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ERROR BOUNDS OF CLENSHAW–CURTIS AND RELATED QUADRATURES FOR FUNCTIONS ANALYTIC ON ELLIPSES*

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Dedicated to our friend Sotirios Notaris on the occasion of his 65th birthday.

Abstract. In this paper we consider error bounds of the Clenshaw–Curtis and related quadrature formulae, with respect to the Legendre weight function on the interval $[-1, 1]$, for an analytic integrand f. For certain spaces of analytic functions, Notaris [12] derived estimates for the Clenshaw–Curtis, the Basu, and the Fejér quadrature formula of the first kind. Inspired by these results we derive another kind of error bounds. As is well known, in the case of analytic integrands, the error of such a quadrature formula can be represented as a contour integral with a complex kernel. We study the kernel of the mentioned quadrature formulae on elliptic contours with foci at the points ± 1 and the sum of semi-axes $\rho > 1$ and derive some error bounds. In addition, we obtain a result about the behavior of the modulus of the corresponding kernels on those ellipses in certain cases. Numerical examples demonstrating the accuracy of such error bounds are included.

Key words. Clenshaw–Curtis quadrature, Basu quadrature, Fejér quadrature of the first kind, error bounds, analytic integrand, ellipse

AMS subject classifications. 65D30, 65D32

1. Introduction.

We consider the integral

$$I(f) = \int_{-1}^{1} f(t)\, dt.$$

In [12], Notaris (on the basis of his results in [10]) analyses the interpolatory quadrature formulae relative to the Legendre weight function on the interval $[-1, 1]$ of the type

$$I(f) = I_N(f) + R_N(f), \qquad I_N(f) = \sum_{\nu=1}^{N} \omega_\nu f(\tau_\nu), \tag{1.1}$$

and derives explicit error bounds for the Clenshaw–Curtis, Basu, and Fejér formula of the first kind. These formulae fall into the category of so-called product integration rules. The nodes $\tau_\nu = \tau_\nu^{(N)}$, ordered decreasingly, are all in the interval $[-1, 1]$, and the weights $\omega_\nu = \omega_\nu^{(N)}$ are real numbers. Formula (1.1) is an interpolatory quadrature formula, and it has degree of exactness d at least $N - 1$, i.e., $R_N(f) = 0$ for all $f \in \mathcal{P}_{N-1}$, where $\mathcal{P}_{N-1}$ is the set of all algebraic polynomials of degree less than or equal to $N - 1$.

The most common method for estimating the error of a quadrature formula is by means of high-order derivatives of the involved function. The disadvantages of such an approach are well known. For functions $f \in C^{d+1}[-1, 1]$, the error term of formula (1.1) can be estimated by

$$|R_N(f)| \le c_d \max_{-1 \le t \le 1} \left| f^{(d+1)}(t) \right|, \qquad c_d = \int_{-1}^{1} |K_d(t)|\, dt, \tag{1.2}$$

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where K_d is the d-th Peano kernel (cf. [1, Section 4.3]). Estimate (1.2), although frequently quoted, is of limited use. One reason is that higher-order derivatives of a function are not readily available, and even if they are, the resulting error bound cannot be applied to functions of lower-order continuity. Moreover, estimates like (1.2) do not lend themselves to comparing quadrature formulae with different degrees of exactness.

Derivative-free error estimates can be obtained by either contour integration or Hilbert space methods. There is a nice survey on them by Notaris [11]. The oldest references mentioning this topic are listed in that paper by Notaris: Fock [4], Hermite [8], Heine [7, p. 16], Davis [2].

Contour integration methods are based on Cauchy's theorem. Here, the error term $R_N(f)$ of formula (1.1) can be estimated by

$$|R_N(f)| \le \frac{l(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_N(z)|\right)\cdot\left(\max_{z\in\Gamma}|f(z)|\right),$$

assuming that f is analytic (holomorphic) in a domain D, Γ is a contour in D surrounding the interval of integration, $l(\Gamma)$ is the length of the curve Γ, and $K_N(z)$ is the so-called kernel. An apparent advantage of this method is that the above estimate can be optimized by appropriately choosing the contour Γ. Contours most frequently used are either concentric circles or confocal ellipses, the latter shrinking to the interval of integration as the sum of the semi-axes approaches 1, and this makes them a preferred choice for functions having a pole in the vicinity of the interval of integration. There is a rich literature on the subject; see the papers, books, and the references in [1, 3, 5, 14].

The Hilbert space method was applied in [12] to the Clenshaw–Curtis quadrature formula, the Basu quadrature formula, and the Fejér quadrature formula of the first kind. The estimates obtained by Hilbert space techniques coincide with the estimates obtained by contour integration on circular contours; see [11].

In this paper we apply a contour integration method with Γ chosen to be an ellipse. Our analysis of the error is based on its well-known representation in terms of a contour integral of an appropriate kernel. Namely, if we use a rule $I_N(f)$ with N nodes to approximate the value of the integral $I(f)$ with an integrand f that is analytic in a neighborhood of this interval, then the error of quadrature admits the integral representation

$$R_N(f) = I(f) - I_N(f) = \frac{1}{2\pi i}\oint_\Gamma K_N(z)\, f(z)\, dz\,, \tag{1.3}$$

where the kernel K_N is given by

$$K_N(z) = \frac{\varrho_N(z)}{\pi_N(z)}\,, \qquad \varrho_N(z) = \int_{-1}^{1}\frac{\pi_N(t)}{z-t}\, w(t)\, dt\,,$$

and where $\pi_N(t) = c\cdot\prod_{\nu=1}^{N}(t-\tau_\nu)$ is a non-monic node polynomial with a constant $c \neq 0$ and Γ denotes any closed smooth contour surrounding the real interval $[-1,1]$. Since they are the level curves for the conformal function that maps the exterior of $[-1,1]$ onto the exterior of the unit circle, elliptic contours $\mathcal{E}_\rho$ with foci at the points ±1 and semi-axes given by $\frac{1}{2}\left(\rho+\rho^{-1}\right)$ and $\frac{1}{2}\left(\rho-\rho^{-1}\right)$, with $\rho > 1$, are often considered suitable for getting estimates of the error of quadrature. Namely, if we define $\phi(z) = z + \sqrt{z^2-1}$, these elliptic level curves are given by the expression

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} : |\phi(z)| = |z+\sqrt{z^2-1}| = \rho \right\}\,,$$

where $\rho > 1$ and the branch of $\sqrt{z^2-1}$ is taken so that $|\phi(z)| > 1$, for $|z| > 1$. In other words,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} : z = \frac{1}{2}\left(u + \frac{1}{u}\right),\ u = \rho e^{i\,\theta},\, \theta \in [0, 2\pi] \right\},$$

where $u = \phi(z)$. Our approach is based on the method used in [15] (see formula (3.1)), with the corresponding expression for the kernel $K_N(z)$ on $\mathcal{E}_\rho$ represented in the form

$$K_N(z) = \frac{A}{u^s}\varphi(u),$$
$$\varphi(u) = 1 + \sum_{k=k_0}^{+\infty} c_k u^{-k}, \qquad c_{k_0} \neq 0, \quad k_0 \geq 1,$$

where A is a real constant and $s \in \mathbb{N}$. Then, for $u = \rho\, e^{i\theta}$ ($\rho > 1$, $\theta \in [0, 2\pi)$),

$$\max_{z \in \mathcal{E}_\rho} |K_N(z)| = \frac{|A|}{\rho^s} \max_{|u|=\rho} |\varphi(u)|,$$

while in all our cases $k_0 = 2$, and the function $\varphi(u)$ will be even in u, $\varphi(u) = \varphi(-u)$, i.e., $\varphi(u) = 1 + \sum_{j=1}^{+\infty} c_{2j} u^{-2j}$, where c_{2j} are real. This allows us to use [15, Corollary 3.5], which claims that under those conditions, the following holds:

- if $c_2 > 0$, then for sufficiently large $\rho > 1$,
$$\max_{|u|=\rho} \varphi(u) = \varphi(\pm\rho);$$
- if $c_2 < 0$, then for sufficiently large $\rho > 1$,
$$\max_{|u|=\rho} \varphi(u) = \varphi(\pm i\rho).$$

2. Results.

2.1. The Fejér formula of the first kind. In the case of the Fejér formula [12, formula (4.30)], we have that the corresponding kernel is equal to ($N = n$ (≥ 2))

$$K_n^F(z) = \frac{\varrho_n^F(z)}{\pi_n^F(z)}, \qquad z \notin [-1, 1],$$

where we can take

$$\pi_n^F(z) = T_n(z) = \frac{u^n + \frac{1}{u^n}}{2}, \qquad z = \frac{u + \frac{1}{u}}{2}, \qquad u = \rho\, e^{i\theta} \ \ (n \geq 0).$$

Then, $\varrho_n^F(z)$ is given by

$$\varrho_n^F(z) = \int_{-1}^{1} \frac{T_n(t)}{z-t}\, dt, \tag{2.1}$$

and $K_n^F(z) = \varrho_n^F(z)/\pi_n^F(z)$.

PROPOSITION 2.1. *Let* $z = \frac{1}{2}(u + \frac{1}{u})$, $u = \rho\, e^{i\theta}$, $\rho > 1$, $T_n(z) = \dfrac{u^n + \frac{1}{u^n}}{2}$, $n \in \mathbb{N}_0$. *For such a complex number* z ($z \in \mathbb{C} \setminus [-1, 1]$) *it holds that*

$$\varrho_n^F(z) = \int_{-1}^{1} \frac{T_n(t)}{z-t}\, dt = T_n(z) \ln \frac{z+1}{z-1} - 4{\sum_{k=1}^{[(n+1)/2]}}' \frac{T_{n-2k+1}(z)}{2k-1}, \tag{2.2}$$

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and

$$\int_{-1}^{1} \frac{\left(t^2-1\right) T_n(t)}{z-t} dt = \left(z^2-1\right) \varrho_n^F(z) + \begin{cases} \frac{2z}{n^2-1}, & n \equiv 0 \pmod 2, \\ \frac{2}{n^2-4}, & n \equiv 1 \pmod 2, \end{cases} \tag{2.3}$$

with the principal value of the logarithmic function $\ln w = \ln |w| + i \arg w$ *(*$-\pi < \arg w \leq \pi$*) and where the notation* $\sum'$ *means that the last term in the sum must be halved if* n *is odd.*

Also, for $n = 2m$ $(m \geq 0)$*,*

$$\varrho_{2m}^F(z)/2 = -2 \sum_{k=0}^{\infty} \frac{2k+1}{4m^2-(2k+1)^2} u^{-2k-1}, \tag{2.4}$$

and, for $n = 2m+1$ $(m \geq 0)$*,*

$$\varrho_{2m+1}^F(z)/2 = -4 \sum_{k=1}^{\infty} \frac{k}{(2m+1)^2-4k^2} u^{-2k}. \tag{2.5}$$

Proof. Following the proofs of [10, Proposition 2.2] and [12, Proposition 2.1], since (2.2) and (2.3) hold for $z = r$, where r is a real number and $|r| > 1$, we conclude that those results hold for any complex number z ($z \in \mathbb{C} \setminus [-1,1]$) by the identity theorem.

The relation (2.4) holds since

$$\begin{aligned}
\varrho_{2m}^F(z)/2 &= (u^{2m}+u^{-2m}) \sum_{k=0}^{\infty} \frac{u^{-2k-1}}{2k+1} - \sum_{k=1}^{m} \frac{u^{2m-2k+1}+u^{-(2m-2k+1)}}{2k-1} \\
&= \sum_{k=0}^{\infty} \frac{u^{2m-2k-1}}{2k+1} + \sum_{k=0}^{\infty} \frac{u^{-2m-2k-1}}{2k+1} - \sum_{k=1}^{m} \frac{u^{2m-2k+1}}{2k-1} - \sum_{k=1}^{m} \frac{u^{-2m+2k-1}}{2k-1} \\
&= \sum_{k=-m}^{\infty} \frac{u^{-2k-1}}{2m+2k+1} + \sum_{k=m}^{\infty} \frac{u^{-2k-1}}{-2m+2k+1} \\
&\qquad - \sum_{k=-m}^{-1} \frac{u^{-2k-1}}{2m+2k+1} - \sum_{k=0}^{m-1} \frac{u^{-2k-1}}{2m-2k-1} \\
&= \sum_{k=0}^{\infty} \frac{u^{-2k-1}}{2m+2k+1} + \sum_{k=0}^{\infty} \frac{u^{-2k-1}}{-2m+2k+1} \\
&= -2 \sum_{k=0}^{\infty} \frac{2k+1}{4m^2-(2k+1)^2} u^{-2k-1}.
\end{aligned}$$

The relation (2.5) holds since

$$\begin{aligned}
\varrho_{2m+1}^F(z)/2 &= (u^{2m+1}+u^{-2m-1}) \sum_{k=0}^{\infty} \frac{u^{-2k-1}}{2k+1} \\
&\qquad - \left(\frac{1}{2m+1} + \sum_{k=1}^{m} \frac{u^{2m-2k+2}+u^{-(2m-2k+2)}}{2k-1} \right) \\
&= \sum_{k=0}^{\infty} \frac{u^{2m-2k}}{2k+1} + \sum_{k=0}^{\infty} \frac{u^{-2m-2k-2}}{2k+1} - \sum_{k=1}^{m+1} \frac{u^{2m-2k+2}}{2k-1} - \sum_{k=1}^{m} \frac{u^{-2m+2k-2}}{2k-1}
\end{aligned}$$

$$= \sum_{k=-m}^{\infty} \frac{u^{-2k}}{2m+2k+1} + \sum_{k=m+1}^{\infty} \frac{u^{-2k}}{-2m+2k-1} - \sum_{k=-m}^{0} \frac{u^{-2k}}{2m+2k+1} + \sum_{k=1}^{m} \frac{u^{-2k}}{-2m+2k-1}$$
$$= \sum_{k=1}^{\infty} \frac{u^{-2k}}{2m+2k+1} + \sum_{k=1}^{\infty} \frac{u^{-2k}}{-2m+2k-1}$$
$$= -4\sum_{k=1}^{\infty} \frac{k}{(2m+1)^2-4k^2} u^{-2k}.$$

The proof is complete. $\square$

This means that for all $m \in \mathbb{N}$,

$$K_{2m}^F(z(u)) = -\frac{4\left(\dfrac{1}{4m^2-1}\cdot\dfrac{1}{u} + \dfrac{3}{4m^2-9}\cdot\dfrac{1}{u^3} + O\left(u^{-5}\right)\right)}{\dfrac{u^{2m}+u^{-2m}}{2}}$$
$$= -\frac{8}{4m^2-1}\cdot\frac{1}{u^{2m+1}}\left(1 + \frac{3(4m^2-1)}{4m^2-9}\cdot\frac{1}{u^2} + O(u^{-4})\right)$$
$$= -\frac{8}{4m^2-1}\cdot\frac{1}{u^{2m+1}}\varphi_{2m}^F(u), \qquad \text{as } |u| \to +\infty,$$

and for all $m \in \mathbb{N}_0$,

$$(2.6)\quad K_{2m+1}^F(z(u)) = -\frac{8\left(\dfrac{1}{(2m+1)^2-4}\cdot\dfrac{1}{u^2} + \dfrac{2}{(2m+1)^2-16}\cdot\dfrac{1}{u^4} + O\left(u^{-6}\right)\right)}{\dfrac{u^{2m+1}+u^{-2m-1}}{2}}$$
$$= -\frac{16}{(2m+1)^2-4}\cdot\frac{1}{u^{2m+3}}\left(1 + \frac{2((2m+1)^2-4)}{(2m+1)^2-16}\cdot\frac{1}{u^2} + O(u^{-4})\right),$$
$$= -\frac{16}{(2m+1)^2-4}\cdot\frac{1}{u^{2m+3}}\varphi_{2m+1}^F(u), \qquad \text{as } |u| \to +\infty,$$

where $z(u) = \frac{1}{2}(u + \frac{1}{u})$. It is easy to verify that the coefficient $\frac{3(4m^2-1)}{4m^2-9}$ is negative only for $m = 1$ ($n = 2$), while it is positive for each $m > 1$, and the coefficient $\frac{2(2m+1)^2-8}{(2m+1)^2-16}$ is positive for each $m > 1$ ($n > 3$), while for $m = 1$ ($n = 3$) it is negative.

It is also obvious that

$$\varphi_{2m}^F(u) \equiv \varphi_{2m}^F(-u)$$

since

$$\pi_{2m}^F(z(u)) \equiv \pi_{2m}^F(z(-u)) \qquad \text{and} \qquad \varrho_{2m}^F(z(u)) \equiv -\varrho_{2m}^F(z(-u)),$$

while

$$\varphi_{2m+1}^F(u) \equiv \varphi_{2m+1}^F(-u)$$

since

$$\pi_{2m+1}^F(z(u)) \equiv -\pi_{2m+1}^F(z(-u)) \qquad \text{and} \qquad \varrho_{2m+1}^F(z(u)) \equiv \varrho_{2m+1}^F(z(-u)),$$

which, on the basis of [15, Corollary 3.5] (c_{2j} are real for the condition there), allows us to state the following theorem:

THEOREM 2.2. *For the Fejér quadrature formula of the first kind, where $n \in \mathbb{N}$, $n \neq 1, 2, 3$, there exists a $\rho^* \in (1, +\infty)$ $(\rho^* = \rho_n^*)$ such that for each $\rho \geq \rho^*$ the modulus of the kernel $\left|K_n^F(z)\right|$ attains its maximum value on the real axis ($\theta = 0$ and $\theta = \pi$), i.e.,*

$$\max_{z \in \mathcal{E}_\rho} \left|K_n^F(z)\right| = \left|K_n^F\left(\frac{1}{2}(\rho+\rho^{-1})\right)\right| = \left|K_n^F\left(-\frac{1}{2}(\rho+\rho^{-1})\right)\right|, \qquad n \geq 4,$$

while for $1 \leq n \leq 3$, there exists a $\rho^* \in (1, +\infty)$ $(\rho^* = \rho_2^*$ or $\rho^* = \rho_3^*)$ such that for each $\rho \geq \rho^*$ the modulus of the kernel $\left|K_n^F(z)\right|$ attains its maximum value on the imaginary semi-axis ($\theta = \pi/2$), i.e., for $n = 1, 2, 3$,

$$\max_{z \in \mathcal{E}_\rho} \left|K_n^F(z)\right| = \left|K_n^F\left(\frac{i}{2}(\rho-\rho^{-1})\right)\right| = \left|K_n^F\left(-\frac{i}{2}(\rho-\rho^{-1})\right)\right|.$$

In the case $n = 1$, as it is mentioned in [12], the Fejér formula of the first kind is the 1-point Gauss formula for the Legendre weight function $\omega(t) = 1$ on $[-1, 1]$, while (2.2) for $n = 1$ becomes

$$\begin{aligned}\varrho_1^F(z) &= \int_{-1}^{1} \frac{T_1(t)}{z-t}\, dt = T_1(z) \ln \frac{z+1}{z-1} - 2\, T_0(z) \\ &= z \ln \frac{z+1}{z-1} - 2 = \left(u + \frac{1}{u}\right) \ln \frac{u+1}{u-1} - 2,\end{aligned}$$

hence,

$$K_1^F(z(u)) = \frac{\varrho_1^F(z(u))}{\pi_1^F(z(u))} = 2\frac{\left(u+\frac{1}{u}\right)\ln\dfrac{u+1}{u-1} - 2}{u + \frac{1}{u}},$$

and, referring to (2.6),

$$\begin{aligned}&\frac{16}{3} \cdot \frac{1}{u^3}\left(1 + \frac{2}{5} \cdot \frac{1}{u^2} + O(u^{-4})\right)\left(1 - \frac{1}{u^2} + O(u^{-4})\right) \\ &\qquad = \frac{16}{3} \cdot \frac{1}{u^3}\left(1 - \frac{3}{5} \cdot \frac{1}{u^2} + O(u^{-4})\right) = \frac{16}{3} \cdot \frac{1}{u^3} \varphi_1^F(u),\end{aligned}$$

for $\varphi_1^F(u)$ satisfying $\varphi_1^F(u) = \varphi_1^F(-u)$. It follows that

$$\max_{z \in \mathcal{E}_\rho} \left|K_1^F(z)\right| = \left|K_1^F\left(\frac{i}{2}(\rho-\rho^{-1})\right)\right| = \left|K_1^F\left(-\frac{i}{2}(\rho-\rho^{-1})\right)\right|.$$

2.2. The Clenshaw–Curtis formula. In the case of the Clenshaw–Curtis quadrature formula [12, Equation (4.1)] ($N = n+2$ ($n \geq 0$)), we have that the corresponding kernel is equal to

$$K_{n+2}^{CC}(z) = \frac{\varrho_{n+2}^{CC}(z)}{\pi_{n+2}^{CC}(z)}, \qquad z \notin [-1,1],$$

where we can take

$$\pi_{n+2}^{CC}(z) = T_{n+2}(z) - T_n(z) = \frac{u^{n+2} + \dfrac{1}{u^{n+2}} - u^n - \dfrac{1}{u^n}}{2}, \quad z = \frac{u+\frac{1}{u}}{2}, \quad u = \rho\, e^{i\theta},$$

and where T_i is the Chebyshev orthogonal polynomial of the first kind of degree i, while

$$\begin{aligned}\varrho_{n+2}^{CC}(z) &= \int_{-1}^{1} \frac{\pi_{n+2}^{CC}(t)}{z-t}\,dt = \int_{-1}^{1} \frac{T_{n+2}(t) - T_n(t)}{z-t}\,dt \\ &= \int_{-1}^{1} \frac{T_{n+2}(t)}{z-t}\,dt - \int_{-1}^{1} \frac{T_n(t)}{z-t} = \varrho_{n+2}^{F}(z) - \varrho_n^{F}(z).\end{aligned}$$

Here we used the notation from (2.1). Now we can state the following:

THEOREM 2.3. *For the Clenshaw–Curtis quadrature formula, where $n \in \mathbb{N}_0$, $n \neq 2,3$, there exists a $\rho^* \in (1,+\infty)$ ($\rho^* = \rho_{n+2}^*$) such that for each $\rho \geq \rho^*$ the modulus of the kernel $\left|K_{n+2}^{CC}(z)\right|$ attains its maximum value on the real axis ($\theta = 0$ and $\theta = \pi$), i.e.,*

$$\max_{z\in\mathcal{E}_\rho}\left|K_{n+2}^{CC}(z)\right| = \left|K_{n+2}^{CC}\left(\frac{1}{2}(\rho+\rho^{-1})\right)\right| = \left|K_{n+2}^{CC}\left(-\frac{1}{2}(\rho+\rho^{-1})\right)\right|, \;\; n \neq 2,3, \;\; n \in \mathbb{N}_0,$$

while for $n = 2$ and $n = 3$ there exists a $\rho^* \in (1,+\infty)$ ($\rho^* = \rho_4^*$ or $\rho^* = \rho_5^*$) such that for each $\rho \geq \rho^*$ the modulus of the kernel $\left|K_n^{CC}(z)\right|$ attains its maximum value on the imaginary semi-axis ($\theta = \pi/2$), i.e., for $n = 4,5$,

$$\max_{z\in\mathcal{E}_\rho}\left|K_n^{CC}(z)\right| = \left|K_n^{CC}\left(\frac{i}{2}(\rho-\rho^{-1})\right)\right| = \left|K_n^{CC}\left(-\frac{i}{2}(\rho-\rho^{-1})\right)\right|.$$

Proof. Recall that $z(u) = \frac{1}{2}(u+\frac{1}{u})$. If $n = 2m$ for some $m \in \mathbb{N}_0$, then

$$\begin{aligned}&\varrho_{2m+2}^{CC}(z(u)) \\ &\quad = \varrho_{2m+2}^{F}(z(u)) - \varrho_{2m}^{F}(z(u)) \\ &\quad = 4\sum_{k=0}^{\infty}\left(\frac{2k+1}{4m^2-(2k+1)^2} - \frac{2k+1}{4(m+1)^2-(2k+1)^2}\right)u^{-2k-1} \\ &\quad = 4\sum_{k=0}^{\infty}\frac{(2k+1)\left(4((m+1)^2-m^2)\right)}{\left(4m^2-(2k+1)^2\right)\left(4(m+1)^2-(2k+1)^2\right)}\,u^{-2k-1} \\ &\quad = 16(2m+1) \\ &\qquad \times \sum_{k=0}^{\infty}\frac{2k+1}{(2m-2k-1)(2m-2k+1)(2m+2k+1)(2m+2k+3)}\,u^{-2k-1}\end{aligned}$$

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$$
\begin{aligned}
&= 16\Bigg(\frac{1}{(2m-1)(2m+1)(2m+3)}\frac{1}{u} \\
&\qquad + \frac{3(2m+1)}{(2m-3)(2m-1)(2m+3)(2m+5)}\frac{1}{u^3} + O\left(u^{-5}\right)\Bigg) \\
&= \frac{16u^{-1}}{(2m-1)(2m+1)(2m+3)}\left(1 + \frac{3(2m+1)^2}{(2m-3)(2m+5)}\frac{1}{u^2} + O\left(u^{-4}\right)\right).
\end{aligned}
$$

On the other hand,

$$
\pi_{2m+2}^{CC}(z(u)) = \frac{u^{2m+2}}{2}\left(1 - \frac{1}{u^2} + O\left(u^{-4}\right)\right), \qquad z(u) = \frac{u + \frac{1}{u}}{2}, \quad |u| \to +\infty.
$$

For the case $m = 0$, we have

$$
\pi_{2}^{CC}(z(u)) = \frac{u^{2}}{2}\left(1 - \frac{2}{u^2} + O\left(u^{-4}\right)\right), \qquad z(u) = \frac{u + \frac{1}{u}}{2}, \quad |u| \to +\infty.
$$

Hence, we have for each $m \in \mathbb{N}$,

$$
\begin{aligned}
K_{2m+2}^{CC}(z(u)) &= \frac{\rho_{2m+2}^{CC}(z)}{\pi_{2m+2}^{CC}(z)} \\
&= \frac{32}{u^{2m+3}(2m-1)(2m+1)(2m+3)} \\
&\qquad \times \left(1 + \left(1 + \frac{3(2m+1)^2}{(2m-3)(2m+5)}\right)\frac{1}{u^2} + O\left(u^{-4}\right)\right) \\
&= \frac{32}{u^{2m+3}(2m-1)(2m+1)(2m+3)}\varphi_{2m+2}^{CC}(u), \quad \text{as } |u| \to +\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
K_{2}^{CC}(z(u)) = \frac{\rho_{2}^{CC}(z)}{\pi_{2}^{CC}(z)} &= \frac{-32}{3u^3}\left(1 + \left(2 - \frac{1}{5}\right)\frac{1}{u^2} + O\left(u^{-4}\right)\right) \\
&= \frac{-32}{3u^3}\left(1 + \frac{9}{5}\frac{1}{u^2} + O\left(u^{-4}\right)\right) \\
&= -\frac{32}{3u^3}\varphi_{2}^{CC}(u), \quad \text{as } |u| \to +\infty.
\end{aligned}
$$

It is also obvious that

$$
\varphi_{2m+2}^{CC}(u) \equiv \varphi_{2m+2}^{CC}(-u)
$$

since

$$
\pi_{2m+2}^{CC}(z(u)) \equiv \pi_{2m+2}^{CC}(z(-u)) \qquad \text{and} \qquad \varrho_{2m+2}^{CC}(z(u)) \equiv -\varrho_{2m+2}^{CC}(z(-u)),
$$

for each $m \in \mathbb{N}_0$, which directly finishes the proof on the basis of [15, Corollary 3.5] (c_{2j} are real for the condition there) since $1 + \frac{3(2m+1)^2}{(2m-3)(2m+5)} > 0$ for each $m \in \mathbb{N}$, $m \neq 1$, while for $m = 1$ ($n = 2$) this expression is negative.

If $n = 2m+1$, $m \in \mathbb{N}_0$, then

$$\begin{aligned}
\varrho^{CC}_{2m+3}(z(u)) &= \varrho^{F}_{2m+3}(z(u)) - \varrho^{F}_{2m+1}(z(u)) \\
&= 8\sum_{k=1}^{\infty}\left(\frac{k}{(2m+1)^2-4k^2} - \frac{k}{(2m+3)^2-4k^2}\right)u^{-2k} \\
&= \sum_{k=1}^{\infty}\frac{8k\left((2m+3)^2-(2m+1)^2\right)}{\left((2m+1)^2-4k^2\right)\left((2m+3)^2-4k^2\right)}u^{-2k} \\
&= 64\sum_{k=1}^{\infty}\frac{k(m+1)}{(2m-2k+1)(2m-2k+3)(2m+2k+1)(2m+2k+3)}u^{-2k} \\
&= 64\left(\frac{m+1}{(2m-1)(2m+1)(2m+3)(2m+5)}\frac{1}{u^2}\right. \\
&\qquad \left. + \frac{2(m+1)}{(2m-3)(2m-1)(2m+5)(2m+7)}\frac{1}{u^4}\right) + O\left(u^{-6}\right) \\
&= 64\left(\frac{(m+1)u^{-2}}{(2m-1)(2m+1)(2m+3)(2m+5)}\right. \\
&\qquad \left. + \frac{2(2m+1)(2m+3)}{(2m-3)(2m+7)}\frac{1}{u^2} + O\left(u^{-4}\right)\right).
\end{aligned}$$

On the other hand,

$$\pi^{CC}_{2m+3}(z(u)) = \frac{u^{2m+3}}{2}\left(1 - \frac{1}{u^2} + O\left(u^{-4}\right)\right), \qquad z(u) = \frac{u+\frac{1}{u}}{2}, \quad |u| \to +\infty.$$

Thus, we have

$$\begin{aligned}
K^{CC}_{2m+3}(z(u)) &= \frac{128(m+1)}{u^{2m+5}(2m-1)(2m+1)(2m+3)(2m+5)} \\
&\qquad \times\left(1 + \left(1 + \frac{2(2m+1)(2m+3)}{(2m-3)(2m+7)}\right)\frac{1}{u^2} + O\left(u^{-4}\right)\right) \\
&= \frac{128(m+1)}{u^{2m+5}(2m-1)(2m+1)(2m+3)(2m+5)}\varphi^{CC}_{2m+3}(u).
\end{aligned}$$

It is also obvious that

$$\varphi^{CC}_{2m+3}(u) \equiv \varphi^{CC}_{2m+3}(-u)$$

since

$$\pi^{CC}_{2m+3}(z(u)) \equiv -\pi^{CC}_{2m+3}(z(-u)) \qquad \text{and} \qquad \varrho^{CC}_{2m+3}(z(u)) \equiv \varrho^{CC}_{2m+3}(z(-u)),$$

which again directly finishes the proof on the basis of [15, Corollary 3.5] (c_{2j} are real for the condition there) since $1 + \frac{2(2m+1)(2m+3)}{(2m-3)(2m+7)} > 0$ for each $m \in \mathbb{N}_0$, $m \neq 1$, while for $m = 1$ ($n = 3$) this expression is negative. ∎

2.3. The Basu formula. In the case of the Basu formula from [12, Equation (4.14)], we have that the corresponding kernel is equal to ($N = n+2$ ($n \geq 0$))

$$K^{B}_{n+2}(z) = \frac{\varrho^{B}_{n+2}(z)}{\pi^{B}_{n+2}(z)}, \qquad z \notin [-1, 1],$$

where we can take

$$\pi^B_{n+2}(z(u)) = \left(z^2-1\right)T_n(z) = \frac{\left(u^2-2+\frac{1}{u^2}\right)\left(u^n+\frac{1}{u^n}\right)}{8}$$

$$(2.7) \qquad = \frac{u^{n+2}}{8}\left(1-\frac{2}{u^2}+O\left(\frac{1}{u^4}\right)\right),$$

$$|u|\to+\infty,\ \ z(u)=\frac{u+\frac{1}{u}}{2},\ \ u=\rho\, e^{i\theta},$$

and

$$\varrho^B_{n+2}(z) = \int_{-1}^{1}\frac{\pi^B_{n+2}(t)}{z-t}\,dt = \int_{-1}^{1}\frac{(t^2-1)T_n(t)}{z-t}\,dt.$$

For the case $n=0$, we have

$$\pi^B_2(z(u)) = \frac{u^2}{4}\left(1-\frac{2}{u^2}+O\left(\frac{1}{u^4}\right)\right), \qquad |u|\to+\infty, \quad z(u)=\frac{u+\frac{1}{u}}{2}, \quad u=\rho\, e^{i\theta}.$$

From (2.3) and (2.4) we have

$$\begin{aligned}
\varrho^B_{2m+2}(z(u)) &= \left(z^2-1\right)\varrho^F_{2m}(z)+\frac{2z}{4m^2-1}\\
&= -\frac{1}{4}\left(u^2-2+\frac{1}{u^2}\right)4\sum_{k=0}^{\infty}\frac{2k+1}{4m^2-(2k+1)^2}u^{-2k-1}+\frac{u+\frac{1}{u}}{4m^2-1}\\
&= -\left(u^2-2+\frac{1}{u^2}\right)\left(\frac{1}{4m^2-1}\cdot\frac{1}{u}+\frac{3}{4m^2-9}\frac{1}{u^3}\right.\\
&\qquad\left.+\frac{5}{4m^2-25}\frac{1}{u^5}+O\left(\frac{1}{u^7}\right)\right)+\frac{u+\frac{1}{u}}{4m^2-1},
\end{aligned}$$

when $|u|\to+\infty$. Further, we obtain that in the expansion of the last expression, the coefficient multiplying u is equal to $-\frac{1}{4m^2-1}+\frac{1}{4m^2-1}=0$, the coefficient multiplying $\frac{1}{u}$ is equal to

$$\frac{2}{4m^2-1}+\frac{1}{4m^2-1}-\frac{3}{4m^2-9}=\frac{3}{4m^2-1}-\frac{3}{4m^2-9}=\frac{-24}{\left(4m^2-1\right)\left(4m^2-9\right)},$$

while the coefficient multiplying $\frac{1}{u^3}$ is equal to

$$\begin{aligned}
-\frac{1}{4m^2-1}+\frac{6}{4m^2-9}-\frac{5}{4m^2-25} &= \frac{-288m^2-120}{\left(4m^2-1\right)\left(4m^2-9\right)\left(4m^2-25\right)}\\
&= -\frac{24\left(12m^2+5\right)}{\left(4m^2-1\right)\left(4m^2-9\right)\left(4m^2-25\right)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\varrho^B_{2m+2}(z(u)) = \Bigg(&-\frac{24}{\left(4m^2-1\right)\left(4m^2-9\right)}\frac{1}{u}\\
&-\frac{24\left(12m^2+5\right)}{\left(4m^2-1\right)\left(4m^2-9\right)\left(4m^2-25\right)}\frac{1}{u^3}+O\left(\frac{1}{u^5}\right)\Bigg)
\end{aligned}$$

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$$= -\frac{24}{(4m^2-1)(4m^2-9)}\frac{1}{u}\left(1+\frac{12m^2+5}{4m^2-25}\frac{1}{u^2}+O\left(\frac{1}{u^4}\right)\right), \quad |u|\to\infty.$$

Finally, using (2.7),

$$\begin{aligned} K^B_{2m+2}(z(u)) &= -\frac{192}{(4m^2-1)(4m^2-9)\,u^{2m+3}} \\ &\quad\times\left(1+\left(\frac{12m^2+5}{4m^2-25}+2\right)\frac{1}{u^2}+O\left(\frac{1}{u^4}\right)\right) \\ &= -\frac{192}{(4m^2-1)(4m^2-9)\,u^{2m+3}}\varphi^B_{2m+2}(u), \qquad \text{as } |u|\to\infty. \end{aligned}$$

For the case $m=0$, we have

$$K^B_2(z(u)) = -\frac{96}{9u^3}\varphi^B_2(u).$$

It is easy to verify that the coefficient $\frac{12m^2+5}{4m^2-25}+2$ is negative only for $m=2$ ($n=4$), while it is positive for each $m\in\mathbb{N}_0$, $m\neq 2$.

For $n=2m+1$, $m\in\mathbb{N}_0$, from (2.3) and (2.5) we have

$$\begin{aligned} &\varrho^B_{2m+3}(z(u)) \\ &\quad= \left(z^2-1\right)\varrho^F_{2m+1}(z(u)) + \frac{2}{(2m+1)^2-4} \\ &\quad= -\frac{1}{4}\left(u^2-2+\frac{1}{u^2}\right)8\sum_{k=1}^{\infty}\frac{k}{(2m+1)^2-4k^2}u^{-2k} + \frac{2}{(2m+1)^2-4} \\ &\quad= 2\left(u^2-2+\frac{1}{u^2}\right)\left(-\frac{1}{(2m+1)^2-4}\cdot\frac{1}{u^2} - \frac{2}{(2m+1)^2-16}\frac{1}{u^4}\right. \\ &\qquad\qquad \left. - \frac{3}{(2m+1)^2-36}\frac{1}{u^6} + O\left(\frac{1}{u^8}\right)\right) + \frac{2}{(2m+1)^2-4}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{as } |u|\to+\infty. \end{aligned}$$

Further, we obtain that in the expansion of the last expression the term that does not depend on u is equal to $-\frac{2}{(2m+1)^2-4}+\frac{2}{(2m+1)^2-4}=0$, the coefficient multiplying $\frac{1}{u^2}$ is equal to

$$\frac{4}{(2m+1)^2-4} - \frac{4}{(2m+1)^2-16} = \frac{-48}{((2m+1)^2-4)\,((2m+1)^2-16)},$$

while the coefficient multiplying $\frac{1}{u^4}$ is equal to

$$\begin{aligned} &-\frac{2}{(2m+1)^2-4} + \frac{8}{(2m+1)^2-16} - \frac{6}{(2m+1)^2-36} \\ &\qquad = \frac{-96\left((2m+1)^2+4\right)}{((2m+1)^2-1)\,((2m+1)^2-16)\,((2m+1)^2-36)}. \end{aligned}$$

Thus,

$$\begin{aligned}&\varrho_{2m+3}^{B}(z(u))\\&=-\frac{48}{((2m+1)^2-4)\,((2m+1)^2-16)}\frac{1}{u^2}\left(1+\frac{2(2m+1)^2+8}{(2m+1)^2-36}\frac{1}{u^2}+O\left(\frac{1}{u^4}\right)\right),\end{aligned}$$

when $|u|\to\infty$. Finally, using (2.7),

$$\begin{aligned}K_{2m+3}^{B}(z(u))&=\frac{-384}{((2m+1)^2-4)\,((2m+1)^2-16)\,u^{2m+5}}\\&\qquad\times\left(1+\left(\frac{2(2m+1)^2+8}{(2m+1)^2-36}+2\right)\frac{1}{u^2}+O\left(\frac{1}{u^4}\right)\right)\\&=\frac{-384}{((2m+1)^2-4)\,((2m+1)^2-16)\,u^{2m+5}}\varphi_{2m+3}^{B}(u),\end{aligned}$$

when $|u|\to\infty$. It is easy to verify that the coefficient $\frac{2(2m+1)^2+8}{(2m+1)^2-36}+2$ is negative only for $m=2$ ($n=5$), while it is positive for each $m\in\mathbb{N}_0$, $m\neq 2$.

It is also obvious that

$$\varphi_{2m+2}^{B}(u)\equiv\varphi_{2m+2}^{B}(-u)$$

since

$$\pi_{2m+2}^{B}(z(u))\equiv\pi_{2m+2}^{B}(z(-u))\qquad\text{and}\qquad\varrho_{2m+2}^{B}(z(u))\equiv-\varrho_{2m+2}^{B}(z(-u)),$$

while

$$\varphi_{2m+3}^{B}(u)\equiv\varphi_{2m+3}^{B}(-u)$$

since

$$\pi_{2m+3}^{B}(z(u))\equiv-\pi_{2m+3}^{B}(z(-u))\qquad\text{and}\qquad\varrho_{2m+3}^{B}(z(u))\equiv\varrho_{2m+3}^{B}(z(-u)),$$

which directly on the basis of [15, Corollary 3.5] (the coefficients in the expansion of $\varphi(u)$ are real in that corollary) allows us to state the following result:

THEOREM 2.4. *For the Basu quadrature formula, where $n\in\mathbb{N}_0$, $n\neq 4,5$, there exists a $\rho^*\in(1,+\infty)$ ($\rho^*=\rho^*_{n+2}$) such that for each $\rho\geq\rho^*$ the modulus of the kernel $\left|K_{n+2}^{B}(z)\right|$ attains its maximum value on the real axis ($\theta=0$ and $\theta=\pi$), i.e.,*

$$\max_{z\in\mathcal{E}_\rho}\left|K_{n+2}^{B}(z)\right|=\left|K_{n+2}^{B}\left(\frac{1}{2}(\rho+\rho^{-1})\right)\right|=\left|K_{n+2}^{B}\left(-\frac{1}{2}(\rho+\rho^{-1})\right)\right|,\;\; n\neq 4,5,\; n\in\mathbb{N}_0,$$

while for $n=4$ and $n=5$ there exists a $\rho^*\in(1,+\infty)$ ($\rho^*=\rho_6^*$ or $\rho^*=\rho_7^*$) such that for each $\rho\geq\rho^*$ the modulus of the kernel $\left|K_{n+2}^{B}(z)\right|$ attains its maximum value on the imaginary semi-axis ($\theta=\pi/2$), for $n=6,7$, i.e.,

$$\max_{z\in\mathcal{E}_\rho}\left|K_{n}^{B}(z)\right|=\left|K_{n}^{B}\left(\frac{i}{2}(\rho-\rho^{-1})\right)\right|=\left|K_{n}^{B}\left(-\frac{i}{2}(\rho-\rho^{-1})\right)\right|.$$

REMARK 2.5. Observe that the Fejér rule with $N=1$ (N is the number of nodes) coincides with the midpoint rule, the Clenshaw–Curtis and the Basu rules with $N=2$ coincide with the trapezoidal rule, and those with $N=3$ with the Simpson rule.

3. Numerical results. Hereafter, for a function g and a compact subset E of the complex plane, the L^∞–norm of g on E is denoted by

$$\|g\|_E = \max_{z\in E} |g(z)|\,.$$

Now, from (1.3), taking $\Gamma = \mathcal{E}_\rho$, we easily get that

$$|R_N(f)| \le \frac{l(\mathcal{E}_\rho)}{2\pi}\, \|K_N\|_{\mathcal{E}_\rho} \cdot \|f\|_{\mathcal{E}_\rho}\,.$$

First, we give a general description of the numerical computations of the error kernels for the Fejér, Clenshaw–Curtis, and Basu rules. $K_n^F(z)$ is given as a fraction with $\rho_n^F(z)$ in (2.2) in the numerator and $T_n(z)$ in the denominator. $K_{n+2}^{CC}(z)$ is given as a fraction with $\rho_{n+2}^F(z) - \rho_n^F(z)$ in the numerator and $T_{n+2}(z) - T_n(z)$ in the denominator. $K_{n+2}^B(z)$ is given as a fraction with (2.3) in the numerator and $(z^2-1)T_n(z)$ in the denominator. Computing $\rho_n^F(z)$ requires $T_k(z)$ and $\log\frac{z+1}{z-1}$. The latter is an analytic function on $z\in\mathbb{C}\setminus[-1,1]$. The recurrence relation is used to compute $T_k(z)$. If $z = (u+u^{-1})/2$, then $T_k(z) = (u^k+u^{-k})/2$.

In the case of the Fejér quadrature formula, for each $n \ge 4$ it holds that $\|K_n^F\|_{\mathcal{E}_\rho} = \left|K_n^F(a_1)\right|$ for ρ large enough, where we set $a_1 = \frac{\rho+\frac{1}{\rho}}{2}$, and it is easy to find that

$$\left|K_n^F(a_1)\right| = 2\left|\frac{\left(\rho^n+\frac{1}{\rho^n}\right)\ln\frac{\rho+1}{\rho-1} - 2{\sum_{k=1}^{\left[\frac{n+1}{2}\right]}}' \frac{\rho^{n-2k+1}+\frac{1}{\rho^{n-2k+1}}}{2k-1}}{\rho^n+\frac{1}{\rho^n}}\right|$$

because this involves just the ordinary absolute value of a real number.

In the case of the Clenshaw–Curtis quadrature formula, for $n=0$, $n=1$, and each $n\ge 4$ it holds that $\|K_{n+2}^{CC}\|_{\mathcal{E}_\rho} = \left|K_{n+2}^{CC}(a_1)\right|$ for ρ large enough, and for the same reason it is easy to find that

$$\begin{aligned}&\left|K_{n+2}^{CC}(a_1)\right| \\ &= 2\left|\frac{\left(\rho^{n+2}+\frac{1}{\rho^{n+2}}-\rho^n-\frac{1}{\rho^n}\right)\ln\frac{\rho+1}{\rho-1} - 2{\sum_{k=1}^{\left[\frac{n+3}{2}\right]}}' \frac{\rho^{n-2k+3}+\frac{1}{\rho^{n-2k+3}}}{2k-1} + 2{\sum_{k=1}^{\left[\frac{n+1}{2}\right]}}' \frac{\rho^{n-2k+1}+\frac{1}{\rho^{n-2k+1}}}{2k-1}}{\rho^{n+2}+\frac{1}{\rho^{n+2}}-\rho^n-\frac{1}{\rho^n}}\right|,\end{aligned}$$

while in the case of the Basu quadrature formula, for $n=0$, $n=1$, $n=2$, $n=3$, and each $n\ge 6$ it holds that $\|K_{n+2}^B\|_{\mathcal{E}_\rho} = \left|K_{n+2}^B(a_1)\right|$ for ρ large enough, and it is easy to find

$$\begin{aligned}&\left|K_{n+2}^B(a_1)\right| \\ &= 2\left|\frac{\left(\rho^2+\frac{1}{\rho^2}-2\right)\left(\left(\rho^n+\frac{1}{\rho^n}\right)\ln\frac{\rho+1}{\rho-1} - 2{\sum_{k=1}^{\left[\frac{n+1}{2}\right]}}' \frac{\rho^{n-2k+1}+\frac{1}{\rho^{n-2k+1}}}{2k-1}\right) + b(n,\rho)}{\left(\rho^2+\frac{1}{\rho^2}-2\right)\left(\rho^n+\frac{1}{\rho^n}\right)}\right|,\end{aligned}$$

where

$$b(n,\rho)=\begin{cases}\dfrac{4(\rho+\frac{1}{\rho})}{n^2-1}, & n\equiv 0 \pmod 2,\\ \dfrac{8}{n^2-4}, & n\equiv 1 \pmod 2.\end{cases}$$

On the other hand, in the case of the Fejér quadrature formula, for $n=2$ and $n=3$ it holds that $\|K_n^F\|_{\mathcal{E}_\rho}=\left|K_n^F\,(i\,b_1)\right|$ for ρ large enough, where we set $b_1=\frac{\rho-\frac{1}{\rho}}{2}$, and it is easy to find that

$$\begin{aligned}&\left|K_n^F\,(i\,b_1)\right|\\ &=2\left|\frac{\left((i\rho)^n+\dfrac{1}{(i\rho)^n}\right)\ln\frac{i\rho+1}{i\rho-1}-2{\sum}'^{\left[\frac{n+1}{2}\right]}_{k=1}\dfrac{(i\rho)^{n-2k+1}+\frac{1}{(i\rho)^{n-2k+1}}}{2k-1}}{(i\rho)^n+\dfrac{1}{(i\rho)^n}}\right|\\ &=\left|\frac{\left(\rho^n+\dfrac{(-1)^n}{\rho^n}\right)(2\arctan\,\rho-\pi)+2{\sum}'^{\left[\frac{n+1}{2}\right]}_{k=1}(-1)^{k-1}\dfrac{\rho^{n-2k+1}+\dfrac{(-1)^{n+1}}{\rho^{n-2k+1}}}{2k-1}}{\rho^n+\dfrac{(-1)^n}{\rho^n}}\right|\end{aligned}$$

because

$$\begin{aligned}\ln\frac{i\rho+1}{i\rho-1}&=\ln\left(\frac{i\rho+1}{i\rho-1}\frac{i\rho+1}{i\rho+1}\right)=\ln\left(\frac{\rho^2-1}{\rho^2+1}-\frac{2\rho}{\rho^2-1}i\right)\\ &=i\arg\left(\frac{\rho^2-1}{\rho^2+1}-\frac{2\rho}{\rho^2+1}i\right)=i\,\arctan\left(\frac{2\rho}{1-\rho^2}\right)=i\,(2\arctan\,\rho-\pi)\,.\end{aligned}$$

Analogously, in the case of the Clenshaw–Curtis quadrature formula, for $n=2$ and $n=3$ it holds that $\|K_{n+2}^{CC}\|_{\mathcal{E}_\rho}=\left|K_{n+2}^{CC}\,(i\,b_1)\right|$ for ρ large enough, and we have

$$\left|K_{n+2}^{CC}\,(i\,b_1)\right|=2\left|\frac{\left(\rho^{n+2}+\frac{(-1)^n}{\rho^{n+2}}+\rho^n+\frac{(-1)^n}{\rho^n}\right)(2\arctan\,\rho-\pi)-2c(n,\rho)}{\rho^{n+2}+\frac{(-1)^n}{\rho^{n+2}}+\rho^n+\frac{(-1)^n}{\rho^n}}\right|,$$

where

$$c(n,\rho)={\sum}'^{\left[\frac{n+3}{2}\right]}_{k=1}(-1)^k\frac{\rho^{n-2k+3}+\dfrac{(-1)^{n+1}}{\rho^{n-2k+3}}}{2k-1}+{\sum}'^{\left[\frac{n+1}{2}\right]}_{k=1}(-1)^k\frac{\rho^{n-2k+1}+\dfrac{(-1)^{n+1}}{\rho^{n-2k+1}}}{2k-1},$$

while in the case of the Basu quadrature formula, for $n=4$ and $n=5$ it holds that $\|K_{n+2}^B\|_{\mathcal{E}_\rho}=\left|K_{n+2}^B\,(i\,b_1)\right|$ for ρ large enough, and we have

$$\begin{aligned}&\|K_{4+2}^B\|_{\mathcal{E}_\rho}=\left|K_6^B\,(i\,b_1)\right|\\ &=2\left|\frac{\left((i\rho)^2+\frac{1}{(i\rho)^2}-2\right)\left(\left((i\rho)^4+\frac{1}{(i\rho)^4}\right)\ln\frac{i\rho+1}{i\rho-1}-2{\sum}'^{2}_{k=1}\frac{(i\rho)^{5-2k}+\frac{1}{(i\rho)^{5-2k}}}{2k-1}\right)+\frac{2\left(i\,\rho+\frac{1}{i\,\rho}\right)}{4^2-1}}{\left((i\rho)^2+\frac{1}{(i\rho)^2}-2\right)\left((i\rho)^4+\frac{1}{(i\rho)^4}\right)}\right|\end{aligned}$$

$$= 2 \left| \frac{\left(\rho^2+\frac{1}{\rho^2}+2\right)\left(\left(\rho^4+\frac{1}{\rho^4}\right)(2\arctan\,\rho-\pi)+2\sum_{k=1}^{\prime 2}(-1)^{(k-1)}\frac{\rho^{5-2k}+\frac{1}{\rho^{5-2k}}}{2k-1}\right)-\frac{2}{15}}{\left(\rho^2+\frac{1}{\rho^2}+2\right)\left(\rho^4+\frac{1}{\rho^4}\right)} \right|$$

and

$$\|K_{5+2}^B\|_{\mathcal{E}_\rho} = \left|K_7^B\,(i\,b_1)\right|$$
$$= 2 \left| \frac{\left((i\rho)^2+\frac{1}{(i\rho)^2}-2\right)\left(\left((i\rho)^5+\frac{1}{(i\rho)^5}\right)\ln\frac{i\rho+1}{i\rho-1}-2\sum_{k=1}^{\prime 3}\frac{(i\rho)^{6-2k}+\frac{1}{(i\rho)^{6-2k}}}{2k-1}\right)+\frac{8}{5^2-4}}{\left((i\rho)^2+\frac{1}{(i\rho)^2}-2\right)\left((i\rho)^5+\frac{1}{(i\rho)^5}\right)} \right|$$
$$= 2 \left| \frac{\left(\rho^2+\frac{1}{\rho^2}+2\right)\left(\left(\rho^5-\frac{1}{\rho^5}\right)(2\arctan\,\rho-\pi)+2\sum_{k=1}^{\prime 3}(-1)^{(k-1)}\frac{\rho^{6-2k}+\frac{1}{\rho^{6-2k}}}{2k-1}\right)+\frac{8}{21}}{\left(\rho^2+\frac{1}{\rho^2}+2\right)\left(\rho^5-\frac{1}{\rho^5}\right)} \right|.$$

The ellipse $\mathcal{E}_\rho$ has length $l(\mathcal{E}_\rho) = 4\varepsilon^{-1}E(\varepsilon)$, where ε is the eccentricity of $\mathcal{E}_\rho$, i.e., $\varepsilon = 2/(\rho+\rho^{-1}) = a_1^{-1}$, and

$$E(\varepsilon) = \int_0^{\pi/2} \sqrt{1-\varepsilon^2\sin^2\theta}\,d\theta$$

is the complete elliptic integral of the second kind. It can be calculated by the MATLAB function `[K,E]=ellipke(M)`.

For the test function

$$f(z) = e^{\omega z}, \qquad \omega > 0,$$

from [12, Section 5] it holds that

$$\|f\|_{\mathcal{E}_\rho} = e^{\omega a_1}.$$

Under the assumption that f is analytic inside $\mathcal{E}_{\rho_{\max}}$, our error bounds for the presented values of n become (in this case the integrand is an entire function, therefore $\rho_{\max} = +\infty$)

$$r_n^F(f) = \inf_{\rho^*<\rho<\rho_{\max}} \left(\frac{2a_1}{\pi}\,E\left(a_1^{-1}\right)\left|K_n^F\,(a_1)\right|\,e^{\omega a_1}\right), \tag{3.1}$$

$$r_{n+2}^{CC}(f) = \inf_{\rho^*<\rho<\rho_{\max}} \left(\frac{2a_1}{\pi}\,E\left(a_1^{-1}\right)\left|K_{n+2}^{CC}\,(a_1)\right|\,e^{\omega a_1}\right), \tag{3.2}$$

and

$$r_{n+2}^{B}(f) = \inf_{\rho^*<\rho<\rho_{\max}} \left(\frac{2a_1}{\pi}\,E\left(a_1^{-1}\right)\left|K_{n+2}^{B}\,(a_1)\right|\,e^{\omega a_1}\right), \tag{3.3}$$

where ρ^* is defined in Theorem 2.3. Our aim is to estimate the (minimal) value of ρ^*, since for smaller values of ρ^* the application of (3.1), (3.2), (3.3) (with the information about the location of the point of $\mathcal{E}_\rho$ at which $|K_N|$ ($N = n, n+2$) attains its maximum) is possible

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TABLE 3.1
The values of the error bounds $r_n^F(f)$, compared with the actual value of the error Error^F *and the value of the integral $I_\omega(f)$, for several values of n, ω.*

ω	n	ρ_{opt}	$r_n^F(f)$	Error$^{\mathrm{F}}$
0.5	5	24.106	8.192(-7)	1.308(-7)
0.5	10	20.693	1.253(-13)	1.069(-14)
1.0	5	12.201	5.534(-5)	8.705(-6)
1.0	10	19.608	6.350(-11)	1.117(-11)
2.0	5	6.353	4.331(-3)	6.471(-4)
2.0	10	10.185	1.015(-7)	1.240(-8)
2.0	15	19.687	3.671(-15)	1.841(-15)
4.0	5	3.512	5.408(-1)	7.096(-2)
4.0	10	5.307	1.522(-4)	1.741(-5)
4.0	15	6.411	2.182(-9)	1.453(-10)
4.0	20	9.523	3.557(-14)	5.316(-15)
8.0	5	2.100	2.336(+2)	2.402(+1)
8.0	10	3.034	6.221(-1)	5.807(-2)
8.0	15	8.604	3.048(-5)	1.965(-5)
8.0	20	11.384	1.956(-8)	1.025(-8)

in a larger region and we can get stronger estimates. Numerical research shows that the corresponding value of ρ^* is fairly close to 1, i.e., $\rho^* < 1.3$ in all reported cases.

Table 3.1 presents the error bounds (3.1) for the integral $I_\omega(f) = \int_{-1}^{1} e^{\omega t}\, dt$ by the Fejér quadrature rule for several values of n, ω.

Table 3.2 presents the error bounds (3.2) for the integral $\int_{-1}^{1} e^{\omega t}\, dt$ by the Clenshaw–Curtis quadrature rule for several values of n, ω. In [12, Tables 4-6], for the same integral, the error estimates (5.3), (5.5), (5.7), (5.8), (5.9), (5.12) there are reported for the same values n, ω. We observe that the bound (5.3) there is the strongest, while our error bound (3.2) lies within the range of the error bounds (5.5), (5.7), (5.8), (5.9), (5.12) or is even slightly stronger than those bounds.

Although superior, the error bound (5.5) in [12] may have a weakness. Namely, as it is stated in [12], just after formula (2.8), the norm of f, required in order to apply (5.5), cannot always be found exactly, and because of that it has to be estimated, and hence the bound (5.5) may becomes less precise.

Table 3.3 presents the error bounds (3.3) for the integral $\int_{-1}^{1} e^{\omega t}\, dt$ with the Basu quadrature rule for several values of n, ω.

The numerical results are computed in the programming language MATLAB by using high-precision arithmetic with about 30 significant digits. One of the MATLAB commands that we used is `fminbnd`, which returns the minimum of a function on an interval and the value ρ_{opt} where that minimum is attained.

4. Concluding remarks. By using Hilbert space methods, Notaris [12] derived certain error bounds for the Clenshaw–Curtis and related quadrature rules with the integrand f being an analytic function in a domain D containing the interval of integration $[-1, 1]$. These are very effective bounds; some of them are the same as the actual error. In this paper we propose another kind of error bounds for the Clenshaw–Curtis and related quadrature rules with the integrand being an analytic function on ellipses containing the interval $[-1, 1]$. We

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TABLE 3.2
The values of the error bounds $r_{n+2}^{CC}(f)$, compared with the actual value of the error Error^{CC} and the value of the integral $I_\omega(f)$, for several values of n, ω.

ω	n	ρ_{opt}	$r_{n+2}^{CC}(f)$	$\mathrm{Error}^{\mathrm{CC}}$
0.5	5	31.697	5.440(-10)	7.777(-11)
0.5	10	12.768	2.814(-18)	1.556(-18)
1.0	5	16.156	1.496(-7)	2.059(-8)
1.0	10	11.419	8.967(-15)	6.490(-15)
2.0	5	8.304	4.522(-5)	6.009(-6)
2.0	10	11.616	1.144(-10)	2.856(-11)
2.0	15	5.948	4.525(-18)	1.346(-18)
4.0	5	4.454	2.035(-2)	2.469(-3)
4.0	10	6.294	1.452(-6)	1.551(-7)
4.0	15	5.490	3.860(-12)	4.199(-13)
4.0	20	4.500	8.426(-18)	7.911(-18)
8.0	5	2.529	2.677(+1)	2.746(0)
8.0	10	3.494	2.043(-2)	1.850(-3)
8.0	15	4.661	2.437(-6)	2.174(-7)
8.0	20	3.656	3.326(-10)	5.902(-11)

TABLE 3.3
The values of the error bounds $r_{n+2}^{B}(f)$, compared with the actual value of the error Error^{B} and the value of the integral $I_\omega(f)$, for several values of n, ω.

ω	n	ρ_{opt}	$r_{n+2}^{B}(f)$	$\mathrm{Error}^{\mathrm{B}}$
0.5	6	28.963	2.895(-10)	3.901(-11)
0.5	10	13.090	4.588(-18)	1.395(-18)
1.0	6	16.184	7.613(-8)	1.043(-8)
1.0	10	12.502	3.145(-14)	5.575(-15)
2.0	6	8.398	2.398(-5)	3.150(-6)
2.0	10	11.146	1.378(-10)	2.469(-11)
2.0	15	8.168	5.913(-17)	6.905(-19)
4.0	6	4.520	1.171(-2)	1.411(-3)
4.0	10	6.297	1.300(-6)	1.376(-7)
4.0	15	5.723	3.593(-12)	2.174(-13)
4.0	20	4.061	3.643(-16)	2.814(-18)
8.0	6	2.441	1.556(+1)	1.699(0)
8.0	10	3.565	2.053(-2)	1.815(-3)
8.0	15	7.931	1.254(-6)	1.166(-7)
8.0	20	8.931	4.063(-10)	2.147(-11)

use well-known contour integration methods. We tested our method and observed that our estimates are in the range of the ones derived by Notaris [12].

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