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# AN EXTENDED TWO-STEP METHOD FOR SOLVING GENERALIZED INVERSE EIGENVALUE PROBLEMS*

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**Abstract.** In this paper, we study numerical algorithms for generalized inverse eigenvalue problems. We propose an extended two-step method, motivated by the technique of K. Aishima, which, compared with other existing methods, requires fewer operations and/or has a higher convergence order. Under appropriate assumptions, the new method is proven to have a third-order root-convergence rate. The numerical results show that the new method is effective and robust.

**Key words.** generalized inverse eigenvalue problems, inverse eigenvalue problems, extended two-step method, Newton's method, cubic root-convergence

**AMS subject classifications.** 65F18, 65F15, 15A18

## 1. Introduction.

The generalized inverse eigenvalue problem (GIEP) is defined as follows: Let $\{A_i\}_{i=0}^n$ and $\{B_i\}_{i=0}^n$ be $2(n+1)$ real symmetric $n \times n$ matrices, and let $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$. We define

$$A(\mathbf{c}) := A_0 + \sum_{i=1}^{n} c_i A_i \qquad \text{and} \qquad B(\mathbf{c}) := B_0 + \sum_{i=1}^{n} c_i B_i, \tag{1.1}$$

with $B(\mathbf{c}) > 0$, and assume that the eigenvalues $\{\lambda_i((A(\mathbf{c}), B(\mathbf{c})))\}_{i=1}^n$ of the generalized eigenvalue problem $A(\mathbf{c})\mathbf{x} = \lambda B(\mathbf{c})\mathbf{x}$ are ordered as

$$\lambda_1((A(\mathbf{c}), B(\mathbf{c}))) \leq \lambda_2((A(\mathbf{c}), B(\mathbf{c}))) \leq \cdots \leq \lambda_n((A(\mathbf{c}), B(\mathbf{c}))).$$

Given real numbers $\{\lambda_i^*\}_{i=1}^n$ such that $\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_n^*$, the aim of solving the GIEP is to find $\mathbf{c}^* \in \mathbb{R}^n$ such that

$$\lambda_i((A(\mathbf{c}^*), B(\mathbf{c}^*))) = \lambda_i^*, \qquad i = 1, \ldots, n. \tag{1.2}$$

A frequently encountered special case of the GIEP is obtained when $B(\mathbf{c}) \equiv I$, where $I$ represents the identity matrix. This problem is known as the inverse eigenvalue problem (for short, IEP), which has been extensively studied; see [1, 2, 4, 5, 14, 20, 24, 26, 29, 31]. In recent years, numerical methods like those in [4, 5, 14, 26, 29] have been proposed to avoid solving the (approximate) Jacobian equation involved in the IEP. Compared with other methods and as shown by many numerical examples, these approaches do not suffer from instability caused by a possible ill-conditioning of the (approximate) Jacobian equation. Wen et al. in [29] proposed a two-step inexact Newton–Chebyshev-like method with cubic root-convergence rate, in which the approximate eigenvectors are obtained by applying the one-step inverse power method. In this case, solving the approximate Jacobian equations is avoided by using the Chebyshev method to approximate the inverse of the Jacobian matrix. Motivated by the technique in [1, 2] and the Ulm–Chebyshev steps for solving general nonlinear equations, an improved two-step method was proposed in [31] for the IEP, where the (approximate) Jacobian equations are also

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avoided, and the proposed method was proved to be convergent with a cubic root-convergence rate.

As a generalization of the IEP, the GIEP has also attracted the attention of researchers. The GIEP arises in various areas of scientific computing and engineering applications, such as the study of a vibrating string in [5], the educational testing problem in [14], the factor analysis in [18], the discrete analogue of inverse Sturm–Liouville problems in [17, 24], and structural design in [15, 16, 19, 20, 27, 30].

We now describe several examples arising in various areas of application.

APPLICATION 1.1. Consider the following boundary value problem

$$-\frac{d^2u}{dx^2} + p(x)u = \lambda u, \qquad u(0) = u(\pi) = 0, \tag{1.3}$$

and the associated inverse Sturm–Liouville problem [17] in which the goal is to recover the potential function $p(x)$ from spectral data. Let us define $h := \frac{\pi}{n+1}$, $u_i = u(ih)$, $p_i = p(ih)$, $i = 1, 2, \ldots, n$, and assume that $\{\lambda_j^*\}_{j=1}^n$ is given. Using finite differences to approximate $u''$, a discretization of (1.3) is

$$\frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + p_i u_i = \lambda_j^* u_i, \qquad i = 1, 2, \ldots, n, \quad u_0 = u_{n+1} = 0,$$

in which $\lambda_j^*$ is an eigenvalue in the set $\{\lambda_j^*\}_{j=1}^n$. We can write this problem in the form of a GIEP with

$$A_0 := \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & & & 0 \\ 0 & -1 & 2 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & & & & & -1 \\ 0 & 0 & \cdots & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

corresponding to the second-order derivative. Regarding $p(x)$, we introduce $A_k$ and $B_k$, for $1 \le k \le n$, defined as follows:

$$A_k = \{a_{ij}^k\}, \qquad a_{ij}^k = \begin{cases} 1 & \text{if } k = i = j, \\ 0 & \text{otherwise,} \end{cases}$$
$$B_0 = I, \qquad B_k = 0, \qquad k = 1, 2, \ldots, n.$$

Note that, for prescribed eigenvalues $\lambda_1^* \le \lambda_2^* \le \cdots \le \lambda_n^*$, the solution $\mathbf{c}^*$ of the inverse eigenvalue problem corresponds to $p(x)$, namely, $c_i^* \approx p(ih)$, for $1 \le i \le n$.

APPLICATION 1.2. It is shown in [27] that a certain physical application can be modeled as a system of equations of motion given by

$$m_i \frac{d^2u_i}{dt^2} = \begin{cases} -k_iu_i + k_{i+1}(u_{i+1} - u_i) & i = 1, \\ -k_i(u_i - u_{i-1}) + k_{i+1}(u_{i+1} - u_i) & i = 2, 3, \ldots, n-1, \\ -k_i(u_i - u_{i-1}) & i = n, \end{cases}$$

where $m_i$ and $k_i$ (for $i = 1, 2, \ldots, n$) are the mass of the $i$th particle and the stiffness of the $i$th spring, respectively. We can write the matrix form of the equations as

$$M\frac{d^2u}{dt^2} = -Ku,$$

in which

$$K=\begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 & 0\\ -k_2 & k_2+k_3 & -k_3 & & & 0\\ 0 & -k_3 & k_3+k_4 & \ddots & & 0\\ \vdots & & \ddots & \ddots & & \vdots\\ 0 & & & & & -k_n\\ 0 & 0 & \cdots & & -k_n & k_n \end{bmatrix}, \quad M=\mathrm{diag}(m_1,m_2,\ldots,m_n).$$

The matrices $M$ and $K$ are called the mass and stiffness matrices, respectively. A fundamental solution $u(t) = e^{i\omega t}$ results in the symmetric eigenvalue problem $Kx = \lambda Mx$, where $\lambda = \omega^2$. To find the stiffness parameters $k_i$ corresponding to given natural frequencies $\omega_1 < \omega_2 < \cdots < \omega_n$, we can write this problem in the form of a GIEP as follows:

$$K = K_0 + \sum_{i=1}^{n} k_i K_i,$$

where

$$\begin{array}{llll} K_0 = 0, & K_1 = e_1 e_1^T, & K_i = (e_{i-1}-e_i)(e_{i-1}-e_i)^T, & i = 2,3,\ldots,n,\\ M_0 = M, & M_i = 0, & & i = 1,2,\ldots,n, \end{array}$$

in which $e_j$ is the $j$th column of the identity matrix.

APPLICATION 1.3. In many practical applications of structures with $l$ elements and $n$ displacement degrees of freedom, such as in truss structure design, by assuming that the vector $\mathbf{c} = (c_1, c_2, \ldots, c_l)$ is a set of design parameters, the global stiffness matrix $A(\mathbf{c})$ and global mass matrix $B(\mathbf{c})$ are defined as

$$A(\mathbf{c}) := K_0 + \sum_{i=1}^{l} c_i K_i \qquad \text{and} \qquad B(\mathbf{c}) := M_0 + \sum_{i=1}^{l} c_i M_i,$$

where $K_i$ and $M_i$ are the stiffness and mass matrices of the $i$th element of the structure. In these problems, the goal is to determine $c_i > 0$, $i = 1, 2, \ldots, l$, such that $\{\lambda_1^*, \lambda_2^*, \ldots, \lambda_l^*\}$ is the set of eigenvalues of the generalized eigenvalue problem $A(\mathbf{c})\mathbf{x} = \lambda B(\mathbf{c})\mathbf{x}$.

On the other hand, various applications of the GIEP have made it a favorite topic for analysis. Many recent studies have focused on the theory of solvability and mathematical properties of general GIEPs; see [6, 8, 22]. The GIEP is equivalent to finding a solution $\mathbf{c}^* \in \mathbb{R}^n$ of the following nonlinear equation:

$$\mathbf{f}(\mathbf{c}) := \begin{bmatrix} \lambda_1((A(\mathbf{c}), B(\mathbf{c}))) - \lambda_1^*\\ \lambda_2((A(\mathbf{c}), B(\mathbf{c}))) - \lambda_2^*\\ \cdots\\ \cdots\\ \lambda_n((A(\mathbf{c}), B(\mathbf{c}))) - \lambda_n^* \end{bmatrix} = \mathbf{0}; \tag{1.4}$$

see [1, 3, 9, 10, 11, 21]. Based on this equivalence, Dai and Lancaster in [9] employed Newton's method for solving the nonlinear equation (1.4) by extending the ideas developed in [14], for which each iteration step of Newton's method requires computing the complete solution of the generalized eigenvalue problem $A(\mathbf{c})\mathbf{x} = \lambda B(\mathbf{c})\mathbf{x}$. Recently, in order to reduce the expense of Newton's method, different Newton-type methods have also been proposed

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in [10, 11], including (inexact) Newton-like methods and the extended Cayley transform method, which solve inexactly both the Jacobian system and the inverse iteration update. In particular, the extended Cayley transform method computes the approximate eigenvectors instead of the exact eigenvectors. Nevertheless, approximate eigenvectors are required to be orthogonal, which leads to solving exactly $n$ linear equations, and so the computational complexity of the Cayley transform method is at least $O(n^3)$.

In order to optimize the calculation of the approximate eigenvectors, K. Aishima in [3] designed an improved method for solving the GIEP (1.2). In 2025, Luo and Shen in [21] designed an Ulm-like method which avoids solving the (approximate) Jacobian equations to find the approximate eigenvectors. However, all the methods mentioned above are quadratically convergent.

In this paper, motivated by the use of the Chebyshev method in [12] for solving general nonlinear equations and Aishima's method in [3] for the GIEP, we propose an extended two-step method for solving the GIEP and show that the proposed algorithm is convergent with a cubic root-convergence rate. The main contributions of this paper are as follows:

(i) we avoid solving approximate Jacobian equations by utilizing Ulm–Chebyshev steps to approximate the inverses of the approximate Jacobian matrices;
(ii) our proposed method employs the technique in [12] instead of the Cayley transform to approximate the eigenvectors;
(iii) our proposed method is convergent with cubic root-convergence rate, which speeds up the convergence rate of other existing methods.

The structure of the paper is as follows: In Section 2 we present the extended two-step method for solving the GIEP. In Section 3.1, we provide some useful foundational knowledge required for this article. In Section 3.2, under appropriate assumptions, we demonstrate that the proposed method is convergent with a cubic root-convergence rate. In Section 4, we report some numerical results, which explain the efficiency of our new method.

**Notation.** In this article, we use the following symbols and definitions: The Euclidean vector norm or its corresponding induced matrix norm and the matrix Frobenius norm are denoted by $\|\cdot\|$ and $\|\cdot\|_F$, respectively. Moreover, $A > 0$ ($A \geq 0$) denotes that $A$ is a symmetric positive definite matrix (symmetric positive semidefinite matrix). Usually, we use $\mathbf{B}(\mathbf{x}, \delta)$ to denote the open ball in $\mathbb{R}^n$ with center $\mathbf{x} \in \mathbb{R}^n$ and radius $\delta$. By $I$ we denote the identity matrix of appropriate dimension. The set of all $n \times n$ real symmetric matrices and all $n \times n$ real diagonal matrices with increasing diagonal entries are denoted by $\mathcal{S}(n)$ and $\mathcal{D}(n)$, respectively. Define

$$\boldsymbol{\lambda}(\mathbf{c}) = (\lambda_1((A(\mathbf{c}), B(\mathbf{c}))), \lambda_2((A(\mathbf{c}), B(\mathbf{c}))), \ldots, \lambda_n((A(\mathbf{c}), B(\mathbf{c}))))^T$$

and $\Lambda_* = \operatorname{diag}(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \in \mathbb{R}^{n\times n}$. Let $\mathbf{c}^*$ be a solution of the GIEP, and define the Jacobian matrix $J(\mathbf{c}^*)$ by

$$[J(\mathbf{c}^*)]_{ij} = \mathbf{q}_i(\mathbf{c}^*)^T \left(A_j - \lambda_i^* \cdot B_j\right) \mathbf{q}_i(\mathbf{c}^*),$$

where $1 \leq i, j \leq n$ and $\{\mathbf{q}_i(\mathbf{c}^*)\}_{i=1}^n$ are the eigenvectors of $(A(\mathbf{c}^*), B(\mathbf{c}^*))$ corresponding to the eigenvalues $\{\lambda_i^*\}_{i=1}^n$. Let $Q_* = (\mathbf{q}_1(\mathbf{c}^*), \mathbf{q}_2(\mathbf{c}^*), \ldots, \mathbf{q}_n(\mathbf{c}^*))$ satisfying

$$\begin{cases} Q_*^T A(\mathbf{c}^*) Q_* = \Lambda_*, \\ Q_*^T B(\mathbf{c}^*) Q_* = I. \end{cases}$$

**2. The extended two-step method.** In this section, motivated by the use of the Chebyshev method in [12] and Aishima's method in [3], we propose an extended two-step method for solving the GIEP (1.2). For $1 \le t \le n$, without loss of generality, we assume that

$$\lambda_1^* = \lambda_2^* = \cdots = \lambda_t^* < \lambda_{t+1}^* < \cdots < \lambda_n^*. \tag{2.1}$$

Suppose that $\mathbf{c}^k$ and $Q_k := (\mathbf{q}_1^k, \mathbf{q}_2^k, \ldots, \mathbf{q}_n^k)$ have been computed as approximations of $\mathbf{c}^*$ and $Q_*$, respectively. Define $Q_k^*$ by

$$Q_k^* := \arg\min_{Q_*} \| \sqrt{B(\mathbf{c}^*)}\,(Q_k - Q_*)\,\|_F. \tag{2.2}$$

Then we obtain

$$\begin{aligned} (Q_k^*)^T A(\mathbf{c}^*) Q_k^* &= \Lambda^*, \\ (Q_k^*)^T B(\mathbf{c}^*) Q_k^* &= I. \end{aligned} \tag{2.3}$$

Thus, by $B(\mathbf{c}^*) > 0$, the matrix $\sqrt{B(\mathbf{c}^*)} Q_k^*$ is orthogonal. Let $F_k \in \mathbb{R}^{n\times n}$ be such that

$$Q_k = Q_k^*(I + F_k). \tag{2.4}$$

Therefore, by the arguments in [3, Lemma 1] it holds that

$$[F_k]_{ij} = [F_k]_{ji}, \qquad (i,j) \in \mathcal{I}_1, \tag{2.5}$$

where $\mathcal{I}_1 := \{(i,j) | 1 \le i,j \le t,\ i \ne j\}$. Furthermore, by (2.3) and (2.4) we get

$$Q_k^T B(\mathbf{c}^*) Q_k = I + F_k + F_k^T + \Delta_1^k, \tag{2.6}$$

$$Q_k^T A(\mathbf{c}^*) Q_k = \Lambda^* + \Lambda^* F_k + F_k^T \Lambda^* + \Delta_2^k, \tag{2.7}$$

where

$$\|\Delta_1^k\| \le \|F_k\|^2 \qquad \text{and} \qquad \|\Delta_2^k\| \le \|\Lambda^*\| \|F_k\|^2. \tag{2.8}$$

By omitting the quadratic terms in (2.6) and (2.7), we may expect that $\bar{F}_k$ and $\bar{\mathbf{c}}^k$, which are approximations of $F_k$ and $\mathbf{c}^*$, respectively, satisfy the following equations:

$$Q_k^T B(\bar{\mathbf{c}}^k) Q_k = I + \bar{F}_k + \bar{F}_k^T, \tag{2.9}$$

$$Q_k^T A(\bar{\mathbf{c}}^k) Q_k = \Lambda^* + \Lambda^* \bar{F}_k + \bar{F}_k^T \Lambda^*. \tag{2.10}$$

Moreover, similar to the proof of (2.5), we obtain

$$[\bar{F}_k]_{ij} = [\bar{F}_k]_{ji}, \qquad (i,j) \in \mathcal{I}_1, \tag{2.11}$$

and combining (2.9) and (2.10), we have

$$\left[Q_k^T B(\bar{\mathbf{c}}^k) Q_k \Lambda^* - Q_k^T A(\bar{\mathbf{c}}^k) Q_k\right]_{ij} = \left[\bar{F}_k \Lambda^* - \Lambda^* \bar{F}_k\right]_{ij}, \qquad (i,j) \in \mathcal{I}_2, \tag{2.12}$$

where $\mathcal{I}_2 := \{(i,j) | 1 \le i,j \le n,\ (i,j) \notin \mathcal{I}_1,\ i \ne j\}$.

Now, our goal is to obtain $\bar{F}_k$. In fact, by (2.9) and (2.11) we get

$$[\bar{F}_k]_{ii} := \frac{(\mathbf{q}_i^k)^T B(\bar{\mathbf{c}}^k) \mathbf{q}_i^k - 1}{2}, \qquad 1 \le i \le n, \tag{2.13}$$

$$[\bar{F}_k]_{ij} := \frac{(\mathbf{q}_i^k)^T B(\bar{\mathbf{c}}^k) \mathbf{q}_j^k}{2}, \qquad (i,j) \in \mathcal{I}_1. \tag{2.14}$$

Finally, from (2.12), we have

$$[\bar{F}_k]_{ij} := \frac{\lambda_j^*(\mathbf{q}_i^k)^T B(\bar{\mathbf{c}}^k)\mathbf{q}_j^k - (\mathbf{q}_i^k)^T A(\bar{\mathbf{c}}^k)\mathbf{q}_j^k}{\lambda_j^* - \lambda_i^*}, \qquad (i,j) \in \mathcal{I}_2. \tag{2.15}$$

Therefore, we compute the matrix $\bar{Q}_k := [\bar{\mathbf{q}}_1^k, \bar{\mathbf{q}}_2^k, \ldots, \bar{\mathbf{q}}_n^k]$ by

$$\bar{Q}_k = Q_k(I - \bar{F}_k), \tag{2.16}$$

where $I - \bar{F}_k$ is the first-order approximation of $(I + \bar{F}_k)^{-1}$ obtained by the Neumann series. Next, let $G_k \in \mathbb{R}^{n\times n}$ be such that

$$\bar{Q}_k = Q_k^*(I + G_k). \tag{2.17}$$

Then, by an argument analogous to that used to obtain (2.5)–(2.15), we have

$$[G_k]_{ij} = [G_k]_{ji}, \qquad (i,j) \in \mathcal{I}_1,$$

$$\bar{Q}_k^T B(\mathbf{c}^*)\bar{Q}_k = I + G_k + G_k^T + \Delta_3^k, \tag{2.18}$$

$$\bar{Q}_k^T A(\mathbf{c}^*)\bar{Q}_k = \Lambda^* + \Lambda^* G_k + G_k^T \Lambda^* + \Delta_4^k, \tag{2.19}$$

where

$$\|\Delta_3^k\| \leq \|G_k\|^2 \qquad \text{and} \qquad \|\Delta_4^k\| \leq \|\Lambda^*\|\|G_k\|^2. \tag{2.20}$$

Let $\bar{G}_k$ and $\mathbf{c}^{k+1}$ be the approximations of $G_k$ and $\mathbf{c}^*$ such that

$$\bar{Q}_k^T B(\mathbf{c}^{k+1})\bar{Q}_k = I + \bar{G}_k + \bar{G}_k^T,$$

$$\bar{Q}_k^T A(\mathbf{c}^{k+1})\bar{Q}_k = \Lambda^* + \Lambda^* \bar{G}_k + \bar{G}_k^T \Lambda^*,$$

$$[\bar{G}_k]_{ij} = [\bar{G}_k]_{ji}, \quad (i,j) \in \mathcal{I}_1,$$

$$[\bar{G}_k]_{ii} := \frac{(\bar{\mathbf{q}}_i^k)^T B(\mathbf{c}^{k+1})\bar{\mathbf{q}}_i^k - 1}{2}, \qquad 1 \leq i \leq n, \tag{2.21}$$

$$[\bar{G}_k]_{ij} := \frac{(\bar{\mathbf{q}}_i^k)^T B(\mathbf{c}^{k+1})\bar{\mathbf{q}}_j^k}{2}, \qquad (i,j) \in \mathcal{I}_1, \tag{2.22}$$

$$[\bar{G}_k]_{ij} := \frac{\lambda_j^*(\bar{\mathbf{q}}_i^k)^T B(\mathbf{c}^{k+1})\bar{\mathbf{q}}_j^k - (\bar{\mathbf{q}}_i^k)^T A(\mathbf{c}^{k+1})\bar{\mathbf{q}}_j^k}{\lambda_j^* - \lambda_i^*}, \qquad (i,j) \in \mathcal{I}_2. \tag{2.23}$$

Finally, $Q_{k+1}$ is computed by

$$Q_{k+1} = \bar{Q}_k(I - \bar{G}_k). \tag{2.24}$$

Therefore, we obtain the following extended two-step method for solving the GIEP (1.2) with the eigenvalues given by (2.1).

**Algorithm I: the extended two-step method**

Step 1. Given $\mathbf{c}^0$ and $Q_0 := (\mathbf{q}_1^0, \mathbf{q}_2^0, \ldots, \mathbf{q}_n^0)$, define $J_0 = J(\mathbf{c}^0)$ and the vector $\mathbf{b}^0$ by:

$$\begin{aligned} [J_0]_{ij} &= (\mathbf{p}_i^0)^T (A_j - \lambda_i^* \cdot B_j)\mathbf{p}_i^0, \qquad & 1 \leq i,j \leq n, \\ [\mathbf{b}]_i^0 &= (\mathbf{p}_i^0)^T (A_0 - \lambda_i^* \cdot B_0)\mathbf{p}_i^0, \qquad & 1 \leq i \leq n. \end{aligned} \tag{2.25}$$

Let $B_0 \in \mathbb{R}^{n\times n}$ be such that

$$\|I - B_0 J(\mathbf{c}^0)\| \leq \mu,$$

where $\mu$ is a positive constant.

Step 2. For $k = 0, 1, 2, \ldots$ until convergence:

(a) Calculate $\bar{\mathbf{c}}^k$ by:

$$\bar{\mathbf{c}}^k = \mathbf{c}^k - B_k(J_k\mathbf{c}^k + \mathbf{b}^k). \tag{2.26}$$

(b) Form the matrix $\bar{F}_k$ by (2.13), (2.14), and (2.15).

(c) Calculate $\bar{Q}_k$ by (2.16).

(d) Calculate $\boldsymbol{\rho}^k$ by:

$$\rho_i = (\bar{\mathbf{q}}_i)^T(A(\bar{\mathbf{c}}^k) - \lambda_i \cdot B(\bar{\mathbf{c}}^k))\bar{\mathbf{q}}_i, \qquad \text{for } 1 \le i \le n.$$

(e) Calculate $\mathbf{c}^{k+1}$ by:

$$\mathbf{c}^{k+1} = \bar{\mathbf{c}}^k - B_k\boldsymbol{\rho}^k. \tag{2.27}$$

(f) Form the matrix $\bar{G}_k$ by (2.21), (2.22), and (2.23).

(g) Calculate $Q_{k+1}$ by (2.24).

(h) Calculate the matrix $J_{k+1}$ and the vector $\mathbf{b}^{k+1}$:

$$\begin{aligned} [J_{k+1}]_{ij} &= (\mathbf{q}_i^{k+1})^T(A_j - \lambda_i^* \cdot B_j)\mathbf{q}_i^{k+1}, \qquad 1 \le i, j \le n, \\ [\mathbf{b}]_i^{k+1} &= (\mathbf{q}_i^{k+1})^T(A_0 - \lambda_i^* \cdot B_0)\mathbf{q}_i^{k+1}, \qquad 1 \le i \le n. \end{aligned}$$

(i) Compute the Chebyshev matrices $B_{k+1}$ by

$$B_{k+1} = B_k + B_k\left(2I - J(\mathbf{c}^{k+1})B_k\right)\left(I - J(\mathbf{c}^{k+1})B_k\right).$$

REMARK 2.1. Although the proposed method shares similarities with the approaches in [2, 3, 21], it differs in that it combines the Ulm–Chebyshev scheme with Aishima's procedure to solve generalized symmetric inverse eigenvalue problems, achieving cubic convergence without requiring higher-order derivatives.

## 3. Convergence analysis.

In this section, we present the convergence analysis for Algorithm 1.

### 3.1. Preliminary results.

Let us present some preliminary results. In particular, the following lemma is known from [21]:

LEMMA 3.1 ([21]). *Let $A, B \in \mathbb{R}^{n\times n}$, with $B$ nonsingular and $\|B^{-1}\|\|A - B\| < 1$. Then $A$ is nonsingular, and it holds that*

$$\|A^{-1}\| \le \frac{\|B^{-1}\|}{1 - \|B^{-1}\|\|A - B\|}.$$

LEMMA 3.2. *Let $A, B \in \mathcal{S}(n)$, and $B \ge \alpha I$, with $\alpha > 0$. Then there exist $\eta > 0$ and $\xi > 0$ such that*

$$\min_{\bar{X}} \|X - \bar{X}\| \le \eta\left(\|A - \bar{A}\| + \|B - \bar{B}\|\right),$$

*where $\bar{A},\ \bar{B} \in \mathcal{S}(n)$, $\bar{B} \ge \alpha I$, $\|A - \bar{A}\| \le \xi$, $\|B - \bar{B}\| \le \xi$,*

$$\begin{cases} X^TAX = \Lambda, \\ X^TBX = I, \end{cases} \quad \text{and} \quad \begin{cases} \bar{X}^T\bar{A}\bar{X} = \bar{\Lambda}, \\ \bar{X}^T\bar{B}\bar{X} = I. \end{cases}$$

*Proof.* Clearly, $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ and $\bar{B}^{-\frac{1}{2}}\bar{A}\bar{B}^{-\frac{1}{2}} \in \mathcal{S}(n)$, and by [22, Lemmas 2.2 and 2.4] we have that

$$\begin{aligned}\|B^{-\frac{1}{2}}AB^{-\frac{1}{2}} - \bar{B}^{-\frac{1}{2}}\bar{A}\bar{B}^{-\frac{1}{2}}\| &\leq \|B^{-1}\|\|A-\bar{A}\| + \|B^{-1}\|\|\bar{A}\|\|\bar{B}^{-1}\|\|B-\bar{B}\| \\ &\leq \frac{1}{\alpha}\|A-\bar{A}\| + \frac{\sigma_{\max}(\bar{A})}{\alpha^2}\|B-\bar{B}\| \\ &\leq \frac{\alpha+\sigma_{\max}(\bar{A})}{\alpha^2}\xi,\end{aligned}$$

which, together with [26, Lemma 4.1], implies that there exists $\eta_1 > 0$ such that

$$\begin{aligned}\min_{\bar{X}}\|X-\bar{X}\| &\leq \eta_1\|B^{-\frac{1}{2}}AB^{-\frac{1}{2}} - \bar{B}^{-\frac{1}{2}}\bar{A}\bar{B}^{-\frac{1}{2}}\| \\ &\leq \frac{\eta_1}{\alpha}\|A-\bar{A}\| + \frac{\eta_1\sigma_{\max}(\bar{A})}{\alpha^2}\|B-\bar{B}\| \\ &\leq \eta\left(\|A-\bar{A}\| + \|B-\bar{B}\|\right),\end{aligned}$$

where $\eta := \max\{\frac{\eta_1}{\alpha}, \frac{\eta_1\sigma_{\max}(\bar{A})}{\alpha^2}\}$. $\square$

Let $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz continuous function, and let $\mathbf{h}'$ be the Jacobian of $\mathbf{h}$ whenever it exists. The set of differentiable points of $\mathbf{h}$ is denoted as $D_{\mathbf{h}}$. Moreover, the B-differential Jacobian of $\mathbf{h}$ at $\mathbf{x} \in \mathbb{R}^n$ is defined by

$$\partial_B\mathbf{h}(\mathbf{x}) := \Big\{U \in \mathbb{R}^{m\times n} \big| U = \lim_{\mathbf{x}_k\to\mathbf{x}} \mathbf{h}'(\mathbf{x}_k),\ \mathbf{x}_k \in D_{\mathbf{h}}\Big\}.$$

Let $\mathbf{h} := \varphi \circ \psi$, where $\psi : \mathbb{R}^n \to \mathbb{R}^t$ is continuously differentiable and $\varphi : \mathbb{R}^t \to \mathbb{R}^m$ is nonsmooth. The relative generalized Jacobian $\partial_{Q|S}\mathbf{h}(\mathbf{x})$ [28] and generalized Jacobian $\partial_Q\mathbf{h}(\mathbf{x})$ [25] are defined by

$$\partial_{Q|S}\mathbf{h}(\mathbf{x}) := \big\{U | U \text{ is a limit of } U_i \in \partial_Q\mathbf{h}(\mathbf{y}_i),\ \mathbf{y}_i \in S,\ \mathbf{y}_i \to \mathbf{x}\big\},$$

and

$$\partial_Q\mathbf{h}(\mathbf{x}) := \partial_B(\varphi(\psi(\mathbf{x})))\psi'(\mathbf{x}).$$

For $\mathbf{c} \in \mathbb{R}^n$, write

$$\Lambda(\mathbf{c}) := \mathrm{diag}(\lambda_1((A(\mathbf{c}), B(\mathbf{c}))), \ldots, \lambda_n((A(\mathbf{c}), B(\mathbf{c})))),$$

and define

$$\mathcal{Q}(\mathbf{c}) := \left\{Q(\mathbf{c}) | Q(\mathbf{c})^T A(\mathbf{c})Q(\mathbf{c}) = \Lambda(\mathbf{c}) \text{ and } Q(\mathbf{c})^T B(\mathbf{c})Q(\mathbf{c}) = I\right\}.$$

By (1.4), we get

$$\partial_Q\mathbf{f}(\mathbf{c}) = \left\{J | [J]_{ij} = \mathbf{q}_i(\mathbf{c})^T(A_j - \lambda_i^* \cdot B_j)\mathbf{q}_i(\mathbf{c}), \text{ where } [\mathbf{q}_1(\mathbf{c}), \ldots, \mathbf{q}_n(\mathbf{c})] \in \mathcal{Q}(\mathbf{c})\right\}.$$

In particular, if $J(\mathbf{c})$ is a singleton, then $\partial_Q\mathbf{f}(\mathbf{c}) = \{J(\mathbf{c})\}$. Let

$$S := \{\mathbf{c} \in \mathbb{R}^n | (A(\mathbf{c}), B(\mathbf{c})) \text{ has distinct eigenvalues}\}.$$

Then we obtain

$$\partial_Q\mathbf{f}(\mathbf{c}) = \{J(\mathbf{c})\} = \{\mathbf{f}'(\mathbf{c})\}. \tag{3.1}$$

Thus, we have

$$\partial_{Q|S}\mathbf{f}(\mathbf{c}) = \{J | J = \lim_{k\to+\infty} J(\mathbf{y}^k), \text{ with } \{\mathbf{y}^k\} \subset S \text{ and } \mathbf{y}^k \to \mathbf{c}\}.$$

The following lemma illustrates a certain continuity property for the relative generalized Jacobian $\partial_{Q|S}\mathbf{f}(\cdot)$; see [21, 26].

LEMMA 3.3 ([21, 26]). *If $\mathbf{c}^* \in clS$ is a solution of the GIEP with the eigenvalues given by (2.1) and each $J \in \partial_{Q|S}\mathbf{f}(\mathbf{c}^*)$ is nonsingular, then there exist two numbers $\delta_1 > 0$ and $\kappa \geq 1$ such that*

$$\sup_{V\in\partial_{Q|S}\mathbf{f}(\mathbf{c})} \|V^{-1}\| \leq \kappa, \qquad \textit{for any} \quad \mathbf{c} \in B(\mathbf{c}^*,\ \delta_1).$$

Let $\bar{\mathbf{c}}^k$, $\mathbf{c}^k$, $Q_k$, $\bar{Q}_k$, $\bar{F}_k$, $\bar{G}_k$, $J_k$, and $B_k$ be generated by Algorithm 1 with the initial point $\mathbf{c}^0$, and let $Q_k^*$ and $F_k$ be defined by (2.2) and (2.4), respectively. Define $H_k$ by

$$Q_k = Q_{k-1}^*(I + H_k), \qquad \text{for } k = 1, 2, \ldots$$

Then, by [3, Lemma 2] we have

$$\|F_k\| \leq 3\|H_k\|, \qquad \text{for } k = 1, 2, \ldots \tag{3.2}$$

Next, we give the following key lemma:

LEMMA 3.4. *Let $\mathbf{c}^* \in clS$ be a solution of the GIEP with the eigenvalues given by (2.1). Then in Algorithm 1, there exists a number $\gamma \geq 1$ such that, for any $k = 0, 1, 2, \ldots$, if $\|F_k\| \leq 1$ and $\|G_k\| \leq 1$, it holds that*

$$\|F_k - \bar{F}_k\| \leq \gamma\left(\|\bar{\mathbf{c}}^k - \mathbf{c}^*\| + \|F_k\|^2\right), \tag{3.3}$$

$$\|G_k\| \leq \gamma\left(\|\bar{\mathbf{c}}^k - \mathbf{c}^*\| + \|F_k\|^2\right), \tag{3.4}$$

$$\|G_k - \bar{G}_k\| \leq \gamma\left(\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + \|G_k\|^2\right), \tag{3.5}$$

*and*

$$\|F_{k+1}\| \leq 3\gamma\left(\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + \|G_k\|^2\right). \tag{3.6}$$

*Proof.* Clearly, by (1.1) we obtain

$$\|A(\mathbf{c}^*) - A(\bar{\mathbf{c}}^k)\| \leq \max_j \|A_j\|\|\mathbf{c}^* - \bar{\mathbf{c}}^k\|, \tag{3.7}$$

$$\|B(\mathbf{c}^*) - B(\bar{\mathbf{c}}^k)\| \leq \max_j \|B_j\|\|\mathbf{c}^* - \bar{\mathbf{c}}^k\|. \tag{3.8}$$

By the orthogonality of $\sqrt{B(\mathbf{c}^*)}Q_k^*$, one has

$$\|Q_k^*\| = \|B(\mathbf{c}^*)^{-\frac{1}{2}}B(\mathbf{c}^*)^{\frac{1}{2}}Q_k^*\| \leq \|B(\mathbf{c}^*)^{-\frac{1}{2}}\|\|B(\mathbf{c}^*)^{\frac{1}{2}}Q_k^*\| \leq \|B(\mathbf{c}^*)^{-\frac{1}{2}}\|.$$

It follows from (2.4) and $\|F_k\| \leq 1$ that

$$\|\mathbf{q}_i^k\| \leq \|Q_k\| \leq (1 + \|F_k\|)\|Q_k^*\| \leq 2\|B(\mathbf{c}^*)^{-\frac{1}{2}}\|, \qquad 1 \leq i \leq n. \tag{3.9}$$

Based on (2.6) and (2.9), we have

$$Q_k^T\left(B(\mathbf{c}^*) - B(\bar{\mathbf{c}}^k)\right)Q_k = \left(F_k - \bar{F}_k\right)^T + \left(F_k - \bar{F}_k\right) + \Delta_1^k. \tag{3.10}$$

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In fact, by (2.5) and (2.11) it holds that

$$\left[F_k - \bar{F}_k\right]_{ij} = \left[F_k - \bar{F}_k\right]_{ji}, \qquad (i,j) \in \mathcal{I}_1,$$

which, as a consequence of (3.10), gives

$$\left|[F_k - \bar{F}_k]_{ij}\right| = \frac{\left|(\mathbf{q}_i^k)^T(B(\mathbf{c}^*) - B(\bar{\mathbf{c}}^k))\mathbf{q}_j^k\right| - \left|[\Delta_1^k]_{ij}\right|}{2}, \qquad \text{for } (i,j) \in \mathcal{I}_1 \text{ or } i = j.$$

Thus, by (2.8), (3.8), and (3.9) we have

$$\begin{aligned} &\left|[F_k - \bar{F}_k]_{ij}\right| \\ &\qquad \leq 2\|B(\mathbf{c}^*)^{-1}\| \cdot \max_j \|B_j\| \|\mathbf{c}^* - \bar{\mathbf{c}}^k\| + \frac{\|F_k\|^2}{2}, \quad \text{for } (i,j) \in \mathcal{I}_1 \text{ or } i = j. \end{aligned} \tag{3.11}$$

On the other hand, for $(i,j) \in \mathcal{I}_2$, by (2.7) and (2.10) we also have

$$\left[Q_k^T(A(\mathbf{c}^*) - A(\bar{\mathbf{c}}^k))Q_k\right]_{ij} = \left[(F_k - \bar{F}_k)^T\Lambda^* + \Lambda^*(F_k - \bar{F}_k) + \Delta_2^k\right]_{ij},$$

which, together with (3.10), yields

$$\begin{aligned} \left|[F_k - \bar{F}_k]_{ij}\right| \leq &\frac{|\lambda_j^*| \cdot |[\Delta_1^k]_{ij}| + |[\Delta_2^k]_{ij}| + \left|(\mathbf{q}_i^k)^T(A(\mathbf{c}^*) - A(\bar{\mathbf{c}}^k))\mathbf{q}_j^k\right|}{\lambda_j^* - \lambda_i^*} \\ &+ \frac{|\lambda_j^*| \cdot \left|(\mathbf{q}_i^k)^T(B(\mathbf{c}^*) - B(\bar{\mathbf{c}}^k))\mathbf{q}_j^k\right|}{\lambda_j^* - \lambda_i^*}. \end{aligned}$$

Define

$$\delta_{\min}^* := \min\left\{|\lambda_i^* - \lambda_j^*|,\ (i,j) \in \mathcal{I}_1,\ i \neq j\right\}. \tag{3.12}$$

Then, by (2.8), (3.7), (3.8), (3.9), and the definition of $\delta_{\min}^*$, we obtain

$$\begin{aligned} &\left|[F_k - \bar{F}_k]_{ij}\right| \\ &\quad \leq \frac{1}{\delta_{\min}^*}\Bigg(2\|\Lambda^*\| \cdot \|F_k\|^2 \\ &\qquad + 2\|B(\mathbf{c}^*)^{-1}\|\Big(\max_j \|A_j\| + \|\Lambda^*\| \cdot \max_j \|B_j\|\Big) \cdot \|\mathbf{c}^* - \bar{\mathbf{c}}^k\|\Bigg), \end{aligned} \tag{3.13}$$

where $(i,j) \in \mathcal{I}_2$. Thus, by (3.11), (3.13), and the definitions of $\gamma_1$ and $\gamma$ in (3.16) and (3.17), respectively, it holds that

$$\|F_k - \bar{F}_k\| \leq \gamma_1\left(\|\bar{\mathbf{c}}^k - \mathbf{c}^*\| + \|F_k\|^2\right) \leq \gamma\left(\|\bar{\mathbf{c}}^k - \mathbf{c}^*\| + \|F_k\|^2\right). \tag{3.14}$$

Meanwhile, by (2.4) and (2.16) we obtain that

$$\bar{Q}_k = Q_k^*(I + F_k)(I - \bar{F}_k).$$

Combining this with (2.17), we get

$$G_k = F_k - \bar{F}_k - F_k\bar{F}_k = F_k - \bar{F}_k + F_k(F_k - \bar{F}_k) - F_k^2.$$

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Then, by $\|F_k\| \leq 1$ and (3.14) we have

$$\begin{aligned}\|G_k\| &\leq (1+\|F_k\|)\|F_k - \bar{F}_k\| + \|F_k\|^2 \\ &\leq 2\gamma_1\|\bar{\mathbf{c}}^k - \mathbf{c}^*\| + (1+2\gamma_1)\|F_k\|^2 \leq \gamma\left(\|\bar{\mathbf{c}}^k - \mathbf{c}^*\| + \|F_k\|^2\right).\end{aligned} \tag{3.15}$$

This together with (3.14) shows that (3.3) and (3.4) hold. Similarly, (3.5) also holds.

On the other hand, from (2.17) and (2.24), we have

$$Q_{k+1} = Q_k^*(I+G_k)(I-\bar{G}_k).$$

Noting that $Q_{k+1} = Q_k^*(I+H_{k+1})$, by an argument analogous to that used to obtain (3.15), we have

$$\|H_{k+1}\| \leq \gamma\left(\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + \|G_k\|^2\right).$$

Thus, (3.6) follows from (3.2), and the proof is complete. □

**3.2. Convergence analysis.** In this section, we carry out the convergence analysis of Algorithm 1. We define (using (3.12))

$$L := \sqrt{n}\eta\|\sqrt{B(\mathbf{c}^*)}\|\left(\max_j\|A_j\| + \max_j\|B_j\|\right),$$

$$\gamma_1 = n\max\left\{2\|B(\mathbf{c}^*)^{-1}\|\max_j\|B_j\|,\ \frac{1}{2},\ \frac{2\|\Lambda^*\|}{\delta^*_{\min}},\ \frac{2\|B(\mathbf{c}^*)^{-1}\|\big(\max_j\|A_j\| + \|\Lambda^*\|\max_j\|B_j\|\big)}{\delta^*_{\min}}\right\}, \tag{3.16}$$

$$\gamma = 1+2\gamma_1, \tag{3.17}$$

$$\alpha_1 := 2\|B(\mathbf{c}^*)^{-\frac{1}{2}}\|\left[\big((1+\gamma)L+\gamma\big) + \gamma\big(1+(1+\gamma)(1+L^2)\big)\right], \tag{3.18}$$

$$\alpha_2 := 4n\|B(\mathbf{c}^*)^{-\frac{1}{2}}\|\left(\max_j\|A_j\| + \|\Lambda^*\|\max_j\|B_j\|\right), \tag{3.19}$$

$$\alpha_3 := 1+8\kappa\alpha_2\|B(\mathbf{c}^*)^{-\frac{1}{2}}\|\big((1+\gamma)L+\gamma\big), \tag{3.20}$$

$$\alpha_4 := 2\sqrt{n}\gamma^2(1+L^2)^2\|\Lambda^*\|, \tag{3.21}$$

$$0 < \tau = \min\left\{1,\ \frac{1}{L},\ \frac{1}{1+4\kappa L^2\sqrt{n}\|\Lambda^*\|},\ \frac{1}{\gamma(1+L^2)},\ \frac{1}{\alpha_3+2\kappa\alpha_4},\ \frac{L}{3\gamma\big((\alpha_3+2\kappa\alpha_4)+\gamma^2(1+L^2)^2\big)},\ \frac{1}{(1+2\kappa\alpha_1\alpha_2)^3}\right\}, \tag{3.22}$$

and

$$0 < \delta = \min\left\{\frac{\tau}{2},\ \delta_1,\ \frac{1}{2n\kappa\alpha_1\alpha_2}\right\},$$

where $\eta$ is defined in Lemma 3.2 and $\delta_1$ and $\kappa$ are defined in Lemma 3.3.

For the proof of the main theorem we need the following proposition:

PROPOSITION 3.5. *Under the same assumptions as in Lemma 3.3, there exist three numbers $0 < \tau \leq 1$, $0 < \delta < \tau$, and $0 \leq \mu \leq \delta$ such that, for $\mathbf{c}^0 \in \mathbf{B}(\mathbf{c}^*, \delta)\bigcap S$ and $B_0 \in \mathbb{R}^{n\times n}$,*

$$\|I - B_0 J(\mathbf{c}^0)\| \leq \mu. \tag{3.23}$$

*Then, for $k = 0, 1, 2, \ldots$, we have*

$$\|F_k\| \leq L\tau\left(\frac{\delta}{\tau}\right)^{3^k}, \tag{3.24}$$

$$\|\mathbf{c}^k - \mathbf{c}^*\| \leq \tau\left(\frac{\delta}{\tau}\right)^{3^k}, \tag{3.25}$$

$$\|I - B_k J_k\| \leq \tau\left(\frac{\delta}{\tau}\right)^{3^k}, \tag{3.26}$$

*and*

$$\|\bar{\mathbf{c}}^k - \mathbf{c}^*\| \leq \tau\left(\frac{\delta}{\tau}\right)^{2\cdot 3^k}. \tag{3.27}$$

*Proof.* Suppose that $\mathbf{c}^0 \in \mathbf{B}(\mathbf{c}^*, \delta) \bigcap S$ and $J_0$ is generated by (2.25). Then, by (3.1) it holds that

$$\partial_{Q|S}\mathbf{f}(\mathbf{c}^0) = \partial_Q \mathbf{f}(\mathbf{c}^0) = \{\mathbf{f}'(\mathbf{c}^0)\} = \{J_0\}.$$

Thus, from the definition of $\delta$ and Lemma 3.3, we have that

$$J_0^{-1} \text{ exists with } \|J_0^{-1}\| \leq \kappa, \tag{3.28}$$

where $\kappa > 0$. Clearly, for $k = 0$, (3.25) and (3.26) hold because $\|\mathbf{c}^0 - \mathbf{c}^*\| \leq \delta$ and $\|I - B_0 J_0\| \leq \mu \leq \delta$. By (2.4) and the orthogonality of $\sqrt{B(\mathbf{c}^*)}Q_0^*$, we have

$$\|F_0\|_F = \|\sqrt{B(\mathbf{c}^*)}Q_0^* F_0\|_F = \|\sqrt{B(\mathbf{c}^*)}(Q_0 - Q_0^*)\|_F,$$

which implies

$$\|F_0\| \leq \|F_0\|_F \leq \sqrt{n}\|\sqrt{B(\mathbf{c}^*)}\|\|Q_0 - Q_0^*\|. \tag{3.29}$$

On the other hand, by (1.1) we have

$$\|A(\mathbf{c}^0) - A(\mathbf{c}^*)\| \leq \max_j \|A_j\|\|\mathbf{c}^0 - \mathbf{c}^*\| \leq \max_j \|A_j\|\delta$$

and

$$\|B(\mathbf{c}^0) - B(\mathbf{c}^*)\| \leq \max_j \|B_j\|\|\mathbf{c}^0 - \mathbf{c}^*\| \leq \max_j \|B_j\|\delta,$$

which by Lemma 3.2 gives

$$\|Q_0 - Q_0^*\| \leq \eta\left(\|A(\mathbf{c}^0) - A(\mathbf{c}^*)\| + \|B(\mathbf{c}^0) - B(\mathbf{c}^*)\|\right) \leq \eta\left(\max_j \|A_j\| + \max_j \|B_j\|\right)\delta.$$

As a consequence of (3.29) and the definition of $L$, we obtain

$$\|F_0\| \leq \sqrt{n}\eta\|\sqrt{B(\mathbf{c}^*)}\|\left(\max_j \|A_j\| + \max_j \|B_j\|\right)\delta \leq L\delta. \tag{3.30}$$

Hence, (3.24) is true for $k = 0$.

Next, we prove that (3.27) holds for $k=0$. By (2.6) and (2.7) we have

$$Q_0^T A(\mathbf{c}^*)Q_0 - Q_0^T B(\mathbf{c}^*)Q_0\Lambda^* = \Lambda^* F_0 - F_0\Lambda^* + \Delta_2^0 - \Delta_1^0\Lambda^*,$$

which gives

$$(\mathbf{q}_i^0)^T\left(A(\mathbf{c}^*) - \lambda_i^* B(\mathbf{c}^*)\right)\mathbf{q}_i^0 = [\Delta_2^0]_{ii} - \lambda_i^*[\Delta_1^0]_{ii}, \qquad \text{for } 1 \le i \le n.$$

Then, by (2.8) we obtain

$$\left|(\mathbf{q}_i^0)^T\left(A(\mathbf{c}^*) - \lambda_i^* B(\mathbf{c}^*)\right)\mathbf{q}_i^0\right| \le 2\|\Lambda^*\|\|F_0\|^2. \tag{3.31}$$

Note that, by (2.25), $[J_0\mathbf{c}^* + \mathbf{b}^0]_i = (\mathbf{q}_i^0)^T(A(\mathbf{c}^*) - \lambda_i^* B(\mathbf{c}^*))\mathbf{q}_i^0$. So, it follows from (3.30) and (3.31) that

$$\|J_0\mathbf{c}^* + \mathbf{b}^0\| \le 2\sqrt{n}\|\Lambda^*\|\|F_0\|^2 \le 2L^2\sqrt{n}\|\Lambda^*\|\delta^2. \tag{3.32}$$

Furthermore, by (3.23) and (3.28) we have

$$\|B_0\| \le \|B_0 J_0\|\|J_0^{-1}\| \le (1 + \|I - B_0 J_0\|)\|J_0^{-1}\| \le (1+\mu)\kappa \le 2\kappa. \tag{3.33}$$

From (2.26), we obtain

$$\bar{\mathbf{c}}^0 - \mathbf{c}^* = (\mathbf{c}^0 - \mathbf{c}^*) - B_0(J_0\mathbf{c}^0 + \mathbf{b}^0) = (I - B_0 J_0)(\mathbf{c}^0 - \mathbf{c}^*) - B_0(J_0\mathbf{c}^* + \mathbf{b}^0).$$

By (3.23), (3.32), (3.33), and the definition of $\delta$, it holds that

$$\begin{aligned}\|\bar{\mathbf{c}}^0 - \mathbf{c}^*\| &\le \|I - B_0 J_0\| \cdot \|\mathbf{c}^0 - \mathbf{c}^*\| + \|B_0\| \cdot \|J_0\mathbf{c}^* + \mathbf{b}^0\| \\ &\le \delta^2 + 4\kappa L^2\sqrt{n}\|\Lambda^*\|\delta^2 \le (1 + 4\kappa L^2\sqrt{n}\|\Lambda^*\|)\delta^2.\end{aligned}$$

Therefore, by the fact that $1 + 4\kappa L^2\sqrt{n}\|\Lambda^*\| \le \frac{1}{\tau}$, we further obtain

$$\|\bar{\mathbf{c}}^0 - \mathbf{c}^*\| \le \frac{\delta^2}{\tau} = \tau\left(\frac{\delta}{\tau}\right)^{2\cdot 3^0}.$$

Now assume that (3.24)–(3.27) are true for all $0 \le k \le m$. Recalling (3.24) and the definition of $\tau$, we obtain

$$\|F_k\| \le 1. \tag{3.34}$$

Then, applying Lemma 3.4, equations (3.3) and (3.4) hold for $0 \le k \le m$. By (3.24), (3.27), (3.34), and the fact that $\tau \le 1$, we have

$$\begin{aligned}\|\bar{F}_k\| &\le \|F_k\| + \|F_k - \bar{F}_k\| \\ &\le \|F_k\| + \gamma\left(\|\bar{\mathbf{c}}^k - \mathbf{c}^*\| + \|F_k\|\right) \le (1+\gamma)\|F_k\| + \gamma\|\bar{\mathbf{c}}^k - \mathbf{c}^*\| \\ &\le (1+\gamma)L\tau\left(\frac{\delta}{\tau}\right)^{3^k} + \gamma\tau\left(\frac{\delta}{\tau}\right)^{2\cdot 3^k} \le \left((1+\gamma)L + \gamma\right)\tau\left(\frac{\delta}{\tau}\right)^{3^k},\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}\|G_k\| &\le \gamma\left(\|\bar{\mathbf{c}}^k - \mathbf{c}^*\| + \|F_k\|^2\right) \le \gamma\tau\left(\frac{\delta}{\tau}\right)^{2\cdot 3^k} + \gamma L^2\tau^2\left(\frac{\delta}{\tau}\right)^{2\cdot 3^k} \\ &\le \gamma(1+L^2)\tau\left(\frac{\delta}{\tau}\right)^{2\cdot 3^k} \le 1,\end{aligned} \tag{3.36}$$

where the last inequality holds because $\tau \le \frac{1}{\gamma(1+L^2)}$. Hence, applying Lemma 3.4 again, we have

$$\|G_k - \bar{G}_k\| \le \gamma\left(\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + \|G_k\|^2\right), \qquad \text{for } 0 \le k \le m, \tag{3.37}$$

and

$$\|F_{m+1}\| \le 3\gamma\left(\|\mathbf{c}^{m+1} - \mathbf{c}^*\| + \|G_m\|^2\right). \tag{3.38}$$

Therefore, for $0 \le k \le m-1$, by (3.36) and (3.37) we have

$$\begin{aligned}
\|\bar{G}_k\| &\le \|G_k\| + \|G_k - \bar{G}_k\| \le \|G_k\| + \gamma\left(\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + \|G_k\|^2\right) \\
&\le (1+\gamma)\|G_k\| + \gamma\|\mathbf{c}^{k+1} - \mathbf{c}^*\| \\
&\le \gamma(1+\gamma)(1+L^2)\tau\left(\frac{\delta}{\tau}\right)^{2\cdot 3^k} + \gamma\tau\left(\frac{\delta}{\tau}\right)^{3^{k+1}} \\
&\le \gamma[1 + (1+\gamma)(1+L^2)]\tau\left(\frac{\delta}{\tau}\right)^{2\cdot 3^k}, \qquad \text{for } 0 \le k \le m-1.
\end{aligned} \tag{3.39}$$

From (2.16) and (2.24) we have

$$Q_{k+1} - \bar{Q}_k = -\bar{Q}_k\bar{G}_k, \tag{3.40}$$

and

$$\bar{Q}_k - Q_k = -Q_k\bar{F}_k. \tag{3.41}$$

Similar to the proof of (3.9), by (3.36) we obtain

$$\|\bar{Q}_k\| \le (1 + \|G_k\|)\|Q_k^*\| \le 2\|B(\mathbf{c}^*)^{-\frac{1}{2}}\|, \qquad 1 \le k \le m, \tag{3.42}$$

which together with (3.9), (3.35), (3.39), (3.40), and (3.41) gives

$$\begin{aligned}
\|Q_{k+1} - Q_k\| &\le \|Q_{k+1} - \bar{Q}_k\| + \|\bar{Q}_k - Q_k\| \le 2\|B(\mathbf{c}^*)^{-\frac{1}{2}}\|\left(\|\bar{F}_k\| + \|\bar{G}_k\|\right) \\
&\le 2\|B(\mathbf{c}^*)^{-\frac{1}{2}}\|\left[\left((1+\gamma)L + \gamma\right) + \gamma\left(1 + (1+\gamma)(1+L^2)\right)\right]\tau\left(\frac{\delta}{\tau}\right)^{3^k} \\
&\le \alpha_1\tau\left(\frac{\delta}{\tau}\right)^{3^k},
\end{aligned}$$

with $\alpha_1$ as in (3.17). Therefore, we further have

$$\begin{aligned}
\|Q_m - Q_0\| &\le \sum_{l=0}^{m-1}\|Q_{l+1} - Q_l\| \\
&\le \alpha_1\tau\left[\left(\frac{\delta}{\tau}\right) + \left(\frac{\delta}{\tau}\right)^3 + \left(\frac{\delta}{\tau}\right)^{3^2} + \cdots + \left(\frac{\delta}{\tau}\right)^{3^{m-1}}\right].
\end{aligned} \tag{3.43}$$

Since $3^n \ge n+1$, for each $n \ge 0$, we obtain from (3.43) and $\frac{\delta}{\tau} \le \frac{1}{2}$ that

$$\begin{aligned}
\|Q_m - Q_0\| &\le \alpha_1\tau\left[\left(\frac{\delta}{\tau}\right) + \left(\frac{\delta}{\tau}\right)^2 + \left(\frac{\delta}{\tau}\right)^3 + \cdots + \left(\frac{\delta}{\tau}\right)^m\right] \\
&\le \alpha_1\delta\frac{\left[1 - \left(\frac{\delta}{\tau}\right)^m\right]}{1 - \left(\frac{\delta}{\tau}\right)} \le \frac{\alpha_1\delta}{1 - \left(\frac{\delta}{\tau}\right)} \le 2\alpha_1\delta.
\end{aligned} \tag{3.44}$$

By (3.9), the Cauchy-Schwarz inequality, and the definition of $\{J_k\}$ in Algorithm 1, we have

$$\begin{aligned}
|[J_m]_{ij} - [J_0)]_{ij}| &= \left|(\mathbf{q}_i^m)^T(A_j - \lambda_i B_j)\mathbf{q}_i^m - (\mathbf{q}_i^0)^T(A_j - \lambda_i B_j)\mathbf{q}_i^0\right| \\
&\le \left|(\mathbf{q}_i^m)^T A_j \mathbf{q}_i^m - (\mathbf{q}_i^0)^T A_j \mathbf{q}_i^0\right| + \lambda_i\left|(\mathbf{q}_i^m)^T B_j \mathbf{q}_i^m - (\mathbf{q}_i^0)^T B_j \mathbf{q}_i^0\right| \\
&= \left|(\mathbf{q}_i^m - \mathbf{q}_i^0)^T A_j \mathbf{q}_i^m - (\mathbf{q}_i^0)^T A_j(\mathbf{q}_i^0 - \mathbf{q}_i^m)\right| \\
&\qquad + \lambda_i\left|(\mathbf{q}_i^m - \mathbf{q}_i^0)^T B_j \mathbf{q}_i^m - (\mathbf{q}_i^0)^T B_j(\mathbf{q}_i^0 - \mathbf{q}_i^m)\right| \\
&\le 4\|B(\mathbf{c}^*)^{-\frac{1}{2}}\|\left(\|A_j\| + \lambda_i\|B_j\|\right)\|\mathbf{q}_i^m - \mathbf{q}_i^0\|, \qquad 1 \le i, j \le n.
\end{aligned}$$

Therefore, using the Frobenius norm $\|\cdot\|_F$, by (3.44) it holds

$$\begin{aligned}
\|J_m - J_0\| &\le \|J_m - J_0\|_F \\
&\le 4n\|B(\mathbf{c}^*)^{-\frac{1}{2}}\|\left(\max_j\|A_j\| + \|\Lambda^*\|\max_j\|B_j\|\right)\|Q_m - Q_0\| \\
&\le 2n\alpha_1\alpha_2\delta,
\end{aligned} \tag{3.45}$$

with $\alpha_2$ as in (3.19). Thanks to (3.28) and $\delta \le \frac{1}{2n\kappa\alpha_1\alpha_2}$ in (3.22), we further have

$$\|J_0^{-1}\| \cdot \|J_m - J_0\| \le 2n\kappa\alpha_1\alpha_2\delta \le \frac{1}{2} < 1.$$

Note that by Lemma 3.1, $J_m^{-1}$ exists, and moreover

$$\|J_m^{-1}\| \le \frac{\|J_0^{-1}\|}{1 - \|J_0^{-1}\| \cdot \|J_m - J_0\|} \le 2\kappa.$$

Thus, by (3.26) we get

$$\|B_m\| \le \|B_m J_m\|\|J_m^{-1}\| \le (1+\|I - B_m J_m\|)\|J_m^{-1}\| \le 2\kappa\left(1 + \tau\left(\frac{\delta}{\tau}\right)^{3^4}\right) \le 4\kappa. \tag{3.46}$$

Based on (2.18) and (2.19), we obtain

$$\bar{Q}_m^T A(\mathbf{c}^*)\bar{Q}_m - \bar{Q}_m^T B(\mathbf{c}^*)\bar{Q}_m\Lambda^* = \Lambda^* G_m - G_m\Lambda^* + \Delta_4^m - \Delta_3^m\Lambda^*,$$

which implies

$$(\bar{\mathbf{q}}_i^m)^T A(\mathbf{c}^*)\bar{\mathbf{q}}_i^m - \lambda_i^*(\bar{\mathbf{q}}_i^m)^T B(\mathbf{c}^*)\bar{\mathbf{q}}_i^m = [\Delta_4^m]_{ii} - \lambda_i^*[\Delta_3^m]_{ii}, \qquad 1 \le i \le n.$$

Then, by (2.20) we have

$$\left|(\bar{\mathbf{q}}_i^m)^T A(\mathbf{c}^*)\bar{\mathbf{q}}_i^m - \lambda_i^*(\bar{\mathbf{q}}_i^m)^T B(\mathbf{c}^*)\bar{\mathbf{q}}_i^m\right| \le 2\|\Lambda^*\| \cdot \|G_m\|^2, \qquad 1 \le i \le n. \tag{3.47}$$

Moreover, defining $\bar{J}_m \in \mathbb{R}^{n\times n}$ and $\bar{\mathbf{b}}^m \in \mathbb{R}^n$ as in (2.25), it holds that

$$[\bar{J}_m\mathbf{c}^* + \bar{\mathbf{b}}^m]_i = (\bar{\mathbf{q}}_i^m)^T A(\mathbf{c}^*)\bar{\mathbf{q}}_i^m - \lambda_i^*(\bar{\mathbf{q}}_i^m)^T B(\mathbf{c}^*)\bar{\mathbf{q}}_i^m.$$

So, by (3.24), (3.27), (3.36), (3.47), and the fact that $\tau \le 1$, we have

$$\begin{aligned}
\|\bar{J}_m\mathbf{c}^* + \bar{\mathbf{b}}^m\| &\le 2\sqrt{n}\|\Lambda^*\|\|G_m\|^2 \\
&\le 2\sqrt{n}\gamma^2\|\Lambda^*\|\left(\|\bar{\mathbf{c}}^k - \mathbf{c}^*\| + \|F_k\|^2\right)^2 \\
&\le 2\sqrt{n}\gamma^2\|\Lambda^*\|\left(\tau\left(\frac{\delta}{\tau}\right)^{2\cdot 3^m} + L^2\tau^2\left(\frac{\delta}{\tau}\right)^{2\cdot 3^k}\right)^2 \\
&\le 2\sqrt{n}\gamma^2(1+L^2)^2\|\Lambda^*\|\tau^2\left(\frac{\delta}{\tau}\right)^{4\cdot 3^k} \le \alpha_4\tau^2\left(\frac{\delta}{\tau}\right)^{4\cdot 3^k},
\end{aligned} \tag{3.48}$$

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with $\alpha_4$ as in (3.20). Moreover, by an argument analogous to that used to obtain (3.45), it holds that

$$\begin{aligned}\|J_m - \bar{J}_m\| &\le 4n\|B(\mathbf{c}^*)^{-\frac{1}{2}}\| \left(\max_j \|A_j\| + \|\Lambda^*\| \max_j \|B_j\|\right) \|Q_m - \bar{Q}_m\| \\ &\le \alpha_2\|Q_m - \bar{Q}_m\|.\end{aligned}$$

Thus, by (3.9), (3.19), (3.35), (3.41), and (3.46), we obtain

$$\begin{aligned}\|I - B_m\bar{J}_m\| &\le \|I - B_mJ_m\| + \|B_m\|\|J_m - \bar{J}_m\| \\ &\le \|I - B_mJ_m\| + \alpha_2\|B_m\|\|Q_m - \bar{Q}_m\| \\ &\le \|I - B_mJ_m\| + \alpha_2\|B_m\|\|Q_m\|\|\bar{F}_m\| \\ &\le \tau\left(\frac{\delta}{\tau}\right)^{3^m} + 8\kappa\alpha_2\|B(\mathbf{c}^*)^{-\frac{1}{2}}\|\big((1+\gamma)L+\gamma\big)\tau\left(\frac{\delta}{\tau}\right)^{3^m} \\ &\le \Big(1 + 8\kappa\alpha_2\|B(\mathbf{c}^*)^{-\frac{1}{2}}\|\big((1+\gamma)L+\gamma\big)\Big)\tau\left(\frac{\delta}{\tau}\right)^{3^m} \\ &:= \alpha_3\tau\left(\frac{\delta}{\tau}\right)^{3^m}.\end{aligned} \tag{3.49}$$

Due to the fact $\bar{J}_m\bar{\mathbf{c}}_m + \bar{\mathbf{b}}^m = \boldsymbol{\rho}^m$ and resorting to (2.27), we derive

$$\mathbf{c}^{m+1} - \mathbf{c}^* = \bar{\mathbf{c}}^m - \mathbf{c}^* - B_m\boldsymbol{\rho}^m = (I - B_m\bar{J}_m)(\bar{\mathbf{c}}^m - \mathbf{c}^*) - B_m(\bar{J}_m\mathbf{c}^* + \mathbf{b}^m),$$

which, together with (3.46), (3.48), and (3.49), gives

$$\begin{aligned}\|\mathbf{c}^{m+1} - \mathbf{c}^*\| &\le \|I - B_m\bar{J}_m\|\|\bar{\mathbf{c}}^m - \mathbf{c}^*\| + \|B_m\|\|\bar{J}_m\mathbf{c}^* + \bar{\mathbf{b}}^m\| \\ &\le \alpha_3\tau^2\left(\frac{\delta}{\tau}\right)^{3^{m+1}} + 2\kappa\alpha_1\tau^2\left(\frac{\delta}{\tau}\right)^{4\cdot 3^k} \\ &\le (\alpha_3 + 2\kappa\alpha_4)\tau^2\left(\frac{\delta}{\tau}\right)^{3^{m+1}} \le \tau\left(\frac{\delta}{\tau}\right)^{3^{m+1}},\end{aligned} \tag{3.50}$$

where the last inequality holds because $\tau \le \frac{1}{\alpha_3+2\kappa\alpha_4}$. Equations (3.36), (3.38), and the fact that $\tau \le \frac{L}{3\gamma\left((\alpha_3+2\kappa\alpha_4)+\gamma^2(1+L^2)^2\right)}$ yield

$$\begin{aligned}\|F_{m+1}\| &\le 3\gamma\left(\|\mathbf{c}^{m+1} - \mathbf{c}^*\| + \|G_m\|^2\right) \\ &\le 3\gamma\left((\alpha_3 + 2\kappa\alpha_1)\tau^2\left(\frac{\delta}{\tau}\right)^{3^{m+1}} + \gamma^2(1+L^2)^2\tau^2\left(\frac{\delta}{\tau}\right)^{4\cdot 3^k}\right) \\ &\le 3\gamma\big((\alpha_3 + 2\kappa\alpha_4) + \gamma^2(1+L^2)^2\big)\tau^2\left(\frac{\delta}{\tau}\right)^{3^{k+1}} \le L\tau\left(\frac{\delta}{\tau}\right)^{3^{k+1}}.\end{aligned} \tag{3.51}$$

Hence, (3.50) and (3.51) show that (3.25) and (3.24) are true for $k = m + 1$. By an argument analogous to that used to obtain (3.45), it can be seen that

$$\|J_{m+1} - J_m\| \le \alpha_2\|Q_{m+1} - Q_m\|. \tag{3.52}$$

On the other hand, by (3.42) we have

$$\|Q_{m+1} - Q_m\| \le \alpha_1\tau\left(\frac{\delta}{\tau}\right)^{3^m},$$

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which together with (3.52) gives

$$\|J_{m+1} - J_m\| \leq \alpha_1\alpha_2\tau\left(\frac{\delta}{\tau}\right)^{3^m}. \tag{3.53}$$

Notice that

$$B_{m+1} = B_m + B_m\left(2I - J_{m+1}B_m\right)\left(I - J_{m+1}B_m\right).$$

Then

$$I - B_{m+1}J_{m+1} = \left(I - B_mJ_{m+1}\right)^3 = \left(I - B_mJ_m + B_m(J_m - J_{m+1})\right)^3.$$

This, together with (3.26), (3.46), and (3.53) implies

$$\begin{aligned}\|I - B_{m+1}J_{m+1}\| &\leq \left(\|I - B_mJ_m\| + \|B_m\|\|J_m - J_{m+1}\|\right)^3\\ &\leq \left(\tau\left(\frac{\delta}{\tau}\right)^{3^m} + 2\kappa\alpha_1\alpha_2\tau\left(\frac{\delta}{\tau}\right)^{3^m}\right)^3\\ &\leq (1+2\kappa\alpha_1\alpha_2)^3\tau^3\left(\frac{\delta}{\tau}\right)^{3^{m+1}} \leq \tau\left(\frac{\delta}{\tau}\right)^{3^{m+1}},\end{aligned}$$

where the inequality holds because $\tau \leq 1$ and $\tau \leq \frac{1}{(1+2\kappa\alpha_1\alpha_2)^3}$. Therefore, (3.26) also holds for $k = m+1$. Note that the proof of (3.27) for $k = m+1$ is completely the same as that of (3.25) for $k = m+1$, and so we omit it here. Thus, we have proved that (3.24)–(3.27) are true for all $k > 0$ by mathematical induction, and the proof is complete. ∎

Next, using the above proposition, we carry out the root-convergence analysis for Algorithm 1.

THEOREM 3.6. *Under the same assumptions as in Proposition 3.5, suppose that Algorithm 1 is applied with the initial point* $\mathbf{c}^0$. *Then, for some* $k$, $\mathbf{c}^k$ *is well defined and converges to the limit* $\mathbf{c}^*$ *with cubic root-convergence .*

*Proof.* Based on the estimate of the sequence $\{\mathbf{c}^k\}$ in (3.50), we can directly use the definition of root-convergence in [23] to derive the rate of the sequence $\{\mathbf{c}^k\}$ as follows:

1. If $p = 1$, then

$$\mathcal{R}_1\{\mathbf{c}^k\} = \limsup_{k\to\infty}\|\mathbf{c}^k - \mathbf{c}^*\|^{\frac{1}{k}} \leq \limsup_{k\to\infty}\tau\left(\frac{\delta}{\tau}\right)^{\frac{3^k}{k}} = 0.$$

2. If $1 < p < 3$, then

$$\mathcal{R}_p\{\mathbf{c}^k\} = \limsup_{k\to\infty}\|\mathbf{c}^k - \mathbf{c}^*\|^{\frac{1}{p^k}} \leq \limsup_{k\to\infty}\tau\left(\frac{\delta}{\tau}\right)^{\frac{3^k}{p^k}} = 0.$$

3. If $p \geq 3$, then

$$\mathcal{R}_p\{\mathbf{c}^k\} = \limsup_{k\to\infty}\|\mathbf{c}^k - \mathbf{c}^*\|^{\frac{1}{p^k}} \leq \limsup_{k\to\infty}\tau\left(\frac{\delta}{\tau}\right)^{\frac{3^k}{p^k}} = \tau \leq 1.$$

Hence, $\mathcal{R}_p\{\mathbf{c}^k\} = 0$, for any $p \in [1,\ 3)$, and $\mathcal{R}_p\{\mathbf{c}^k\} \leq 1$, for any $p \in [3,\ +\infty)$. Then, according to [23, (2.3.1)], it can be deduced that $O_{\mathcal{R}}\{\mathbf{c}^*\} \geq 3$, and the proof is complete. ∎

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**4. Numerical experiments.** In this section, we provide some numerical examples to compare Algorithm 1 with other existing methods in [1, 3, 9, 10, 11, 21], demonstrating the effectiveness of Algorithm 1 for solving the GIEPs.

For convenience, we denote the algorithm of Aishima in [1, 3] as Algorithm AA, the Newton method in [9] as Algorithm NM, the inexact Newton-like method in [10] as Algorithm INLM, the Cayley transform method in [11] as Algorithm CTM, and the Ulm-like algorithm in [21] as Algorithm ULA.

All linear equations that appear in all algorithms, were solved by the QMR method [13] via the MATLAB `qmr` function, in which the maximal number of iterations is set to 400. In the following examples, our experimental results are reported in tables, where the averaged residual values of $\|\mathbf{c}^k - \mathbf{c}^*\|$ over ten runs are given, and "it." denotes the averaged numbers of outer iterations. For all algorithms, the outer (Newton) iterations are stopped when

$$\sqrt{\sum_{1}^{n} |\mathbf{q}_i(\mathbf{c}^k)^T(A(\mathbf{c}^k) - \lambda_i^* \cdot B(\mathbf{c}^k))\mathbf{q}_i(\mathbf{c}^k)|^2} \leq 10^{-12}.$$

EXAMPLE 4.1 (see [11]). In this example, we construct the matrices $A(\mathbf{c})$ and $B(\mathbf{c})$ given in (1.1) by defining the matrices $\{A_i\}_{i=0}^n$ and $\{B_k\}_{k=0}^n$ as

$$A_0 = O,\ A_1 = I,\ A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \cdots, A_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and

$$B_0 = 0, \qquad B_k = \{b_{ij}^k\}, \qquad b_{ij}^k = \begin{cases} 1 & \text{if } i = j = k, \\ 0 & \text{otherwise.} \end{cases}$$

We consider three problems: $n = 100,\ 200,\ 300$. For any $n$, the vector $\tilde{\mathbf{c}}^*$ is generated such that $|\lambda_{l+1}((A(\tilde{\mathbf{c}}^*), B(\tilde{\mathbf{c}}^*))) - \lambda_l((A(\tilde{\mathbf{c}}^*), B(\tilde{\mathbf{c}}^*)))| < \eta$, with $1 \leq l \leq n-1$, where

$$\eta = \begin{cases} 5 \times 10^{-5} & n = 100, \\ 1 \times 10^{-5} & n = 200, \\ 1 \times 10^{-6} & n = 300. \end{cases}$$

Set

$$\lambda_i^* = \begin{cases} \lambda_l((A(\tilde{\mathbf{c}}^*), B(\tilde{\mathbf{c}}^*))) & i = l,\ l+1, \\ \lambda_i((A(\tilde{\mathbf{c}}^*), B(\tilde{\mathbf{c}}^*))) & \text{otherwise.} \end{cases}$$

Thus, we choose $\{\lambda_i^*\}_{i=1}^n$ as the prescribed eigenvalues; here, multiple eigenvalues are present. Because all algorithms are locally convergent, the initial guess $\mathbf{c}_0$ is obtained by chopping the components of $\tilde{\mathbf{c}}^*$ to six decimal places for $n = 100,\ 200,\ 300$. The residual values for this example are given in Table 4.1, and the averaged CPU time in seconds of all algorithms are given in Table 4.2.

EXAMPLE 4.2 ([7]). Here we consider the usefulness of GIEPs in structural design. Figure 4.1 displays a 10-bar truss structure with Young's modulus $E = 6.95 \times 10^{10}\ \mathrm{N/m}^2$, weight

TABLE 4.1
*Averaged residual values of $\|\mathbf{c}^k - \mathbf{c}^*\|$ for the ten tests for Example 4.1.*

| $n$ | it. | Algor. NM | Algor. INLM $\beta = 1.8$ | Algor. CTM | Algor. I | Algor. AA | Algor. ULA |
|---|---|---|---|---|---|---|---|
| 100 | 0 | 6.8e-4 | 6.8e-4 | 6.8e-4 | 6.8e-4 | 6.8e-4 | 6.8e-4 |
| | 1 | 3.4e-5 | 3.3e-5 | 3.6e-5 | 3.5e-7 | 4.6e-6 | 4.5e-6 |
| | 2 | 1.8e-6 | 2.6e-6 | 2.4e-7 | 2.9e-13 | 1.5e-8 | 4.3e-8 |
| | 3 | 6.4e-8 | 7.9e-8 | 2.8e-9 | | 5.9e-12 | 5.8e-12 |
| | 4 | 4.2e-11 | 3.1e-10 | 4.1e-12 | | 2.3e-14 | 4.1e-15 |
| | 5 | 5.6e-13 | 7.2e-13 | 8.7e-15 | | | |
| 200 | 0 | 1.9e-5 | 1.9e-5 | 1.9e-5 | 1.9e-5 | 1.9e-5 | 1.9e-5 |
| | 1 | 2.0e-5 | 2.4e-5 | 3.2e-5 | 4.9e-8 | 2.1e-6 | 4.2e-7 |
| | 2 | 8.6e-6 | 7.5e-6 | 3.6e-7 | 4.1e-17 | 8.2e-8 | 9.4e-9 |
| | 3 | 3.2e-8 | 1.1e-8 | 5.9e-9 | | 6.2e-10 | 5.6e-11 |
| | 4 | 5.8e-11 | 9.6e-11 | 1.2e-12 | | 4.9e-13 | 7.9e-13 |
| | 5 | 5.6e-13 | 4.2e-13 | 8.4e-14 | | | |
| 300 | 0 | 4.6e-5 | 4.6e-5 | 4.6e-5 | 4.6e-5 | 4.6e-5 | 4.6e-5 |
| | 1 | 8.1e-6 | 2.2e-6 | 5.2e-6 | 4.9e-8 | 7.3e-7 | 2.5e-7 |
| | 2 | 4.9e-7 | 4.8e-7 | 6.2e-7 | 2.4e-17 | 3.1e-9 | 7.1e-9 |
| | 3 | 1.8e-9 | 4.2e-9 | 5.8e-10 | | 4.3e-11 | 9.9e-11 |
| | 4 | 5.6e-11 | 6.7e-11 | 4.8e-12 | | 3.2e-13 | 5.8e-14 |
| | 5 | 9.4e-13 | 8.8e-13 | 1.2e-14 | | | |

density $\bar{\rho} = 2650\,\mathrm{kg/m}^3$, length of the bars $l = 10\,\mathrm{m}$, acceleration of gravity $g = 9.81\,\mathrm{m/s}^2$, nonstructural mass at all nodes $m_0 = 425\,\mathrm{kg}$. The design parameters are the areas of the cross sections of the bars. Since the number of design parameters exceeds the order of the global stiffness and mass matrices by two, the areas of the cross sections of the 7th and 8th bars are fixed with the values $0.000865\,\mathrm{m}^2$ and $0.000165\,\mathrm{m}^2$, respectively. The global stiffness and mass matrices of the structure are

$$A(\mathbf{c}) = A_0 + \sum_{i=1}^{8} c_i A_i \qquad \text{and} \qquad B(\mathbf{c}) = B_0 + \sum_{i=1}^{8} c_i B_i,$$

where $A_0$, $B_0$, $A_i$, and $B_i$ $(i = 1, 2, \ldots, 8)$ are $8 \times 8$ symmetric matrices and $c_1, \ldots, c_8$ are the areas of the cross sections of the bars 1–6, 9, 10, respectively.

The frequencies of the structure are prescribed to be $\omega_j = 5j$ $(j = 1, 2, \ldots, 8)$, i.e., the given eigenvalues are $\lambda_j = (2j\omega)^2$ $(j = 1, 2, \ldots, 8)$. We assume that the areas of the cross sections of the bars are

$$\mathbf{c}^* = 10^{-3}(1.7024, 0.4203, 1.6994, 1.6010, 0.7056, 1.1164, 0.4391, 1.1192),$$

and the starting point $\mathbf{c}^0$ is obtained by perturbing $\mathbf{c}^*$ as $\mathbf{c}_0 = \mathbf{c}^* + \frac{1}{80}\texttt{randn(8,1)}$. The residual values for this example are given in Table 4.3.

EXAMPLE 4.3 (see [8]). Let $n = 4k + 4$, and consider an $n$-bar truss structure (see Figure 4.2) with Young's modulus $E = 6.95 \times 10^{10}\,\mathrm{N/m}^2$, weight density $\bar{\rho} = 2650\,\mathrm{kg/m}^3$, acceleration of gravity $g = 9.81\,\mathrm{m/s}^2$, nonstructural mass at all nodes $m_0 = 425\,\mathrm{kg}$, and the length of the horizontal and vertical bars $l = 1\,\mathrm{m}$. We use $c_i$ to denote the area of the cross section of the $i$th bar. The stiffness and the mass matrices of the structure can be expressed as

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TABLE 4.2
*Averaged CPU time in seconds of all algorithms for the ten tests for Example 4.1.*

| n | 50 | 100 | 150 | 200 | 250 | 300 |
|---|---|---|---|---|---|---|
| Algorithms NM | 10.11e-1 | 9.22 | 47.04 | 129.86 | 398.03 | 812.21 |
| Algorithms INLM with $\beta = 1.8$ | 9.56e-1 | 8.89 | 46.69 | 120.57 | 382.85 | 801.29 |
| Algorithms CTM | 9.23e-1 | 8.64 | 46.24 | 113.47 | 342.65 | 796.43 |
| Algorithms AA | 9.13e-1 | 8.54 | 46.19 | 110.57 | 322.85 | 785.29 |
| Algorithms ULA | 8.56e-1 | 8.42 | 42.04 | 99.86 | 301.03 | 712.21 |
| Algorithms I | 7.21e-1 | 7.31 | 35.39 | 76.68 | 213.34 | 521.37 |

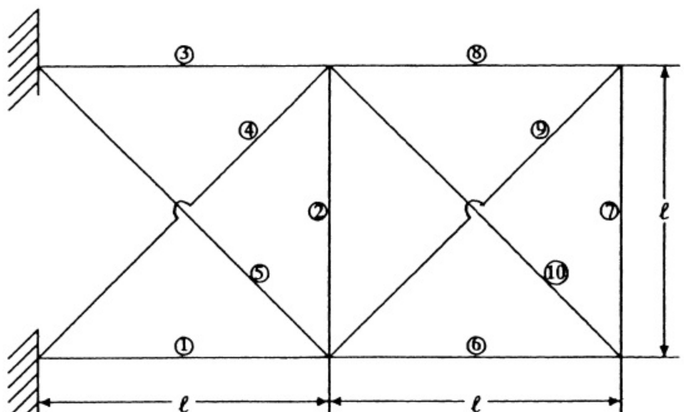


FIG. 4.1. *The 10-bar truss structure. Figure from [7]. ©Copyrighted content. Creative commons license may not apply.*

$$A(\mathbf{c}) = A_0 + \sum_{i=1}^{n} c_i A_i \qquad \text{and} \qquad B(\mathbf{c}) = B_0 + \sum_{i=1}^{n} c_i B_i,$$

where $A_0$, $B_0$, $A_i$, and $B_i$ $(i = 1, 2, \ldots, n)$ are $8 \times 8$ symmetric matrices. We assume that all the areas of the cross sections of the bars are $1 \times 10^{-3}\,\mathrm{m}^2$, i.e., $\mathbf{c}^* = 10^{-3}(1, 1, \ldots, 1) \in \mathbb{R}^n$. Since both algorithms are locally convergent, the initial guess is $\mathbf{c}_0 = \mathbf{c}^* + \frac{1}{10n}$`randn(n,1)`. The residual values for this example are given in Table 4.4.

EXAMPLE 4.4. For Application 1.1, we consider the two problem sizes $n = 100$ and 200. Let $c_i = e^{3ih}$, for $1 \le i \le n$. Then we obtain the prescribed eigenvalues $\lambda_1^* < \lambda_2^* < \cdots < \lambda_n^*$ using the initial guess $\mathbf{c}_0 = \mathbf{c}^* + \frac{1}{10}$`randn(n,1)`. The residual values for this example are given in Table 4.5.

EXAMPLE 4.5. For Application 1.1, we consider the problem size $n = 300$ and the initial guess $\mathbf{c}_0 = \mathbf{c}^* + \zeta$, where $\mathbf{c}^* = [1, 2, 3, \ldots, n-1, n]^T$ and $\zeta = 0.5$ `ones(n,1)`. The residual values for this example are given in Table 4.6.

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TABLE 4.3
*Averaged residual values of* $\|\mathbf{c}^k - \mathbf{c}^*\|$ *for Example 4.2.*

| it. | Algor. NM | Algor. INLM $\beta = 1.8$ | Algor. CTM | Algor. I | Algor. AA | Algor. ULA |
|---|---|---|---|---|---|---|
| 0 | 6.2e-2 | 6.2e-2 | 6.2e-2 | 6.2e-2 | 6.2e-2 | 6.2e-2 |
| 1 | 1.9e-3 | 6.3e-3 | 4.6e-3 | 4.5e-6 | 3.4e-3 | 4.3e-3 |
| 2 | 8.5e-5 | 2.3e-5 | 3.6e-5 | 8.3e-14 | 2.9e-5 | 7.7e-5 |
| 3 | 6.9e-9 | 8.9e-9 | 2.5e-9 | | 9.8e-8 | 6.2e-9 |
| 4 | 4.3e-12 | 4.2e-12 | 4.9e-12 | | 1.9e-13 | 5.9e-13 |
| 5 | 6.9e-14 | 6.4e-14 | 3.7e-14 | | | |

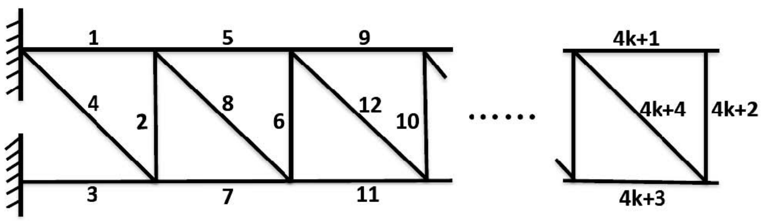


FIG. 4.2. *The n-bar truss structure. Figure from [8]. ©Copyrighted content. Creative commons license may not apply.*

EXAMPLE 4.6. We present a practical engineering application in vibrations; see [5]. We consider the vibration of a taut string with $n$ beads. Figure 4.3 displays such a model for the case where $n = 4$. Here, we assume that the $n$ beads are placed along the string, where the ends of the string are clamped. The mass of the $j$th bead is denoted by $m_j$. The horizontal lengths between the masses $m_j$ and $m_{j+1}$ (and between each bead at each end and the clamped support) are set to be the constant $L$. The horizontal tension is set to be the constant $T$. Then the equation of motion is governed by

$$m_j y_j''(t) = T\frac{y_{j+1} - y_j}{L} - T\frac{y_j - y_{j-1}}{L}, \qquad j = 1, \dots, n, \tag{4.1}$$

where $y_0 = y_{n+1} = 0$. In other words, the ends of the string are fixed. The matrix form of (4.1) is given by

$$y''(t) = -CJy(t), \tag{4.2}$$

where $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$, $C = \text{diag}(m_1, m_2, \dots, m_n)$, and $J$ is the discrete Laplacian matrix

$$J = \frac{T}{L}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathcal{S}^n.$$

The general solution of (4.2) is given in terms of the eigenvalue problem

$$Jy = \lambda Cy,$$

TABLE 4.4
*Averaged residual values of* $\|\mathbf{c}^k - \mathbf{c}^*\|$ *for Example 4.3.*

| $n$ | it. | Algor. NM | Algor. INLM $\beta = 1.8$ | Algor. CTM | Algor. I | Algor. AA | Algor. ULA |
|---|---|---|---|---|---|---|---|
| 100 | 0 | 1.1e+0 | 1.1e+0 | 1.1e+0 | 1.1e+0 | 1.1e+0 | 1.1e+0 |
| | 1 | 2.3e-1 | 2.4e-1 | 2.1e-1 | 2.2e-1 | 2.4e-1 | 2.7e-1 |
| | 2 | 3.6e-2 | 3.5e-2 | 3.2e-2 | 3.8e-3 | 2.9e-2 | 2.8e-2 |
| | 3 | 4.2e-3 | 4.3e-3 | 4.6e-3 | 4.6e-7 | 5.9e-3 | 5.8e-3 |
| | 4 | 3.3e-6 | 3.9e-6 | 3.1e-6 | 3.0e-16 | 6.2e-5 | 3.9e-5 |
| | 5 | 2.4e-9 | 2.3e-9 | 3.1e-9 | | 4.3e-9 | 8.2e-9 |
| | 6 | 7.2e-15 | 8.6e-15 | 9.2e-15 | | 9.8e-16 | 9.6e-16 |
| 200 | 0 | 3.4e+0 | 3.4e+0 | 3.4e+0 | 3.4e+0 | 3.4e+0 | 3.4e+0 |
| | 1 | 9.4e-1 | 8.3e-1 | 7.3e-1 | 4.3e-1 | 5.2e-1 | 5.8e-1 |
| | 2 | 3.3e-2 | 3.4e-2 | 3.2e-2 | 3.8e-3 | 3.9e-2 | 3.8e-2 |
| | 3 | 5.1e-3 | 7.2e-3 | 6.3e-3 | 4.6e-7 | 5.3e-3 | 5.4e-3 |
| | 4 | 1.9e-5 | 5.3e-5 | 5.8e-5 | 5.6e-16 | 5.6e-5 | 5.9e-5 |
| | 5 | 9.5e-9 | 8.4e-9 | 3.1e-9 | | 2.3e-9 | 3.9e-9 |
| | 6 | 2.4e-15 | 4.3e-15 | 5.6e-15 | | 1.8e-15 | 9.6e-16 |

TABLE 4.5
*Averaged residual values of* $\|\mathbf{c}^k - \mathbf{c}^*\|$ *for Example 4.4.*

| $n$ | it. | Algor. NM | Algor. INLM $\beta = 1.8$ | Algor. CTM | Algor. I | Algor. AA | Algor. ULA |
|---|---|---|---|---|---|---|---|
| 100 | 0 | 1.7e-0 | 1.7e-0 | 1.7e-0 | 1.7e-0 | 1.7e-0 | 1.7e-0 |
| | 1 | 2.5e-1 | 2.5e-1 | 2.1e-1 | 2.4e-2 | 1.9e-1 | 1.4e-1 |
| | 2 | 7.1e-2 | 7.2e-2 | 7.5e-2 | 7.5e-7 | 5.9e-2 | 5.4e-2 |
| | 3 | 2.2e-5 | 2.9e-5 | 2.0e-5 | 2.2e-15 | 5.1e-5 | 2.9e-5 |
| | 4 | 1.7e-9 | 1.2e-9 | 2.4e-9 | | 7.2e-9 | 4.1e-9 |
| | 5 | 9.1e-15 | 4.5e-15 | 8.1e-15 | | 7.4e-15 | 6.5e-15 |
| 200 | 0 | 7.2e-0 | 7.2e-0 | 7.2e-0 | 7.2e-0 | 7.2e-0 | 7.2e-0 |
| | 1 | 2.2e-1 | 2.7e-1 | 3.1e-1 | 6.4e-2 | 2.9e-1 | 4.4e-1 |
| | 2 | 5.0e-2 | 6.1e-2 | 5.1e-2 | 7.5e-7 | 5.2e-2 | 5.7e-2 |
| | 3 | 0.9e-5 | 5.6e-5 | 5.4e-5 | 4.6e-16 | 5.5e-5 | 5.9e-5 |
| | 4 | 9.5e-9 | 4.7e-9 | 2.0e-9 | | 9.2e-8 | 2.9e-9 |
| | 5 | 1.7e-15 | 9.9e-14 | 5.5e-15 | | 8.4e-15 | 7.5e-15 |

where $\lambda$ is the square of the natural frequency of the vibration system and the nonzero vector $y$ accounts for the interplay between the masses. The inverse problem for the beaded string is to compute the masses $\{m_j\}_{j=1}^n$ so that the resulting system has a prescribed set of natural frequencies. It is easy to verify that the eigenvalues of $J$ are given by

$$\lambda_j(J) = \frac{4T}{L}\left(\sin\frac{j\pi}{n+1}\right)^2, \qquad j = 1, 2, \ldots, n.$$

Thus, the inverse problem is converted into the form of the GIEP, where $A_0 = J$, $A_j = 0$, $B_0 = 0$, and $B_j = e_j e_j^T$, for $j = 1, 2, \ldots, n$. We consider an inverse problem for the beaded

TABLE 4.6
*Averaged residual values of* $\|\mathbf{c}^k - \mathbf{c}^*\|$ *for Example 4.5.*

| $n$ | it. | Algor. NM | Algor. INLM $\beta = 1.8$ | Algor. CTM | Algor. I | Algor. AA | Algor. ULA |
|---|---|---|---|---|---|---|---|
| 300 | 0 | 9.2e+0 | 9.2e+0 | 9.2e+0 | 9.2e+0 | 9.2e+0 | 9.2e+0 |
| | 1 | 2.1e+0 | 2.9e+0 | 3.1e+0 | 1.1e-1 | 2.8e+0 | 2.2e+0 |
| | 2 | 8.8e-1 | 8.1e-1 | 9.1e-1 | 2.5e-3 | 5.2e-1 | 4.6e-1 |
| | 3 | 3.9e-3 | 3.1e-3 | 2.8e-3 | 4.9e-7 | 7.9e-3 | 8.2e-3 |
| | 4 | 6.4e-5 | 6.7e-5 | 3.7e-5 | 5.8e-16 | 3.5e-5 | 3.6e-5 |
| | 5 | 3.2e-9 | 9.9e-8 | 5.4e-9 | | 4.9e-9 | 4.5e-9 |
| | 6 | 9.5e-14 | 8.4e-14 | 3.1e-15 | | 9.5e-14 | 3.2e-15 |

string with $n = 6$ beads, where

$$\begin{aligned}
&(m_1, m_2, m_3, m_4, m_5, m_6) \\
&\qquad = (0.017804,\ 0.030783,\ 0.017804, 0.017804,\ 0.030783,\ 0.017804)\,(\mathrm{kg}), \\
&(n+1)L = 1.12395\,(\mathrm{meter}), \\
&\qquad T = 166.0370\,(\mathrm{Newton}), \\
&\qquad \boldsymbol{\lambda}^* = (9113.978, 30746.32, 83621.69, 148694.4, 148694.4, 193537.0)^T, \\
&\qquad c^* = (58081.57, 33592.71, 58081.57, 58081.57, 33592.71, 58081.57)^T.
\end{aligned}$$

We report our numerical results for the starting point $\mathbf{c}_0 = \mathbf{c}^* + \frac{1}{60}\texttt{randn(6,1)}$. The information in Table 4.7 presents the average values of $\|\mathbf{c}^k - \mathbf{c}^*\|$, and Table 4.8 gives the computed masses for the beaded string.

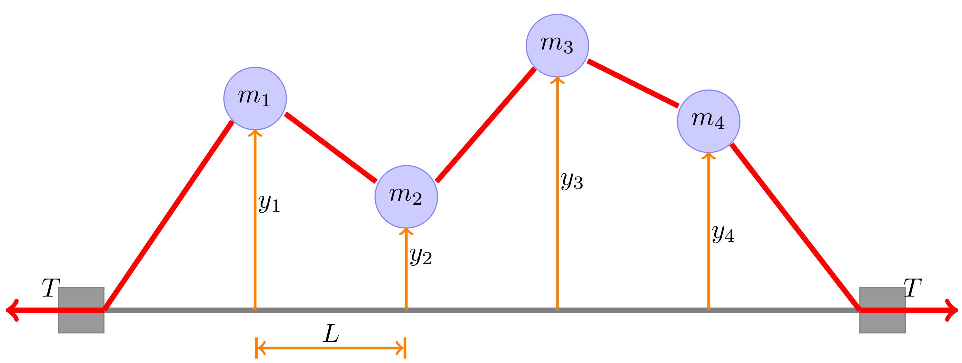


FIG. 4.3. *A string with* $n = 4$ *beads.*

From Tables 4.1 and 4.3–4.7, we know that Algorithm 1 requires a lower number of iterations compared to Algorithms NM, CTM, INLM, AA, and ULA in [1, 3, 9, 10, 11, 21]. Table 4.2 shows that Algorithm 1 demonstrates a more cost-effective CPU time compared to the other methods. Table 4.8 shows that the desired masses are recovered. All of these numerical observations agree with our prediction and further validate our theoretical results.

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TABLE 4.7
*Averaged residual values of* $\|\mathbf{c}^k - \mathbf{c}^*\|$ *for Example 4.6.*

| it. | Algor. NM | Algor. INLM $\beta = 1.8$ | Algor. CTM | Algor. I | Algor. AA | Algor. ULA |
|---|---|---|---|---|---|---|
| 0 | 5.6e-2 | 5.6e-2 | 5.6e-2 | 5.6e-2 | 5.6e-2 | 5.6e-2 |
| 1 | 3.2e-3 | 3.9e-3 | 6.1e-3 | 4.9e-6 | 2.1e-3 | 2.2e-3 |
| 2 | 5.9e-5 | 6.1e-5 | 4.8e-5 | 9.9e-14 | 3.5e-5 | 5.8e-5 |
| 3 | 7.2e-8 | 7.9e-8 | 2.9e-9 | | 4.9e-8 | 5.1e-9 |
| 4 | 3.5e-10 | 4.4e-10 | 5.3e-11 | | 3.6e-13 | 5.6e-14 |
| 5 | 8.7e-13 | 6.9e-13 | 1.2e-14 | | | |

TABLE 4.8
*Recovered masses for Example 4.6.*

| | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ |
|---|---|---|---|---|---|---|
| true | 0.017804 | 0.030783 | 0.017804 | 0.017804 | 0.030783 | 0.017804 |
| recovered | 0.017804 | 0.030783 | 0.017804 | 0.017804 | 0.030783 | 0.017804 |

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