Electronic Transactions on Numerical Analysis.
Volume 65, pp. 207–225, 2026.

ISSN 1068–9613.
DOI: 10.1553/etna_vol65s207

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# A SIMPLIFIED ITERATED LAVRENTIEV REGULARIZATION METHOD FOR NONLINEAR ILL-POSED MONOTONE OPERATOR EQUATIONS UNDER A HEURISTIC RULE*

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**Abstract.** In this paper, we propose a simplified iterative Lavrentiev regularization approach for computing stable approximate solutions of the nonlinear ill-posed operator equation $\mathcal{F}(x) = y$, where $\mathcal{F} : D(\mathcal{F}) \subset X \to X$ is a nonlinear monotone operator defined on a Hilbert space $X$. For iterative regularization methods, the choice of a suitable stopping rule is a key issue since it strongly influences the stability and accuracy of the computed solution. In many practical situations, the exact level of noise present in the data is either unknown or unreliable, which makes classical a priori and a posteriori stopping rules difficult to apply. To address this difficulty, Q. Jin and W. Wang introduced in 2018 a heuristic stopping rule for the iteratively regularized Gauss–Newton method. Motivated by their work, we propose a heuristic selection rule for a simplified version of the Lavrentiev regularization method. The main advantage of the proposed scheme is that it requires the computation of the Fréchet derivative of the operator $\mathcal{F}$ only once, namely at an initial approximation $x_0$ of the exact solution $x^\dagger$. Under suitable assumptions that control the nonlinearity of the operator, we derive error estimates for the proposed method. Finally, we illustrate the practical behavior of the method by applying it to a nonlinear integral operator problem.

**Key words.** Lavrentiev regularization, nonlinear ill-posed problems, heuristic parameter choice rules

**AMS subject classifications.** 65J20, 47J06

## 1. Introduction.

In this paper, we study nonlinear ill-posed operator equations of the form

$$\mathcal{F}(x) = y, \tag{1.1}$$

where $\mathcal{F} : D(\mathcal{F}) \subset X \to X$ is a nonlinear operator defined on a real Hilbert space $X$. Throughout the paper, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the inner product and the induced norm on $X$, respectively. We assume that the operator $\mathcal{F}$ is monotone, that is,

$$\langle \mathcal{F}(x) - \mathcal{F}(\tilde{x}), x - \tilde{x} \rangle \geq 0 \qquad \text{for all } x, \tilde{x} \in D(\mathcal{F}).$$

We assume that equation (1.1) admits a unique solution $x^\dagger$. In practice, however, the exact right-hand side $y$ is not available. Instead, one has access only to noisy data $y^\delta$ satisfying

$$\|y - y^\delta\| \leq \delta, \tag{1.2}$$

where $\delta > 0$ denotes the noise level. Since the problem (1.1) is ill-posed, small perturbations in the data may cause large deviations in the solution. Consequently, regularization techniques are required to obtain stable approximations of the exact solution $x^\dagger$.

Classical regularization methods for nonlinear ill-posed problems include Tikhonov regularization and its variants, and they have been extensively studied in the literature (see, for example, [5, 6, 7, 16]). For the special class of monotone operators, Lavrentiev regularization provides a simpler and computationally efficient alternative [17, 18]. In Lavrentiev regularization, an approximate solution is obtained by solving

$$\mathcal{F}(x) - y^\delta + \alpha(x - x_0) = 0, \tag{1.3}$$

*Received November 29, 2025. Accepted April 17, 2026. Published online on June 18, 2026. Recommended by Sergei Pereverzyev.
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where $\alpha > 0$ is a regularization parameter and $x_0$ is an initial guess of the exact solution. If the operator $\mathcal{F}$ is Fréchet differentiable in a neighborhood of $x^\dagger$, then equation (1.3) can be formally rewritten by linearizing $\mathcal{F}$ at the current iterate. This leads to the representation

$$x = x_0 + (\mathcal{F}'(x) + \alpha I)^{-1}\big[y^\delta - \mathcal{F}(x) + \mathcal{F}'(x)(x - x_0)\big],$$

where $\mathcal{F}'(x)$ denotes the Fréchet derivative of $\mathcal{F}$ at $x$.

Iterative regularization methods of Lavrentiev type have been widely studied for solving nonlinear ill-posed operator equations. A classical and important contribution in this direction was made by Bakushinsky and Smirnova [4], who introduced an iterative version of the Lavrentiev regularization scheme in which the Fréchet derivative of the forward operator is updated at every iteration. Their iterative method is defined as

$$x_{k+1}^\delta = x_k^\delta - \big(A_k^\delta + \alpha_k I\big)^{-1}\Big[\mathcal{F}(x_k^\delta) - y^\delta + \alpha_k(x_k^\delta - x_0)\Big], \tag{1.4}$$

where $A_k^\delta := \mathcal{F}'(x_k^\delta)$ and $(\alpha_k)$ is a sequence of positive regularization parameters satisfying $\lim_{k\to\infty} \alpha_k = 0$.

To determine a suitable stopping index, Bakushinsky and Smirnova employed a generalized discrepancy principle (see, for example, [2, 3]), according to which the stopping index $k_\delta$ is chosen as the smallest integer such that

$$\|\mathcal{F}(x_{k_\delta}^\delta) - y^\delta\|^2 \le \tau\delta < \|\mathcal{F}(x_k^\delta) - y^\delta\|^2, \qquad 0 \le k < k_\delta,$$

for some fixed constant $\tau > 1$. Under appropriate assumptions, they proved that the corresponding approximate solution $x_{k_\delta}^\delta$ converges to the exact solution $x^\dagger$ as the noise level $\delta \to 0$.

Although this approach enjoys strong theoretical guarantees, the repeated evaluation of the Fréchet derivative at each iteration can be computationally expensive. Motivated by this issue, Mahale and Nair [15] later established convergence and order-optimal error estimates for such iterative Lavrentiev-type schemes under suitable nonlinearity conditions on the forward operator.

In order to further reduce the computational cost, Mahale [14] proposed a simplified iterated Lavrentiev regularization method in which the Fréchet derivative is frozen at the initial guess $x_0$. The resulting iteration is given by

$$x_{k+1}^\delta = x_k^\delta - \big(\mathcal{A}_0 + \alpha_k I\big)^{-1}\Big[\mathcal{F}(x_k^\delta) - y^\delta + \alpha_k(x_k^\delta - x_0)\Big], \tag{1.5}$$

where $\mathcal{A}_0 := \mathcal{F}'(x_0)$ and $(\alpha_k)$ is a sequence of positive regularization parameters satisfying

$$\alpha_k > 0, \qquad 1 \le \frac{\alpha_k}{\alpha_{k+1}} \le \mu, \qquad \text{and} \qquad \lim_{k\to\infty} \alpha_k = 0. \tag{1.6}$$

Under a general source condition, order-optimal convergence rates for this simplified scheme were obtained by choosing the stopping index $k_\delta$ as the smallest non-negative integer satisfying the discrepancy-type condition

$$\Big\|\alpha_{k_\delta}\big(\mathcal{F}'(x_0) + \alpha_{k_\delta} I\big)^{-1/2}\big(\mathcal{F}(x_{k_\delta}^\delta) - y^\delta\big)\Big\| \le \tau_0\delta, \tag{1.7}$$

where $\tau_0 > 1$ is a prescribed constant.

The principal advantage of this simplified regularization strategy is that the Fréchet derivative is evaluated only once at the initial point $x_0$. This significantly reduces the computational burden while preserving the stability and convergence properties of the original iterated Lavrentiev method.

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A crucial component of any iterative regularization method is the choice of a suitable stopping rule. It has been observed that most classical a priori and a posteriori stopping rules, including, for example, the discrepancy principle, require accurate knowledge of the noise level $\delta$. In many practical applications, however, such information may be unavailable or unreliable due to experimental errors or physical limitations. The use of inaccurate noise estimates in regularization algorithms may lead to poor approximations of the exact solution.

To overcome this difficulty, heuristic stopping rules that do not require explicit knowledge of the noise level have been proposed and studied. A general heuristic strategy for selecting regularization parameters was introduced by Hanke and Raus [8]. Their approach is based on a posteriori error bounds in which the actual error $\|x_\alpha^\delta - x^\dagger\|$ can be estimated by the quantity $\|\mathcal{F}(x_\alpha^\delta) - y^\delta\|/\sqrt{\alpha}$. This idea led to the selection rule

$$\alpha_* = \arg\min_{\alpha>0} \frac{\|\mathcal{F}(x_\alpha^\delta) - y^\delta\|}{\sqrt{\alpha}},$$

which has been successfully applied to Tikhonov regularization, the Landweber iteration, and related methods.

Several heuristic parameter choice rules have since been developed, including generalized cross-validation [20], the L-curve method [9], the quasi-optimality criterion [13, 19], and the Hanke–Raus rule [8, 11]. These methods have been further analyzed and extended by various authors (see, for example, [7, 11, 20]). Heuristic rules are particularly advantageous when the noise level is unknown or when the noise is dominated by high-frequency components.

More recently, Jin and Wang [12] proposed a heuristic stopping rule for the iteratively regularized Gauss–Newton method. Their rule is defined using the functional

$$\theta(k, y^\delta) = \frac{\|\mathcal{F}(x_k^\delta) - y^\delta\|^2}{\alpha_k},$$

and the stopping index is chosen as

$$k_* \in \arg\min\{\theta(k, y^\delta) : k = 0, 1, \ldots, k_\infty\},$$

where $k_\infty$ denotes the largest index for which the iterates remain in the domain of $\mathcal{F}$. Convergence analysis and error estimates under various source conditions were established in [12].

Motivated by these developments, the aim of the present paper is to formulate and analyze a heuristic stopping rule for the simplified iterated Lavrentiev regularization method. In contrast to Newton-type methods, our approach exploits the monotonicity of the operator and requires the evaluation of the Fréchet derivative only at the initial guess. We obtain a posteriori error estimates for the proposed method under a Hölder-type source condition using suitable nonlinearity conditions on the operator $\mathcal{F}$. To demonstrate the validity of our method, we apply it to an integral operator equation in Section 5.

The paper is organized as follows. In Section 2, we introduce the assumptions and formulate the heuristic stopping rule. Error estimates are derived in the subsequent sections, and the theoretical results are illustrated by considering numerical examples involving nonlinear integral operator equations.

## 2. Preliminaries, assumptions, and the formulation of the heuristic rule.

Before introducing the heuristic stopping rule for the ill-posed operator equation, we define an integer $k_\infty := k_\infty(y^\delta)$ by

$$k_\infty := \max\{k : x_l^\delta \in D(\mathcal{F}) \text{ for all } 0 \le l \le k\}.$$

That is, $k_\infty$ represents the largest index for which all iterates $x_k^\delta$ remain in the domain of the operator $\mathcal{F}$. The value of $k_\infty$ may be finite or infinite. This definition guarantees that the iterative sequence $(x_k^\delta)$ generated by (1.5) stays inside $D(\mathcal{F})$ up to the index $k_\infty$.

We now introduce the heuristic rule used in this work.

RULE 1. Let

$$\theta(k, y^\delta) := \frac{\|\mathcal{F}(x_k^\delta) - y^\delta\|^2}{\alpha_k^2}.$$

We define the stopping index $k_* = k_*(y^\delta)$ as

$$k_* \in \arg\min \left\{ \theta(k, y^\delta) : k = 0, 1, \ldots, k_\infty \right\}. \tag{2.1}$$

The approximation to the exact solution $x^\dagger$ is then given by $x_{k_*}^\delta$. To establish the well-definedness of this heuristic rule and to support the convergence analysis, we impose the following condition on the noise level:

ASSUMPTION 2.1. *There exists a constant* $0 < \kappa < 1$ *such that*

$$\|y^\delta - v\| \geq \kappa \|y^\delta - y\|,$$

*for all* $v \in \{\mathcal{F}(x) : x \in S(y^\delta)\}$, *where*

$$S(y^\delta) := \{x_k^\delta : 0 \leq k \leq k_\infty\}$$

*is the sequence generated by the iterative method* (1.5) *using the noisy data* $y^\delta$.

This assumption has previously been employed in [12] for the error analysis of iterative regularized Gauss–Newton methods. The following lemma shows that the heuristic rule defined in (2.1) is well defined.

LEMMA 2.2. *Let Assumption 2.1 hold, and suppose that* $y^\delta \neq y$. *Then there exists a finite integer* $k_*$ *satisfying* (2.1).

*Proof.* If $k_\infty$ is finite, then the result follows immediately. We therefore consider the case $k_\infty = \infty$. From Assumption 2.1 and the properties of the regularization parameters $\alpha_k$ given in (1.6), we obtain

$$\theta(k, y^\delta) = \frac{\|\mathcal{F}(x_k^\delta) - y^\delta\|^2}{\alpha_k^2} \geq \frac{\kappa^2 \|y^\delta - y\|^2}{\alpha_k^2} \longrightarrow \infty \qquad \text{as } k \to \infty.$$

Hence, the function $\theta(k, y^\delta)$ attains its minimum at some finite index $k_*$, which completes the proof. ◻

**2.1. A posteriori error estimates under Hölder-type source conditions.** In this section, we study the well-definedness of the iterative method and derive error estimates under a Hölder-type source condition on the exact solution $x^\dagger$. For this purpose, we first introduce a nonlinearity assumption on the operator $\mathcal{F}$.

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ASSUMPTION 2.3. *There exists a radius $r > 0$ such that the ball*

$$\mathcal{B}_r(x^\dagger) := \{x \in X : \|x - x^\dagger\| < r\}$$

*is contained in the domain $D(\mathcal{F})$. Moreover, the operator $\mathcal{F}$ is Fréchet differentiable at every point $x \in \mathcal{B}_r(x^\dagger)$. The derivative $\mathcal{F}'(x_0)$ is scaled such that $\mathcal{F}'(x_0) \leq \alpha_0$. In addition, there exists a constant $\mathcal{K} > 0$ satisfying*

$$\|\mathcal{F}(x) - \mathcal{F}(\bar{x}) - \mathcal{F}'(x_0)(x - \bar{x})\| \leq \frac{\mathcal{K}}{2}\|x - \bar{x}\| \, \|\mathcal{F}'(x_0)(x - \bar{x})\|$$

*for all $x, \bar{x} \in \mathcal{B}_r(x^\dagger)$.*

Next, we impose a source condition for the initial error.

ASSUMPTION 2.4. *There exist an element $v \in X$ and a constant $p > 0$ such that*

$$x_0 - x^\dagger = \mathcal{A}_0^p v,$$

*where $v \in N(\mathcal{F}'(x^\dagger))$, and the fractional power $\mathcal{A}_0^p$ is defined by*

$$\mathcal{A}_0^p v := \frac{\sin(p\pi)}{\pi} \int_0^\infty s^{p-1}(\mathcal{A}_0 + sI)^{-1}\mathcal{A}_0 v \, ds, \qquad v \in X.$$

The above assumption is also usually referred to as a *Hölder-type source condition*.

For convenience, we introduce the following scalar functions commonly used in Lavrentiev-type regularization:

$$r_\alpha(\lambda) := \frac{\alpha}{\alpha + \lambda}, \qquad g_\alpha(\lambda) := \frac{1}{\alpha + \lambda}, \qquad \lambda \geq 0.$$

Using standard functional calculus for monotone operators, the corresponding operator functions $r_\alpha(\mathcal{A}_0)$ and $g_\alpha(\mathcal{A}_0)$ are defined in the usual way. Such operator functions are widely used in the analysis of iterative regularization methods; see, for example [4].

For the convergence analysis, we define $\hat{k}$ as the smallest integer satisfying

$$\alpha_{\hat{k}} \leq \frac{c_0\delta}{\|e_0\|} \leq \alpha_k, \qquad 0 \leq k \leq \hat{k}, \tag{2.2}$$

where $\alpha_0 > \frac{c_0\delta}{\|e_0\|}$. The conditions imposed on $(\alpha_k)$ in (1.6) ensure the existence of such an index $\hat{k}$.

PROPOSITION 2.5. *Let $A, B \in \mathcal{L}(X, X)$ be monotone operators. Then, for $\alpha > 0$, $p \in (0, 1]$, and $s \geq 0$, the following estimates hold:*

$$\|(A + \alpha I)^{-1}A^p\| \leq \alpha^{p-1},$$

*and*

$$\|B^s x\| \leq c_s \|B^{s+1}x\|^{\frac{s}{s+1}} \|x\|^{\frac{1}{s+1}},$$

*where $c_s = s^{-s/(s+1)} + s^{1/(s+1)} \leq 2$.*

*Proof.* The proof can be found in [17, Proposition 2.1]. ∎

We now establish the well-definedness of the iterates generated by (1.5) when the stopping index is chosen according to (2.2).

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LEMMA 2.6. *Let Assumptions 2.3, (1.2), and (1.6) hold, and let $\hat{k}$ be defined by (2.2). If*

$$\frac{1}{c_0} + \mathcal{K}\gamma\|e_0\| < 1,$$

*then*

$$\|e_k^\delta\| \leq 2\|e_0\|, \qquad \|\mathcal{A}_0 e_k^\delta\| \leq \gamma\alpha_k\|e_0\|, \tag{2.3}$$

*for all $0 \leq k \leq \hat{k}$, where*

$$\gamma = \frac{\left(1 + \frac{1}{c_0}\right)\mu}{1 - \mathcal{K}\mu\|e_0\|}.$$

*Proof.* The assertion is established by means of mathematical induction. For the initial step $k = 0$, the stated estimates are immediate. We now assume that the conclusion holds for some index $k$ with $0 \leq k \leq \hat{k}$ and verify that it remains valid for the subsequent iterate $k + 1$.

Recalling the iteration scheme defined in (1.5), we obtain

$$x_{k+1}^\delta = x_k^\delta - (\alpha_k I + \mathcal{A}_0)^{-1}\big(\mathcal{F}(x_k^\delta) - y^\delta + \alpha_k(x_k^\delta - x_0)\big). \tag{2.4}$$

By subtracting the exact solution $x^\dagger$ from both sides, the above relation can be rewritten as

$$\begin{aligned} x_{k+1}^\delta - x^\dagger &= (x_k^\delta - x^\dagger) - (\alpha_k I + \mathcal{A}_0)^{-1}\big(\alpha_k(x_k^\delta - x_0) + \mathcal{F}(x_k^\delta) - y^\delta\big) \\ &= (\alpha_k I + \mathcal{A}_0)^{-1}\Big((\alpha_k I + \mathcal{A}_0)(x_k^\delta - x^\dagger) - \big(\mathcal{F}(x_k^\delta) - y^\delta - \alpha_k(x_k^\delta - x_0)\big)\Big) \\ &= (\alpha_k I + \mathcal{A}_0)^{-1}\Big(\alpha_k(x_0 - x^\dagger) - \big(\mathcal{F}(x_k^\delta) - y^\delta - \mathcal{A}_0(x_k^\delta - x^\dagger)\big)\Big). \end{aligned}$$

Introducing the notation $e_k^\delta = x_k^\delta - x^\dagger$ and using the definition of the function $r_{\alpha_k}$, we arrive at

$$x_{k+1}^\delta - x^\dagger = r_{\alpha_k}(\mathcal{A}_0)(x_0 - x^\dagger) - (\alpha_k I + \mathcal{A}_0)^{-1}\big(\mathcal{F}(x_k^\delta) - y^\delta - \mathcal{A}_0 e_k^\delta\big).$$

Consequently, the error representation takes the form

$$\begin{aligned} x_{k+1}^\delta - x^\dagger &= r_{\alpha_k}(\mathcal{A}_0)(x_0 - x^\dagger) \\ &\quad - (\alpha_k I + \mathcal{A}_0)^{-1}\big(\mathcal{F}(x_k^\delta) - y - \mathcal{A}_0 e_k^\delta\big) + (\alpha_k I + \mathcal{A}_0)^{-1}(y - y^\delta). \end{aligned}$$

Taking norms on both sides and employing Assumption 2.3, together with the standard resolvent estimates for $(\alpha_k I + \mathcal{A}_0)^{-1}$ that arise in Lavrentiev regularization for monotone operators (see, for example, [4, 10]), as well as condition (2.2), we obtain

$$\begin{aligned} \|e_{k+1}^\delta\| \leq & \|r_{\alpha_k}(\mathcal{A}_0)e_0\| + \|(\alpha_k I + \mathcal{A}_0)^{-1}(y - y^\delta)\| \\ & + \|(\alpha_k I + \mathcal{A}_0)^{-1}\big(\mathcal{F}(x_k^\delta) - y - \mathcal{A}_0 e_k^\delta\big)\|. \end{aligned}$$

We also make use of the classical bounds ([4, 10]).

$$\|(\alpha_k I + \mathcal{A}_0)^{-1}\alpha_k\| \leq 1 \quad \text{and} \quad \|(\alpha_k I + \mathcal{A}_0)^{-1}\| \leq \frac{1}{\alpha_k}.$$

Combining these inequalities with (1.2), (2.2), and Assumption 2.3, it follows that

$$\|e_{k+1}^\delta\| \leq \|e_0\| + \frac{\delta}{\alpha_k} + \frac{\mathcal{K}}{2\alpha_k}\|e_k^\delta\|\,\|\mathcal{A}_0 e_k^\delta\| \leq \left(1 + \frac{1}{c_0}\right)\|e_0\| + \frac{\mathcal{K}}{2\alpha_k}\|e_k^\delta\|\,\|\mathcal{A}_0 e_k^\delta\|.$$

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Invoking the induction hypothesis together with estimate (2.3), we deduce

$$\|e_{k+1}^\delta\| \le \left(1+\frac{1}{c_0}\right)\|e_0\| + \gamma\mathcal{K}\|e_0\|^2 \le \Big(1+\frac{1}{c_0}+\gamma\mathcal{K}\|e_0\|\Big)\|e_0\|.$$

The assumption of Lemma 2.6 therefore implies

$$\|e_{k+1}^\delta\| \le 2\|e_0\|.$$

By induction, this yields

$$\|e_k^\delta\| \le 2\|e_0\|, \qquad \text{for all } \ 0 \le k \le \hat{k}.$$

In particular, whenever $r > 2\|e_0\|$, we have $x_k^\delta \in B_r(x^\dagger)$ for every $0 \le k \le \hat{k}$.

Finally, returning to (2.4), we observe that

$$\mathcal{A}_0 e_{k+1}^\delta = \mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0)e_0 - \mathcal{A}_0(\alpha_k I + \mathcal{A}_0)^{-1}\big(\mathcal{F}(x_k^\delta) - y^\delta - \mathcal{A}_0 e_k^\delta\big).$$

Applying Assumption 2.3 together with (1.2) and (2.2), we infer

$$\|\mathcal{A}_0 e_{k+1}^\delta\| \le \alpha_k\|e_0\| + \delta + \frac{\mathcal{K}}{2}\|e_k^\delta\|\,\|\mathcal{A}_0 e_k^\delta\| \le \alpha_k\|e_0\| + \frac{\alpha_k}{c_0}\|e_0\| + \frac{\mathcal{K}}{2}\|e_k^\delta\|\,\|\mathcal{A}_0 e_k^\delta\|.$$

Using again the induction hypothesis and the condition of Lemma 2.6, we obtain

$$\begin{aligned}\|\mathcal{A}_0 e_{k+1}^\delta\| &\le \Big(1+\frac{1}{c_0}+\gamma\mathcal{K}\|e_0\|\Big)\alpha_k\|e_0\| \\ &\le \Big(1+\frac{1}{c_0}+\gamma\mathcal{K}\|e_0\|\Big)\mu\alpha_{k+1}\|e_0\| \le \gamma\alpha_{k+1}\|e_0\|.\end{aligned}$$

Here,

$$\gamma = \frac{(1+\frac{1}{c_0})\mu}{1-\mu\mathcal{K}\|e_0\|}.$$

Thus, by induction, we conclude that

$$\|\mathcal{A}_0 e_k^\delta\| \le \gamma\alpha_k\|e_0\|, \qquad \text{for all } \ 0 \le k \le \hat{k}.$$

This completes the proof. □

LEMMA 2.7. *Assume that the conditions of Lemma 2.6 hold and that the index $\hat{k}$ is chosen according to (2.2). Then the residual at $\hat{k}$ satisfies*

$$\|\mathcal{F}(x_{\hat{k}}^\delta) - y^\delta\| \le C_\star\,\delta,$$

*where $C_\star = 1 + 2\gamma c_0$.*

*Proof.* Using Lemma 2.6, the choice of $\hat{k}$ in (2.2), and Assumption 2.3, we obtain

$$\begin{aligned}\|\mathcal{F}(x_{\hat{k}}^\delta) - y^\delta\| &\le \|y^\delta - y\| + \|\mathcal{A}_0 e_{\hat{k}}^\delta\| + \|\mathcal{F}(x_{\hat{k}}^\delta) - y - \mathcal{A}_0 e_{\hat{k}}^\delta\| \\ &\le \delta + \gamma\alpha_{\hat{k}}\|e_0\| + \frac{\mathcal{K}}{2}\|e_{\hat{k}}^\delta\|\,\|\mathcal{A}_0 e_{\hat{k}}^\delta\| \\ &\le \delta + \gamma(1+\mathcal{K}\|e_0\|)\alpha_{\hat{k}}\|e_0\| \le (1+2\gamma c_0)\delta.\end{aligned}$$

This proves the claim. □

Next, we show that the heuristic stopping rule defined in (2.1) is well defined, and we derive a lower bound for the corresponding regularization parameter.

LEMMA 2.8. *Let the conditions of Lemma 2.6 and Assumption 2.1 be satisfied. Then the index $k_*$ determined by the heuristic rule (2.1) is finite. Moreover,*

$$\alpha_{k_*} \geq \frac{c_0'\,\delta}{\|e_0\|}, \qquad \text{where} \qquad c_0' = \frac{\kappa c_0}{C_\star \mu},$$

*and $C_\star$ is defined in Lemma 2.7.*

*Proof.* By the definition of $k_*$ and Lemma 2.7, we have

$$\theta(k_*, y^\delta) \leq \theta(\hat{k}, y^\delta) = \frac{\|\mathcal{F}(x^\delta_{\hat{k}}) - y^\delta\|^2}{\alpha^2_{\hat{k}}} \leq \frac{{C_\star}^2\delta^2\mu^2}{\alpha^2_{\hat{k}-1}} \leq \frac{{C_\star}^2\mu^2\|e_0\|^2}{c_0^2}.$$

Assumption 2.1 yields

$$\frac{\kappa^2\delta^2}{\alpha^2_{k_*}} \leq \theta(k_*, y^\delta) \leq \frac{{C_\star}^2\mu^2\|e_0\|^2}{c_0^2}.$$

This implies

$$\alpha_{k_*} \geq \frac{\kappa c_0}{C_\star \mu}\frac{\delta}{\|e_0\|}.$$

The result follows. □

To ensure that the iterates are well defined up to the stopping index, we introduce an auxiliary integer $k_*^\delta$. Let $k_*^\delta$ be the smallest positive integer such that

$$\alpha_{k_*^\delta} \leq \frac{c_0'\delta}{\|e_0\|} \leq \alpha_k, \qquad 0 \leq k \leq k_*^\delta. \tag{2.5}$$

By Lemma 2.8, we have $k_* \leq k_*^\delta$ and $0 < c_0' < 1$.

The next lemma establishes the well-definedness of the iterates $x_k^\delta$ for all $0 \leq k \leq k_*^\delta$.

LEMMA 2.9. *Let Assumption 2.3 and the conditions of Lemma 2.8 hold. Suppose that the sequence $(\alpha_k)$ satisfies (2.5) and*

$$\frac{1 + \gamma_1 \mathcal{K}\|e_0\|}{c_0'} < 1.$$

*Then, for all $0 \leq k \leq k_*^\delta$,*

$$\|e_k^\delta\| \leq \frac{2\|e_0\|}{c_0'}, \qquad \|\mathcal{A}_0 e_k^\delta\| \leq \frac{\gamma_1 \alpha_k \|e_0\|}{c_0'}, \tag{2.6}$$

*where*

$$\gamma_1 = \frac{(1 + c_0')\mu}{c_0' - \mu\mathcal{K}\|e_0\|}.$$

*Proof.* From the basic estimate and the choice of $\alpha_k$, we have

$$\|e^\delta_{k+1}\| \leq \left(1 + \frac{1}{c_0'}\right)\|e_0\| + \frac{\mathcal{K}}{2\alpha_k}\|e_k^\delta\|\,\|\mathcal{A}_0 e_k^\delta\|.$$

Using the induction hypothesis and the assumption of the lemma gives

$$\|e_{k+1}^{\delta}\| \le \frac{2\|e_0\|}{c_0'}.$$

By induction,

$$\|e_k^{\delta}\| \le \frac{2\|e_0\|}{c_0'}, \qquad 0 \le k \le k_*^{\delta}.$$

If $r > \frac{2\|e_0\|}{c_0'}$, then all iterates remain in $\mathcal{B}_r(x^{\dagger})$. A similar argument, using Assumption 2.3, yields

$$\|\mathcal{A}_0 e_{k+1}^{\delta}\| \le \frac{\gamma_1 \alpha_{k+1}\|e_0\|}{c_0'}.$$

Again, induction completes the proof. ☐

**3. Some estimates under a general source condition.** In this section, we establish several auxiliary estimates that play an important role in the proof of the main theorem.

LEMMA 3.1. *Assume that the conditions of Lemma 2.8 are satisfied. Then, for every* $0 \le k < k_*^{\delta}$, *the following inequalities hold:*

$$\|e_{k+1}^{\delta} - r_{\alpha_k}(\mathcal{A}_0)e_0\| \le C_0\|e_k^{\delta}\| + \frac{\delta}{\alpha_k}, \tag{3.1}$$

*and*

$$\|\mathcal{A}_0 e_{k+1}^{\delta} - \mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0)e_0\| \le C_1\|\mathcal{A}_0 e_k^{\delta}\| + \delta, \tag{3.2}$$

*where*

$$C_0 = \frac{\mathcal{K}\gamma_1\|e_0\|}{2c_0'}, \qquad C_1 = \frac{\mathcal{K}\|e_0\|}{c_0'}.$$

*Proof.* From the definition of the iterative scheme, we can write

$$e_{k+1}^{\delta} = r_{\alpha_k}(\mathcal{A}_0)e_0 - g_{\alpha_k}(\mathcal{A}_0)\big(\mathcal{F}(x_k^{\delta}) - y^{\delta} - \mathcal{A}_0 e_k^{\delta}\big).$$

Rearranging the terms gives

$$e_{k+1}^{\delta} - r_{\alpha_k}(\mathcal{A}_0)e_0 = g_{\alpha_k}(\mathcal{A}_0)(y - y^{\delta}) - g_{\alpha_k}(\mathcal{A}_0)\big(\mathcal{F}(x_k^{\delta}) - y - \mathcal{A}_0 e_k^{\delta}\big).$$

Taking norms and using Assumption 2.3, estimate (1.2), inequality (2.6), and the bound $\|(\alpha_k I + \mathcal{A}_0)^{-1}\| \le \alpha_k^{-1}$, we obtain

$$\begin{aligned}\|e_{k+1}^{\delta} - r_{\alpha_k}(\mathcal{A}_0)e_0\| &\le \|g_{\alpha_k}(\mathcal{A}_0)(y - y^{\delta})\| + \|g_{\alpha_k}(\mathcal{A}_0)(\mathcal{F}(x_k^{\delta}) - y - \mathcal{A}_0 e_k^{\delta})\| \\ &\le \frac{\mathcal{K}}{2\alpha_k}\|e_k^{\delta}\|\,\|\mathcal{A}_0 e_k^{\delta}\| + \frac{\delta}{\alpha_k} \le \frac{\mathcal{K}\gamma_1\|e_0\|}{2c_0'}\|e_k^{\delta}\| + \frac{\delta}{\alpha_k}.\end{aligned}$$

Setting $C_0 = \frac{\mathcal{K}\gamma_1\|e_0\|}{2c_0'}$ yields (3.1).

Next, applying $\mathcal{A}_0$ to the iteration formula gives

$$\mathcal{A}_0 e_{k+1}^{\delta} = \mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0)e_0 - g_{\alpha_k}(\mathcal{A}_0)\mathcal{A}_0\big(\mathcal{F}(x_k^{\delta}) - y^{\delta} - \mathcal{A}_0 e_k^{\delta}\big),$$

which implies

$$\mathcal{A}_0 e_{k+1}^\delta - \mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0) e_0 = g_{\alpha_k}(\mathcal{A}_0)\mathcal{A}_0(y - y^\delta) - g_{\alpha_k}(\mathcal{A}_0)\mathcal{A}_0\big(\mathcal{F}(x_k^\delta) - y - \mathcal{A}_0 e_k^\delta\big).$$

Taking norms and using Assumption 2.3 together with (1.2) and (2.6), we get

$$\|\mathcal{A}_0 e_{k+1}^\delta - \mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0) e_0\| \le \delta + \frac{\mathcal{K}}{2}\|e_k^\delta\|\,\|\mathcal{A}_0 e_k^\delta\| \le \delta + \frac{\mathcal{K}\|e_0\|}{c_0'}\|\mathcal{A}_0 e_k^\delta\|.$$

With $C_1 = \frac{\mathcal{K}\|e_0\|}{c_0'}$, inequality (3.2) follows. ∎

LEMMA 3.2. *Assume that the conditions of Lemma 2.9 hold and that*

$$\max\{\mathcal{C}C_1,\ C_1(1+\mu)\} < 1.$$

*Then, for all $0 \le k \le k_*^\delta$, the following estimates are valid:*

$$\|\mathcal{A}_0 e_k^\delta\| \le \mathcal{C}\big(\|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0) e_0\| + \delta\big), \tag{3.3}$$

*and*

$$\|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0) e_0\| \le \mathcal{C}_3\big(\|\mathcal{F}(x_k^\delta) - y^\delta\| + \delta\big), \tag{3.4}$$

*where*

$$\mathcal{C} = \frac{1+\mu}{1 - C_1(1+\mu)}, \qquad \mathcal{C}_2 = \max\Big\{\frac{1}{1-\mathcal{C}C_1}, \frac{1+\mathcal{C}C_1}{1-\mathcal{C}C_1}\Big\},$$
$$\mathcal{C}_3 = \Big(1 + \frac{c_0'}{c_0' - \mathcal{K}\|e_0\|}\Big)\mathcal{C}_2.$$

*Proof.* From inequality (3.2), we directly obtain

$$\|\mathcal{A}_0 e_{k+1}^\delta\| \le \|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0) e_0\| + \delta + C_1\|\mathcal{A}_0 e_k^\delta\|. \tag{3.5}$$

For convenience, define

$$\sigma_k := \|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0) e_0\| + \delta, \qquad \eta_k := \frac{\|\mathcal{A}_0 e_k^\delta\|}{\sigma_k}.$$

Dividing (3.5) by $\sigma_{k+1}$ yields

$$\eta_{k+1} \le \frac{\sigma_k}{\sigma_{k+1}} + C_1 \frac{\sigma_k}{\sigma_{k+1}}\eta_k.$$

Using the parameter condition (1.6), we have

$$\|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0) e_0\| \le \mu\|\mathcal{A}_0 r_{\alpha_{k+1}}(\mathcal{A}_0) e_0\|,$$

which implies

$$\frac{\sigma_k}{\sigma_{k+1}} \le 1 + \mu.$$

Hence,

$$\eta_{k+1} \le (1+\mu) + C_1(1+\mu)\eta_k. \tag{3.6}$$

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The scaling condition $\|\mathcal{A}_0\| \le \alpha_0$ gives

$$\|\mathcal{A}_0 e_0\| \le 2\|\mathcal{A}_0 r_{\alpha_0}(\mathcal{A}_0)e_0\| \le \frac{1+\mu}{1-C_1(1+\mu)}\big(\|\mathcal{A}_0 r_{\alpha_0}(\mathcal{A}_0)e_0\| + \delta\big),$$

and therefore

$$\eta_0 \le \frac{1+\mu}{1-C_1(1+\mu)}.$$

Applying induction to (3.6), we conclude that

$$\eta_k \le \frac{1+\mu}{1-C_1(1+\mu)}, \qquad 0 \le k \le k_*^\delta.$$

This proves inequality (3.3).

Next, from (3.2) we also have

$$\|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0)e_0\| \le \|\mathcal{A}_0 e_{k+1}^\delta\| + \delta + C_1\|\mathcal{A}_0 e_k^\delta\|.$$

Combining this with (3.3) gives

$$(1-\mathcal{C}C_1)\|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0)e_0\| \le \|\mathcal{A}_0 e_{k+1}^\delta\| + (1+\mathcal{C}C_1)\delta,$$

which leads to

$$\|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0)e_0\| \le \mathcal{C}_2\big(\|\mathcal{A}_0 e_{k+1}^\delta\| + \delta\big). \tag{3.7}$$

Using Assumption 2.3, estimate (2.6), and Lemma 2.8, we obtain

$$\|\mathcal{A}_0 e_{k+1}^\delta\| \le \frac{\mathcal{K}\|e_0\|}{c_0'}\|\mathcal{A}_0 e_{k+1}^\delta\| + \|\mathcal{F}(x_{k+1}^\delta) - y^\delta\| + \delta.$$

Rearranging terms yields

$$\|\mathcal{A}_0 e_{k+1}^\delta\| \le \frac{c_0'}{c_0' - \mathcal{K}\|e_0\|}\big(\|\mathcal{F}(x_{k+1}^\delta) - y^\delta\| + \delta\big). \tag{3.8}$$

Combining (3.7) and (3.8) gives

$$\|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0)e_0\| \le \mathcal{C}_3\big(\|\mathcal{F}(x_{k+1}^\delta) - y^\delta\| + \delta\big).$$

Since $\|\mathcal{A}_0 r_{\alpha_{k+1}}(\mathcal{A}_0)e_0\| \le \|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0)e_0\|$, inequality (3.4) holds for all $0 < k \le k_*^\delta$. For $k = 0$, the result follows from the scaling condition $\|\mathcal{A}_0\| \le \alpha_0$. □

In the next lemma, we derive an explicit bound for the error $\|e_k^\delta\|$ for all iteration indices $0 \le k \le k_*^\delta$.

LEMMA 3.3. *Assume that Assumption 2.4 holds and that the assumptions of Lemmas 2.9, 3.1, and 3.2 are satisfied. Let $k_*^\delta$ be chosen according to the stopping rule (2.5), and suppose that $C_0(1+\mu) < 1$. Then, for all $0 \le k \le k_*^\delta$, the following estimate holds:*

$$\|e_k^\delta\| \le \mathcal{C}_5\left\{\|v\|\left(\frac{\|\mathcal{F}(x_k^\delta) - y^\delta\| + \delta}{\|v\|}\right)^{\frac{p}{p+1}} + \frac{\delta}{\alpha_k}\right\},$$

*where*

$$\mathcal{C}_5 = \max\left\{c_p\mathcal{C}_4\mathcal{C}_3^{\frac{p}{p+1}},\ c_p\mathcal{C}_4\right\}, \qquad \mathcal{C}_4 = \frac{1+\mu}{1-C_0(1+\mu)},$$

*and the constant $\mathcal{C}_3$ is defined as in Lemma 3.2.*

*Proof.* From inequality (3.1), we have

$$\|e_{k+1}^\delta\| \le \|r_{\alpha_k}(\mathcal{A}_0)e_0\| + \frac{\delta}{\alpha_k} + C_0\|e_k^\delta\| \le \theta_k + C_0\|e_k^\delta\|,$$

where we define

$$\theta_k := \|r_{\alpha_k}(\mathcal{A}_0)e_0\| + \frac{\delta}{\alpha_k}.$$

Therefore,

$$\|e_{k+1}^\delta\| \le \theta_k + C_0\|e_k^\delta\|. \tag{3.9}$$

From the parameter condition (1.6), it follows that $\theta_k \le (1+\mu)\theta_{k+1}$. Dividing both sides of (3.9) by $\theta_{k+1}$ and setting

$$\ell_k := \frac{\|e_k^\delta\|}{\theta_k},$$

we obtain

$$\ell_{k+1} \le (1+\mu) + C_0(1+\mu)\ell_k. \tag{3.10}$$

The scaling condition $\|\mathcal{A}_0\| \le \alpha_0$ implies

$$\|e_0\| \le 2\|r_{\alpha_0}(\mathcal{A}_0)e_0\| \le 2\theta_0.$$

Hence,

$$\ell_0 \le 2 \le \frac{1+\mu}{1-C_0(1+\mu)}.$$

Applying induction to inequality (3.10), we conclude that

$$\ell_k \le \frac{1+\mu}{1-C_0(1+\mu)}, \qquad \text{for all } \ 0 \le k \le k_*^\delta.$$

This gives

$$\|e_k^\delta\| \le \mathcal{C}_4\left\{\|r_{\alpha_k}(\mathcal{A}_0)e_0\| + \frac{\delta}{\alpha_k}\right\}, \qquad 0 \le k \le k_*^\delta. \tag{3.11}$$

We now estimate the term $\|r_{\alpha_k}(\mathcal{A}_0)e_0\|$. By Assumption 2.4 and Proposition 2.5, we obtain

$$\begin{aligned}\|r_{\alpha_k}(\mathcal{A}_0)e_0\| &= \|r_{\alpha_k}(\mathcal{A}_0)\mathcal{A}_0^p v\| = \|\mathcal{A}_0^p r_{\alpha_k}(\mathcal{A}_0)v\| \\ &\le c_p\|\mathcal{A}_0^{p+1} r_{\alpha_k}(\mathcal{A}_0)v\|^{\frac{p}{p+1}}\|r_{\alpha_k}(\mathcal{A}_0)v\|^{\frac{1}{p+1}} \\ &\le c_p\|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0)e_0\|^{\frac{p}{p+1}}\|v\|^{\frac{1}{p+1}}.\end{aligned}$$

Using inequality (3.4), this yields

$$\|r_{\alpha_k}(\mathcal{A}_0)e_0\| \le c_p\|v\|^{\frac{1}{p+1}}\left(\mathcal{C}_3\{\|\mathcal{F}(x_k^\delta) - y^\delta\| + \delta\}\right)^{\frac{p}{p+1}}. \tag{3.12}$$

Finally, combining (3.11) and (3.12), we obtain

$$\begin{aligned}\|e_k^\delta\| &\le \mathcal{C}_4\left(c_p\|v\|^{\frac{1}{p+1}}\left(\mathcal{C}_3\{\|\mathcal{F}(x_k^\delta) - y^\delta\| + \delta\}\right)^{\frac{p}{p+1}} + \frac{\delta}{\alpha_k}\right) \\ &\le \mathcal{C}_5\left\{\|v\|\left(\frac{\|\mathcal{F}(x_k^\delta) - y^\delta\| + \delta}{\|v\|}\right)^{\frac{p}{p+1}} + \frac{\delta}{\alpha_k}\right\}.\end{aligned}$$

This completes the proof. $\square$

**4. Main result.** We now present the main theoretical result of this paper, which establishes an a posteriori error estimate for the proposed method under a general Hölder-type source condition.

THEOREM 4.1. *Assume that all the hypotheses of Lemma 3.3 are satisfied, and let the stopping index $k_*$ be determined by the heuristic rule (2.1). Then the approximate solution $x^\delta_{k_*}$ satisfies the error bound*

$$\|x^\delta_{k_*} - x^\dagger\| \le \xi\, \|v\| \left( \frac{\delta_* + \delta}{\|v\|} \right)^{\frac{p}{p+1}},$$

*where $\delta = \|y - y^\delta\|$, $\delta_* = \|\mathcal{F}(x^\delta_{k_*}) - y^\delta\|$, and*

$$\xi = \mathcal{C}_5 \left( 1 + \frac{\tau\mu(1+\kappa^{-1})}{\mathcal{C}_6} \right).$$

*Here, $\mathcal{C}_5$ is the constant from Lemma 3.3, and*

$$\mathcal{C}_6 = \frac{1}{\mathcal{C}(\mathcal{K}\|e_0\| + 1)}.$$

*Proof.* We begin by applying Lemma 3.3 at the stopping index $k = k_*$. This yields

$$\begin{aligned} \|x^\delta_{k_*} - x^\dagger\| &\le \mathcal{C}_5 \left\{ \|v\| \left( \frac{\|\mathcal{F}(x^\delta_{k_*}) - y^\delta\| + \delta}{\|v\|} \right)^{\frac{p}{p+1}} + \frac{\delta}{\alpha_{k_*}} \right\} \\ &\le \mathcal{C}_5 \left\{ \|v\| \left( \frac{\delta_* + \delta}{\|v\|} \right)^{\frac{p}{p+1}} + \theta(k_*, y^\delta)^{1/2} \right\}. \end{aligned} \tag{4.1}$$

We now estimate the quantity $\theta(k_*, y^\delta)$. Let $k_\delta$ be the smallest integer such that

$$\|\mathcal{F}(x^\delta_{k_\delta}) - y^\delta\| \le \tau\delta < \|\mathcal{F}(x^\delta_k) - y^\delta\|, \qquad 0 \le k < k_\delta.$$

By Lemma 2.7, the index $k_\delta$ is well defined and satisfies $k_\delta \le \hat{k} \le k^\delta_*$. Using Assumption 2.3 and Lemma 2.6, we estimate

$$\begin{aligned} \|\mathcal{F}(x^\delta_{k_\delta - 1}) - y^\delta\| &\le \|\mathcal{F}(x^\delta_{k_\delta - 1}) - y - \mathcal{A}_0 e^\delta_{k_\delta - 1}\| + \delta + \|\mathcal{A}_0 e^\delta_{k_\delta - 1}\| \\ &\le \frac{\mathcal{K}}{2} \|e^\delta_{k_\delta - 1}\| \|\mathcal{A}_0 e^\delta_{k_\delta - 1}\| + \delta + \|\mathcal{A}_0 e^\delta_{k_\delta - 1}\| \\ &\le (\mathcal{K}\|e_0\| + 1) \|\mathcal{A}_0 e^\delta_{k_\delta - 1}\| + \delta. \end{aligned}$$

Since $\tau\delta \le \|\mathcal{F}(x^\delta_{k_\delta - 1}) - y^\delta\|$, and using estimate (3.3), we obtain

$$\begin{aligned} \tau\delta &\le \mathcal{C}(\mathcal{K}\|e_0\| + 1) \big( \|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0) e_0\| + \delta \big) + \delta \\ &\le \mathcal{C}(\mathcal{K}\|e_0\| + 1) \|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0) e_0\| + \big( \mathcal{C}(\mathcal{K}\|e_0\| + 1) + 1 \big) \delta. \end{aligned}$$

Choosing $\tau \ge \mathcal{C}(\mathcal{K}\|e_0\| + 1) + 2$, we conclude that

$$\delta \le \mathcal{C}(\mathcal{K}\|e_0\| + 1) \|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0) e_0\|.$$

Using the source condition from Assumption 2.4, we further obtain

$$\begin{aligned} \delta &\le \mathcal{C}(\mathcal{K}\|e_0\| + 1) \|\mathcal{A}_0 r_{\alpha_k}(\mathcal{A}_0) \mathcal{A}_0^p v\| \le \mathcal{C}(\mathcal{K}\|e_0\| + 1) \|\mathcal{A}_0^{p+1} r_{\alpha_k}(\mathcal{A}_0) v\| \\ &\le \mathcal{C}(\mathcal{K}\|e_0\| + 1) \|v\|\, \alpha^{p+1}_{k_\delta - 1}. \end{aligned}$$

Defining $\mathcal{C}_6 = \frac{1}{\mathcal{C}(\mathcal{K}\|e_0\|+1)}$, we get

$$\alpha_{k_\delta-1}^{p+1} \geq \frac{\mathcal{C}_6\delta}{\|v\|}, \qquad \frac{\delta}{\alpha_{k_\delta-1}} \leq \|v\|\left(\frac{\delta}{\mathcal{C}_6\|v\|}\right)^{\frac{p}{p+1}}.$$

Using the monotonicity of $(\alpha_k)$ and the inequality above, we obtain

$$\frac{\delta}{\alpha_{k_\delta}} \leq \frac{\mu\|v\|}{\mathcal{C}_6}\left(\frac{\delta+\delta_*}{\|v\|}\right)^{\frac{p}{p+1}}.$$

From the definition of $\theta(k, y^\delta)$, it follows that

$$\theta(k_*, y^\delta) \leq \theta(k_\delta, y^\delta) \leq \frac{\tau^2\mu^2\|v\|^2}{\mathcal{C}_6^2}\left(\frac{\delta+\delta_*}{\|v\|}\right)^{\frac{2p}{p+1}}. \tag{4.2}$$

Substituting (4.2) into (4.1) and simplifying, we arrive at

$$\|x_{k_*}^\delta - x^\dagger\| \leq \mathcal{C}_5\left(1 + \frac{\tau\mu(1+\kappa^{-1})}{\mathcal{C}_6}\right)\|v\|\left(\frac{\delta_*+\delta}{\|v\|}\right)^{\frac{p}{p+1}}.$$

Setting $\xi = \mathcal{C}_5\big(1 + \frac{\tau\mu(1+\kappa^{-1})}{\mathcal{C}_6}\big)$ completes the proof. ∎

REMARK 4.2. The error estimate obtained in Theorem 4.1 is order optimal whenever the residual $\delta_* = \|\mathcal{F}(x_{k_*}^\delta) - y^\delta\|$ is of the same order as the noise level $\delta$. In this case, the convergence rate coincides with the optimal rate established in [15] under the discrepancy principle.

REMARK 4.3. We emphasize that the stopping rule used in Theorem 4.1 is a heuristic rule and does not rely on the exact knowledge of the noise level $\delta$. It is well known that heuristic stopping rules cannot guarantee convergence in the classical sense for arbitrary noise realizations without additional assumptions, a limitation commonly referred to as Bakushinsky's veto (see, for example, [1]). Therefore, Theorem 4.1 should not be interpreted as a classical convergence result for a regularization method. Instead, it provides a conditional a posteriori error estimate for the simplified iterated Lavrentiev regularization method under the stated assumptions.

The quantity $\delta_* := \|\mathcal{F}(x_{k_*}^\delta) - y^\delta\|$ serves as an a posteriori reliability indicator for the computed approximation. While $\delta_*$ is useful for assessing the quality of the numerical solution, it is not guaranteed to converge to zero as $\delta \to 0$. Similar interpretations of heuristic stopping rules can be found, for instance, in [12].

**5. Numerical results.** In this section, we perform some numerical experiments to support the theoretical results established herein and to illustrate the behavior of the proposed simplified Lavrentiev regularization method combined with a heuristic stopping rule. It should be noted that the numerical experiments are not intended to demonstrate computational efficiency or execution time. Rather, they are designed to explain how the heuristic stopping rule behaves when the noise level is unknown and how the reconstruction is influenced when the Fréchet derivative is frozen at the initial point. To this end, we first compare the heuristic stopping rule with a discrepancy-type stopping rule. Then, we investigate the effect of freezing the Fréchet derivative at the initial guess $x_0$. Finally, we examine the sensitivity of the heuristic rule with respect to different types of noise.

For the numerical experiments, we consider a nonlinear ill-posed operator equation of the following form

$$\mathcal{F}(x)(t) = \int_0^1 G(t,s)x^3(s)\,ds = y(t),$$

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where the kernel $G(t,s)$ is given by

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Clearly, the operator $\mathcal{F}$ is monotone, and its Fréchet derivative is given by

$$[\mathcal{F}'(x)v](t) = 3\int_0^1 G(t,s)x^2(s)v(s)\,ds, \qquad x, v \in L^2(0,1), \quad t \in (0,1).$$

For the above problem, we choose the exact data

$$y_{\text{exact}}(t) = \frac{1}{110}(t - t^{11}),$$

which correspond to the exact solution $x^\dagger(t) = t^3$. However, in practice, exact data are often unavailable, and only noisy data can be used. Therefore, we consider noisy data of the form

$$y^\delta(t) = y_{\text{exact}}(t) + \delta\,\eta(t),$$

where the noise function $\eta$ is normalized such that $\|\eta\| = 1$, ensuring that $\|y - y^\delta\| \le \delta$.

In the numerical computations, the initial guess is taken as $x_0(t) = t^3 - \frac{3}{56}t^8 + \frac{3}{56}t$, and the regularization parameters are chosen as $\alpha_k = 0.001\left(\frac{1}{2}\right)^k$. Further, the interval $[0,1]$ is discretized into 100 equal subintervals, and integrals are approximated using the trapezoidal rule.

In Table 5.1, we compare the numerical results obtained using the discrepancy-type stopping rule (1.7) and the proposed heuristic stopping rule (Rule 1) for different noise levels $\delta$ and various values of the parameter $\tau$. The results show that both stopping rules are capable of producing stable and accurate approximate solutions. However, an important advantage of the heuristic stopping rule is that it does not require any explicit knowledge of the noise level. In contrast, the discrepancy-type stopping rule relies on an accurate estimate of the noise level, and its performance may deteriorate in situations where this information is unavailable or inaccurately estimated. This comparison clearly highlights the practical advantage of the heuristic approach.

TABLE 5.1
*Comparison of numerical results obtained using rule (1.7) and Rule 1.*

| | Stopping rule (1.7) | | | | | | Rule 1 | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\tau = 1.2$ | | $\tau = 1.5$ | | $\tau = 1.9$ | | | | |
| $\delta$ | $k_\delta$ | error | $k_\delta$ | error | $k_\delta$ | error | $k_*$ | $\delta_*$ | error |
| 0.01e-2 | 3 | 0.1116 | 2 | 0.0773 | 1 | 0.0444 | 1 | 0.06e-2 | 0.0444 |
| 0.05e-3 | 3 | 0.0710 | 3 | 0.0710 | 2 | 0.0411 | 2 | 0.05e-3 | 0.0411 |
| 0.01e-3 | 5 | 0.0486 | 4 | 0.0294 | 3 | 0.0160 | 3 | 0.07e-3 | 0.0160 |
| 0.05e-4 | 6 | 0.0428 | 5 | 0.0276 | 3 | 0.0081 | 3 | 0.74e-4 | 0.0081 |
| 0.01e-4 | 7 | 0.0196 | 6 | 0.0110 | 3 | 0.0018 | 5 | 0.17e-4 | 0.0059 |
| 0.05e-5 | 8 | 0.0180 | 7 | 0.0104 | 3 | 0.0011 | 6 | 0.85e-5 | 0.0056 |
| 0.01e-5 | 9 | 0.0075 | 8 | 0.0040 | 4 | 0.00048 | 8 | 0.21e-5 | 0.0040 |

The behavior of the stopping rules is further illustrated in Figure 5.1. Figure 5.1(a) and Figure 5.1(b) display the quantities $\|\mathcal{F}(x_k^\delta) - y^\delta\|$ and $\theta(k, y^\delta) = \|\mathcal{F}(x_k^\delta) - y^\delta\|^2/\alpha_k^2$ as functions of the iteration index $k$. Figure 5.1(c) displays the reconstruction obtained using the heuristic stopping rule, while Figure5.1(d)–(f) display the reconstructions obtained using the discrepancy-type stopping rule for $\tau = 1.2$, $1.5$, and $1.9$, respectively.

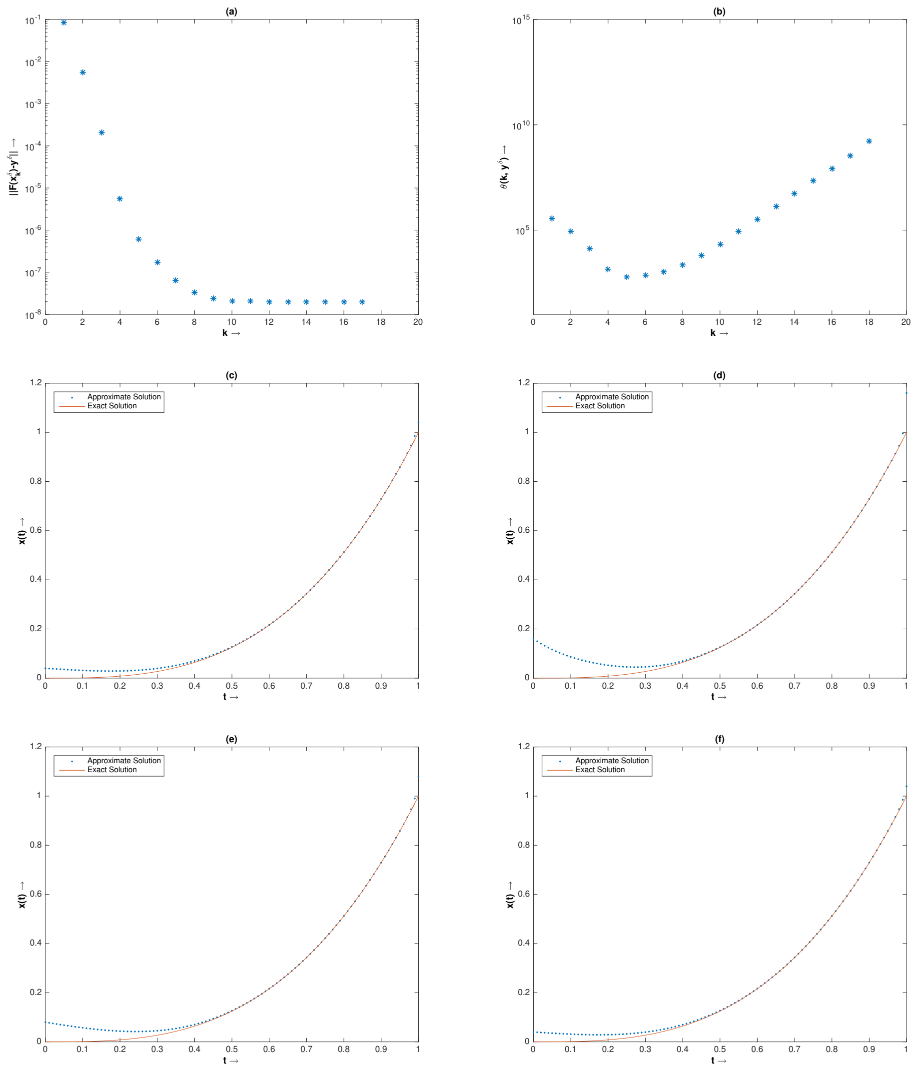


FIG. 5.1. *Graphs for (a)* $\|\mathcal{F}(x_k^\delta) - y^\delta\|$ *versus k, (b)* $\theta(k, y^\delta)$ *versus k, (c) reconstruction using the heuristic rule, (d)–(f) reconstructions using the stopping rule* (1.7) *for* $\tau = 1.2$, $1.5$, *and* $1.9$.

Next, we study the effect of freezing the Fréchet derivative at the initial guess $x_0$. The numerical results are reported in Table 5.2. This table compares the heuristic stopping rule applied to the simplified method (1.5), in which the Fréchet derivative is computed only at $x_0$, with the method (1.4), where the derivative is updated at each iteration. The results indicate that freezing the Fréchet derivative does not significantly affect the reconstruction accuracy, thereby supporting the main idea of the simplified approach.

TABLE 5.2
*Numerical results for different methods obtained using the heuristic stopping rule.*

| | Heuristic Rule for simplified method (1.5) | | | Heuristic Rule for method (1.4) | | |
|---|---|---|---|---|---|---|
| $\delta$ | $k_*$ | $\delta_*$ | error | $k_*$ | $\delta_*$ | error |
| 0.01e-2 | 1 | 0.06e-2 | 0.0444 | 1 | 0.06e-2 | 0.0444 |
| 0.05e-3 | 2 | 0.05e-3 | 0.0411 | 2 | 0.02e-2 | 0.0394 |
| 0.01e-3 | 3 | 0.07e-3 | 0.0160 | 2 | 0.01e-2 | 0.0086 |
| 0.05e-4 | 3 | 0.74e-4 | 0.0081 | 3 | 0.57e-4 | 0.0082 |
| 0.01e-4 | 5 | 0.17e-4 | 0.0059 | 3 | 0.57e-4 | 0.0017 |
| 0.05e-5 | 6 | 0.85e-5 | 0.0056 | 3 | 5.75e-5 | 0.00094 |
| 0.01e-5 | 8 | 0.21e-5 | 0.0040 | 3 | 5.78e-5 | 0.00038 |

Finally, we analyze the influence of different noise types on the heuristic stopping rule. Gaussian (white) noise is defined by $\eta(t) = \frac{\xi(t)}{\|\xi\|}$, where $\xi(t)$ is a Gaussian random variable with zero mean and unit variance. Oscillatory noise is given by $\eta(t) = \frac{\sin(\omega\pi t)}{\|\sin(\omega\pi t)\|}$ with a sufficiently large frequency parameter $\omega$. Localized noise is defined as $\eta(t) = \frac{\chi_{(a,b)}(t)}{\|\chi_{(a,b)}\|}$, where $\chi_{(a,b)}$ denotes the characteristic function of a small subinterval $(a,b) \subset (0,1)$. Finally, smooth low-frequency noise is defined by $\eta(t) = \frac{\cos(2\pi t)}{\|\cos(2\pi t)\|}$. In all numerical experiments, the algorithmic parameters are kept fixed, and only the noise structure is varied.

The numerical results for Gaussian and oscillatory noise are presented in Table 5.3, while the results for localized and smooth noise are reported in Table 5.4. These tables show that the heuristic stopping rule performs robustly for Gaussian noise and smooth noise, whereas for oscillatory and localized noise, the stopping index becomes more sensitive, and the reconstruction error increases. This behavior is consistent with the known theoretical limitations of heuristic parameter choice rules.

TABLE 5.3
*Numerical results for Gaussian and oscillatory noise.*

| | Gaussian noise | | | Oscillatory noise | | |
|---|---|---|---|---|---|---|
| $\delta$ | $k_*$ | $\delta_*$ | error | $k_*$ | $\delta_*$ | error |
| 0.01e-2 | 2 | 0.32e-3 | 0.189 | 1 | 0.69e-3 | 0.0103 |
| 0.05e-3 | 2 | 0.22e-3 | 0.0095 | 2 | 0.23e-3 | 0.0097 |
| 0.01e-3 | 3 | 0.81e-4 | 0.0037 | 3 | 0.83e-4 | 0.0038 |
| 0.05e-4 | 3 | 0.78e-4 | 0.0019 | 3 | 0.78e-4 | 0.0019 |
| 0.01e-4 | 5 | 0.18e-4 | 0.0014 | 4 | 0.36e-4 | 0.00077 |
| 0.05e-5 | 5 | 0.17e-4 | 0.00070 | 5 | 0.17e-4 | 0.00071 |
| 0.01e-5 | 6 | 0.87e-5 | 0.00030 | 6 | 0.87e-5 | 0.00030 |

Overall, the numerical experiments are in good agreement with the theoretical analysis presented in this paper. They confirm that the heuristic stopping rule can select a reasonable stopping index without requiring prior knowledge of the noise level and that freezing the Fréchet derivative at the initial point does not significantly degrade the reconstruction quality. At the same time, the results clearly illustrate the limitations of heuristic stopping rules under certain noise structures, thereby providing a balanced and realistic numerical validation of the proposed method.

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TABLE 5.4
*Numerical results for localized and smooth noise.*

| | Localized noise | | | Smooth noise | | |
|---|---|---|---|---|---|---|
| $\delta$ | $k_*$ | $\delta_*$ | error | $k_*$ | $\delta_*$ | error |
| 0.01e-2 | 2 | 0.25e-3 | 0.0157 | 2 | 0.21e-3 | 0.0131 |
| 0.05e-3 | 2 | 0.20e-3 | 0.0080 | 3 | 0.85e-4 | 0.0129 |
| 0.01e-3 | 3 | 0.78e-4 | 0.0030 | 4 | 0.36e-4 | 0.0051 |
| 0.05e-4 | 4 | 0.37e-4 | 0.0027 | 5 | 0.17e-4 | 0.0050 |
| 0.01e-4 | 5 | 0.17e-4 | 0.00094 | 7 | 0.43e-5 | 0.0036 |
| 0.05e-5 | 5 | 0.17e-4 | 0.00049 | 8 | 0.21e-5 | 0.0034 |
| 0.01e-5 | 6 | 0.87e-5 | 0.00020 | 10 | 0.53e-6 | 0.0024 |

**Acknowledgment.** The author is grateful to Dr. Shashikant Kumar for valuable technical discussions and support during the revision process. The author also expresses gratitude to the anonymous reviewers for their constructive feedback, which significantly improved the quality of this work.

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