bmatrix} c_i & s_i \\ -s_i & c_i \end{bmatrix} \begin{bmatrix} L_n^{(i-1)}(i, i) \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{1 + L_n^{(i-1)^2}(i, i)} \\ 0 \end{bmatrix}.$$

It turns out that the matrix $L_n^{(n)} \in \mathbb{R}^{n \times (n+1)}$ is lower triangular. Furthermore,

$$\mathbf{g}_n = G_1^T G_2^T \cdots G_n^T \mathbf{e}_{n+1} = \begin{bmatrix} (-1)^{n-1} \prod_{j=1}^n s_j \\ (-1)^{n-2} c_1 \prod_{j=2}^n s_j \\ \vdots \\ c_{n-2} s_{n-1} s_n \\ -c_{n-1} s_n \\ c_n \end{bmatrix}$$

is a multiple of $\mathbf{c}_n(x)$, as can be seen by dividing $\mathbf{g}_n$ by its first component. Its computation requires $12n$ floating-point operations.

Taking into account that $C_0^{(\alpha)'}(x) = 0$, the vector

$$\hat{\mathbf{c}}'_n(x) = [C_1^{(\alpha)'}(x), C_2^{(\alpha)'}(x), \dots, C_n^{(\alpha)'}(x)] \in \mathbb{R}^n$$

is a solution of the lower-triangular banded linear system

$$\begin{bmatrix} 1 & & & & \\ -x & 1 & & & \\ \gamma_2^{(\alpha)} & \ddots & \ddots & & \\ & \ddots & -x & 1 & \\ & & \gamma_{n-1}^{(\alpha)} & -x & 1 \end{bmatrix} \begin{bmatrix} \mathcal{C}_1^{(\alpha)'}(x) \\ \mathcal{C}_2^{(\alpha)'}(x) \\ \vdots \\ \mathcal{C}_{n-1}^{(\alpha)'}(x) \\ \mathcal{C}_n^{(\alpha)'}(x) \end{bmatrix} = \begin{bmatrix} \mathcal{C}_0^{(\alpha)}(x) \\ \mathcal{C}_1^{(\alpha)}(x) \\ \vdots \\ \mathcal{C}_{n-2}^{(\alpha)}(x) \\ \mathcal{C}_{n-1}^{(\alpha)}(x) \end{bmatrix},$$

which can be seen by differentiating (4.1). Therefore, $\tilde{\mathbf{c}}'_n(x)$ can be obtained by forward substitution with $4n$ floating-point operations and $\mathcal{O}(n)$ memory.

In the second step, the linear systems (2.2) and (2.4) are solved. Without loss of generality, let us assume that n is even. Furthermore, for the sake of brevity, we only describe how to solve (2.2). The solution of (2.4) can be computed in a similar way.

Let us consider

$$M_n(\lambda) = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ -d_0(\lambda) & 0 & 1 & & \\ & -d_1(\lambda) & 0 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & -d_{n-2}(\lambda) & 0 & 1 \end{bmatrix},$$

$$N_n(\lambda) = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ -\xi_0(\lambda) & 0 & 1 & & \\ & -\xi_1(\lambda) & 0 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & -\xi_{n-2}(\lambda) & 0 & 1 \end{bmatrix},$$

$$\mathbf{q}_n(x) = [\mathcal{Q}_0^{(\alpha)}(x), \mathcal{Q}_1^{(\alpha)}(x), \mathcal{Q}_2^{(\alpha)}(x), \dots, \mathcal{Q}_n^{(\alpha)}(x)]^T \in \mathbb{R}^{n+1}.$$

Then, we can write (2.2) in matrix form as

$$M_n(\lambda)\mathbf{q}_n(x) = N_n(\lambda)\mathbf{c}_n(x).$$

Let P_{n+1} be the permutation matrix of order $n+1$, obtained from the identity matrix I_{n+1} of order $n+1$ by taking the odd columns first and then the even ones, i.e., in `Matlab` notation,

$$P_{n+1} = I_{n+1}(:, [1:2:n+1, 2:2:n+1]).$$

Then,

$$P_{n+1}^T M_n(\lambda) P_{n+1} = \begin{bmatrix} M_n^{(e)}(\lambda) & \\ & M_n^{(o)}(\lambda) \end{bmatrix}, \quad P_{n+1}^T N_n(\lambda) P_{n+1} = \begin{bmatrix} N_n^{(e)}(\lambda) & \\ & N_n^{(o)}(\lambda) \end{bmatrix},$$

where $M_n^{(e)}(\lambda)$, $N_n^{(e)}(\lambda)$, $M_n^{(o)}$, and $N_n^{(o)}$ are the following bidiagonal matrices:

$$M_n^{(e)}(\lambda) = \begin{bmatrix} 1 & & & & \\ -d_0(\lambda) & 1 & & & \\ & -d_2(\lambda) & \ddots & & \\ & & \ddots & 1 & \\ & & & -d_n(\lambda) & 1 \end{bmatrix},$$

$$\begin{aligned}
 M_n^{(o)}(\lambda) &= \begin{bmatrix} 1 & & & & \\ -d_1(\lambda) & 1 & & & \\ & -d_3(\lambda) & \ddots & & \\ & & \ddots & 1 & \\ & & & -d_{n-1}(\lambda) & 1 \end{bmatrix}, \\
 N_n^{(e)}(\lambda) &= \begin{bmatrix} 1 & & & & \\ -\xi_0(\lambda) & 1 & & & \\ & -\xi_2(\lambda) & \ddots & & \\ & & \ddots & 1 & \\ & & & -\xi_n(\lambda) & 1 \end{bmatrix}, \\
 N_n^{(o)}(\lambda) &= \begin{bmatrix} 1 & & & & \\ -\xi_1(\lambda) & 1 & & & \\ & -\xi_3(\lambda) & \ddots & & \\ & & \ddots & 1 & \\ & & & -\xi_{n-1}(\lambda) & 1 \end{bmatrix}.
 \end{aligned}$$

Moreover,

$$P_{n+1}^T \mathbf{q}_n(x) = \begin{bmatrix} \mathbf{q}_n^{(e)}(x) \\ \mathbf{q}_n^{(o)}(x) \end{bmatrix}, \quad P_{n+1}^T \mathbf{c}_n(x) = \begin{bmatrix} \mathbf{c}_n^{(e)}(x) \\ \mathbf{c}_n^{(o)}(x) \end{bmatrix},$$

where

$$\begin{aligned}
 \mathbf{q}_n^{(e)}(x) &= \begin{bmatrix} \mathcal{Q}_0^{(\alpha)}(x) \\ \mathcal{Q}_2^{(\alpha)}(x) \\ \vdots \\ \mathcal{Q}_{n-2}^{(\alpha)}(x) \\ \mathcal{Q}_n^{(\alpha)}(x) \end{bmatrix}, & \mathbf{q}_n^{(o)}(x) &= \begin{bmatrix} \mathcal{Q}_1^{(\alpha)}(x) \\ \mathcal{Q}_3^{(\alpha)}(x) \\ \vdots \\ \mathcal{Q}_{n-3}^{(\alpha)}(x) \\ \mathcal{Q}_{n-1}^{(\alpha)}(x) \end{bmatrix}, \\
 \mathbf{c}_n^{(e)}(x) &= \begin{bmatrix} \mathcal{C}_0^{(\alpha)}(x) \\ \mathcal{C}_2^{(\alpha)}(x) \\ \vdots \\ \mathcal{C}_{n-2}^{(\alpha)}(x) \\ \mathcal{C}_n^{(\alpha)}(x) \end{bmatrix}, & \mathbf{c}_n^{(o)}(x) &= \begin{bmatrix} \mathcal{C}_1^{(\alpha)}(x) \\ \mathcal{C}_3^{(\alpha)}(x) \\ \vdots \\ \mathcal{C}_{n-3}^{(\alpha)}(x) \\ \mathcal{C}_{n-1}^{(\alpha)}(x) \end{bmatrix}.
 \end{aligned}$$

Therefore, the linear system (2.2) is decoupled into two lower-bidiagonal linear systems of half the size,¹

$$(4.2) \quad \begin{cases} M_n^{(e)}(\lambda) \mathbf{q}_n^{(e)}(x) = N_n^{(e)}(\lambda) \mathbf{c}_n^{(e)}(x), \\ M_n^{(o)}(\lambda) \mathbf{q}_n^{(o)}(x) = N_n^{(o)}(\lambda) \mathbf{c}_n^{(o)}(x), \end{cases}$$

that can be solved simultaneously in parallel. The computation of the right-hand sides of (4.2), i.e., $N_n^{(e)}(\lambda) \mathbf{c}_n^{(e)}(x)$ and $N_n^{(o)}(\lambda) \mathbf{c}_n^{(o)}(x)$, requires $2n$ floating-point operations. The two

¹The two linear subsystems are both of order $(n+1)/2$ if n is odd, whereas they are of order $(n+1)/2$ and $n/2$, respectively, if n is even.

lower-bidiagonal linear systems in (4.2) can be solved by forward substitution in a backward stable manner [17], requiring $2n$ floating-point operations. Hence, one step of the Ehrlich–Aberth method, using this implementation and denoted in the sequel by EA_2 , has computational complexity $\mathcal{O}(n^2)$.

We point out here that the backward stability of the substeps of an algorithm does not imply its overall stability. Therefore, we will try to assess the accuracy of the computed result in Section 5 by comparing it with a “nearly exact” result, computed in extended precision.

4.1. Stopping criteria. In [3, 4], an implementation of the Ehrlich–Aberth method is described for computing all the zeros of polynomials expressed as $p_n(x) = \sum_{i=1}^n a_i x^i$, $a_i \in \mathbb{C}$. The evaluation of the polynomial at a point is performed by means of Horner’s scheme, and the adopted stopping criterion relies on the Newton step iteration, yielding a forward error bound.

The aforementioned stopping criterion cannot be adopted for the implementation of the Ehrlich–Aberth method proposed in this work since it relies on recurrence relations. Therefore, we consider the following stopping criteria: an element $z_{n,j}^{(i)}$, $j \in \{1, \dots, n\}$, $i \geq 1$, of the sequence generated by (3.1) is considered to be a zero $x_{n,j}$ of $Q_n^{(\alpha)}(x)$ if either

$$\Delta z_{n,j} = |z_{n,j}^{(i)} - z_{n,j}^{(i-1)}| < \text{tol} \quad \text{or} \quad |\mathcal{Q}_n^{(\alpha)}(z_{n,j}^{(i)}) / \mathcal{Q}_n^{(\alpha)'}(z_{n,j}^{(i)})| < \text{tol},$$

with $\text{tol} = 2n\varepsilon$, where ε is the machine precision and n is the degree of the polynomial. The first inequality provides a bound for the forward error of the current iteration of the Ehrlich–Aberth method, and the second one provides a bound for the forward error of the Newton step.

5. Numerical examples. In this section we report some results obtained when computing the zeros of Gegenbauer–Sobolev polynomials of degrees $n = 100, 200$, and 300 by using the proposed methods for different values of α and λ . All computations were performed in Matlab R2022a with machine precision $\varepsilon \approx 1.11\text{e-}16$.²

For all the considered polynomials, we assume that the exact zeros of $Q_n^{(\alpha)}(x)$ are the values obtained by implementing algorithm AE_1 using the multiple-precision package *Advanpix* Multiprecision Computing Toolbox for Matlab [2] with 200 digits.

In Tables 5.1, 5.2, and 5.3, the results of algorithm AE_1 are reported for computing the zeros of $Q_n^{(\alpha)}(x)$, for

$$\begin{aligned} n &= 100, 200, 300 \\ \alpha &= -0.4, -0.2, 0, 0.2, 0.4, 0.6, \quad \text{and} \\ \lambda &= 10^{-6}, 10^{-4}, 10^{-2}, 10^0, 10^2, 10^4. \end{aligned}$$

In particular, for each considered α and λ , five values are displayed:

- first row: maximum of the absolute errors of the computed zeros;
- second row: maximum of the absolute values of $Q_n^{(\alpha)}(x)$ at the computed zeros;
- third row: maximum of the absolute values of the Newton ratios $Q_n^{(\alpha)}(x) / Q_n^{(\alpha)'}(x)$ at the computed zeros;
- fourth row: maximum of the absolute values of $\Delta z_{n,j}$, $j = 1, \dots, n$;
- fifth row: average number of iterations for computing all the zeros.

²A Matlab implementation is available at
<https://etna.ricam.oeaw.ac.at/volumes/2021-2030/vol165/addition/p517.php>.

For the sake of brevity, the results obtained by algorithm AE_2 are omitted since they are very similar to those obtained by algorithm AE_1 . It can be observed that, in all the considered cases, the absolute errors of the computed zeros are of the order of machine precision. Moreover, the values of $\mathcal{Q}_n^{(\alpha)}(x)$ at the computed zeros are very small.

Regarding the maximum of the absolute values of the Newton ratios $\mathcal{Q}_n^{(\alpha)}(x)/\mathcal{Q}_n^{(\alpha)'}(x)$ and the maximum of the absolute values of $\Delta z_{n,j}$, $j = 1, \dots, n$, these values are almost equal since the Newton ratios in the neighborhood of a zero are very small.

Finally, the average number of iterations required to compute each zero is always less than 4 and is almost independent of α and λ , thereby exhibiting a computational complexity of order $\mathcal{O}(n^2)$.

TABLE 5.1
Results obtained for computing the the zeros of $\mathcal{Q}_{100}^{(\alpha)}(x)$ by algorithm AE_1 for different values of α and λ .

$n = 100, \text{tol} \approx 2.22\text{e-}14$						
$\lambda \backslash \alpha$	-0.4	-0.2	0.0	0.2	0.4	0.6
10^{-6}	1.11e-16	1.11e-16	1.11e-16	2.22e-16	2.22e-16	1.11e-16
	9.15e-42	7.26e-42	9.07e-42	4.92e-42	4.93e-42	4.93e-42
	2.15e-14	2.03e-14	2.21e-14	1.18e-14	1.18e-14	1.18e-14
	2.15e-14	2.03e-14	2.22e-14	1.18e-14	1.18e-14	1.18e-14
	3.76	3.72	3.88	3.89	3.89	3.89
10^{-4}	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	1.26e-41	5.62e-42	7.55e-42	3.95e-42	3.95e-42	3.95e-42
	1.55e-14	1.78e-14	2.10e-14	1.09e-14	1.09e-14	1.09e-14
	1.55e-14	1.78e-14	2.10e-14	1.09e-14	1.09e-14	1.09e-14
	3.76	3.71	3.86	3.87	3.87	3.87
10^{-2}	1.11e-16	1.11e-16	3.46e-18	1.11e-16	1.11e-16	2.77e-17
	2.58e-42	4.34e-42	6.30e-42	3.16e-42	3.17e-42	3.17e-42
	1.11e-14	1.55e-14	2.00e-14	1.00e-14	1.00e-14	1.00e-14
	1.11e-14	1.55e-14	2.00e-14	1.00e-14	1.00e-14	1.00e-14
	3.77	3.71	3.86	3.87	3.87	3.87
10^0	2.22e-16	1.11e-16	1.11e-16	1.11e-16	2.22e-16	1.11e-16
	1.95e-41	1.22e-41	2.26e-41	2.26e-41	2.28e-41	2.26e-41
	1.50e-14	1.35e-14	1.98e-14	2.15e-14	2.15e-14	2.15e-14
	1.50e-14	1.36e-14	1.97e-14	2.15e-14	2.15e-14	2.15e-14
	3.75	3.70	3.85	3.85	3.85	3.85
10^2	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	1.44e-41	2.58e-42	4.39e-42	4.96e-42	4.97e-42	4.97e-42
	1.20e-14	1.18e-14	1.81e-14	2.04e-14	2.04e-14	2.04e-14
	1.20e-14	1.18e-14	1.80e-14	2.04e-14	2.04e-14	2.04e-14
	3.76	3.69	3.84	3.84	3.84	3.84
10^4	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	4.55e-41	4.35e-42	3.66e-42	4.13e-42	4.13e-42	4.13e-42
	1.91e-14	2.07e-14	1.71e-14	1.93e-14	1.93e-14	1.93e-14
	1.90e-14	2.07e-14	1.71e-14	1.93e-14	1.93e-14	1.93e-14
	3.75	3.67	3.84	3.84	3.84	3.84

TABLE 5.2
Results obtained for computing the the zeros of $Q_{200}^{(\alpha)}(x)$ by algorithm $\mathbb{AE}_1$ for different values of α and λ .

$n = 200, \text{tol} \approx 4.44\text{e-}14$						
$\lambda \backslash \alpha$	-0.4	-0.2	0.0	0.2	0.4	0.6
10^{-6}	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	2.50e-71	2.58e-71	2.70e-71	2.76e-71	2.76e-71	2.76e-71
	4.16e-14	4.20e-14	4.18e-14	4.27e-14	4.28e-14	4.27e-14
	4.16e-14	4.20e-14	4.18e-14	4.27e-14	4.27e-14	4.27e-14
	3.55	3.80	3.85	3.85	3.85	3.85
10^{-4}	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	2.19e-71	2.18e-71	2.28e-71	2.33e-71	2.33e-71	2.33e-71
	4.24e-14	4.04e-14	4.02e-14	4.12e-14	4.12e-14	4.12e-14
	4.24e-14	4.04e-14	4.03e-14	4.11e-14	4.11e-14	4.12e-14
	3.54	3.80	3.84	3.84	3.84	3.84
10^{-2}	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	4.73e-71	1.85e-71	1.93e-71	1.98e-71	1.98e-71	1.98e-71
	4.43e-14	3.88e-14	3.88e-14	3.97e-14	3.97e-14	3.97e-14
	4.42e-14	3.88e-14	3.88e-14	3.98e-14	3.98e-14	3.97e-14
	3.54	3.80	3.84	3.84	3.84	3.84
10^0	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	2.22e-16
	9.45e-71	4.22e-71	3.99e-71	3.97e-71	3.97e-71	3.96e-71
	4.25e-14	4.26e-14	3.74e-14	3.83e-14	3.83e-14	3.83e-14
	4.25e-14	4.26e-14	3.74e-14	3.83e-14	3.83e-14	3.83e-14
	3.55	3.78	3.83	3.83	3.83	3.83
10^2	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	4.69e-71	1.49e-71	1.69e-71	1.41e-71	1.41e-71	1.41e-71
	4.41e-14	4.08e-14	4.44e-14	3.69e-14	3.69e-14	3.70e-14
	4.41e-14	4.08e-14	4.44e-14	3.69e-14	3.69e-14	3.70e-14
	3.55	3.78	3.82	3.83	3.83	3.83
10^4	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	8.09e-71	1.25e-71	1.42e-71	1.45e-71	1.45e-71	1.45e-71
	3.90e-14	3.91e-14	4.26e-14	4.35e-14	4.35e-14	4.35e-14
	3.90e-14	3.90e-14	4.26e-14	4.35e-14	4.35e-14	4.35e-14
	3.56	3.78	3.82	3.82	3.82	3.82

6. Conclusions. Two variants of the Ehrlich–Aberth method are considered for computing all the zeros of a Gegenbauer–Sobolev polynomial $Q_n^{(\alpha)}(x)$ of degree n . The proposed algorithms rely on the computation of $Q_n^{(\alpha)}(x)$ and its derivative $Q_n^{(\alpha)'}(x)$, exploiting a connection formula that links the sequence of Gegenbauer–Sobolev polynomials $\{Q_j^{(\alpha)}(x)\}_{j \geq 0}$ with the sequence of Gegenbauer polynomials $\{C_j^{(\alpha)}(x)\}_{j \geq 0}$.

The choice of the initial guesses for the approximations of the zeros of $Q_n^{(\alpha)}(x)$ is crucial for guaranteeing fast convergence. For this purpose, some interlacing properties satisfied by the zeros of the Gegenbauer–Sobolev polynomials were proved, and the initial guesses were chosen by properly combining these properties. Extensive numerical tests show that the proposed algorithms have computational complexity $\mathcal{O}(n^2)$ and are very accurate.

TABLE 5.3
 Results obtained for computing the the zeros of $Q_{300}^{(\alpha)}(x)$ by algorithm $\mathbb{A}E_1$ for different values of α and λ .

$n = 300, \text{tol} \approx 6.66\text{e-}14$						
$\lambda \backslash \alpha$	-0.4	-0.2	0.0	0.2	0.4	0.6
10^{-6}	1.11e-16	2.22e-16	1.11e-16	1.11e-16	5.55e-17	1.11e-16
	3.26e-101	4.58e-101	4.57e-101	4.62e-101	4.62e-101	4.61e-101
	4.27e-14	6.12e-14	5.96e-14	6.01e-14	6.01e-14	6.01e-14
	4.27e-14	6.12e-14	5.96e-14	6.01e-14	6.01e-14	6.01e-14
	3.18	3.80	3.82	3.82	3.82	3.82
10^{-4}	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	5.28e-101	3.91e-101	3.90e-101	3.94e-101	3.94e-101	3.94e-101
	5.99e-14	5.96e-14	5.81e-14	5.86e-14	5.86e-14	5.86e-14
	5.99e-14	5.96e-14	5.81e-14	5.86e-14	5.86e-14	5.86e-14
	3.17	3.80	3.82	3.82	3.82	3.82
10^{-2}	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	7.43e-101	3.34e-101	3.33e-101	3.36e-101	3.36e-101	3.34e-101
	5.48e-14	5.80e-14	5.66e-14	5.71e-14	5.71e-14	5.71e-14
	5.48e-14	5.80e-14	5.66e-14	5.71e-14	5.71e-14	5.71e-14
	3.17	3.80	3.82	3.82	3.82	3.82
10^0	2.22e-16	2.22e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	5.10e-101	2.85e-101	2.84e-101	2.87e-101	2.87e-101	2.87e-101
	4.21e-14	5.64e-14	5.51e-14	5.56e-14	5.56e-14	5.56e-14
	4.21e-14	5.65e-14	5.51e-14	5.56e-14	5.56e-14	5.56e-14
	3.16	3.79	3.81	3.81	3.81	3.81
10^2	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	8.72e-101	2.44e-101	2.42e-101	2.45e-101	2.45e-101	2.45e-101
	6.37e-14	5.49e-14	5.37e-14	5.42e-14	5.42e-14	5.42e-14
	6.37e-14	5.49e-14	5.37e-14	5.42e-14	5.42e-14	5.42e-14
	3.16	3.79	3.81	3.81	3.81	3.81
10^4	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16	1.11e-16
	7.71e-101	2.08e-101	2.60e-101	2.63e-101	2.63e-101	2.63e-101
	5.35e-14	5.35e-14	6.52e-14	6.58e-14	6.58e-14	6.58e-14
	5.35e-14	5.35e-14	6.52e-14	6.58e-14	6.58e-14	6.58e-14
	3.16	3.79	3.81	3.81	3.81	3.81

Supplementary material. A Matlab implementation of the algorithms are available as supplementary material in form of a compressed file entitled GegenSob.zip at (see the file License.md for copyright issues) at the following link:

<https://etna.ricam.oeaw.ac.at/volumes/2021-2030/vol165/addition/p517.php>

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