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REGULARIZATION OF PORT-HAMILTONIAN DESCRIPTOR SYSTEMS*

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Abstract. We study the regularization problem for port-Hamiltonian descriptor systems by proportional and/or derivative output feedback. Necessary and sufficient conditions are given which guarantee that there exist output feedbacks such that the closed-loop system is regular, has index at most one, and is still port-Hamiltonian with desired rank properties. All results are derived based on condensed forms that may be computed numerically reliably using orthogonal transformations.

Key words. Port-Hamiltonian descriptor system, output feedback, structure-preserving regularization, index reduction

AMS subject classifications. 93B05, 93B40, 93B52, 65F35

1. Introduction.

Port-Hamiltonian systems (see [22, 24, 26]) provide an ideal model class for the automated modeling, control, and simulation of real-world physical systems. Physical properties are encoded in the structure of the mathematical model; see [22, 26]. These physical properties of the systems include balance and conservation laws, the shaping of energy storage and energy dissipation, as well as the interpretation of controller systems as virtual system components. When algebraic constraints are present in the physical system, it is more appropriate to use the class of port-Hamiltonian descriptor systems; see [19] for the most general definition in the linear time-invariant case, [3] for linear time-invariant systems, [17, 25] for nonlinear systems, and [18] for a survey. Due to automated modeling procedures that include redundant equations, the resulting port-Hamiltonian descriptor systems may not be regular, i.e., for a given input, the solutions to initial-value problems may not exist or are not unique. Furthermore, the system may be of index higher than one, i.e., the solution depends on derivatives of the input function. Then, impulses may arise in the response of the system if the control is not sufficiently smooth.

How to deal with non-regular or high-index problems is well understood for general descriptor systems. If possible, feedback is used to make the system regular and of index at most one. This topic has been studied extensively in many different versions. For general linear time-invariant systems, the existence of regularizing state or output feedbacks is discussed in [21, 23]. The construction of these feedbacks with orthogonal transformations is discussed in [5, 6, 7, 9, 10, 20], and a survey is given in [4]. For linear time-varying descriptor systems, a summary is presented in [15] and for general nonlinear descriptor systems in [8, 15]. Various applications (see, e.g., [1, 2, 24]) strongly motivate us to study the regularization problem for port-Hamiltonian descriptor systems since existing regularization procedures in general do not preserve the port-Hamiltonian structure. Hence, it is important from a theoretical and practical point of view to develop a theory and numerically reliable methods for the regularization and index reduction preserving the port-Hamiltonian structure. For a special class of linear time-invariant port-Hamiltonian descriptor systems and when only proportional feedback is used, this research topic has recently been studied in [12]. In this paper, we discuss this topic

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for general linear time-invariant port-Hamiltonian descriptor systems, and we also include derivative output feedback.

The paper is organized as follows. The regularization problem via output feedback for port-Hamiltonian descriptor systems is introduced in Section 2 as well as some preliminaries. The main results are presented in Section 3, and concluding remarks are given in Section 4.

2. Preliminaries and problem statement. We study port-Hamiltonian descriptor systems of the form

$$E\dot{x} = Ax + Bu, \tag{2.1}$$
$$y = Cx, \tag{2.2}$$

on a time interval $\mathbb{I} \subset \mathbb{R}$, where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, and $C \in \mathbb{R}^{m,n}$, $x : \mathbb{I} \to \mathbb{R}^n$ is the state, $y : \mathbb{I} \to \mathbb{R}^m$ is the output, $u : \mathbb{I} \to \mathbb{R}^m$ is the input or control of the system. For a system of the form (2.1)–(2.2) to be port-Hamiltonian, the coefficient matrices have to satisfy

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} (J-R)Q & G-P \\ (G+P)^T Q & 0 \end{bmatrix}, \tag{2.3}$$

with $J, R, Q \in \mathbb{R}^{n,n}, G, P \in \mathbb{R}^{n,m}$, $J = -J^T$, and the further relations

$$Q^T E = E^T Q \geq 0, \qquad Q^T RQ = Q^T R^T Q \geq 0, \qquad Q^T P = 0; \tag{2.4}$$

see [18].

The quadratic function $\mathcal{H}(x) = \frac{1}{2} x^T E^T Q x$ is called the *Hamiltonian*, which can be interpreted as the energy stored in the system. It satisfies the *power balance equation* (see [17, 18])

$$\frac{d}{dt}\mathcal{H}(x) = -\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q^T RQ & Q^T P \\ P^T Q & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + y^T u$$

along any solution x and for any input u. Here, the first term on the right-hand side is the dissipated energy, and the second term is the supplied energy.

REMARK 2.1. It is well-known that port-Hamiltonian descriptor systems satisfy the dissipation inequality

$$\frac{d}{dt}\mathcal{H}(x) \leq y^T u,$$

which is referred to as passivity in control. For this to hold in the case of systems without feedthrough term, it is necessary that

$$\begin{bmatrix} Q^T RQ & Q^T P \\ P^T Q & 0 \end{bmatrix} \geq 0,$$

which implies that $Q^T P = 0$ and $Q^T RQ \geq 0$. This implies in particular that the uncontrolled system (with $u = 0$) is semidissipative.

Note that E and Q are allowed to be singular. Further note that the present methods are also directly applicable for problems with a feedthrough term, which can always be formulated as (2.1); see [18]. However, we consider only problems without a feedthrough term.

The response of the descriptor system (2.1) can be described in terms of the eigenstructure of the *matrix pencil* $\alpha E - \beta A$, which we abbreviate by (E, A). The system is called *regular* if the pencil (E, A) is regular, i.e.,

$$\det(\alpha E - \beta A) \neq 0 \quad \text{for some } (\alpha, \beta) \in \mathbb{C}^2.$$

The *generalized eigenvalues* of a regular matrix pencil (E, A) are defined as the pairs $(\alpha_j, \beta_j) \in \mathbb{C}^2 \backslash \{0, 0\}$ such that

$$\det(\alpha_j E - \beta_j A) = 0, \quad j = 1, 2, \dots, n.$$

If $\beta_j \neq 0$, then the eigenvalue pairs are said to be *finite* with values given by $\lambda_j = \alpha_j / \beta_j$, and otherwise, if $\beta_j = 0$, then the pair is said to be an *infinite* eigenvalue. The maximum number of finite eigenvalues that a pencil (E, A) can have is less than or equal to the rank of E.

In this paper, we denote a full column rank matrix with its columns spanning the right nullspace of the matrix M by $\mathcal{S}_\infty(M)$ and with its columns spanning the left nullspace of M by $\mathcal{T}_\infty(M)$, respectively.

If the system (2.1) is regular, then for given consistent initial values, i.e., initial values that satisfy the algebraic equations present in the system, the existence and uniqueness of classical solutions to the dynamical equations are guaranteed; see [15]. In the regular case, the solutions can be characterized in terms of the Kronecker Canonical Form (KCF), which states that there exist nonsingular matrices X and Y (representing left and right generalized eigenvectors and generalized eigenvectors of the system pencil, respectively) such that

$$X(sE - A)Y = \begin{bmatrix} sI - J & 0 \\ 0 & sN - I \end{bmatrix}, \tag{2.5}$$

with $s \in \mathbb{C}$, where the eigenvalues of J coincide with the finite eigenvalues of the pencil and N is a nilpotent matrix such that for $i > 0$ it holds that $N^i = 0$, $N^{i-1} \neq 0$. The matrix N corresponds to the infinite eigenvalues. The *index* of a descriptor system, denoted by $\operatorname{ind}(E, A)$, is defined as the degree i of nilpotency of the matrix N, i.e., the index of the system is the dimension of the largest block associated with an infinite eigenvalue in the KCF (2.5). Note that the system (2.1) is regular and has index at most one if and only if it has exactly $\operatorname{rank}(E)$ finite eigenvalues.

Due to the port-Hamiltonian structure, the index of a regular port-Hamiltonian descriptor system of the form (2.1) is at most 2 [16, 18, 19], but its index may indeed be 2.

As a special case for port-Hamiltonian descriptor systems with $Q = I$ the identity, the regularization problem by proportional output feedback has been studied recently in [12]. In this paper, we extend these results to the more general form of port-Hamiltonian descriptor systems given in (2.1).

We consider proportional and derivative output feedbacks of the form

$$u = Fy - K\dot{y} + v,$$

where $F, K \in \mathbb{R}^{m,m}$ are feedback matrices so that the resulting system has the form

$$\begin{aligned} (E + BKC)\dot{x} &= (A + BFC)x + Bv, \\ y &= Cx, \end{aligned}$$

with desired properties. Here, proportional feedback control is achieved with $K = 0$, and derivative feedback control corresponds to the case $F = 0$. Proportional feedback changes the

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system matrix A, while derivative feedback alters the matrix E. Therefore, different properties of the system can be achieved using different feedback combinations.

We study the following problems:

a) *Regularization problem for the descriptor system* (2.1) *by proportional output feedback:* Determine a matrix F such that the pencil $(E, A + BFC)$ is regular, $\text{ind}(E, A + BFC) \leq 1$, and the resulting system

$$
\begin{aligned}
E\dot{x}(t) &= (A + BFC)x(t) + Bv(t), \\
y(t) &= Cx(t),
\end{aligned} \tag{2.6}
$$

is still port-Hamiltonian, i.e., satisfies (2.3) with the properties (2.4).

b) *Regularization problem for the descriptor system* (2.1) *by derivative output feedback:* Determine a matrix K such that the pencil $(E + BKC, A)$ is regular, $\text{ind}(E + BKC, A) \leq 1$, and the resulting system

$$
\begin{aligned}
(E + BKC)\dot{x}(t) &= Ax(t) + Bv(t), \\
y(t) &= Cx(t),
\end{aligned} \tag{2.7}
$$

is still port-Hamiltonian, i.e., satisfies (2.3) with the properties (2.4).

c) *Regularization problem for the descriptor system* (2.1) *by derivative and proportional output feedback:* Determine matrices F and K such that the pencil $(E + BKC, A + BFC)$ is regular, $\text{ind}(E + BKC, A + BFC) \leq 1$, and the resulting system

$$
\begin{aligned}
(E + BKC)\dot{x}(t) &= (A + BFC)x(t) + Bv(t), \\
y(t) &= Cx(t),
\end{aligned} \tag{2.8}
$$

is still port-Hamiltonian, i.e., satisfies (2.3) with the properties (2.4).

3. Main results.

One of the main difficulties for solving the regularization problem of port-Hamiltonian descriptor systems is to preserve the port-Hamiltonian structure by feedbacks.

The following results present necessary and sufficient solvability conditions for the regularization of the port-Hamiltonian descriptor system (2.1) by proportional output feedback and derivative output feedback, respectively. The following result presents the answer to Problem a).

THEOREM 3.1. *Let the system* (2.1) *be port-Hamiltonian. Then there exists a matrix F such that the pencil $(E, A + BFC)$ is regular, $\text{ind}(E, A + BFC) \leq 1$, and the closed-loop system* (2.6) *is port-Hamiltonian if and only if*

$$
\text{rank} \begin{bmatrix} E & A\mathcal{S}_\infty(E) \\ 0 & C\mathcal{S}_\infty(E) \end{bmatrix} = n. \tag{3.1}
$$

Proof. Condition (3.1) follows from the necessary conditions for the regularization problem of a general descriptor system (2.1) by output feedback without the port-Hamiltonian requirement; see [9, Theorem 3.3]. Hence, the necessity follows.

To show the sufficiency, we give a constructive proof, which can be implemented as numerical method, to show that under the condition (3.1) there exists a matrix $F \in \mathbb{R}^{m,m}$ such that $(E, A + BFC)$ is regular and of index at most one and the closed-loop system (2.6) is port-Hamiltonian.

We first compute orthogonal matrices U and V such that

$$UEV = \begin{matrix} & n_1 & n_2 & n_3 \\ n_1 & E_{11} & 0 & 0 \\ n_2 & 0 & 0 & 0 \\ n_3 & 0 & 0 & 0 \end{matrix}, \qquad UAV = \begin{matrix} & n_1 & n_2 & n_3 \\ n_1 & A_{11} & A_{12} & A_{13} \\ n_2 & A_{21} & A_{22} & 0 \\ n_3 & A_{31} & 0 & 0 \end{matrix},$$

with $n_1 + n_2 + n_3 = n$, where E_{11} and A_{22} are nonsingular. It is well-known (see, e.g., [12]) that for a port-Hamiltonian descriptor system of the form (2.1), the matrix Q in (2.4) satisfies

$$E^T Q = Q^T E \geq 0, \quad -A^T Q - Q^T A \geq 0, \quad C - B^T Q = 0. \tag{3.2}$$

If Q is partitioned analogously, then the first inequality in (3.2) implies that $Q_{12} = 0$ and $Q_{13} = 0$ since E_{11} is nonsingular. Then, from the second inequality it follows that $A_{22}Q_{23} = 0$ and hence $Q_{23} = 0$ since A_{22} is nonsingular, so that we have

$$UQV = \begin{matrix} & n_1 & n_2 & n_3 \\ n_1 & Q_{11} & 0 & 0 \\ n_2 & Q_{21} & Q_{22} & 0 \\ n_3 & Q_{31} & Q_{32} & Q_{33} \end{matrix}.$$

Partition the matrices

$$UB = \begin{matrix} n_1 \\ n_2 \\ n_3 \end{matrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \quad CV = \begin{matrix} n_1 & n_2 & n_3 \\ [C_1 & C_2 & C_3] \end{matrix}$$

accordingly. Using condition (3.1) it follows that C_3 is of full column rank, so we can compute an orthogonal matrix $W \in \mathbb{R}^{m,m}$ such that

$$W^T \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} = \begin{matrix} & n_1 & n_2 & n_3 \\ m-n_3 & C_{11} & C_{12} & 0 \\ n_3 & C_{21} & C_{22} & C_{23} \end{matrix},$$

where C_{23} is nonsingular because $\operatorname{rank}(C_{23}) = \operatorname{rank}(C_3)$. This follows from condition (3.1) since multiplication with $\mathcal{S}_\infty(E)$ projects onto the last two block columns.

Partition

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} W = \begin{matrix} & m-n_3 & n_3 \\ n_1 & B_{11} & B_{12} \\ n_2 & B_{21} & B_{22} \\ n_3 & B_{31} & B_{32} \end{matrix}.$$

Then the condition $C = B^T Q$ implies that

$$B_{32}^T Q_{33} = C_{23}, \qquad B_{31}^T Q_{33} = 0,$$

which, together with the nonsingularity of C_{23}, gives that B_{32} and Q_{33} are nonsingular and $B_{31} = 0$.

Then, for

$$F = -W \begin{bmatrix} 0 & 0 \\ 0 & F_{22} \end{bmatrix} W^T$$

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with

$$F_{22} \in \mathbb{R}^{n_3 \times n_3}, \qquad F_{22} = F_{22}^T > 0,$$

a direct calculation yields that $(E, A+BFC)$ is regular and of index at most one. Furthermore,

$$A + BFC = (J - R)Q + BFB^TQ = [J - (R - BFB^T)]Q,$$

and

$$Q^T(R - BFB^T)Q = Q^T(R - BFB^T)^TQ = Q^TRQ - Q^TBFB^TQ \geq 0.$$

Hence, the closed-loop system (2.6) is port-Hamiltonian. ∎

For derivative feedback we have an analogous result which presents the answer to Problem b).

THEOREM 3.2. *Let the system* (2.1) *be port-Hamiltonian. Then there exists a matrix* K *such that the pencil* $(E + BKC, A)$ *is regular,* $\operatorname{ind}(E + BKC, A) \leq 1$*,*

$$\operatorname{rank}(E + BKC) = \max_{\hat{K} \in \mathbb{R}^{m,m}} \operatorname{rank}(E + B\hat{K}C),$$

and the closed-loop system (2.7) *is port-Hamiltonian if and only if*

$$\operatorname{rank} \begin{bmatrix} E & A\mathcal{S}_\infty\left(\begin{bmatrix} E \\ C \end{bmatrix}\right) \\ C & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} E & A\mathcal{S}_\infty\left(\begin{bmatrix} E \\ C \end{bmatrix}\right) & B \end{bmatrix} = n. \tag{3.3}$$

Proof. Condition (3.3) follows from the necessary conditions for the regularization problem of a general descriptor systems (2.1) by derivative output feedback without a port-Hamiltonian requirement; see [9, Theorem 3.1]. Hence, the necessity follows.

For the port-Hamiltonian descriptor system (2.1), it holds that

$$\operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix} \leq \operatorname{rank} \begin{bmatrix} E & B \end{bmatrix}.$$

This follows since $C = B^TQ$ and $E^TQ = Q^TE \geq 0$. Thus,

$$\max_{\hat{K} \in \mathbb{R}^{m,m}} \operatorname{rank}(E + B\hat{K}C) = \operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix}.$$

To show the sufficiency, we present a construction of such a regularizing feedback that can be implemented as a numerical method. For this, we first separate the part of the system that cannot be influenced by derivative output feedback and that is already of index at most one by determining orthogonal matrices U and V such that

$$UEV = \begin{array}{c} \\ \hat{n}_1 \\ \hat{n}_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix} \end{array}, \quad UAV = \begin{array}{c} \\ \hat{n}_1 \\ \hat{n}_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \end{array}, \quad CV = \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{bmatrix} C_1 & 0 \end{bmatrix} \end{array},$$

with $n_1 + n_2 = \hat{n}_1 + \hat{n}_2 = n$, where

$$\operatorname{rank} \begin{bmatrix} E_{11} \\ C_1 \end{bmatrix} = n_1,$$

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and with the index-at-most-one system

$$\operatorname{rank}(sE_{22} - A_{22}) = \hat{n}_2, \qquad \text{for all } s \in \mathbb{C}.$$

This can be achieved by first forming a full column-rank decomposition of C and then using the staircase algorithm in the kernel to separate the infinite Jordan blocks of size one; see [13]. Partition the matrix

$$UB = \begin{matrix} \hat{n}_1 \\ \hat{n}_2 \end{matrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

accordingly. Then condition (3.3) is equivalent to

$$n_2 = \hat{n}_2, \qquad E_{22} = 0, \qquad \operatorname{rank}(A_{22}) = n_2, \qquad \operatorname{rank} \begin{bmatrix} E_{11} & B_1 \end{bmatrix} = n_1.$$

Thus, condition (3.3) implies that

$$\operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n_1.$$

Because $\operatorname{rank} \begin{bmatrix} E_{11} \\ C_1 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} E_{11} & B_1 \end{bmatrix} = n_1$, it is easy to determine a matrix K (for example, by taking a scaled identity of size n_1) such that

$$K = K^T \geq 0, \qquad \operatorname{rank}(E_{11} + B_1 K C_1) = n_1.$$

Obviously, the matrix pencil $(E + BKC, A)$ is regular, $\operatorname{ind}(E + BKC, A) \leq 1$, and it holds that $\operatorname{rank}(E + BKC) = \operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix}$. Moreover, with Q as in (3.2), we have

$$\begin{aligned}(E + BKC)^T Q = E^T Q + C^T K^T B^T Q &= E^T Q + Q^T BKB^T Q \\ &= Q^T (E + BKC) \geq 0.\end{aligned}$$

Hence, the closed-loop system (2.7) is port-Hamiltonian. □

The necessary and sufficient conditions in Theorems 3.1 and 3.2 are easily verified, and the desired feedback matrices F and K can be smoothly computed by applying orthogonal transformations that can be implemented numerically stable.

Note that the necessary and sufficient conditions for the regularization of the descriptor system (2.1) by proportional output feedback, presented in [9] without the port-Hamiltonian requirement, are the condition (3.1) and

$$\operatorname{rank} \begin{bmatrix} E & A\mathcal{S}_\infty(E) & B \end{bmatrix} = n. \tag{3.4}$$

Because of the port-Hamiltonian structure, condition (3.1) implies condition (3.4), such that condition (3.4) is not needed for the port-Hamiltonian descriptor system (2.1).

In the following construction, we assume without loss of generality that C in (2.1) is of full row rank, i.e., $\operatorname{rank}(C) = m$.

In order to derive solvability conditions for the regularization of the port-Hamiltonian descriptor system (2.1) by derivative and proportional output feedback, we need the following two lemmas:

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LEMMA 3.3. *Let the system* (2.1) *be port-Hamiltonian and* rank $\begin{bmatrix} E \\ C \end{bmatrix} = n$. *Then there exist orthogonal matrices* U, V, *and* W *such that*

$$UEV = \begin{array}{c} {\scriptstyle n-r_b} \\ {\scriptstyle r_e+r_b-n} \\ {\scriptstyle n-r_e} \end{array} \overset{\begin{array}{ccc} {\scriptstyle n-r_b} & {\scriptstyle r_e+r_b-n} & {\scriptstyle n-r_e} \end{array}}{\begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}},$$

$$UBW = \begin{array}{c} {\scriptstyle n-r_b} \\ {\scriptstyle r_e+r_b-n} \\ {\scriptstyle n-r_e} \end{array} \overset{\begin{array}{cc} {\scriptstyle r_e+r_b-n} & {\scriptstyle n-r_e} \end{array}}{\begin{bmatrix} 0 & B_{12} \\ B_{21} & B_{22} \\ 0 & B_{32} \end{bmatrix}}, \tag{3.5}$$

$$W^TCV = \begin{array}{c} {\scriptstyle r_e+r_b-n} \\ {\scriptstyle n-r_e} \end{array} \overset{\begin{array}{ccc} {\scriptstyle n-r_b} & {\scriptstyle r_e+r_b-n} & {\scriptstyle n-r_e} \end{array}}{\begin{bmatrix} 0 & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \end{bmatrix}},$$

where B_{21}, C_{12}, B_{32}, C_{23}, *and* E_{11} *are nonsingular.*

The proof is given in the Appendix A.1.

Now consider orthogonal matrices $\mathcal{U}$ and $\mathcal{V}$ with the partitioning

$$\mathcal{U} = \begin{array}{c} {\scriptstyle n-r_b} \\ {\scriptstyle r_e+r_b-n} \end{array} \overset{\begin{array}{cc} {\scriptstyle n-r_b} & {\scriptstyle r_e+r_b-n} \end{array}}{\begin{bmatrix} \mathcal{U}_{11} & \mathcal{U}_{12} \\ \mathcal{U}_{21} & \mathcal{U}_{22} \end{bmatrix}}, \quad \mathcal{V} = \begin{array}{c} {\scriptstyle n-r_b} \\ {\scriptstyle r_e+r_b-n} \end{array} \overset{\begin{array}{cc} {\scriptstyle n-r_b} & {\scriptstyle r_e+r_b-n} \end{array}}{\begin{bmatrix} \mathcal{V}_{11} & \mathcal{V}_{12} \\ \mathcal{V}_{21} & \mathcal{V}_{22} \end{bmatrix}},$$

which satisfy

$$\begin{aligned} \mathcal{U}\begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} &= \begin{bmatrix} \mathcal{E}_{11} \\ 0 \end{bmatrix}, \qquad \operatorname{rank}(\mathcal{E}_{11}) = n - r_b, \qquad \text{and} \\ \begin{bmatrix} E_{11} & E_{12} \end{bmatrix}\mathcal{V} &= \begin{bmatrix} \tilde{\mathcal{E}}_{11} & 0 \end{bmatrix}, \qquad \operatorname{rank}(\tilde{\mathcal{E}}_{11}) = n - r_b. \end{aligned}$$

Set

$$\begin{aligned} &\hat{E}_{22} = (\mathcal{U}_{21}E_{12} + \mathcal{U}_{22}E_{22})\mathcal{V}_{22}, \quad \hat{B}_{21} = \mathcal{U}_{22}B_{21}, \qquad \hat{B}_{22} = \mathcal{U}_{21}B_{12} + \mathcal{U}_{22}B_{22}, \\ &\hat{C}_{12} = C_{12}\mathcal{V}_{22}, \qquad \hat{C}_{22} = C_{21}\mathcal{V}_{12} + C_{22}\mathcal{V}_{22}, \end{aligned}$$

and

$$\begin{bmatrix} I & & \\ \mathcal{U}_{21} & \mathcal{U}_{22} & \\ & & I \end{bmatrix} UAV \begin{bmatrix} I & \mathcal{V}_{12} & \\ & \mathcal{V}_{22} & \\ & & I \end{bmatrix} = \begin{array}{c} {\scriptstyle n-r_b} \\ {\scriptstyle r_e+r_b-n} \\ {\scriptstyle n-r_e} \end{array} \overset{\begin{array}{ccc} {\scriptstyle n-r_b} & {\scriptstyle r_e+r_b-n} & {\scriptstyle n-r_e} \end{array}}{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}.$$

Then determine orthogonal matrices Z and $\mathcal{Z}$ such that

$$Z\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}\mathcal{Z} = \begin{array}{c} {\scriptstyle \mu} \\ {\scriptstyle r_b-\mu} \end{array} \overset{\begin{array}{cc} {\scriptstyle \mu} & {\scriptstyle r_b-\mu} \end{array}}{\begin{bmatrix} \mathcal{A}_{22} & 0 \\ 0 & 0 \end{bmatrix}},$$

where $\mathcal{A}_{22}$ is nonsingular.

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LEMMA 3.4. *Consider a port-Hamiltonian system of the form* (2.1)*, and suppose that* $\operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n$*. Let* U*,* V*, and* W *be orthogonal matrices that transform the system to the condensed form* (3.5)*, and let* $\mathcal{W}$ *be orthogonal such that*

$$Z \begin{bmatrix} \hat{B}_{21} & \hat{B}_{22} \\ 0 & B_{32} \end{bmatrix} \mathcal{W} = \begin{array}{c} \\ \mu \\ r_b-\mu \end{array} \begin{array}{c} \begin{array}{cc} \mu & r_b-\mu \end{array} \\ \begin{bmatrix} \mathcal{B}_{21} & \mathcal{B}_{22} \\ 0 & \mathcal{B}_{32} \end{bmatrix} \end{array},$$

where $\mathcal{B}_{21}$*,* $\mathcal{B}_{32}$ *are nonsingular. Then,*

$$\mathcal{W}^T \begin{bmatrix} \hat{C}_{12} & 0 \\ \hat{C}_{22} & C_{23} \end{bmatrix} \mathcal{Z} = \begin{array}{c} \\ \mu \\ r_b-\mu \end{array} \begin{array}{c} \begin{array}{cc} \mu & r_b-\mu \end{array} \\ \begin{bmatrix} \mathcal{C}_{12} & 0 \\ \mathcal{C}_{22} & \mathcal{C}_{23} \end{bmatrix} \end{array},$$

where $\mathcal{C}_{12}$ *and* $\mathcal{C}_{23}$ *are nonsingular.*

The proof is given in the Appendix A.2.

We are now ready to present the necessary and sufficient solvability conditions for the regularization of port-Hamiltonian descriptor systems by derivative output feedback and derivative and proportional output feedback (with a given rank of $E + BKC$). The following result presents the answer to Problem c).

THEOREM 3.5. *Suppose that the port-Hamiltonian system* (2.1) *is* completely observable*, i.e.,* $\operatorname{rank}\begin{bmatrix} \alpha E - \beta A \\ C \end{bmatrix} = n$ *for any* $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{0, 0\}$*. Then the following assertions hold:*

(i) There exists a matrix K *such that* $(E + BKC, A)$ *is regular,*

$$\operatorname{ind}(E + BKC, A) \leq 1, \qquad \operatorname{rank}(E + BKC) = r,$$

and the closed-loop system (2.7) *is port-Hamiltonian if and only if*

$$n - \operatorname{rank}[\mathcal{T}_\infty^T(E\mathcal{S}_\infty(C))A\mathcal{S}_\infty(\mathcal{T}_\infty^T(B)E)] \leq r \leq n \tag{3.6}$$

and $n - r$ *is even when* $\operatorname{rank}[\mathcal{T}_\infty^T(E\mathcal{S}_\infty(C))A\mathcal{S}_\infty(\mathcal{T}_\infty^T(B)E)] > 0$ *is even and*

$$\left(\mathcal{T}_\infty^T(E\mathcal{S}_\infty(C))B\right)^{-1} [\mathcal{T}_\infty^T(E\mathcal{S}_\infty(C))A\mathcal{S}_\infty(\mathcal{T}_\infty^T(B)E)] \left(C\mathcal{S}_\infty(\mathcal{T}_\infty^T(B)E)\right)^{-1}$$

is skew-symmetric.

(ii) There exist matrices F *and* K *such that* $(E + BKC, A + BFC)$ *is regular,*

$$\operatorname{ind}(E + BKC, A + BFC) \leq 1, \qquad \operatorname{rank}(E + BKC) = r,$$

and the system (2.8) *is port-Hamiltonian if and only if*

$$n - r_b \leq r \leq n.$$

Proof. Since (2.1) is completely observable, we have $\operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n$. Using (A.1) we have that

$$\operatorname{rank}[\mathcal{T}_\infty^T(E\mathcal{S}_\infty(C))A\mathcal{S}_\infty(\mathcal{T}_\infty^T(B)E)] = \operatorname{rank}\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} = \mu$$

and

$$\begin{aligned}&\left(\mathcal{T}_\infty^T(E\mathcal{S}_\infty(C))B\right)^{-1}[\mathcal{T}_\infty^T(E\mathcal{S}_\infty(C))A\mathcal{S}_\infty(\mathcal{T}_\infty^T(B)E)]\left(C\mathcal{S}_\infty(\mathcal{T}_\infty^T(B)E)\right)^{-1}\\ &\quad=\begin{bmatrix}\hat{B}_{21} & \hat{B}_{22}\\ 0 & B_{32}\end{bmatrix}^{-1}\begin{bmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{bmatrix}\begin{bmatrix}\hat{C}_{12} & 0\\ \hat{C}_{22} & C_{23}\end{bmatrix}^{-1}\\ &\quad=\mathcal{W}\begin{bmatrix}\mathcal{B}_{21}^{-1}\mathcal{A}_{22}\mathcal{C}_{12}^{-1} & 0\\ 0 & 0\end{bmatrix}\mathcal{W}^T.\end{aligned}$$

Since $-A^TQ - Q^TA \geq 0$, using (A.2) we have that

$$\mathcal{B}_{12}^{-1}\mathcal{A}_{22}\mathcal{C}_{12}^{-1} + (\mathcal{B}_{12}^{-1}\mathcal{A}_{22}\mathcal{C}_{12}^{-1})^T \leq 0.$$

Hence, there exists an orthogonal matrix $\hat{\mathcal{P}}$ such that

$$\hat{\mathcal{P}}(\mathcal{B}_{12}^{-1}\mathcal{A}_{22}\mathcal{C}_{12}^{-1})\hat{\mathcal{P}}^T = \hat{\mathcal{A}}_{22} \tag{3.7}$$

is in real Schur form (see [14]) and $\hat{\mathcal{A}}_{22} + \hat{\mathcal{A}}_{22}^T \leq 0$. Set

$$\begin{aligned}\hat{X} &= \begin{bmatrix}I & & \\ & \hat{\mathcal{P}} & \\ & & I\end{bmatrix}\begin{bmatrix}I & & \\ & \mathcal{B}_{21} & \mathcal{B}_{22}\\ & 0 & \mathcal{B}_{32}\end{bmatrix}^{-1}\begin{bmatrix}I & \\ & Z\end{bmatrix}X,\\ \hat{Y} &= Y\begin{bmatrix}I & \\ & \mathcal{Z}\end{bmatrix}\begin{bmatrix}I & & \\ & \mathcal{C}_{12} & 0\\ & \mathcal{C}_{22} & \mathcal{C}_{23}\end{bmatrix}^{-1}\begin{bmatrix}I & & \\ & \hat{\mathcal{P}}^T & \\ & & I\end{bmatrix}.\end{aligned}$$

Then

$$\hat{X}Q\hat{Y} = \begin{bmatrix}\mathcal{Q}_{11} & & \\ & I & \\ & & I\end{bmatrix},$$

and, for $K, F \in \mathbb{R}^{m\times m}$, form the matrices

$$\begin{aligned}W^TKW &= \begin{bmatrix}\mathcal{K}_{11} & 0\\ 0 & 0\end{bmatrix} + (\mathcal{W}\hat{\mathcal{P}})\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix}(\mathcal{W}\hat{\mathcal{P}})^T,\\ W^TFW &= (\mathcal{W}\hat{\mathcal{P}})\begin{bmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{bmatrix}(\mathcal{W}\hat{\mathcal{P}})^T,\end{aligned}$$

where $\mathcal{K}_{11}, K_{11}, F_{11} \in \mathbb{R}^{(n-r_b)\times(n-r_b)}$, $B_{21}\mathcal{K}_{11}C_{12} = E_{22}$, Then we have

$$\hat{X}(E+BKC)\hat{Y} = \begin{bmatrix}E_{11} & 0 & 0\\ 0 & K_{11} & K_{12}\\ 0 & K_{21} & K_{22}\end{bmatrix}, \qquad \hat{X}A\hat{Y} = \begin{bmatrix}A_{11} & \hat{\mathcal{A}}_{12} & \hat{\mathcal{A}}_{13}\\ \hat{\mathcal{A}}_{21} & \hat{\mathcal{A}}_{22} & 0\\ \hat{\mathcal{A}}_{31} & 0 & 0\end{bmatrix},$$

$$\hat{X}(A+BFC)\hat{Y} = \begin{bmatrix}A_{11} & \hat{\mathcal{A}}_{12} & \hat{\mathcal{A}}_{13}\\ \hat{\mathcal{A}}_{21} & \hat{\mathcal{A}}_{22}+F_{11} & F_{12}\\ \hat{\mathcal{A}}_{31} & F_{21} & F_{22}\end{bmatrix}.$$

(i) Then $(E + BKC, A)$ is regular, $\operatorname{ind}(E + BKC, A) \leq 1$, and $\operatorname{rank}(E + BKC) = r$ if and only if

$$\left(\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \begin{bmatrix} \hat{\mathcal{A}}_{22} & 0 \\ 0 & 0 \end{bmatrix} \right)$$

is regular, of index at most one, and

$$\operatorname{rank} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = r - (n - r_b) = r + r_b - n.$$

To show the other parts, observe first that if $(E+BKC, A)$ is regular, $\operatorname{ind}(E+BKC, A) \leq 1$, and the closed-loop system (2.7) is port-Hamiltonian, then it follows that

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \geq 0,$$

hence the following assertions hold:

$$\begin{aligned} \operatorname{rank}(K_{22}) &= \operatorname{rank} \begin{bmatrix} K_{21} & K_{22} \end{bmatrix} = r_b - \mu, \\ r &= n - r_b + \operatorname{rank} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \geq n - r_b + \operatorname{rank}(K_{22}) \\ &= n - r_b + r_b - \mu = n - \mu, \\ K_{21} &= K_{12}^T, \qquad K_{11} - K_{12} K_{22}^{-1} K_{12}^T \geq 0, \end{aligned}$$

and furthermore, $\mathcal{S}_\infty (K_{11} - K_{12} K_{22}^{-1} K_{12}^T)^T \hat{\mathcal{A}}_{22} \mathcal{S}_\infty (K_{11} - K_{12} K_{22}^{-1} K_{12}^T)$ is nonsingular.

Since any real skew-symmetric matrix with odd dimension is singular, it follows that when $\mu > 0$ and $\mathcal{A}_{22} + \mathcal{A}_{22}^T = 0$, then we have that μ is even and

$$\operatorname{rank}(\mathcal{S}_\infty (K_{11} - K_{12} K_{22}^{-1} K_{12}^T)^T \hat{\mathcal{A}}_{22} \mathcal{S}_\infty (K_{11} - K_{12} K_{22}^{-1} K_{12}^T)$$

is also even, i.e., $n - r$ is even.

Next, let r be any integer satisfying (3.6), i.e., $n - \mu \leq r \leq n$. We have that $\hat{\mathcal{A}}_{22}$ is in real Schur form

$$\hat{\mathcal{A}}_{22} = \begin{bmatrix} \mathcal{T}_1 & * & * & * \\ & \ddots & * & * \\ & & \mathcal{T}_k & * \\ & & & \mathcal{D} \end{bmatrix},$$

where $\mathcal{D}$ is in real Schur form, $\mathcal{D} + \mathcal{D}^T \neq 0$, and

$$\mathcal{T}_i = \begin{bmatrix} 0 & t_i \\ -t_i & 0 \end{bmatrix}, \qquad t_i \neq 0, \quad i = 1, \ldots, k.$$

Choose $K_{12} = 0$, $K_{21} = 0$, $K_{22} = I$, and let K_{11} be constructed as follows:

If $2k = \mu > 0$, then $\mathcal{A}_{22} + \mathcal{A}_{22}^T = 0$, $r + \mu - n$ is even, and if $r + \mu - n = 2s$, then set

$$K_{11} = \begin{bmatrix} I_{2s} & \\ & 0 \end{bmatrix}.$$

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Otherwise, if $2k < \mu$, then we choose K_{11} as in the following cases:

$$K_{11} = \begin{bmatrix} I_{2s} & \\ & 0 \end{bmatrix}, \qquad \text{when } r+\mu-n = 2s \leq 2k;$$

$$K_{11} = \begin{bmatrix} I_{2s} & & \\ & 0 & \\ & & 1 \end{bmatrix}, \qquad \text{when } r+\mu-n = 2s+1,\ s \leq k;$$

$$K_{11} = \begin{bmatrix} I_{2k} & & \\ & I_{r+\mu-n-2k} & \\ & & 0 \end{bmatrix}, \quad \text{when } r+\mu > 2k.$$

Then the pair $(K_{11}, \hat{\mathcal{A}}_{22})$ is regular, $\operatorname{ind}(K_{11}, \hat{\mathcal{A}}_{22}) \leq 1$, and $\operatorname{rank}(K_{11}) = r+\mu-n$, and thus $(E+BKC, A)$ is regular, $\operatorname{ind}(E+BKC, A) \leq 1$, and $\operatorname{rank}(E+BKC) = r$. Moreover,

$$\begin{bmatrix} E_{11} & & \\ & K_{11} & \\ & & I \end{bmatrix} \begin{bmatrix} \mathcal{Q}_{11} & & \\ & I & \\ & & I \end{bmatrix} = \begin{bmatrix} \mathcal{Q}_{11} & & \\ & I & \\ & & I \end{bmatrix}^T \begin{bmatrix} E_{11} & & \\ & K_{11} & \\ & & I \end{bmatrix} \geq 0, \tag{3.8}$$

or equivalently,

$$(E+BKC)^T Q = Q^T(E+BKC) \geq 0,$$

and thus the closed-loop system (2.7) is port-Hamiltonian.

(ii) Obviously, $\operatorname{rank}(E+BKC) \geq n - r_b$ for any $K \in \mathbb{R}^{m\times m}$. Let r be an integer with $n - r_b \leq r \leq n$, and choose $K_{12} = 0$, $K_{21} = 0$, $K_{22} = I$, $F_{12} = 0$, $F_{21} = 0$, $F_{11} = -I$, $F_{22} = -I$, and

$$F_{11} = \begin{bmatrix} I_{r+\mu-n} & \\ & 0 \end{bmatrix}.$$

It then follows that $(K_{11}, \hat{\mathcal{A}}_{22} + F_{11})$ is regular, $\operatorname{ind}(K_{11}, \hat{\mathcal{A}}_{22} + F_{11}) \leq 1$, and $\operatorname{rank}(K_{11}) = r+\mu-n$. Hence, $\operatorname{rank}(E+BKC) = r$, $(E+BKC, A+BFC)$ is regular, and $\operatorname{ind}(E+BKC, A+BFC) \leq 1$. In addition, (3.8) holds, and thus $(E+BKC)^T Q \geq 0$. Moreover,

$$A + BFC = (J-R)Q + BFB^TQ = [J - (R - BFB^T)]Q,$$

and

$$Q^T(R - BFB^T)Q = Q^TRQ - Q^TBFB^TQ \geq 0.$$

Therefore, the closed-loop system (2.8) is port-Hamiltonian. □

The condensed form (3.5) is computed using only orthogonal transformations, and so it can be implemented as a numerically reliable algorithm. The Schur form (3.7) can be computed as follows:

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Algorithm.

i) Compute the QR factorization

$$\begin{bmatrix} \mathcal{C}_{12} \\ \mathcal{A}_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix},$$

where $\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$ is orthogonal and R is nonsingular. Then L_{11} and L_{22} are nonsingular (see, e.g., [11]), and

$$\mathcal{A}_{22}\mathcal{C}_{12}^{-1} = L_{21}L_{11}^{-1} = -L_{22}^{-T}L_{12}^{T}, \qquad \mathcal{B}_{21}^{-1}\mathcal{A}_{22}\mathcal{C}_{12}^{-1} = -\mathcal{B}_{21}^{-1}(L_{22}^{T})^{-1}L_{12}^{T}.$$

ii) Compute the QR factorization

$$\begin{bmatrix} -L_{22}^{T}\mathcal{B}_{21} & L_{12}^{T} \end{bmatrix} = \begin{bmatrix} \mathcal{R} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix},$$

where $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$ is orthogonal and $\mathcal{R}$ is nonsingular. Then again $\mathcal{L}_{11}$ is nonsingular (see [11]), and

$$\mathcal{B}_{21}^{-1}\mathcal{A}_{22}\mathcal{C}_{12}^{-1} = \mathcal{L}_{11}^{-1}\mathcal{L}_{12}.$$

Compute $\mathcal{L}_{11}^{-1}\mathcal{L}_{12}$ using the Cosine-Sine decomposition (see [14]) of $\mathcal{L}_{11}$ and $\mathcal{L}_{12}$ and then compute the real Schur form (3.7).

All computations, and hence the numerical construction of the matrices K and F in Theorem 3.2, can be implemented in a numerically reliable way.

REMARK 3.6. An important feature of Theorems 3.1, 3.2, 3.5, and the construction of the feedback matrices is that the computation of the matrix Q in (2.4) is not directly involved because this would be computationally difficult.

4. Concluding remarks. We have presented necessary and sufficient conditions for the regularization of port-Hamiltonian descriptor systems by derivative and/or proportional output feedback. All results are derived based on condensed forms that can be implemented using only orthogonal transformations and hence as numerically reliable algorithms.

Appendix A. Proofs of the Lemmas.

A.1. Proof of Lemma 3.3.

Proof. We again give a constructive proof that can be directly implemented as a numerical method. First, observe that

$$m = r_c = \operatorname{rank}(C) = \operatorname{rank}(B^TQ) \le \operatorname{rank}(B) = r_b \le m,$$

so that

$$r_c = \operatorname{rank}(C) = \operatorname{rank}(B) = r_b = m.$$

The construction of U, V, and W proceeds as follows: Determine orthogonal matrices U_1 and V_1 such that

$$E^{(1)} = U_1EV_1 = \begin{matrix} & \begin{matrix} r_e & n-r_e \end{matrix} \\ \begin{matrix} r_e \\ n-r_e \end{matrix} & \begin{bmatrix} E_{11}^{(1)} & 0 \\ 0 & 0 \end{bmatrix} \end{matrix}, \qquad \operatorname{rank}(E_{11}^{(1)}) = r_e.$$

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Partition the matrices analogously as

$$B^{(1)} = U_1 B = \begin{matrix} r_e \\ n-r_e \end{matrix} \begin{bmatrix} B_1^{(1)} \\ B_2^{(1)} \end{bmatrix}, \qquad C^{(1)} = CV_1 = \begin{matrix} & r_e & n-r_e \\ & \begin{bmatrix} C_1^{(1)} & C_2^{(1)} \end{bmatrix} \end{matrix}.$$

Note that by $\operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n$, $C = B^T Q$, and $E^T Q = Q^T E \geq 0$, it follows that

$$\operatorname{rank}(C_2^{(1)}) = n - r_e$$

and

$$Q^{(1)} = U_1 Q V_1 = \begin{matrix} r_e \\ n-r_e \end{matrix} \overset{\begin{matrix} r_e & n-r_e \end{matrix}}{\begin{bmatrix} Q_{11}^{(1)} & 0 \\ Q_{21}^{(1)} & Q_{22}^{(1)} \end{bmatrix}}.$$

Compute an orthogonal matrix W such that

$$B_2^{(1)} W = \overset{\begin{matrix} r_b+r_e-n & n-r_e \end{matrix}}{\begin{bmatrix} 0 & B_{22}^{(2)} \end{bmatrix}}$$

and partition the matrix

$$B^{(2)} = \begin{bmatrix} B_1^{(1)} \\ B_2^{(1)} \end{bmatrix} W = \begin{matrix} r_e \\ n-r_e \end{matrix} \overset{\begin{matrix} r_b+r_e-n & n-r_e \end{matrix}}{\begin{bmatrix} B_{11}^{(2)} & B_{12}^{(2)} \\ 0 & B_{22}^{(2)} \end{bmatrix}}$$

accordingly. Then by $C = B^T Q$ it follows that

$$\begin{aligned} C^{(2)} &= W^T \begin{bmatrix} C_1^{(1)} & C_2^{(1)} \end{bmatrix} = \begin{matrix} r_b+r_e-n \\ n-r_e \end{matrix} \overset{\begin{matrix} r_e & n-r_e \end{matrix}}{\begin{bmatrix} C_{11}^{(2)} & 0 \\ C_{21}^{(2)} & C_{22}^{(2)} \end{bmatrix}}, \\ C_{22}^{(2)} &= (B_{22}^{(2)})^T Q_{22}^{(1)}. \end{aligned}$$

Since $\operatorname{rank}(C_2^{(1)}) = n - r_e$, it follows that $\operatorname{rank}(C_{22}^{(2)}) = \operatorname{rank}(C_2^{(1)}) = n - r_e$, and consequently, $\operatorname{rank}(B_{22}^{(2)}) = n - r_e$ and $Q_{22}^{(1)}$ is nonsingular. Note that $\operatorname{rank}(C) = \operatorname{rank}(B) = m$, so we obtain

$$\operatorname{rank}(C_{11}^{(2)}) = \operatorname{rank}(B_{11}^{(2)}) = r_e + r_b - n.$$

Compute orthogonal matrices U_2 and V_2 such that

$$U_2 B_{11}^{(2)} = \begin{bmatrix} 0 \\ B_{21} \end{bmatrix}, \qquad C_{11}^{(2)} V_2 = \begin{bmatrix} 0 & C_{12} \end{bmatrix},$$

where $B_{21}, C_{12} \in \mathbb{R}^{(r_e+r_b-n)\times(r_e+r_b-n)}$ are nonsingular.

Partition the matrices

$$\begin{matrix} n-r_b \\ r_e+r_b-n \end{matrix} \overset{\begin{matrix} n-r_b & r_e+r_b-n \end{matrix}}{\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}} = U_2 E_{11}^{(1)} V_2,$$

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$$U_2B_{12}^{(2)} = \begin{matrix} n-r_b \\ r_e+r_b-n \end{matrix}\begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix}, \qquad C_{21}^{(2)}V_2 = \begin{matrix} & n-r_b & r_e+r_b-n \\ & \big[\; C_{21} & C_{22} \;\big] \end{matrix},$$

and

$$B_{32} = B_{22}^{(2)}, \qquad C_{23} = C_{22}^{(2)}, \qquad U = \begin{bmatrix} U_2 & \\ & I \end{bmatrix} U_1, \qquad V = V_1 \begin{bmatrix} V_2 & \\ & I \end{bmatrix}$$

accordingly.

Using $E^TQ \geq 0$ and $C = B^TQ$, it follows that

$$UQV = \begin{matrix} n-r_b \\ r_e+r_b-n \\ n-r_e \end{matrix}\overset{\begin{matrix} n-r_b & r_e+r_b-n & n-r_e \end{matrix}}{\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ 0 & Q_{22} & 0 \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}},$$

as well as

$$C_{12} = B_{21}^TQ_{22}, \qquad C_{23} = B_{32}^TQ_{33},$$

which together with the nonsingularity of C_{12}, C_{23}, B_{21}, and B_{32} implies that Q_{22} and Q_{33} are nonsingular. In the following, we show that E_{11} is nonsingular by employing that

$$\begin{bmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \geq 0.$$

We may assume, without loss of generality, that

$$Q_{22} = I, \qquad Q_{11} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q_{12} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

and

$$E_{11} = \begin{bmatrix} E_{11}^1 & E_{11}^2 \\ E_{11}^3 & E_{11}^4 \end{bmatrix}, \qquad E_{12} = \begin{bmatrix} E_{12}^1 \\ E_{12}^2 \end{bmatrix}, \qquad E_{21} = \begin{bmatrix} E_{21}^1 & E_{21}^2 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} E_{11}^1 & E_{11}^2 & E_{12}^1 \\ 0 & 0 & 0 \\ \hat{E}_{21}^1 & \hat{E}_{21}^2 & \hat{E}_{22} \end{bmatrix} \geq 0,$$

and thus,

$$\operatorname{rank}(E_{11}^1) = \operatorname{rank}\begin{bmatrix} E_{11}^1 & E_{11}^2 & E_{12}^1 \end{bmatrix}, \qquad E_{11}^1 \text{ is nonsingular},$$

and

$$E_{11}^2 = 0, \qquad \hat{E}_{21}^2 = Z_1^TE_{11}^2 + Z_2^TE_{11}^4 + E_{21}^2 = Z_2^TE_{11}^4 + E_{21}^2 = 0,$$

i.e.,

$$E_{11}^2 = 0, \quad E_{21}^2 = -Z_2^TE_{11}^4, \quad \operatorname{rank}(E_{11}^4) = \operatorname{rank}\begin{bmatrix} E_{11}^4 \\ E_{12}^2 \end{bmatrix}, \quad \text{and } E_{11}^4 \text{ is nonsingular}.$$

Hence, E_{11} is nonsingular. □

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A.2. Proof of Lemma 3.4.

Proof. Set

$$X = \begin{bmatrix} I & 0 & -B_{12}B_{32}^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & & \\ \mathcal{U}_{21} & \mathcal{U}_{22} & \\ & & I \end{bmatrix} U,$$

$$Y = V \begin{bmatrix} I & \mathcal{V}_{12} & \\ & \mathcal{V}_{22} & \\ & & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -C_{23}^{-1}C_{21} & 0 & I \end{bmatrix}.$$

Then

$$XEY = \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & \hat{E}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad XAY = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & A_{22} & A_{23} \\ \tilde{A}_{31} & A_{32} & A_{33} \end{bmatrix},$$

$$XBW = \begin{bmatrix} 0 & 0 \\ \hat{B}_{21} & \hat{B}_{22} \\ 0 & B_{32} \end{bmatrix}, \qquad W^TCY = \begin{bmatrix} 0 & \hat{C}_{12} & 0 \\ 0 & \hat{C}_{22} & C_{23} \end{bmatrix}. \tag{A.1}$$

Note that E_{11} is nonsingular, so $\mathcal{U}_{22}$ and $\mathcal{V}_{22}$ are nonsingular; see [11]. Hence, $\hat{B}_{21}$ and $\hat{C}_{12}$ are nonsingular. It follows from $E^TQ \geq 0$ and $C = B^TQ$ that

$$X^{-T}QY = \begin{matrix} \\ n-r_b \\ r_e+r_b-n \\ n-r_e \end{matrix} \begin{matrix} n-r_b & r_e+r_b-n & n-r_e \\ \left[\begin{matrix} \mathcal{Q}_{11} \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ \mathcal{Q}_{22} \\ \mathcal{Q}_{32} \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ \mathcal{Q}_{33} \end{matrix}\right] \end{matrix},$$

where $\mathcal{Q}_{22}$ and $\mathcal{Q}_{33}$ are nonsingular, and

$$E_{11}^T\mathcal{Q}_{11} \geq 0, \qquad \hat{E}_{22}^T\mathcal{Q}_{22} \geq 0.$$

From $C = B^TQ$ and $A^TQ + Q^TA \leq 0$, it follows that

$$\begin{bmatrix} \hat{C}_{12} & 0 \\ \hat{C}_{22} & C_{23} \end{bmatrix} = \begin{bmatrix} \hat{B}_{21} & \hat{B}_{22} \\ 0 & B_{32} \end{bmatrix}^T \begin{bmatrix} \mathcal{Q}_{22} & 0 \\ \mathcal{Q}_{32} & \mathcal{Q}_{33} \end{bmatrix}, \tag{A.2}$$

$$\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^T \begin{bmatrix} \mathcal{Q}_{22} & 0 \\ \mathcal{Q}_{32} & \mathcal{Q}_{33} \end{bmatrix} + \begin{bmatrix} \mathcal{Q}_{22} & 0 \\ \mathcal{Q}_{32} & \mathcal{Q}_{33} \end{bmatrix}^T \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \leq 0.$$

This implies that

$$Z \begin{bmatrix} \mathcal{Q}_{22} & 0 \\ \mathcal{Q}_{32} & \mathcal{Q}_{33} \end{bmatrix} \mathcal{Z} = \begin{matrix} \\ \mu \\ r_b-\mu \end{matrix} \begin{matrix} \mu & r_b-\mu \\ \left[\begin{matrix} \hat{\mathcal{Q}}_{22} \\ \hat{\mathcal{Q}}_{32} \end{matrix}\right. & \left.\begin{matrix} 0 \\ \hat{\mathcal{Q}}_{33} \end{matrix}\right] \end{matrix}.$$

Then it follows that

$$\mathcal{W}^T \begin{bmatrix} \hat{C}_{12} & 0 \\ \hat{C}_{22} & C_{23} \end{bmatrix} \mathcal{Z} = \left(Z \begin{bmatrix} \hat{B}_{21} & \hat{B}_{22} \\ 0 & B_{32} \end{bmatrix} \mathcal{W} \right)^T \begin{bmatrix} \hat{\mathcal{Q}}_{22} & 0 \\ \hat{\mathcal{Q}}_{32} & \hat{\mathcal{Q}}_{33} \end{bmatrix} = \begin{bmatrix} \mathcal{C}_{12} & 0 \\ \mathcal{C}_{22} & \mathcal{C}_{23} \end{bmatrix},$$

where $\mathcal{C}_{12} = \mathcal{B}_{21}^T\hat{\mathcal{Q}}_{22}$ and $\mathcal{C}_{23} = \mathcal{B}_{32}^T\hat{\mathcal{Q}}_{33}$ are nonsingular and furthermore, $\hat{\mathcal{Q}}_{22}$ and $\hat{\mathcal{Q}}_{33}$ are also nonsingular. ∎

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