

COMBINED BOUNDS FOR THE STANDARD AND GENERALIZED SINGULAR VALUE DECOMPOSITIONS*

XIAO SHAN CHEN[†], XUE QING ZOU[†], AND SEAK-WENG VONG[‡]

Abstract. In this paper, we present some new combined bounds for the standard singular value decomposition, extending some existing results. Furthermore, a combined perturbation bound for the generalized singular value decomposition of Grassmann matrix pairs is given, from which we obtain a Hoffman–Wielandt-type bound for generalized singular values.

Key words. combined bounds, standard singular value decomposition, generalized singular value decomposition, Frobenius norm

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1. Introduction. The standard singular value decomposition (SVD) and the generalized singular value decomposition (GSVD) are very useful tools in many matrix computation problems and practical applications, such as the Kronecker canonical form of a general matrix pencil, constrained and total least-squares problems, and so on; see [4, 5, 6, 7, 16, 17, 21, 23] and the references therein. The purpose of this paper is to present some new combined perturbation bounds for the standard SVD, as well as combined perturbation bounds for the GSVD.

As is well known, for any $A \in \mathbb{C}^{m \times n}$, its standard SVD may be written as (see, e.g., [4, 18])

$$(1.1) \quad A = U \Sigma V^H, \quad \text{with } \Sigma = \text{diag}(\sigma_1, \sigma_2, \dots) \text{ and } \sigma_i \geq 0, \ i = 1, 2, \dots,$$

where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary matrices and $\sigma_1, \sigma_2, \dots$ are the singular values of A .

The GSVD of two matrices having the same number of columns is a generalization of the standard SVD. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{p \times n}$. Let $\{A, B\}$ be an (m, p, n) -Grassmann matrix pair (GMP), i.e., $\text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = n$. Then there exist unitary matrices $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{p \times p}$, and a nonsingular matrix $Q \in \mathbb{C}^{n \times n}$ such that [8, 16, 19, 21]

$$(1.2) \quad U^H A Q = \Sigma_a, \quad V^H B Q = \Sigma_b,$$

where

$$(1.3) \quad \Sigma_a = \text{diag}(\alpha_1, \alpha_2, \dots), \quad \Sigma_b = \text{diag}(\dots, \beta_{n-1}, \beta_n),$$

meaning that Σ_a is zero except for the diagonal starting in the top-left corner (the leading diagonal) and Σ_b is zero except for the diagonal ending in the bottom-right corner (the trailing

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[†]Corresponding author: X. S. Chen. School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, People's Republic of China (chenxs33@163.com, 15150007571@163.com).

[‡]Department of Mathematics, University of Macau, Macao (swvong@umac.mo).



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diagonal). It holds that

$$\begin{aligned}
 \alpha_{m+1} &= \cdots = \alpha_n = 0 && \text{if } m \leq n, \\
 \beta_1 &= \cdots = \beta_{n-p} = 0 && \text{if } p \leq n, \quad \text{and} \\
 \alpha_i, \beta_i &\geq 0, \quad \alpha_i^2 + \beta_i^2 = 1, && i = 1, 2, \dots, n.
 \end{aligned}$$

The decomposition (1.2) is called the GSVD of the (m, p, n) -GMP $\{A, B\}$, and $\{(\alpha_i, \beta_i), i = 1, 2, \dots, n\}$ are its generalized singular values, which are denoted by $\sigma\{A, B\}$.

There is a large body of literature on perturbation theories for the SVD and GSVD. Mirsky [14] generalized the Hoffman–Wielandt theorem to singular values, and Wedin [22] generalized the famous Davis–Kahan $\sin \theta$ -theorem, relating invariant subspaces of two selfadjoint operators, to cover the singular value decomposition. Relative perturbation bounds for singular values and singular subspaces have been presented in the literature; see [1, 2, 10, 11, 12, 13]. Sun [19] was the first to give perturbation bounds for generalized singular values and generalized singular subspaces, and Paige [15] and Li [8] improved some results given by Sun [19]. Recently, Li and Sun [12] used Jordan–Wielandt matrices and combined perturbation bounds for eigenspaces and eigenvalues of Hermitian matrices to derive some combined absolute and relative perturbation bounds for the singular values and singular subspaces of a square matrix.

The novelty and main benefit of this paper are the application of the inequality (2.1), given in Section 2, from which one directly obtains combined perturbation bounds for the singular subspaces and singular values of a general matrix. In the sequel, we also readily derive several new perturbation bounds. Furthermore, the technique described in [12] cannot be applied to the GSVD. In contrast, our technique can be utilized to derive combined perturbation bounds for the generalized singular subspaces and generalized singular values of a Grassmann matrix pair. Hence, this paper makes the following two contributions:

1. We employ techniques different from those described by Li and Sun [12]. Our combined bounds for the SVD are derived based on a fundamental conclusion (see Lemma 2.2), whereas Li and Sun [12] use Jordan–Wielandt matrices.
2. Some new perturbation bounds for the SVD are given (see Theorem 3.4 and Theorem 3.6), and combined perturbation bounds for the GSVD are also derived; see Theorem 4.1 and Theorem 4.3.

The rest of this paper is organized as follows. In Section 2, we give several useful lemmas and introduce properties of the angle between two subspaces of the same dimension. In Section 3, some new absolute and relative perturbation bounds for the SVD are presented. Section 4 gives combined bounds for generalized singular values and generalized singular subspaces. Some concluding remarks are given in Section 5.

Throughout this paper, we use the following notation: $\mathbb{C}^{m \times n}$ denotes the set of $m \times n$ complex matrices. The symbol I_n is the $n \times n$ identity matrix, and when no ambiguity can arise, the subscript n is omitted. For $X \in \mathbb{C}^{m \times n}$, $\|X\|_2$ and $\|X\|_F$ denote the spectral norm and Frobenius norm of X , respectively, X^H is the conjugate transpose, $X^\dagger$ is the Moore–Penrose inverse, and $P_X = XX^\dagger$ is the orthogonal projector onto the column space $\mathcal{R}(X)$ of X . The chordal metric is defined for $(\alpha, \beta) \neq (0, 0) \neq (\tilde{\alpha}, \tilde{\beta})$, with $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} \in \mathbb{C}$, by

$$\rho((\alpha, \beta), (\tilde{\alpha}, \tilde{\beta})) \stackrel{\text{def}}{=} \frac{|\tilde{\beta}\alpha - \tilde{\alpha}\beta|}{\sqrt{|\alpha|^2 + |\beta|^2} \sqrt{|\tilde{\alpha}|^2 + |\tilde{\beta}|^2}}.$$

2. Preliminaries. In this section, several useful lemmas are given. Lemma 2.2 plays a major role in deriving perturbation bounds for the SVD and GSVD. We also introduce properties of the angle between two subspaces with the same dimension.

The following result is usually used to derive perturbation bounds for both eigenvalue and generalized eigenvalue problems (see [3, 9, 10, 12]):

LEMMA 2.1 ([3, Theorem 3.2]). *Let $T \in \mathbb{C}^{n \times n}$ and $\Lambda_i = \text{diag}(\lambda_1^{(i)}, \dots, \lambda_n^{(i)}) \in \mathbb{C}^{n \times n}$, $i = 1, 2, 3, 4$. Then there exists a permutation τ of $\{1, 2, \dots, n\}$ such that*

$$\sigma_{\min}^2(T) \sum_{i=1}^n \left| \lambda_i^{(1)} \lambda_{\tau(i)}^{(2)} - \lambda_i^{(3)} \lambda_{\tau(i)}^{(4)} \right|^2 \leq \|\Lambda_1 T \Lambda_2 - \Lambda_3 T \Lambda_4\|_F^2,$$

where $\sigma_{\min}(T)$ denotes the smallest singular value of the matrix T .

We apply the following two lemmas to establish combined perturbation bounds for the standard SVD and GSVD, respectively. The next Lemma 2.2 may be regarded as a generalization of Lemma 2.1.

LEMMA 2.2. *Let $G, H \in \mathbb{C}^{n \times n}$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, $\Delta = \text{diag}(\delta_1, \dots, \delta_n)$, $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_n)$, and $\Omega = \text{diag}(\omega_1, \dots, \omega_n)$, with $\lambda_i, \beta_i, \gamma_i, \omega_i \geq 0$, $i = 1, 2, \dots, n$. Then there exists a permutation τ of $\{1, 2, \dots, n\}$ such that*

$$(2.1) \quad (\sigma_{\min}^2(G) + \sigma_{\min}^2(H)) \sum_{i=1}^n |\lambda_i \delta_{\tau(i)} - \gamma_i \omega_{\tau(i)}|^2 \leq \|\Lambda G \Delta - \Gamma H \Omega\|_F^2 + \|\Lambda H \Delta - \Gamma G \Omega\|_F^2.$$

Proof. Let $G = (g_{ij})$ and $H = (h_{ij})$. Then we have

$$(2.2) \quad \begin{aligned} & \|\Lambda G \Delta - \Gamma H \Omega\|_F^2 + \|\Lambda H \Delta - \Gamma G \Omega\|_F^2 \\ &= \sum_{i,j=1}^n \left(|\lambda_i \delta_j g_{ij} - \gamma_i \omega_j h_{ij}|^2 + |\lambda_i \delta_j h_{ij} - \gamma_i \omega_j g_{ij}|^2 \right) \\ &= \sum_{i,j=1}^n \left((\lambda_i^2 \delta_j^2 + \gamma_i^2 \omega_j^2)(|g_{ij}|^2 + |h_{ij}|^2) - 4\lambda_i \delta_j \gamma_i \omega_j \text{Re}(g_{ij} h_{ij}) \right) \\ &\geq \sum_{i,j=1}^n \left((\lambda_i^2 \delta_j^2 + \gamma_i^2 \omega_j^2)(|g_{ij}|^2 + |h_{ij}|^2) - 2\lambda_i \delta_j \gamma_i \omega_j (|g_{ij}|^2 + |h_{ij}|^2) \right) \\ &= \sum_{i,j=1}^n (\lambda_i \delta_j - \gamma_i \omega_j)^2 (|g_{ij}|^2 + |h_{ij}|^2) \\ &= \sum_{i,j=1}^n \left((\lambda_i \delta_j - \gamma_i \omega_j) |g_{ij}| \right)^2 + \sum_{i,j=1}^n \left((\lambda_i \delta_j - \gamma_i \omega_j) |h_{ij}| \right)^2 \\ &= \|\Lambda G \Delta - \Gamma G \Omega\|_F^2 + \|\Lambda H \Delta - \Gamma H \Omega\|_F^2, \end{aligned}$$

where the first inequality is obtained from the facts that $2\text{Re}(g_{ij} h_{ij}) \leq |g_{ij}|^2 + |h_{ij}|^2$ and that $\lambda_i, \delta_i, \gamma_i$, and ω_i are nonnegative numbers. Therefore, by Lemma 2.1, there exist two permutations τ_1 and τ_2 of $\{1, 2, \dots, n\}$ such that

$$\sigma_{\min}^2(G) \sum_{i=1}^n |\lambda_i \delta_{\tau_1(i)} - \gamma_i \omega_{\tau_1(i)}|^2 \leq \|\Lambda G \Delta - \Gamma G \Omega\|_F^2$$

and

$$\sigma_{\min}^2(H) \sum_{i=1}^n |\lambda_i \delta_{\tau_2(i)} - \gamma_i \omega_{\tau_2(i)}|^2 \leq \|\Lambda H \Delta - \Gamma H \Omega\|_F^2,$$

which implies that there is a permutation τ of $\{1, 2, \dots, n\}$ such that

$$(\sigma_{\min}^2(G) + \sigma_{\min}^2(H)) \sum_{i=1}^n |\lambda_i \delta_{\tau(i)} - \gamma_i \omega_{\tau(i)}|^2 \leq \|\Lambda G \Delta - \Gamma G \Omega\|_F^2 + \|\Lambda H \Delta - \Gamma H \Omega\|_F^2.$$

Combining the above inequality with (2.2) yields the desired result (2.1). $\square$

REMARK 2.3. Using Lemma 2.2, we may obtain a result given by Li [8]. In fact, if G and H are $n \times n$ unitary matrices, then $\sigma_{\min}(G) = \sigma_{\min}(H) = 1$. Hence, (2.1) reduces to

$$\sum_{i=1}^n |\lambda_i \delta_{\tau(i)} - \gamma_i \omega_{\tau(i)}|^2 \leq \frac{1}{2} (\|\Lambda G \Delta - \Gamma H \Omega\|_F^2 + \|\Lambda H \Delta - \Gamma G \Omega\|_F^2),$$

which is the result in [8, Lemma 4.3].

The following lemma is a consequence of [8, Formula (7.15)], which is used to derive perturbation bounds for singular subspaces and generalized singular subspaces.

LEMMA 2.4 ([8]). *Let $G, H \in \mathbb{C}^{m \times n}$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$, $\Delta = \text{diag}(\delta_1, \dots, \delta_n)$, $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_m)$, and $\Omega = \text{diag}(\omega_1, \dots, \omega_n)$, with $\lambda_i, \beta_j, \gamma_i, \omega_j \geq 0$, $i = 1, \dots, m$, $j = 1, \dots, n$. Then we have*

$$\begin{aligned} & \left(\min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{|\lambda_i \delta_j - \gamma_i \omega_j|\} \right)^2 \left(\|G\|_F^2 + \|H\|_F^2 \right) \\ & \leq \|\Lambda G \Delta - \Gamma H \Omega\|_F^2 + \|\Lambda H \Delta - \Gamma G \Omega\|_F^2. \end{aligned}$$

The perturbation of singular subspaces and generalized singular subspaces requires the definition of the angles between two subspaces with the same dimension. Let $U_1, \tilde{U}_1 \in \mathbb{C}^{n \times \ell}$ ($1 \leq \ell \leq n-1$), with $U_1^H U_1 = \tilde{U}_1^H \tilde{U}_1 = I_\ell$. Then the canonical angle between the subspaces $\mathcal{R}(U_1)$ and $\mathcal{R}(\tilde{U}_1)$ is defined by (see [18, 20])

$$\Theta(\mathcal{R}(U_1), \mathcal{R}(\tilde{U}_1)) = \arccos \left(U_1^H \tilde{U}_1 \tilde{U}_1^H U_1 \right)^{1/2}.$$

The following results are useful in Sections 3–4:

LEMMA 2.5. *Let $U = [U_1 \ U_2]$ and $\tilde{U} = [\tilde{U}_1 \ \tilde{U}_2]$ be $n \times n$ unitary matrices with $U_1, \tilde{U}_1 \in \mathbb{C}^{n \times \ell}$ ($1 \leq \ell \leq n-1$). Then, for any unitarily invariant norm $\|\cdot\|$, we have*

$$(2.3) \quad \|\sin \Theta(\mathcal{R}(U_1), \mathcal{R}(\tilde{U}_1))\| = \|U_2^H \tilde{U}_1\| = \|\tilde{U}_2^H U_1\|$$

and

$$(2.4) \quad \sigma_{\min}^2(U_1^H \tilde{U}_1) = \sigma_{\min}^2(\tilde{U}_1^H U_1) = 1 - \|\sin \Theta(\mathcal{R}(U_1), \mathcal{R}(\tilde{U}_1))\|_2^2.$$

In particular, identity (2.3) holds for $\|\cdot\|_2$ and $\|\cdot\|_F$.

Proof. The proof of (2.3) can be found in [20, Lemma 7.1]. The C-S decomposition theorem (see [18, 20]) yields (2.4). $\square$

3. Combined bounds for the standard SVD. In this section, we present some absolute and relative perturbation bounds for the SVD. Let $A \in \mathbb{C}^{m \times n}$ have the SVD (1.1), and let $\ell \in \mathbb{N}$ satisfy

$$(3.1) \quad 1 \leq \ell < \min\{m, n\}.$$

From (1.1) we formally obtain

$$(3.2) \quad A \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{matrix} \ell & m-\ell \\ m-\ell & \end{matrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

where

$$(3.3) \quad \Sigma_1 = \text{diag}(\sigma_1, \dots, \sigma_\ell), \quad \Sigma_2 = \text{diag}(\sigma_{\ell+1}, \sigma_{\ell+2}, \dots).$$

In the sequel, the perturbed matrix $\tilde{A} \in \mathbb{C}^{m \times n}$ and the corresponding decompositions (1.1) and (3.2) are marked with tildes; e.g., the matrix corresponding to U_1 is written as $\tilde{U}_1$.

For simplicity, let Ψ and Φ denote the canonical angles between the column subspaces $\mathcal{R}(U_1)$ and $\mathcal{R}(\tilde{U}_1)$ and between the column subspaces $\mathcal{R}(V_1)$ and $\mathcal{R}(\tilde{V}_1)$, respectively; that is,

$$\Psi = \Theta(\mathcal{R}(U_1), \mathcal{R}(\tilde{U}_1)), \quad \Phi = \Theta(\mathcal{R}(V_1), \mathcal{R}(\tilde{V}_1)).$$

We present the following combined bound for the SVDs. For the case of square matrices, Li and Sun [12] used Jordan–Wielandt matrices to prove the result. Here, we use Lemma 2.2 and Lemma 2.4 to provide a concise proof.

THEOREM 3.1. *Let $A, \tilde{A} = A - E \in \mathbb{C}^{m \times n}$ have the SVDs (3.2) and (3.3), and let $\ell \in \mathbb{N}$ satisfy (3.1). Then there exists a permutation τ of $\{1, 2, \dots, \ell\}$ such that*

$$(3.4) \quad \begin{aligned} & \left(2 - \|\sin \Psi\|_2^2 - \|\sin \Phi\|_2^2\right) \sum_{i=1}^{\ell} |\sigma_i - \tilde{\sigma}_{\tau(i)}|^2 + \delta^2 \left(\|\sin \Psi\|_F^2 + \|\sin \Phi\|_F^2\right) \\ & \leq \|E\tilde{V}_1\|_F^2 + \|E^H \tilde{U}_1\|_F^2, \end{aligned}$$

where

$$\delta = \begin{cases} \min_{\substack{\ell+1 \leq i \leq n \\ 1 \leq j \leq \ell}} \{|\sigma_i - \tilde{\sigma}_j|\} & \text{if } m = n, \\ \min \left\{ \min_{\substack{\ell+1 \leq i \leq n \\ 1 \leq j \leq \ell}} \{|\sigma_i - \tilde{\sigma}_j|\}, \min_{1 \leq j \leq \ell} \{\tilde{\sigma}_j\} \right\} & \text{if } m > n, \\ \min \left\{ \min_{\substack{\ell+1 \leq i \leq m \\ 1 \leq j \leq \ell}} \{|\sigma_i - \tilde{\sigma}_j|\}, \min_{1 \leq j \leq \ell} \{\tilde{\sigma}_j\} \right\} & \text{if } m < n. \end{cases}$$

Proof. By the SVDs of A and $\tilde{A}$, we have

$$U^H E \tilde{V} = U^H (A - \tilde{A}) \tilde{V} = \Sigma V^H \tilde{V} - U^H \tilde{U} \tilde{\Sigma},$$

which can be rewritten in block form as

$$(3.5) \quad \begin{bmatrix} U_1^H E \tilde{V}_1 & U_1^H E \tilde{V}_2 \\ U_2^H E \tilde{V}_1 & U_2^H E \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} \Sigma_1 V_1^H \tilde{V}_1 - U_1^H \tilde{U}_1 \tilde{\Sigma}_1 & \Sigma_1 V_1^H \tilde{V}_2 - U_1^H \tilde{U}_2 \tilde{\Sigma}_2 \\ \Sigma_2 V_2^H \tilde{V}_1 - U_2^H \tilde{U}_1 \tilde{\Sigma}_1 & \Sigma_2 V_2^H \tilde{V}_2 - U_2^H \tilde{U}_2 \tilde{\Sigma}_2 \end{bmatrix}.$$

Similarly, we have

$$(3.6) \quad \begin{bmatrix} V_1^H E^H \tilde{U}_1 & V_1^H E^H \tilde{U}_2 \\ V_2^H E^H \tilde{U}_1 & V_2^H E^H \tilde{U}_2 \end{bmatrix} = \begin{bmatrix} \Sigma_1 U_1^H \tilde{U}_1 - V_1^H \tilde{V}_1 \tilde{\Sigma}_1 & \Sigma_1 U_1^H \tilde{U}_2 - V_1^H \tilde{V}_2 \tilde{\Sigma}_2^H \\ \Sigma_2^H U_2^H \tilde{U}_1 - V_2^H \tilde{V}_1 \tilde{\Sigma}_1 & \Sigma_2^H U_2^H \tilde{U}_2 - V_2^H \tilde{V}_2 \tilde{\Sigma}_2^H \end{bmatrix}.$$

It follows from (3.5) and (3.6) that

$$\begin{aligned}
(3.7) \quad & \|E\tilde{V}_1\|_F^2 + \|E^H\tilde{U}_1\|_F^2 \\
&= \|\Sigma_1 V_1^H \tilde{V}_1 - U_1^H \tilde{U}_1 \tilde{\Sigma}_1\|_F^2 + \|\Sigma_1 U_1^H \tilde{U}_1 - V_1^H \tilde{V}_1 \tilde{\Sigma}_1\|_F^2 \\
&\quad + \|\Sigma_2 V_2^H \tilde{V}_1 - U_2^H \tilde{U}_1 \tilde{\Sigma}_1\|_F^2 + \|\Sigma_2^H U_2^H \tilde{U}_1 - V_2^H \tilde{V}_1 \tilde{\Sigma}_1\|_F^2.
\end{aligned}$$

Applying Lemma 2.2 to $\|\Sigma_1 V_1^H \tilde{V}_1 - U_1^H \tilde{U}_1 \tilde{\Sigma}_1\|_F^2 + \|\Sigma_1 U_1^H \tilde{U}_1 - V_1^H \tilde{V}_1 \tilde{\Sigma}_1\|_F^2$, it follows that there exists a permutation τ of $\{1, 2, \dots, \ell\}$ such that

$$\begin{aligned}
(3.8) \quad & \left(\sigma_{\min}^2(V_1^H \tilde{V}_1) + \sigma_{\min}^2(U_1^H \tilde{U}_1) \right) \sum_{i=1}^{\ell} |\sigma_i - \tilde{\sigma}_{\tau(i)}|^2 \\
&\leq \|\Sigma_1 V_1^H \tilde{V}_1 - U_1^H \tilde{U}_1 \tilde{\Sigma}_1\|_F^2 + \|\Sigma_1 U_1^H \tilde{U}_1 - V_1^H \tilde{V}_1 \tilde{\Sigma}_1\|_F^2.
\end{aligned}$$

If $m = n$, then $\Sigma_2 = \text{diag}(\sigma_{\ell+1}, \dots, \sigma_n) \in \mathbb{C}^{(n-\ell) \times (n-\ell)}$. By Lemma 2.4, we get

$$\begin{aligned}
(3.9) \quad & \|\Sigma_2 V_2^H \tilde{V}_1 - U_2^H \tilde{U}_1 \tilde{\Sigma}_1\|_F^2 + \|\Sigma_2 U_2^H \tilde{U}_1 - V_2^H \tilde{V}_1 \tilde{\Sigma}_1\|_F^2 \\
&\geq \min_{\substack{\ell+1 \leq i \leq n \\ 1 \leq j \leq \ell}} \{|\sigma_i - \tilde{\sigma}_j|^2\} \left(\|U_2^H \tilde{U}_1\|_F^2 + \|V_2^H \tilde{V}_1\|_F^2 \right) \\
&= \delta^2 \left(\|U_2^H \tilde{U}_1\|_F^2 + \|V_2^H \tilde{V}_1\|_F^2 \right).
\end{aligned}$$

From Lemma 2.5 and (3.7)–(3.9), we obtain the desired bound (3.4).

If $m > n$, then let $\Sigma_2 = \begin{bmatrix} \Lambda \\ 0 \end{bmatrix} \in \mathbb{C}^{(m-\ell) \times (n-\ell)}$, with $\Lambda = \text{diag}(\sigma_{\ell+1}, \dots, \sigma_n)$, and let $U_2 = \begin{bmatrix} U_{21} & U_{22} \end{bmatrix} \in \mathbb{C}^{(m) \times (m-\ell)}$, with $U_{21} \in \mathbb{C}^{m \times (n-\ell)}$, $U_{22} \in \mathbb{C}^{m \times (m-n)}$. Then, by Lemma 2.4, we have

$$\begin{aligned}
& \|\Sigma_2 V_2^H \tilde{V}_1 - U_2^H \tilde{U}_1 \tilde{\Sigma}_1\|_F^2 + \|\Sigma_2^H U_2^H \tilde{U}_1 - V_2^H \tilde{V}_1 \tilde{\Sigma}_1\|_F^2 \\
&= \|\Lambda V_2^H \tilde{V}_1 - U_{21}^H \tilde{U}_1 \tilde{\Sigma}_1\|_F^2 + \|U_{22}^H \tilde{U}_1 \tilde{\Sigma}_1\|_F^2 + \|\Lambda U_{21}^H \tilde{U}_1 - V_2^H \tilde{V}_1 \tilde{\Sigma}_1\|_F^2 \\
&\geq \min_{\substack{\ell+1 \leq i \leq n \\ 1 \leq j \leq \ell}} \{|\sigma_i - \tilde{\sigma}_j|^2\} \left(\|U_{21}^H \tilde{U}_1\|_F^2 + \|V_2^H \tilde{V}_1\|_F^2 \right) + \min_{1 \leq j \leq \ell} \{\tilde{\sigma}_j^2\} \|U_{22}^H \tilde{U}_1\|_F^2 \\
&\geq \delta^2 \left(\|U_{21}^H \tilde{U}_1\|_F^2 + \|V_2^H \tilde{V}_1\|_F^2 + \|U_{22}^H \tilde{U}_1\|_F^2 \right) \\
&= \delta^2 \left(\|U_2^H \tilde{U}_1\|_F^2 + \|V_2^H \tilde{V}_1\|_F^2 \right).
\end{aligned}$$

Combining the above inequality with (3.7)–(3.8) and using Lemma 2.5 yield the combined bound (3.4).

If $m < n$, then let $\Sigma_2 = \begin{bmatrix} \Lambda & 0 \end{bmatrix} \in \mathbb{C}^{(m-\ell) \times (n-\ell)}$, with $\Lambda = \text{diag}(\sigma_1, \dots, \sigma_m)$, and let $V_2 = \begin{bmatrix} V_{21} & V_{22} \end{bmatrix} \in \mathbb{C}^{n \times \ell}$, with $V_{21} \in \mathbb{C}^{n \times (m-\ell)}$ and $V_{22} \in \mathbb{C}^{n \times (n-m)}$. Similarly, by Lemma 2.4, we get

$$\begin{aligned}
& \|\Sigma_2 V_2^H \tilde{V}_1 - U_2^H \tilde{U}_1 \tilde{\Sigma}_1\|_F^2 + \|\Sigma_2^H U_2^H \tilde{U}_1 - V_2^H \tilde{V}_1 \tilde{\Sigma}_1\|_F^2 \\
&= \|\Lambda V_{21}^H \tilde{V}_1 - U_2^H \tilde{U}_1 \tilde{\Sigma}_1\|_F^2 + \|\Lambda U_2^H \tilde{U}_1 - V_{21}^H \tilde{V}_1 \tilde{\Sigma}_1\|_F^2 + \|V_{22}^H \tilde{V}_1 \tilde{\Sigma}_1\|_F^2 \\
&\geq \delta^2 \left(\|U_2^H \tilde{U}_1\|_F^2 + \|V_2^H \tilde{V}_1\|_F^2 \right),
\end{aligned}$$

which also leads to the desired bound (3.4). $\square$

REMARK 3.2. Using Jordan–Wielandt matrices, Li and Sun [12] have shown the combined bound (3.4) for the case of square matrices; see [12, Theorem 3.1]. Here, we apply different techniques to provide a proof for the general case.

REMARK 3.3. If $\delta > 0$, then it follows from (3.4) that

$$\|\sin \Psi\|_F^2 + \|\sin \Phi\|_F^2 \leq \frac{\|E\tilde{V}_1\|_F^2 + \|E^H\tilde{U}_1\|_F^2}{\delta^2},$$

which was also stated by Wedin in [22] for $m \geq n$.

Next, we provide some combined relative bounds for the SVDs. From the singular value decompositions, we obtain the left and right polar decompositions of A and $\tilde{A}$ as follows (see [4]):

$$A = P_l Q = Q P_r \quad \text{and} \quad \tilde{A} = \tilde{P}_l \tilde{Q} = \tilde{Q} \tilde{P}_r,$$

respectively, where Q and $\tilde{Q}$ are the unitary factors of A and $\tilde{A}$, respectively, and $P_l, \tilde{P}_l$ and $P_r, \tilde{P}_r$ are the left and right Hermitian semidefinite factors of A and $\tilde{A}$, respectively.

THEOREM 3.4. *Let $A, \tilde{A} = A - E \in \mathbb{C}^{n \times n}$ have the SVDs (3.2) and (3.3), with $\text{rank}(A) = \text{rank}(\tilde{A}) = n$, and let $\ell \in \mathbb{N}$ satisfy (3.1). Then there exists a permutation τ of $\{1, 2, \dots, \ell\}$ such that*

$$\begin{aligned}
 (3.10) \quad & (2 - \|\sin \Psi\|_2^2 - \|\sin \Phi\|_2^2) \sum_{i=1}^{\ell} \left(\frac{\sigma_i - \tilde{\sigma}_{\tau(i)}}{\sigma_i^s \tilde{\sigma}_{\tau(i)}^t} \right)^2 \\
 & + \left(\delta^{(s,t)} \right)^2 (\|\sin \Psi\|_F^2 + \|\sin \Phi\|_F^2) \\
 & \leq \|P_l^{-s} E \tilde{P}_r^{-t} \tilde{V}_1\|_F^2 + \|\tilde{U}_1^H \tilde{P}_l^{-t} E P_r^{-s}\|_F^2,
 \end{aligned}$$

where s, t are nonnegative numbers and

$$(3.11) \quad \delta^{(s,t)} = \min_{\substack{\ell+1 \leq i \leq n \\ 1 \leq j \leq \ell}} \left\{ \frac{|\sigma_i - \tilde{\sigma}_j|}{\sigma_i^s \tilde{\sigma}_j^t} \right\}.$$

Proof. It follows easily from $\text{rank}(A) = \text{rank}(\tilde{A}) = n$ that $P_l, P_r, \tilde{P}_l$, and $\tilde{P}_r$ are $n \times n$ Hermitian positive definite matrices, and we have [4]

$$P_l = U \Sigma U^H, \quad P_r = V \Sigma V^H, \quad \tilde{P}_l = \tilde{U} \tilde{\Sigma} \tilde{U}^H, \quad \text{and} \quad \tilde{P}_r = \tilde{V} \tilde{\Sigma} \tilde{V}^H.$$

It follows from (3.2)–(3.3) that

$$\begin{aligned}
 (3.12) \quad & \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} P_l^{-s} E \tilde{P}_r^{-t} \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \end{bmatrix} \\
 & = \begin{bmatrix} \Sigma_1^{1-s} V_1^H \tilde{V}_1 \tilde{\Sigma}_1^{-t} - \Sigma_1^{-s} U_1^H \tilde{U}_1 \tilde{\Sigma}_1^{1-t} & \Sigma_1^{1-s} V_1^H \tilde{V}_2 \tilde{\Sigma}_2^{-t} - \Sigma_1^{-s} U_1^H \tilde{U}_2 \tilde{\Sigma}_2^{1-t} \\ \Sigma_2^{1-s} V_2^H \tilde{V}_1 \tilde{\Sigma}_1^{-t} - \Sigma_2^{-s} U_2^H \tilde{U}_1 \tilde{\Sigma}_1^{1-t} & \Sigma_2^{1-s} V_2^H \tilde{V}_2 \tilde{\Sigma}_2^{-t} - \Sigma_2^{-s} U_2^H \tilde{U}_2 \tilde{\Sigma}_2^{1-t} \end{bmatrix}
 \end{aligned}$$

and

$$\begin{aligned}
 (3.13) \quad & \begin{bmatrix} \tilde{U}_1^H \\ \tilde{U}_2^H \end{bmatrix} \tilde{P}_l^{-t} E P_r^{-s} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \\
 & = \begin{bmatrix} \tilde{\Sigma}_1^{-t} \tilde{U}_1^H U_1 \Sigma_1^{1-s} - \tilde{\Sigma}_1^{1-t} \tilde{V}_1^H V_1 \Sigma_1^{-s} & \tilde{\Sigma}_1^{-t} \tilde{U}_1^H U_2 \Sigma_2^{1-s} - \tilde{\Sigma}_1^{1-t} \tilde{V}_1^H V_2 \Sigma_2^{-s} \\ \tilde{\Sigma}_2^{-t} \tilde{U}_2^H U_1 \Sigma_1^{1-s} - \tilde{\Sigma}_2^{1-t} \tilde{V}_2^H V_1 \Sigma_1^{-s} & \tilde{\Sigma}_2^{-t} \tilde{U}_2^H U_2 \Sigma_2^{1-s} - \tilde{\Sigma}_2^{1-t} \tilde{V}_2^H V_2 \Sigma_2^{-s} \end{bmatrix}.
 \end{aligned}$$

By (3.12) and (3.13) we get

$$\begin{aligned}
 & \left\| P_l^{-s} E \tilde{P}_r^{-t} \tilde{V}_1 \right\|_F^2 + \left\| \tilde{U}_1^H \tilde{P}_l^{-t} E P_r^{-s} \right\|_F^2 \\
 &= \left\| \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} P_l^{-s} E \tilde{P}_r^{-t} \tilde{V}_1 \right\|_F^2 + \left\| \tilde{U}_1^H \tilde{P}_l^{-t} E P_r^{-s} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \right\|_F^2 \\
 (3.14) \quad &= \left\| \Sigma_1^{1-s} V_1^H \tilde{V}_1 \tilde{\Sigma}_1^{-t} - \Sigma_1^{-s} U_1^H \tilde{U}_1 \tilde{\Sigma}_1^{1-t} \right\|_F^2 \\
 &\quad + \left\| \Sigma_2^{1-s} V_2^H \tilde{V}_1 \tilde{\Sigma}_1^{-t} - \Sigma_2^{-s} U_2^H \tilde{U}_1 \tilde{\Sigma}_1^{1-t} \right\|_F^2 \\
 &\quad + \left\| \tilde{\Sigma}_1^{-t} \tilde{U}_1^H U_1 \Sigma_1^{1-s} - \tilde{\Sigma}_1^{1-t} \tilde{V}_1^H V_1 \Sigma_1^{-s} \right\|_F^2 \\
 &\quad + \left\| \tilde{\Sigma}_1^{-t} \tilde{U}_1^H U_2 \Sigma_2^{1-s} - \tilde{\Sigma}_1^{1-t} \tilde{V}_1^H V_2 \Sigma_2^{-s} \right\|_F^2.
 \end{aligned}$$

By Lemma 2.2, there exists a permutation τ of $\{1, 2, \dots, \ell\}$ such that

$$\begin{aligned}
 & \left\| \Sigma_1^{1-s} V_1^H \tilde{V}_1 \tilde{\Sigma}_1^{-t} - \Sigma_1^{-s} U_1^H \tilde{U}_1 \tilde{\Sigma}_1^{1-t} \right\|_F^2 \\
 & \quad + \left\| \tilde{\Sigma}_1^{-t} \tilde{U}_1^H U_1 \Sigma_1^{1-s} - \tilde{\Sigma}_1^{1-t} \tilde{V}_1^H V_1 \Sigma_1^{-s} \right\|_F^2 \\
 (3.15) \quad &= \left\| \Sigma_1^{1-s} V_1^H \tilde{V}_1 \tilde{\Sigma}_1^{-t} - \Sigma_1^{-s} U_1^H \tilde{U}_1 \tilde{\Sigma}_1^{1-t} \right\|_F^2 \\
 & \quad + \left\| \Sigma_1^{1-s} U_1^H \tilde{U}_1 \tilde{\Sigma}_1^{-t} - \Sigma_1^{-s} V_1^H \tilde{V}_1 \tilde{\Sigma}_1^{1-t} \right\|_F^2 \\
 & \geq \left(\sigma_{\min}(V_1^H \tilde{V}_1) + \sigma_{\min}(U_1^H \tilde{U}_1) \right) \sum_{i=1}^{\ell} \left(\frac{\sigma_i - \tilde{\sigma}_{\tau(i)}}{\sigma_i^s \tilde{\sigma}_{\tau(i)}^t} \right)^2.
 \end{aligned}$$

It follows from Lemma 2.4 that

$$\begin{aligned}
 & \left\| \Sigma_2^{1-s} V_2^H \tilde{V}_1 \tilde{\Sigma}_1^{-t} - \Sigma_2^{-s} U_2^H \tilde{U}_1 \tilde{\Sigma}_1^{1-t} \right\|_F^2 \\
 & \quad + \left\| \tilde{\Sigma}_1^{-t} \tilde{U}_1^H U_2 \Sigma_2^{1-s} - \tilde{\Sigma}_1^{1-t} \tilde{V}_1^H V_2 \Sigma_2^{-s} \right\|_F^2 \\
 (3.16) \quad &= \left\| \Sigma_2^{1-s} V_2^H \tilde{V}_1 \tilde{\Sigma}_1^{-t} - \Sigma_2^{-s} U_2^H \tilde{U}_1 \tilde{\Sigma}_1^{1-t} \right\|_F^2 \\
 & \quad + \left\| \Sigma_2^{1-s} U_2^H \tilde{U}_1 \tilde{\Sigma}_1^{-t} - \Sigma_2^{-s} V_2^H \tilde{V}_1 \tilde{\Sigma}_1^{1-t} \right\|_F^2 \\
 & \geq \left(\min_{\substack{\ell+1 \leq i \leq n \\ 1 \leq j \leq \ell}} \left\{ \frac{|\sigma_i - \tilde{\sigma}_j|}{\sigma_i^s \tilde{\sigma}_j^t} \right\} \right)^2 \left(\left\| V_2^H \tilde{V}_1 \right\|_F^2 + \left\| U_2^H \tilde{U}_1 \right\|_F^2 \right) \\
 & = \left(\delta^{(s,t)} \right)^2 \left(\left\| V_2^H \tilde{V}_1 \right\|_F^2 + \left\| U_2^H \tilde{U}_1 \right\|_F^2 \right).
 \end{aligned}$$

By (3.14)–(3.16) and Lemma 2.5, we obtain the combined bound (3.10). $\square$

REMARK 3.5. If $\delta^{(s,t)} > 0$, then, by (3.10), we get

$$\begin{aligned}
 \left\| \sin \Psi \right\|_F^2 + \left\| \sin \Phi \right\|_F^2 &\leq \left(\frac{1}{\delta^{(s,t)}} \right)^2 \left(\left\| P_l^{-s} E \tilde{P}_r^{-t} \tilde{V}_1 \right\|_F^2 + \left\| \tilde{U}_1^H \tilde{P}_l^{-t} E P_r^{-s} \right\|_F^2 \right) \\
 &\leq \left(\frac{1}{\delta^{(s,t)}} \right)^2 \left(\left\| P_l^{-s} E \tilde{P}_r^{-t} \right\|_F^2 + \left\| \tilde{P}_l^{-t} E P_r^{-s} \right\|_F^2 \right),
 \end{aligned}$$

which is an estimate that was given by Dopico, Moro, and Molera [2] for the case $s = t = \frac{1}{2}$.

The following results are relative perturbation bounds for the singular values:

THEOREM 3.6. *Let $A, \tilde{A} = A - E \in \mathbb{C}^{m \times n}$ have the SVDs (3.2) and (3.3), with $\text{rank}(A) = \text{rank}(\tilde{A}) = n$. Then there exists a permutation τ of $\{1, 2, \dots, n\}$ such that*

$$(3.17) \quad \left(2 - \|\sin \Theta(\mathcal{R}(A), \mathcal{R}(\tilde{A}))\|_2^2\right) \sum_{i=1}^n \left(\frac{\sigma_i - \tilde{\sigma}_{\tau(i)}}{\sigma_i^s \tilde{\sigma}_{\tau(i)}^t}\right)^2 \leq \left\| (P_l^\dagger)^s E \tilde{P}_r^{-t} \right\|_F^2 + \left\| (\tilde{P}_l^\dagger)^t E P_r^{-s} \right\|_F^2,$$

where s, t are nonnegative numbers. In particular, when $m = n$, we have

$$(3.18) \quad \sum_{i=1}^n \left(\frac{\sigma_i - \tilde{\sigma}_{\tau(i)}}{\sigma_i^s \tilde{\sigma}_{\tau(i)}^t}\right)^2 \leq \frac{1}{2} \left(\left\| P_l^{-s} E \tilde{P}_r^{-t} \right\|_F^2 + \left\| \tilde{P}_l^{-t} E P_r^{-s} \right\|_F^2 \right).$$

Proof. The condition $\text{rank}(A) = \text{rank}(\tilde{A}) = n$ implies that $m \geq n$. First, we prove inequality (3.17) for $m > n$. Let $\ell = n$. Then the SVDs (3.2) and (3.3) reduce to

$$(3.19) \quad A = U_1 \Sigma_1 V^H \quad \text{and} \quad \tilde{A} = \tilde{U}_1 \tilde{\Sigma}_1 \tilde{V}^H,$$

respectively, where $U_1, \tilde{U}_1 \in \mathbb{C}^{m \times n}$ and $\Sigma_1 = \text{diag}(\sigma_1, \dots, \sigma_n)$ and $\tilde{\Sigma}_1 = \text{diag}(\tilde{\sigma}_1, \dots, \tilde{\sigma}_n)$. It follows from (3.19) that

$$P_l = U_1 \Sigma_1 U_1^H, \quad P_l^\dagger = U_1 \Sigma_1^{-1} U_1^H, \quad \tilde{P}_l = \tilde{U}_1 \tilde{\Sigma}_1 \tilde{U}_1^H, \quad \tilde{P}_l^\dagger = \tilde{U}_1 \tilde{\Sigma}_1^{-1} \tilde{U}_1^H$$

and

$$P_r = V \Sigma_1 V^H, \quad \tilde{P}_r = \tilde{V} \tilde{\Sigma}_1 \tilde{V}^H, \quad \mathcal{R}(A) = \mathcal{R}(U_1), \quad \mathcal{R}(\tilde{A}) = \mathcal{R}(\tilde{U}_1).$$

Therefore,

$$(3.20) \quad U_1^H (P_l^\dagger)^s E \tilde{P}_r^{-t} \tilde{V} = \Sigma_1^{1-s} V^H \tilde{V} \tilde{\Sigma}_1^{-t} - \Sigma_1^{-s} U_1^H \tilde{U}_1 \tilde{\Sigma}_1^{1-t}$$

and

$$(3.21) \quad \tilde{U}_1^H (\tilde{P}_l^\dagger)^t E P_r^{-s} V = \tilde{\Sigma}_1^{-t} \tilde{U}_1^H U_1 \Sigma_1^{1-s} - \tilde{\Sigma}_1^{1-t} \tilde{V}^H V \Sigma_1^{-s}.$$

Noting that $U_2^H (P_l^\dagger)^s = 0$ and $\tilde{U}_2^H (\tilde{P}_l^\dagger)^t = 0$, we obtain from (3.20) and (3.21) that

$$(3.22) \quad \begin{aligned} & \left\| (P_l^\dagger)^s E \tilde{P}_r^{-t} \right\|_F^2 + \left\| (\tilde{P}_l^\dagger)^t E P_r^{-s} \right\|_F^2 \\ &= \left\| U_1^H (P_l^\dagger)^s E \tilde{P}_r^{-t} \tilde{V} \right\|_F^2 + \left\| \tilde{U}_1^H (\tilde{P}_l^\dagger)^t E P_r^{-s} V \right\|_F^2 \\ &= \left\| \Sigma_1^{1-s} V^H \tilde{V} \tilde{\Sigma}_1^{-t} - \Sigma_1^{-s} U_1^H \tilde{U}_1 \tilde{\Sigma}_1^{1-t} \right\|_F^2 \\ &\quad + \left\| \Sigma_1^{1-s} U_1^H \tilde{U}_1 \tilde{\Sigma}_1^{-t} - \Sigma_1^{-s} V^H \tilde{V} \tilde{\Sigma}_1^{1-t} \right\|_F^2. \end{aligned}$$

It follows from Lemma 2.2 that there exists a permutation τ of $\{1, 2, \dots, n\}$ such that

$$(3.23) \quad \begin{aligned} & \left(\sigma_{\min}^2(U_1^H \tilde{U}_1) + \sigma_{\min}^2(V^H \tilde{V}) \right) \sum_{i=1}^n \left(\frac{\sigma_i - \tilde{\sigma}_{\tau(i)}}{\sigma_i^s \tilde{\sigma}_{\tau(i)}^t} \right)^2 \\ & \leq \left\| \Sigma_1^{1-s} V^H \tilde{V} \tilde{\Sigma}_1^{-t} - \Sigma_1^{-s} U_1^H \tilde{U}_1 \tilde{\Sigma}_1^{1-t} \right\|_F^2 \\ & \quad + \left\| \Sigma_1^{1-s} U_1^H \tilde{U}_1 \tilde{\Sigma}_1^{-t} - \Sigma_1^{-s} V^H \tilde{V} \tilde{\Sigma}_1^{1-t} \right\|_F^2. \end{aligned}$$

By Lemma 2.5, (3.22), and (3.23), we get inequality (3.17).

When $m = n$, we have

$$\|\sin \Theta(\mathcal{R}(A), \mathcal{R}(\tilde{A}))\|_2 = 0, \quad P_l^\dagger = P_l^{-1}, \quad \text{and} \quad \tilde{P}_l^\dagger = \tilde{P}_l^{-1}.$$

Hence, the bound (3.18) follows from (3.17). $\square$

REMARK 3.7. If we take $s = 1$ and $t = 0$, then (3.18) reduces to

$$\begin{aligned} \sum_{i=1}^n \left(\frac{\sigma_i - \tilde{\sigma}_{\tau(i)}}{\sigma_i} \right)^2 &\leq \frac{1}{2} \left(\|P_l^{-1}E\|_F^2 + \|EP_r^{-1}\|_F^2 \right) \\ &= \frac{1}{2} \left(\|A^{-1}E\|_F^2 + \|EA^{-1}\|_F^2 \right), \end{aligned}$$

which was proved by Chen in [1].

REMARK 3.8. For nonsingular matrices A and $\tilde{A} = A - E \in \mathbb{C}^{n \times n}$, with SVDs (3.2)–(3.3), Li and Sun [12] derived the relative combined bound

$$\begin{aligned} (3.24) \quad & \left(2 - \|\sin \Psi\|_2^2 - \|\sin \Phi\|_2^2 \right) \\ & \times \sum_{i=1}^n \left(\frac{\sigma_i - \tilde{\sigma}_{\tau(i)}}{\sigma_i^s \tilde{\sigma}_{\tau(i)}^t} \right)^2 + \left(\left(\delta^{(s,t)} \right)^2 + \left(\varsigma^{(s,t)} \right)^2 \right) \\ & \times \left(\|\sin \Psi\|_F^2 + \|\sin \Phi\|_F^2 \right) \\ & \leq \|P_l^{-s} E \tilde{P}_r^{-t}\|_F^2 + \|\tilde{P}_l^{-t} E P_r^{-s}\|_F^2, \end{aligned}$$

where $\delta^{(s,t)}$ is defined by (3.11) and

$$\varsigma^{(s,t)} = \min_{\substack{1 \leq i \leq \ell \\ \ell+1 \leq j \leq n}} \left\{ \frac{|\sigma_i - \tilde{\sigma}_j|}{\sigma_i^s \tilde{\sigma}_j^t} \right\}.$$

Obviously, (3.24) also yields the bound (3.18).

The following example illustrates that $\|\sin \Theta(\mathcal{R}(A), \mathcal{R}(\tilde{A}))\|_2^2$ in (3.17) cannot be removed for $m > n$.

EXAMPLE 3.9. Let

$$A = \begin{bmatrix} 0.1 & 0 \\ 0 & 20 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{A} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}.$$

Then taking $s = 2$, $t = 1$, a simple calculation gives

$$\frac{1}{2} \left(\|(P_l^\dagger)^2 E \tilde{P}_r^{-1}\|_F^2 + \|\tilde{P}_l^\dagger E P_r^{-2}\|_F^2 \right) = 7.3417 \times 10^3.$$

However,

$$\left(\frac{0.1 - 3}{0.1^2 \times 3} \right)^2 + \left(\frac{20 - 4}{20^2 \times 4} \right)^2 = 9.3444 \times 10^3 > 7.3417 \times 10^3$$

and

$$\left(\frac{0.1 - 4}{0.1^2 \times 4} \right)^2 + \left(\frac{20 - 3}{20^2 \times 3} \right)^2 = 9.5063 \times 10^3 > 7.3417 \times 10^3,$$

which shows that the bound

$$\sum_{i=1}^n \left(\frac{\sigma_i - \tilde{\sigma}_{\tau(i)}}{\sigma_i^s \tilde{\sigma}_{\tau(i)}^t} \right)^2 \leq \frac{1}{2} \left(\left\| (P_l^\dagger)^s E \tilde{P}_r^{-t} \right\|_F^2 + \left\| (\tilde{P}_l^\dagger)^t E P_r^{-s} \right\|_F^2 \right)$$

is not valid for $m > n$. However, it holds that

$$\begin{aligned} \left(1 - \frac{1}{2} \left\| \sin \Theta(\mathcal{R}(A), \mathcal{R}(\tilde{A})) \right\|_2^2 \right) & \left[\left(\frac{0.1 - 3}{0.1^2 \times 3} \right)^2 + \left(\frac{20 - 4}{20^2 \times 4} \right)^2 \right] \\ & = 6.7488 \times 10^3 < 7.3417 \times 10^3, \end{aligned}$$

which is consistent with the bound (3.17).

4. Combined perturbation bounds for the GSVD. In this section, we present combined perturbation bounds for generalized singular values and generalized singular subspaces.

Let $\{A, B\}$ be an (m, p, n) -GMP, whose GSVD is given by (1.2) and (1.3). We take an $\ell \in \mathbb{N}$ satisfying

$$(4.1) \quad \max\{n - p, 0\} < \ell < \min\{m, n\}.$$

From (1.2) we formally obtain

$$(4.2) \quad A \begin{bmatrix} \ell & n-\ell \\ Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} \ell & m-\ell \\ U_1 & U_2 \end{bmatrix} \begin{matrix} \ell \\ m-\ell \end{matrix} \begin{bmatrix} \Sigma_{a1} & 0 \\ 0 & \Sigma_{a2} \end{bmatrix}$$

and

$$(4.3) \quad B \begin{bmatrix} \ell & n-\ell \\ Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} p+\ell-n & n-\ell \\ V_1 & V_2 \end{bmatrix} \begin{matrix} p+\ell-n \\ n-\ell \end{matrix} \begin{bmatrix} \Sigma_{b1} & 0 \\ 0 & \Sigma_{b2} \end{bmatrix}.$$

Let the (m, p, n) -GMP $\{\tilde{A}, \tilde{B}\}$ be a perturbed matrix pair. Using the same decomposition as in (1.2), (1.3), (4.2), and (4.3), we use the notation that matrices marked with tildes corresponding to the perturbed matrix pair; e.g., the matrix corresponding to U_1 is denoted by $\tilde{U}_1$. Similar to the notation in Section 3, let

$$\Psi = \Theta(\mathcal{R}(U_1), \mathcal{R}(\tilde{U}_1)), \quad \Phi = \Theta(\mathcal{R}(V_1), \mathcal{R}(\tilde{V}_1)).$$

For the square case, i.e., $m = n = p$, we present the following combined bound for the GSVD:

THEOREM 4.1. *Let $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$ be two (n, n, n) -GMPs with GSVDs (4.2)–(4.3), respectively, and let $\ell \in \mathbb{N}$ satisfy (4.1). Then there exists a permutation τ of $\{1, 2, \dots, \ell\}$ such that*

$$\begin{aligned} (4.4) \quad & (2 - \|\sin \Psi\|_2^2 - \|\sin \Phi\|_2^2) \sum_{i=1}^{\ell} \left| \alpha_i \tilde{\beta}_{\tau(i)} - \beta_i \tilde{\alpha}_{\tau(i)} \right|^2 \\ & + \eta^2 (\|\sin \Psi\|_F^2 + \|\sin \Phi\|_F^2) \leq \left\| (P_Z - P_{\tilde{Z}}) \begin{bmatrix} \tilde{U}_1 & 0 \\ 0 & \tilde{V}_1 \end{bmatrix} \right\|_F^2, \end{aligned}$$

where

$$(4.5) \quad Z = \begin{bmatrix} A^T & B^T \end{bmatrix}^T, \quad \tilde{Z} = \begin{bmatrix} \tilde{A}^T & \tilde{B}^T \end{bmatrix}^T, \quad \eta = \min_{\substack{\ell+1 \leq i \leq n \\ 1 \leq j \leq \ell}} \{|\alpha_i \tilde{\beta}_j - \beta_i \tilde{\alpha}_j|\}.$$

Proof. Let $\Sigma_A = \Lambda$, $\Sigma_B = \Omega$, $\Sigma_{\tilde{A}} = \tilde{\Lambda}$, and $\Sigma_{\tilde{B}} = \tilde{\Omega}$. From the GSVDs of $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$, Li obtained the following two identities (see [8, Equations (4.14) and (4.15)]):

$$(4.6) \quad \Lambda V^H \tilde{V} \tilde{\Omega} - \Omega U^H \tilde{U} \tilde{\Lambda} = \begin{bmatrix} \Omega & -\Lambda \end{bmatrix} \begin{bmatrix} U^H & \\ & V^H \end{bmatrix} (P_Z - P_{\tilde{Z}}) \begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix} \begin{bmatrix} \tilde{\Lambda} \\ \tilde{\Omega} \end{bmatrix}$$

and

$$(4.7) \quad \Lambda U^H \tilde{U} \tilde{\Omega} - \Omega V^H \tilde{V} \tilde{\Lambda} = \begin{bmatrix} \Lambda & \Omega \end{bmatrix} \begin{bmatrix} U^H & \\ & V^H \end{bmatrix} (P_Z - P_{\tilde{Z}}) \begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix} \begin{bmatrix} \tilde{\Omega} \\ -\tilde{\Lambda} \end{bmatrix}.$$

Let $\Sigma_{a1} = \Lambda_1$, $\Sigma_{a2} = \Lambda_2$, $\Sigma_{b1} = \Omega_1$, $\Sigma_{b2} = \Omega_2$ and $\Sigma_{\tilde{a}1} = \tilde{\Lambda}_1$, $\Sigma_{\tilde{a}2} = \tilde{\Lambda}_2$, $\Sigma_{\tilde{b}1} = \tilde{\Omega}_1$, $\Sigma_{\tilde{b}2} = \tilde{\Omega}_2$. Then the left-hand sides of (4.6) and (4.7) can be written in the following block form:

$$(4.8) \quad \begin{aligned} \Lambda V^H \tilde{V} \tilde{\Omega} - \Omega U^H \tilde{U} \tilde{\Lambda} &= \begin{bmatrix} \Lambda_1 V_1^H \tilde{V}_1 \tilde{\Omega}_1 - \Omega_1 U_1^H \tilde{U}_1 \tilde{\Lambda}_1 & \Lambda_1 V_1^H \tilde{V}_2 \tilde{\Omega}_2 - \Omega_1 U_1^H \tilde{U}_2 \tilde{\Lambda}_2 \\ \Lambda_2 V_2^H \tilde{V}_1 \tilde{\Omega}_1 - \Omega_2 U_2^H \tilde{U}_1 \tilde{\Lambda}_1 & \Lambda_2 V_2^H \tilde{V}_2 \tilde{\Omega}_2 - \Omega_2 U_2^H \tilde{U}_2 \tilde{\Lambda}_2 \end{bmatrix}, \\ \Lambda U^H \tilde{U} \tilde{\Omega} - \Omega V^H \tilde{V} \tilde{\Lambda} &= \begin{bmatrix} \Lambda_1 U_1^H \tilde{U}_1 \tilde{\Omega}_1 - \Omega_1 V_1^H \tilde{V}_1 \tilde{\Lambda}_1 & \Lambda_1 U_1^H \tilde{U}_2 \tilde{\Omega}_2 - \Omega_1 V_1^H \tilde{V}_2 \tilde{\Lambda}_2 \\ \Lambda_2 U_2^H \tilde{U}_1 \tilde{\Omega}_1 - \Omega_2 V_2^H \tilde{V}_1 \tilde{\Lambda}_1 & \Lambda_2 U_2^H \tilde{U}_2 \tilde{\Omega}_2 - \Omega_2 V_2^H \tilde{V}_2 \tilde{\Lambda}_2 \end{bmatrix}, \end{aligned}$$

respectively. Considering the $(2, 1)$ -blocks of the two partitioned matrices in (4.8), it follows from Lemmas 2.4 and 2.5 that

$$(4.9) \quad \begin{aligned} &\left\| \Lambda_2 V_2^H \tilde{V}_1 \tilde{\Omega}_1 - \Omega_2 U_2^H \tilde{U}_1 \tilde{\Lambda}_1 \right\|_F^2 + \left\| \Lambda_2 U_2^H \tilde{U}_1 \tilde{\Omega}_1 - \Omega_2 V_2^H \tilde{V}_1 \tilde{\Lambda}_1 \right\|_F^2 \\ &\geq \left(\min_{\substack{\ell+1 \leq i \leq n \\ 1 \leq j \leq \ell}} \{|\alpha_i \tilde{\beta}_j - \beta_i \tilde{\alpha}_j|\} \right)^2 \left(\|\tilde{U}_2^H U_1\|_F^2 + \|\tilde{V}_2^H V_1\|_F^2 \right) \\ &= \eta^2 \left(\|\sin \Psi\|_F^2 + \|\sin \Phi\|_F^2 \right). \end{aligned}$$

Considering the $(1, 1)$ -blocks of the two partitioned matrices in (4.8) and using Lemma 2.2 and Lemma 2.5, there exists a permutation τ of $\{1, \dots, \ell\}$ such that

$$(4.10) \quad \begin{aligned} &\left\| \Lambda_1 V_1^H \tilde{V}_1 \tilde{\Omega}_1 - \Omega_1 U_1^H \tilde{U}_1 \tilde{\Lambda}_1 \right\|_F^2 + \left\| \Lambda_1 U_1^H \tilde{U}_1 \tilde{\Omega}_1 - \Omega_1 V_1^H \tilde{V}_1 \tilde{\Lambda}_1 \right\|_F^2 \\ &\geq \left(\sigma_{\min}^2(U_1^H \tilde{U}_1) + \sigma_{\min}^2(V_1^H \tilde{V}_1) \right) \sum_{i=1}^{\ell} \left| \alpha_i \tilde{\beta}_{\tau(i)} - \beta_i \tilde{\alpha}_{\tau(i)} \right|^2 \\ &= \left(2 - \|\sin \Psi\|_2^2 - \|\sin \Phi\|_2^2 \right) \sum_{i=1}^{\ell} \left| \alpha_i \tilde{\beta}_{\tau(i)} - \beta_i \tilde{\alpha}_{\tau(i)} \right|^2. \end{aligned}$$

From (4.6)–(4.8) we obtain

$$\begin{aligned}
 & \left\| \Lambda_1 V_1^H \tilde{V}_1 \tilde{\Omega}_1 - \Omega_1 U_1^H \tilde{U}_1 \tilde{\Lambda}_1 \right\|_F^2 + \left\| \Lambda_1 U_1^H \tilde{U}_1 \tilde{\Omega}_1 - \Omega_1 V_1^H \tilde{V}_1 \tilde{\Lambda}_1 \right\|_F^2 \\
 & \quad + \left\| \Lambda_2 V_2^H \tilde{V}_1 \tilde{\Omega}_1 - \Omega_2 U_2^H \tilde{U}_1 \tilde{\Lambda}_1 \right\|_F^2 + \left\| \Lambda_2 U_2^H \tilde{U}_1 \tilde{\Omega}_1 - \Omega_2 V_2^H \tilde{V}_1 \tilde{\Lambda}_1 \right\|_F^2 \\
 (4.11) \quad & = \left\| \Lambda V^H \tilde{V}_1 \tilde{\Omega}_1 - \Omega U^H \tilde{U}_1 \tilde{\Lambda}_1 \right\|_F^2 + \left\| \Lambda U^H \tilde{U}_1 \tilde{\Omega}_1 - \Omega V^H \tilde{V}_1 \tilde{\Lambda}_1 \right\|_F^2 \\
 & \leq \left\| \begin{bmatrix} \Omega & -\Lambda \\ \Lambda & \Omega \end{bmatrix} \begin{bmatrix} U^H & \\ & V^H \end{bmatrix} (P_Z - P_{\tilde{Z}}) \begin{bmatrix} \tilde{U}_1 & \\ & \tilde{V}_1 \end{bmatrix} \begin{bmatrix} \tilde{\Lambda}_1 & \tilde{\Omega}_1 \\ \tilde{\Omega}_1 & -\tilde{\Lambda}_1 \end{bmatrix} \right\|_F^2 \\
 & = \left\| (P_Z - P_{\tilde{Z}}) \begin{bmatrix} \tilde{U}_1 & \\ & \tilde{V}_1 \end{bmatrix} \right\|_F^2,
 \end{aligned}$$

where the last equality follows from the fact that the matrices

$$\begin{bmatrix} \Omega & -\Lambda \\ \Lambda & \Omega \end{bmatrix}, \quad \begin{bmatrix} U^H & \\ & V^H \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \tilde{\Lambda} & \tilde{\Omega} \\ \tilde{\Omega} & -\tilde{\Lambda} \end{bmatrix}$$

are unitary. Combining (4.9) and (4.10) with (4.11) yields the desired combined bound (4.4). $\square$

In order to prove a combined bound for the nonsquare case, we need the following lemma:

LEMMA 4.2 ([8, Proposition 2.3]). *Let (α_i, β_i) and (γ_j, δ_j) , $i, j = 1, 2, \dots, n$, be $2n$ generalized singular values. Let $p, q \geq 0$ be two integers, and define*

$$\begin{aligned}
 (\alpha_{n+i}, \beta_{n+i}) &= (\gamma_{n+i}, \delta_{n+i}) = (1, 0), & i &= 1, 2, \dots, p, \\
 (\alpha_{n+p+j}, \beta_{n+p+j}) &= (\gamma_{n+p+j}, \delta_{n+p+j}) = (0, 1), & j &= 1, 2, \dots, q.
 \end{aligned}$$

Then, for any permutation μ of $\{1, 2, \dots, n + p + q\}$, there exists a permutation τ of $\{1, 2, \dots, n\}$ such that

$$\begin{aligned}
 & \left\| \text{diag} \left(\rho((\alpha_1, \beta_1), (\gamma_{\mu(1)}, \delta_{\mu(1)})), \dots, \rho((\alpha_{n+p+q}, \beta_{n+p+q}), (\gamma_{\mu(n+p+q)}, \delta_{\mu(n+p+q)})) \right) \right\|_F \\
 & \geq \left\| \text{diag} \left(\rho((\alpha_1, \beta_1), (\gamma_{\tau(1)}, \delta_{\tau(1)})), \dots, \rho((\alpha_n, \beta_n), (\gamma_{\tau(n)}, \delta_{\tau(n)})) \right) \right\|_F.
 \end{aligned}$$

The next result is a combined bound for the nonsquare case:

THEOREM 4.3. *Let $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$ be two (m, p, n) -GMPs with the GSVDs (4.2)–(4.3), respectively, and let $\ell \in \mathbb{N}$ satisfy (4.1). Then there exists a permutation τ of $\{1, 2, \dots, \ell\}$ such that*

$$\begin{aligned}
 (4.12) \quad & \left(2 - \|\sin \Psi\|_2^2 - \|\sin \Phi\|_2^2 \right) \sum_{i=1}^{\ell} \left| \alpha_i \tilde{\beta}_{\tau(i)} - \beta_i \tilde{\alpha}_{\tau(i)} \right|^2 \\
 & + \tilde{\eta}^2 (\|\sin \Psi\|_F^2 + \|\sin \Phi\|_F^2) \leq \left\| (P_Z - P_{\tilde{Z}}) \begin{bmatrix} \tilde{U}_1 & 0 \\ 0 & \tilde{V}_1 \end{bmatrix} \right\|_F^2,
 \end{aligned}$$

where $Z, \tilde{Z}$, and η are defined by (4.5) and

$$\tilde{\eta} = \begin{cases} \min \left\{ \eta, \min_{1 \leq j \leq \ell} \{\tilde{\beta}_j\}, \min_{\ell+1 \leq i \leq n} \{\alpha_i\} \right\} & \text{if } m > n, p > n, \\ \min \left\{ \eta, \min_{1 \leq j \leq \ell} \{\tilde{\beta}_j\} \right\} & \text{if } m \leq n < p, \\ \min \left\{ \eta, \min_{\ell+1 \leq i \leq n} \{\alpha_i\} \right\} & \text{if } m > n \geq p, \\ \eta & \text{if } m < n, p < n. \end{cases}$$

Proof. When A or B is nonsquare, we reduce the problem to the square case by suitably augmenting the matrices with zero block and unit matrices of appropriate dimensions. Next, we prove the combined perturbation bound (4.12) for the case $m > n$ and $p > n$; the remaining cases can be proved similarly. For the case $m > n$ and $p > n$, let

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} I_{p-n} & 0_{p-n,n} & 0_{p-n,m-n} \\ 0_{m,p-n} & A & 0_{m,m-n} \end{bmatrix}, & \mathcal{B} &= \begin{bmatrix} 0_{p,p-n} & B & 0_{p,m-n} \\ 0_{m-n,p-n} & 0_{m-n,n} & I_{m-n} \end{bmatrix}, \\ \tilde{\mathcal{A}} &= \begin{bmatrix} I_{p-n} & 0_{p-n,n} & 0_{p-n,m-n} \\ 0_{m,p-n} & \tilde{A} & 0_{m,m-n} \end{bmatrix}, & \tilde{\mathcal{B}} &= \begin{bmatrix} 0_{p,p-n} & \tilde{B} & 0_{p,m-n} \\ 0_{m-n,p-n} & 0_{m-n,n} & I_{m-n} \end{bmatrix}. \end{aligned}$$

Then, $\{\mathcal{A}, \mathcal{B}\}$ and $\{\tilde{\mathcal{A}}, \tilde{\mathcal{B}}\}$ are Grassmann matrix pairs of order $m + p - n$, and it follows from (4.2)–(4.3) that the GSVD of $\{\mathcal{A}, \mathcal{B}\}$ is

$$(4.13) \quad \begin{aligned} \mathcal{A}\mathcal{Q} &= \begin{bmatrix} p-n+\ell & m-\ell \\ \mathcal{U}_1 & \mathcal{U}_2 \end{bmatrix} \begin{matrix} p-n+\ell \\ m-\ell \end{matrix} \begin{bmatrix} p-n+\ell & m-\ell \\ \Delta_1 & \Delta_2 \end{bmatrix}, \\ \mathcal{B}\mathcal{Q} &= \begin{bmatrix} p-n+\ell & m-\ell \\ \mathcal{V}_1 & \mathcal{V}_2 \end{bmatrix} \begin{matrix} p-n+\ell \\ m-\ell \end{matrix} \begin{bmatrix} p-n+\ell & m-\ell \\ \Gamma_1 & \Gamma_2 \end{bmatrix}, \end{aligned}$$

where

$$(4.14) \quad \begin{aligned} \mathcal{Q} &= \begin{bmatrix} I_{p-n} & & & \\ & Q_1 & & \\ & & \ddots & \\ & & & Q_2 \\ & & & & I_{m-n} \end{bmatrix}, \\ \mathcal{U}_1 &= \begin{bmatrix} I_{p-n} & 0 \\ 0 & U_1 \end{bmatrix}, & \mathcal{U}_2 &= \begin{bmatrix} 0 \\ U_2 \end{bmatrix}, \\ \mathcal{V}_1 &= \begin{bmatrix} V_1 \\ 0 \end{bmatrix}, & \mathcal{V}_2 &= \begin{bmatrix} V_2 & 0 \\ 0 & I_{m-n} \end{bmatrix}, \\ \Delta_1 &= \begin{bmatrix} I_{p-n} & \\ & \Sigma_{a1} \end{bmatrix}, & \Delta_2 &= \begin{bmatrix} \Sigma_{a2} & 0_{m-\ell,m-n} \end{bmatrix}, \\ \Gamma_1 &= \begin{bmatrix} 0_{p-n+\ell,p-n} & \Sigma_{b1} \end{bmatrix}, & \Gamma_2 &= \begin{bmatrix} \Sigma_{b2} & \\ & I_{m-n} \end{bmatrix}. \end{aligned}$$

Let the perturbed Grassmann matrix pair $\{\tilde{\mathcal{A}}, \tilde{\mathcal{B}}\}$ have the same decomposition (4.13)–(4.14), where the corresponding matrices are marked with tildes. Then we have

$$\begin{aligned} \sigma\{\Delta_1, \Gamma_1\} &= \underbrace{\{(1, 0), \dots, (1, 0)\}}_{p-n} \cup \sigma\{\Sigma_{a1}, \Sigma_{b1}\}, \\ \sigma\{\Delta_2, \Gamma_2\} &= \sigma\{\Sigma_{a2}, \Sigma_{b2}\} \cup \underbrace{\{(0, 1), \dots, (0, 1)\}}_{m-n} \end{aligned}$$

and

$$\begin{aligned} \sigma\{\tilde{\Delta}_1, \tilde{\Gamma}_1\} &= \underbrace{\{(1, 0), \dots, (1, 0)\}}_{p-n} \cup \sigma\{\Sigma_{\tilde{a}1}, \Sigma_{\tilde{b}1}\}, \\ \sigma\{\tilde{\Delta}_2, \tilde{\Gamma}_2\} &= \sigma\{\Sigma_{\tilde{a}2}, \Sigma_{\tilde{b}2}\} \cup \underbrace{\{(0, 1), \dots, (0, 1)\}}_{m-n}. \end{aligned}$$

For the GMPs $\{\mathcal{A}, \mathcal{B}\}$ and $\{\tilde{\mathcal{A}}, \tilde{\mathcal{B}}\}$, we use Lemma 4.2 and bound (4.4) for square matrices to obtain

$$\begin{aligned}
 (4.15) \quad & \left(2 - \|\sin \Theta(\mathcal{R}(\tilde{\mathcal{U}}_1), \mathcal{R}(\mathcal{U}_1))\|_2^2 - \|\sin \Theta(\mathcal{R}(\tilde{\mathcal{V}}_1), \mathcal{R}(\mathcal{V}_1))\|_2^2\right) \\
 & \times \sum_{i=1}^{\ell} |\alpha_i \tilde{\beta}_{\tau(i)} - \beta_i \tilde{\alpha}_{\tau(i)}|^2 \\
 & + \tilde{\eta}^2 (\|\sin \Theta(\mathcal{R}(\tilde{\mathcal{U}}_1), \mathcal{R}(\mathcal{U}_1))\|_F^2 \\
 & + \|\sin \Theta(\mathcal{R}(\tilde{\mathcal{V}}_1), \mathcal{R}(\mathcal{V}_1))\|_F^2) \leq \left\| (P_Z - P_{\tilde{Z}}) \begin{bmatrix} \tilde{\mathcal{U}}_1 & 0 \\ 0 & \tilde{\mathcal{V}}_1 \end{bmatrix} \right\|_F^2
 \end{aligned}$$

for a permutation τ of $\{1, 2, \dots, \ell\}$, where

$$\mathcal{Z} = [\mathcal{A}^T \quad \mathcal{B}^T]^T, \quad \tilde{\mathcal{Z}} = [\tilde{\mathcal{A}}^T \quad \tilde{\mathcal{B}}^T]^T \in \mathbb{C}^{2(m+p-n) \times (m+p-n)},$$

and

$$\begin{aligned}
 (4.16) \quad & \tilde{\eta} = \min \left\{ \min_{\substack{\ell+1 \leq i \leq n \\ 1 \leq j \leq \ell}} \{|\alpha_i \tilde{\beta}_j - \beta_i \tilde{\alpha}_j|\}, \min_{1 \leq j \leq \ell} \{|0 \times \tilde{\alpha}_j - 1 \times \tilde{\beta}_j|\}, \right. \\
 & \left. \min_{\ell+1 \leq i \leq n} \{|0 \times \beta_i - 1 \times \alpha_i|\} \right\} \\
 & = \min \left\{ \eta, \min_{1 \leq j \leq \ell} \{\tilde{\beta}_j\}, \min_{\ell+1 \leq i \leq n} \{\alpha_i\} \right\}.
 \end{aligned}$$

It is easy to verify from the definitions of $\mathcal{U}_i, \mathcal{V}_i$ and $\tilde{\mathcal{U}}_i, \tilde{\mathcal{V}}_i$ together with Lemma 2.5 that, for any unitarily invariant norm $\|\cdot\|$,

$$(4.17) \quad \|\sin \Theta(\mathcal{R}(\tilde{\mathcal{U}}_1), \mathcal{R}(\mathcal{U}_1))\| = \|\tilde{\mathcal{U}}_2^H \mathcal{U}_1\| = \|\tilde{U}_2^H U_1\| = \|\sin \Psi\|$$

and

$$(4.18) \quad \|\sin \Theta(\mathcal{R}(\tilde{\mathcal{V}}_1), \mathcal{R}(\mathcal{V}_1))\| = \|\tilde{\mathcal{V}}_2^H \mathcal{V}_1\| = \|\tilde{V}_2^H V_1\| = \|\sin \Phi\|.$$

In particular, the identities in (4.17) and (4.18) also hold for $\|\cdot\|_2$ and $\|\cdot\|_F$. A simple calculation gives

$$P_Z = \mathcal{Z} [(\mathcal{Z}^H \mathcal{Z})^{-1} \mathcal{Z}^H] = \begin{bmatrix} I_{p-n} & & \\ & P_Z & \\ & & I_{m-n} \end{bmatrix}, \quad P_{\tilde{Z}} = \begin{bmatrix} I_{p-n} & & \\ & P_{\tilde{Z}} & \\ & & I_{m-n} \end{bmatrix},$$

which leads to

$$(4.19) \quad \left\| (P_Z - P_{\tilde{Z}}) \begin{bmatrix} \tilde{\mathcal{U}}_1 & \\ & \tilde{\mathcal{V}}_1 \end{bmatrix} \right\|_F = \left\| (P_Z - P_{\tilde{Z}}) \begin{bmatrix} \tilde{U}_1 & \\ & \tilde{V}_1 \end{bmatrix} \right\|_F.$$

Combining (4.15)–(4.19) yields (4.12). $\square$

REMARK 4.4. The combined perturbation bounds (4.4) and (4.12) can also be used to derive the Hoffman–Wielandt-type bound for generalized singular values given by Li [8]. In fact, when $\ell = n$, then $\|\sin \Psi\|_2 = \|\sin \Phi\|_2 = 0$ and $\|\sin \Psi\|_F = \|\sin \Phi\|_F = 0$. Inequalities (4.4) and (4.12) reduce to

$$\sqrt{\sum_{i=1}^n |\alpha_i \tilde{\beta}_{\tau(i)} - \beta_i \tilde{\alpha}_{\tau(i)}|^2} \leq \frac{1}{\sqrt{2}} \|P_Z - P_{\tilde{Z}}\|_F,$$

which is the result of [8, Theorem 3.2].

REMARK 4.5. The combined bound (3.4) can be obtained from (4.4) and (4.12) using a special limiting procedure described by Sun [19] and Li [9]. For example, in the square case, let $A = U\Sigma V^H$ and $\tilde{A} = A - E = \tilde{U}\tilde{\Sigma}\tilde{V}^H$ be the singular value decompositions of A and $\tilde{A} \in \mathbb{C}^{n \times n}$, respectively, where

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n), \quad \tilde{\Sigma} = \text{diag}(\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_n).$$

Then, for $t > 0$, the Grassmann matrix pair $\{A, tI\}$ has the following generalized singular value decompositions:

$$U^H A Q = \Sigma_A, \quad V^H (tI) Q = \Sigma_B,$$

where

$$\begin{aligned} Q &= V \text{diag} \left(\frac{1}{\sqrt{\sigma_1^2 + t^2}}, \dots, \frac{1}{\sqrt{\sigma_n^2 + t^2}} \right), \\ \Sigma_A &= \text{diag} \left(\frac{\sigma_1}{\sqrt{\sigma_1^2 + t^2}}, \dots, \frac{\sigma_n}{\sqrt{\sigma_n^2 + t^2}} \right), \\ \Sigma_B &= \text{diag} \left(\frac{t}{\sqrt{\sigma_1^2 + t^2}}, \dots, \frac{t}{\sqrt{\sigma_n^2 + t^2}} \right). \end{aligned}$$

Let

$$U = [U_1 \ U_2], \quad V = [V_1 \ V_2], \quad \Sigma_A = \text{diag}(\Lambda_1, \Lambda_2), \quad \Sigma_B = \text{diag}(\Omega_1, \Omega_2),$$

where $U_1, V_1 \in \mathbb{C}^{n \times \ell}$ and $\Lambda_1, \Omega_1 \in \mathbb{C}^{\ell \times \ell}$.

Let the Grassmann matrix pair $\{\tilde{A}, tI\}$ have the same decompositions as above, where the corresponding matrices are marked with tildes. Then, by (4.4), we get

$$\begin{aligned} &\eta^2 \left(\|\sin \Psi\|_F^2 + \|\sin \Phi\|_F^2 \right) + \left(2 - \|\sin \Psi\|_2^2 - \|\sin \Phi\|_2^2 \right) \\ (4.20) \quad &\times \sum_{i=1}^{\ell} \left| \frac{\sigma_i}{\sqrt{\sigma_i^2 + t^2}} \frac{t}{\sqrt{\tilde{\sigma}_{\tau(i)}^2 + t^2}} - \frac{t}{\sqrt{\sigma_i^2 + t^2}} \frac{\tilde{\sigma}_{\tau(i)}}{\sqrt{\tilde{\sigma}_{\tau(i)}^2 + t^2}} \right|^2 \\ &\leq \left\| (P_Z - P_{\tilde{Z}}) \begin{bmatrix} \tilde{U}_1 \\ \tilde{V}_1 \end{bmatrix} \right\|^2, \end{aligned}$$

where $Z = [A^T \ tI]^T$, $\tilde{Z} = [\tilde{A}^T \ tI]^T$, and

$$\eta = \min_{\substack{\ell+1 \leq i \leq n \\ 1 \leq j \leq \ell}} \left| \frac{\sigma_i}{\sqrt{\sigma_i^2 + t^2}} \frac{t}{\sqrt{\tilde{\sigma}_j^2 + t^2}} - \frac{t}{\sqrt{\sigma_i^2 + t^2}} \frac{\tilde{\sigma}_j}{\sqrt{\tilde{\sigma}_j^2 + t^2}} \right|.$$

Since (see [8])

$$t(P_Z - P_{\tilde{Z}}) \rightarrow \begin{bmatrix} 0 & A - \tilde{A} \\ A^H - \tilde{A}^H & 0 \end{bmatrix}, \quad \text{as } t \rightarrow +\infty,$$

multiplying both sides of (4.20) by t^2 and letting t tend to infinity lead to the combined bound (3.4) for $m = n$, that is, the result of [12, Theorem 3.1].

5. Conclusions. In this paper, we obtain some new combined bounds for the standard singular value decomposition based on Lemma 2.2. We extend some existing perturbation results given in [1, 2, 12] and obtain combined bounds for the generalized singular value decomposition.

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