

FRACTAL APPROXIMATIONS ON THE REAL PROJECTIVE PLANE*

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Abstract. The study of the fractal theory in Euclidean spaces has recently emerged as an intriguing research area. The concept of fractal interpolation yields a method to approximate functions that are both self-affine or non-self-affine and consequently allows substantial flexibility and diversity of the fractal modeling problem. In this article, we introduce non-affine fractal functions on the non-Euclidean real projective plane. To do so, a real projective plane with a linear structure is considered. Then we study some classical approximation results for it. After considering a suitable iterated functions system (IFS) on the real projective plane, we construct non-affine fractal functions on it. Some fractal versions of classical approximation results are proved for the projective plane. Moreover, we prove that the attractor of an IFS on the dual space of the real projective plane is also the graph of a fractal function.

Key words. real projective iterated function system, fractal interpolation, α -fractal functions, approximations, dual fractal functions

AMS subject classifications. 28A80, 41A05

1. Introduction. Classical approximation theory is typically used to approximate smooth functions and sometimes even infinitely differentiable functions. However, in many cases, e.g., when dealing with real-world sampled signals like financial series, speech signals, heart rate signal graphs, etc., we need to cope with irregular curves. Traditional approximation techniques may be insufficient in providing good representations of irregular curves. Fractal functions serve as the foundational framework for a constructive approximation theory aimed at non-differentiable functions.

Barnsley [3] was the first to introduce a type of fractal functions as continuous functions that interpolate given data sets. There, the author's focus was on defining self-affine fractal functions. After that, Barnsley et al. [5] defined fractal functions that are both self-affine as well as non-self-affine in nature. Navascués [16] introduced a comprehensive method for defining non-affine fractal variations of classical trigonometric approximants. Subsequently, the author studied fractal polynomial interpolation in Euclidean spaces [15]. Approximation of a classical function by a fractal function has been developed both in theory and applications by many authors (see [1, 2, 17, 20, 22, 25, 26] and the references therein). Most of the authors have focused their research on fractal functions in Euclidean spaces [4, 11, 12, 18, 21]. Recently, Barnsley et al. [6] introduced an iterated function system (IFS) on the real projective space using projective transformations. Firstly, the authors characterized the conditions under which a projective IFS possesses an attractor that avoids a hyperplane. Secondly, the authors established the fact that a projective IFS can have at most one attractor. In the same article, the authors mentioned that the projective IFSs can be modeled in such a way that their attractors can be either smooth objects like arcs of circles and parabolas or rough objects like a fractal interpolation function (FIF). Also, Vince [23] studied Möbius IFSs consisting of Möbius transformations on the extended complex plane or, equivalently, on the Riemann sphere. There, the main result is a characterization, in terms of some topological, geometric, and dynamical properties, of Möbius IFSs that possess an attractor.

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Motivated by those works, in our earlier research [14] we presented the concept of a projective FIF on the real projective plane. Following that, we proceeded to ascertain the fractal dimension of the graph associated with a certain projective FIF [13]. In the real world, many geometric features such as tornadoes, radars, wormholes, and Boy's surfaces can be visualized as subsets of projective spaces. These shapes may not be well approximated by classical or existing fractal approximation techniques or even by an affine projective FIF. In this paper, we mainly investigate the non-affine real projective fractal function (RPF) on a real projective interval. After that, we study approximation results using the constructed non-affine RPF on a projective interval. Regarding this, we primarily establish classical approximation results on a projective interval.

Projective duality is a fundamental concept in projective geometry that relates points and hyperplanes in a one-to-one correspondence. This duality can be used to study projective IFSs and subsequently to generate fractals. Barnsley et al. [6] described the setup of the hyperplane IFS on the dual space of the projective space. The authors also provided some properties of the hyperplane IFS and proved the existence of its attractor. In this paper, we design an IFS on the dual of the real projective plane so that its attractor is the graph of a non-affine fractal function. However, our configuration is used to create the dual real projective iterated function system (RPIFS), which represents a distinct approach compared to Barnsley's framework.

This article extends the concept of α -fractal function, studied in the references [15, 16, 17, 18, 19, 24], to the real projective space. The paper is organized as follows: in Section 2 we discuss some preliminaries of the topic. In Section 3, after setting up a linear structure on the projective plane (which avoids a hyperplane), we prove that the space of all projective polynomials on a projective interval is dense in the space of all continuous functions on it. In Section 4 we introduce the construction of non-affine RPF and prove linearity and boundedness of certain fractal operators. In Section 5 we study approximation results using the constructed non-affine RPF on a projective interval. Finally, in Section 6 we prove the existence of dual non-affine fractal functions on the dual of the projective plane. In general, the current article invites researchers to explore the celebrated fractal approximation theory in a non-Euclidean projective space.

2. Notations and preliminary results. To make this article easier to understand, we provide some basic definitions and notations in this section. The reader may consult the references [6, 8, 10].

Basic Notation

\mathbb{R}^n	: n -dimensional Euclidean space.
\mathbb{RP}^2	: Two dimensional real projective space.
\mathbb{H}_x	: Hyperplane in \mathbb{RP}^2 .
$\mathbb{RP}^2 \setminus \mathbb{H}_x$: Projective space which avoids the hyperplane \mathbb{H}_x .
$\mathbb{P}_{\mathbb{I} \times \{0\}}$: Projective interval.
$\mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$: Set of all continuous functions on $\mathbb{P}_{\mathbb{I} \times \{0\}}$.
$\mathcal{P}_n(\mathbb{P}_{\mathbb{I} \times \{0\}})$: Set of all projective polynomials of degree less than or equal to n on $\mathbb{P}_{\mathbb{I} \times \{0\}}$.
$\mathcal{P}(\mathbb{P}_{\mathbb{I} \times \{0\}})$: Set of all projective polynomials on $\mathbb{P}_{\mathbb{I} \times \{0\}}$.
$\mathbb{RP}^2 \setminus \mathbb{H}_x$: Dual space of $\mathbb{RP}^2 \setminus \mathbb{H}_x$.

2.1. Projective space. A projective space over the field \mathbb{F} is a triplet $(\mathbb{P}, \mathbf{V}, \pi)$, where \mathbb{P} is a set, \mathbf{V} is a finite-dimensional vector space over the field \mathbb{F} , $\dim \mathbf{V} \geq 2$, and π is a map

$$\pi : \mathbf{V} \setminus \{\mathbf{0}\} \rightarrow \mathbb{P}$$

satisfying:

1. π is onto,
2. for any $x, y \in \mathbf{V} \setminus \{\mathbf{0}\}$, $\pi(x) = \pi(y)$ if and only if there is a nonzero $\lambda \in \mathbb{F}$ such that $x = \lambda y$.

Equivalently, in the above situation it is said that \mathbf{V} and π (or just π) define a structure of a projective space on \mathbb{P} . The elements of \mathbb{P} are called points and in some cases also elements of the projective space [8].

In particular, if we consider $\mathbb{F} = \mathbb{R}$, $\mathbf{V} = \mathbb{R}^3$, \mathbb{P} the set of all one-dimensional subspaces or (vector) lines in \mathbb{R}^3 , and for $x \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, $\pi(x) = l_x \in \mathbb{P}$, where l_x is the line that passes through the origin and the point x , then the triplet $(\mathbb{P}, \mathbf{V}, \pi)$ is known as the real projective plane, simply denoted by \mathbb{RP}^2 . Another way of looking at this construction is to define a group action of the multiplicative group \mathbb{R}^* (nonzero reals) on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ by scalar multiplication. Then, $\mathbb{RP}^2 \cong (\mathbb{R}^3 \setminus \{\mathbf{0}\})/\mathbb{R}^*$.

The equivalence class of a point (x_1, x_2, x_3) in \mathbb{RP}^2 is customarily denoted by $(x_1 : x_2 : x_3)$. Throughout the sequel, we consider the real projective plane \mathbb{RP}^2 . Take the canonical map

$$\nu : \mathbb{R}^3 \setminus \{\mathbf{0}\} \rightarrow \mathbb{RP}^2$$

that maps each nonzero vector $(x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ to its class element $(x_1 : x_2 : x_3) \in \mathbb{RP}^2$. The points $(x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ with $\nu(x_1, x_2, x_3) = x$ are referred to as the homogeneous coordinates of an element $x \in \mathbb{RP}^2$. If $x, y \in \mathbb{RP}^2$ have the homogeneous coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3) , respectively, and $\sum_{k=1}^3 x_k y_k = 0$, then we say that x is orthogonal to y and write $x \perp y$.

DEFINITION 2.1 (see [6, 10]). A hyperplane in \mathbb{RP}^2 is a set of the form

$$\mathbb{H}_x = \{y \in \mathbb{RP}^2 : x \perp y\} \subseteq \mathbb{RP}^2$$

for some $x \in \mathbb{RP}^2$.

DEFINITION 2.2 (see [6]). A set $\mathbb{K} \subseteq \mathbb{RP}^2$ is said to avoid a hyperplane if there exists a hyperplane $\mathbb{H}_x \subseteq \mathbb{RP}^2$ such that $\mathbb{H}_x \cap \mathbb{K} = \emptyset$.

2.2. Hausdorff metric and iterated function systems. In this section, we introduce the concept of real projective iterated function systems.

DEFINITION 2.3 (Hausdorff metric). Let (X, d) be a metric space and $K \subset X$, and let $\mathcal{H}(K)$ denote the space of all nonempty compact subsets of K . Then the Hausdorff distance between two elements A and B in $\mathcal{H}(K)$, denoted by H_d , is defined by

$$H_d(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y) \right\}, \quad \text{for all } A, B \in \mathcal{H}(K).$$

If (K, d) is a complete metric space, then $(\mathcal{H}(K), H_d)$ is also a complete metric space [4, 11].

DEFINITION 2.4 (see [4]). For a given complete metric space (X, d) , consider the continuous maps $W_n : X \rightarrow X$, for $n = 1, 2, \dots, N$. Then, the collection

$$\mathcal{I} = \{X; W_n : n = 1, 2, \dots, N\}$$

is called an iterated function system (IFS). If each function $W_n : X \rightarrow X$ is contractive with contraction factor $0 \leq c_n < 1$, then the system is called a hyperbolic IFS.

The Hutchinson operator $W : \mathcal{H}(K) \rightarrow \mathcal{H}(K)$ is defined by

$$W(B) = \bigcup_{n=1}^m W_n(B), \quad \text{for all } B \in \mathcal{H}(K).$$

It can be seen that W is a contraction map with respect to the metric H_d with contractivity factor $c = \max_n \{c_n : n = 1, 2, \dots, N\}$ whenever each W_n is a contraction map on (X, d) with contractivity factors c_n , for $n = 1, 2, \dots, N$. Hence, by the Banach fixed point theorem, W has a unique fixed point F , which is called an *attractor* of the IFS $\mathcal{I} = \{X; W_n : n \in N_m\}$. In particular, if the W_n are the projective transformations on the real projective plane, then $\mathcal{I} = \{\mathbb{RP}^2; W_n : n = 1, 2, \dots, N\}$ is called a *real projective iterated function system* or RPIFS [6].

3. Approximation on the real projective plane. In this section, we mainly prove results concerning the approximation of functions on the real projective plane. To do so, we first provide a linear structure of a real projective plane that avoids a hyperplane. For more details, the reader may consult the reference [14].

3.1. Decomposition of the real projective plane. The decomposition of the projective plane is essential for the subsequent sections in order to establish and prove the results. Let $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ be the standard order basis of \mathbb{R}^3 , and let \mathbb{H}_{e_i} be the hyperplane perpendicular to e_i , for $i = 1, 2, 3$. One may consider the space $\mathbb{RP}^2 \setminus \mathbb{H}_{e_i}$, which avoids the hyperplane \mathbb{H}_{e_i} , for $i = 1, 2, 3$, respectively. In the sequel, we in particular take the space $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$, where \mathbb{H}_{e_3} (that is, the xy -plane in \mathbb{R}^3) denotes the hyperplane orthogonal to e_3 (that is, the z -axis in \mathbb{R}^3) and define two operations \oplus and \odot as follows: For all $(x : y : z), (x' : y' : z') \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ and for all $a \in \mathbb{R}$,

$$\begin{aligned} (x : y : z) \oplus (x' : y' : z') &:= (xz' + x'z : yz' + y'z : zz'), \\ a \odot (x : y : z) &:= (ax : ay : z), \end{aligned}$$

where ‘+’ denotes the usual addition. Both the operations \oplus and \odot are well defined, and it can be easily verified that $(\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}, \oplus, \odot)$ forms a vector space over \mathbb{R} . Note that $(0 : 0 : 1)$ is the additive identity.

We use the notation \ominus to indicate the difference between two elements of $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$. That is, if $(x_1 : y_1 : z_1), (x_2 : y_2 : z_2) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$, then

$$(x_1 : y_1 : z_1) \ominus (x_2 : y_2 : z_2) := (x_1z_2 - x_2z_1 : y_1z_2 - y_2z_1 : z_1z_2),$$

where ‘−’ denotes the usual subtraction. So, each element $(x : y : z)$ in $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ can be expressed as $(x : y : z) = (x : 0 : z) \oplus (0 : y : z)$. Let $\mathbb{H} := \{(x : 0 : z) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}\}$ and $\mathbb{K} := \{(0 : y : z) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}\}$. Then, $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ can be expressed as

$$\mathbb{RP}^2 \setminus \mathbb{H}_{e_3} = \mathbb{H} \oplus \mathbb{K}.$$

Now, for all $(x : y : z) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$, define a real projective norm on $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ as

$$\|(x : y : z)\|_{\mathbb{P}} := \frac{\sqrt{x^2 + y^2}}{|z|}.$$

For all $(x : y : z), (x' : y' : z') \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$, the above norm induces a metric on $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$, called the real projective metric and denoted by $d_{\mathbb{P}}$, given by

$$d_{\mathbb{P}}((x : y : z), (x' : y' : z')) := \|(x : y : z) \ominus (x' : y' : z')\|_{\mathbb{P}}.$$

The metric space $(\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}, d_{\mathbb{P}})$ is complete. Some useful notations and an order relation on \mathbb{H} and \mathbb{K} , respectively, are as follows:

For $(x_1 : 0 : z_1), (x_2 : 0 : z_2) \in \mathbb{H}$, we say that

$$\begin{aligned} (x_1 : 0 : z_1) \preceq (x_2 : 0 : z_2) & \quad \text{if and only if } x_1 z_2 \leq x_2 z_1, \quad \text{and} \\ (x_1 : 0 : z_1) \prec (x_2 : 0 : z_2) & \quad \text{if and only if } x_1 z_2 < x_2 z_1. \end{aligned}$$

Similarly for $(0 : y_1 : z_1), (0 : y_2 : z_2) \in \mathbb{K}$, we define

$$\begin{aligned} (0 : y_1 : z_1) \preceq (0 : y_2 : z_2) & \quad \text{if and only if } y_1 z_2 \leq y_2 z_1, \quad \text{and} \\ (0 : y_1 : z_1) \prec (0 : y_2 : z_2) & \quad \text{if and only if } y_1 z_2 < y_2 z_1. \end{aligned}$$

Also, the product of two elements $(x_1 : y_1 : z_1)$ and $(x_2 : y_2 : z_2)$ in $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ is defined by

$$(x_1 : y_1 : z_1) \star (x_2 : y_2 : z_2) := (x_1 x_2 : y_1 y_2 : z_1 z_2).$$

DEFINITION 3.1 (Projective intervals on \mathbb{H} and \mathbb{K}). *Let $(u_1 : 0 : w_1), (u_2 : 0 : w_2) \in \mathbb{H}$ be such that $(u_1 : 0 : w_1) \prec (u_2 : 0 : w_2)$. Then, the projective interval on \mathbb{H} , denoted by $\mathbb{P}_{\mathbb{I} \times \{0\}}$, is defined as*

$$\mathbb{P}_{\mathbb{I} \times \{0\}} := \left\{ (x : 0 : z) \in \mathbb{H} : (u_1 : 0 : w_1) \preceq (x : 0 : z) \preceq (u_2 : 0 : w_2) \right\}.$$

Similarly, one can define the projective interval on \mathbb{K} .

DEFINITION 3.2 (Projective rectangle). *Let $(u_1 : 0 : w_1), (u_2 : 0 : w_2) \in \mathbb{H}$ and $(0 : y_1 : z_1), (0 : y_2 : z_2) \in \mathbb{K}$ be such that $(u_1 : 0 : w_1) \prec (u_2 : 0 : w_2)$ and $(0 : y_1 : z_1) \prec (0 : y_2 : z_2)$. Then, the projective rectangle on $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ is defined by*

$$\begin{aligned} \mathbb{P}_{\mathbb{I} \times \mathbb{J}} := \left\{ (x : y : z) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} : (u_1 : 0 : w_1) \preceq (x : 0 : z) \preceq (u_2 : 0 : w_2) \right. \\ \left. \text{and } (0 : y_1 : z_1) \preceq (0 : y : z) \preceq (0 : y_2 : z_2) \right\}. \end{aligned}$$

It is an easy exercise to show that the projective intervals and the projective rectangles are compact subsets of $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$. Let

$$\mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}] = \left\{ f : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{K} \text{ is continuous} \right\}.$$

Then, for $f \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$, define $\|f\|_{\mathbb{P}_{\infty}} := \sup\{\|f(x : 0 : z)\|_{\mathbb{P}} : (x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}\}$. The compactness of $\mathbb{P}_{\mathbb{I} \times \{0\}}$ ensures that $\|f\|_{\mathbb{P}_{\infty}}$ is well defined.

REMARK 3.3. The space $\mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ forms a normed linear space with addition defined by $(f \oplus g)(x : 0 : z) = f(x : 0 : z) \oplus g(x : 0 : z)$ and multiplication defined by $(\alpha \odot f)(x : 0 : z) = \alpha \odot f(x : 0 : z)$.

LEMMA 3.4 (see [14]). $(\mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}], \|\cdot\|_{\mathbb{P}_{\infty}})$ is a complete normed linear space.

3.2. Projective polynomial approximation on projective intervals. To state our main results, we at first define projective polynomials on the projective intervals. For $n \geq 0$, the n th-degree projective polynomial $p_n : \mathbb{H} \rightarrow \mathbb{K}$ is defined by

$$p_n(x : 0 : z) = (0 : a_n x^n + a_{n-1} x^{n-1} z + \cdots + a_1 x^1 z^{n-1} + a_0 z^n : z^n),$$

where $a_i \in \mathbb{R}$ and $a_n \neq 0$. Since $p_n(\lambda x : 0 : \lambda z) = p_n(x : 0 : z)$ for all $\lambda \neq 0$, the p_n are well defined. The projective polynomial p_n can also be expressed as

$$\begin{aligned} p_n(x : 0 : z) \\ = (a_n \odot (0 : x^n : z^n)) \oplus (a_{n-1} \odot (0 : x^{n-1} : z^{n-1})) \oplus \cdots \oplus (a_0 \odot (0 : 1 : 1)). \end{aligned}$$

Let $\mathcal{P}_n(\mathbb{H})$ denote the set of all projective polynomials of degree less than or equal to n , and let $\mathcal{P}(\mathbb{H}) := \bigcup_{n=1}^{\infty} \mathcal{P}_n(\mathbb{H})$.

REMARK 3.5.

1. The space $\mathcal{P}_n(\mathbb{H})$ forms a vector space with respect to the addition \oplus and multiplication \odot with the standard basis

$$\{(0 : 1 : 1), (0 : x : z), (0 : x^2 : z^2), \dots, (0 : x^n : z^n)\}.$$

The dimension of $\mathcal{P}_n(\mathbb{H})$ is $n + 1$.

2. The space $\mathcal{P}(\mathbb{H})$ is an infinite-dimensional vector space.

The Stone–Weierstrass theorem states that if f is a real-valued continuous function on a closed interval of \mathbb{R} , then f can be uniformly approximated by polynomials [7]. Now we are in a position to state a Stone–Weierstrass-type theorem for functions on the projective interval.

THEOREM 3.6 (Stone–Weierstrass-type theorem). *Let*

$$\mathbb{P}_{\mathbb{I} \times \{0\}} := \{(x : 0 : z) \in \mathbb{H} : (0 : 0 : 1) \preceq (x : 0 : z) \preceq (1 : 0 : 1)\}$$

and $f : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{K}$ be a continuous function. Then, there exists a sequence of projective polynomials $p_n : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{K}$ such that $p_n \rightarrow f$ uniformly.

Before proceeding to the proof, we provide some definitions and notations here.

DEFINITION 3.7 (Bernstein-type projective polynomials). *For each $n \geq 1$, we define the n th Bernstein-type projective polynomial of a function $f \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ as*

$$(3.1) \quad B_n[f](x : 0 : z) := \oplus \sum_{j=0}^n (0 : \binom{n}{j} x^j (z-x)^{n-j} : z^n) \star f(j : 0 : n),$$

where $\oplus \sum_{j=0}^n$ denotes the sum over \oplus from 0 to n .

The primary purpose of defining the Bernstein-type projective polynomials is to prove that they are uniformly convergent to the function f .

LEMMA 3.8. *Let $f : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{K}$ be any continuous function and $B_n[f]$ be the n th Bernstein-type projective polynomial of the function f . Then, for all $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$,*

$$\|f(x : 0 : z) \ominus B_n[f](x : 0 : z)\|_{\mathbb{P}} \leq \sum_{j=0}^n \|(f(x : 0 : z) \ominus f(j : 0 : n))\|_{\mathbb{P}} \|P_{nj}(x : 0 : z)\|_{\mathbb{P}},$$

where $P_{nj}(x : 0 : z) = (0 : \binom{n}{j} x^j (z-x)^{n-j} : z^n)$.

Proof. Since $P_{nj}(x : 0 : z) = (0 : \binom{n}{j} x^j (z - x)^{n-j} : z^n)$, equation (3.1) can be written as

$$(3.2) \quad B_n[f](x : 0 : z) := \oplus \sum_{j=0}^n P_{nj}(z : 0 : x) \star f(j : 0 : n).$$

Now we have

$$(3.3) \quad \begin{aligned} \oplus \sum_{j=0}^n P_{nj}(x : 0 : z) &= \oplus \sum_{j=0}^n (0 : \binom{n}{j} x^j (z - x)^{n-j} : z^n) \\ &= (0 : \sum_{j=0}^n \binom{n}{j} x^j (z - x)^{n-j} : z^n) \\ &= (0 : z^n : z^n) = (0 : 1 : 1). \end{aligned}$$

For $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$, using (3.2) and (3.3), we get

$$\begin{aligned} &f(x : 0 : z) \ominus B_n[f](x : 0 : z) \\ &= f(x : 0 : z) \star (0 : 1 : 1) \ominus B_n[f](x : 0 : z) \\ &= f(x : 0 : z) \star \left(\oplus \sum_{j=0}^n P_{nj}(x : 0 : z) \right) \ominus \left(\oplus \sum_{j=0}^n P_{nj}(x : 0 : z) \star f(j : 0 : n) \right) \\ &= \oplus \sum_{j=0}^n (f(x : 0 : z) \ominus f(j : 0 : n)) \star P_{nj}(x : 0 : z). \end{aligned}$$

Therefore,

$$\|f(x : 0 : z) \ominus B_n[f](x : 0 : z)\|_{\mathbb{P}} \leq \sum_{j=0}^n \|(f(x : 0 : z) \ominus f(j : 0 : n))\|_{\mathbb{P}} \|P_{nj}(x : 0 : z)\|_{\mathbb{P}}.$$

This completes the proof. \square

Proof of Theorem 3.6. Let $\epsilon > 0$. Since f is continuous on a compact set $\mathbb{P}_{\mathbb{I} \times \{0\}}$, f is uniformly continuous, and there exists a $\delta > 0$ such that for all $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$, with $\|(x : 0 : z) \ominus (j : 0 : n)\|_{\mathbb{P}} < \delta$, we have $\|f(x : 0 : z) \ominus f(j : 0 : n)\|_{\mathbb{P}} < \frac{\epsilon}{2}$. Now, for each $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$, split the points $(j : 0 : n)$ in $\mathbb{P}_{\mathbb{I} \times \{0\}}$ into two sets as follows:

$$\begin{aligned} K_1 &= \{j : j = 0, 1, \dots, n \text{ such that } \|(x : 0 : z) \ominus (j : 0 : n)\|_{\mathbb{P}} < \delta\}, \\ K_2 &= \{j : j = 0, 1, \dots, n \text{ such that } \|(x : 0 : z) \ominus (j : 0 : n)\|_{\mathbb{P}} \geq \delta\}. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{j \in K_1} \|(f(x : 0 : z) \ominus f(j : 0 : n))\|_{\mathbb{P}} \|P_{nj}(x : 0 : z)\|_{\mathbb{P}} &\leq \frac{\epsilon}{2} \sum_{j \in K_1} \|P_{nj}(x : 0 : z)\|_{\mathbb{P}} \\ &\leq \frac{\epsilon}{2} \sum_{j=0}^n \|P_{nj}(x : 0 : z)\|_{\mathbb{P}}. \end{aligned}$$

Now, from the definition of P_{nj} , we get

$$\sum_{j=0}^n \|P_{nj}(x : 0 : z)\|_{\mathbb{P}} = \sum_{j=0}^n \binom{n}{j} \frac{x^j (z - x)^{n-j}}{z^n} = 1.$$

Consequently,

$$(3.4) \quad \sum_{j \in K_1} \|(f(x : 0 : z) \ominus f(j : 0 : n))\|_{\mathbb{P}} \|P_{nj}(x : 0 : z)\|_{\mathbb{P}} \leq \frac{\epsilon}{2}.$$

Since f is bounded on $\mathbb{P}_{\mathbb{I} \times \{0\}}$, set $M := \|f\|_{\mathbb{P}_{\infty}}$. Then,

$$\begin{aligned} & \sum_{j \in K_2} \|(f(x : 0 : z) \ominus f(j : 0 : n))\|_{\mathbb{P}} \|P_{nj}(x : 0 : z)\|_{\mathbb{P}} \\ & \leq 2M \sum_{j \in K_2} \|P_{nj}(x : 0 : z)\|_{\mathbb{P}} \leq 2M \sum_{j=0}^n \|P_{nj}(x : 0 : z)\|_{\mathbb{P}} \\ & \leq 2M \sum_{j=0}^n \|P_{nj}(x : 0 : z)\|_{\mathbb{P}} \left(\frac{\|(x : 0 : z) \ominus (j : 0 : n)\|_{\mathbb{P}}}{\delta} \right)^2 \\ & = \frac{2M}{\delta^2} \sum_{j=0}^n \|P_{nj}(x : 0 : z)\|_{\mathbb{P}} (\|(nx - jz : 0 : nz)\|_{\mathbb{P}})^2 \\ & = \frac{2M}{\delta^2} \sum_{j=0}^n \binom{n}{j} \frac{x^j (z-x)^{n-j}}{z^n} \left(\frac{x}{z} - \frac{j}{n} \right)^2 \\ & = \frac{2M}{\delta^2} \sum_{j=0}^n \binom{n}{j} \left(\frac{x}{z} \right)^j \left(1 - \frac{x}{z} \right)^{n-j} \left(\left(\frac{x}{z} \right)^2 - 2 \frac{x}{z} \frac{j}{n} + \left(\frac{j}{n} \right)^2 \right). \end{aligned}$$

For notational simplicity, we set $X = \frac{x}{z}$ and use the formula $\frac{j}{n} \binom{n}{j} = \binom{n-1}{j-1}$. Then,

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} \left(\frac{x}{z} \right)^j \left(1 - \frac{x}{z} \right)^{n-j} \left(\left(\frac{x}{z} \right)^2 - 2 \frac{x}{z} \frac{j}{n} + \left(\frac{j}{n} \right)^2 \right) \\ & = X^2 \sum_{j=0}^n \binom{n}{j} X^j (1-X)^{n-j} - 2X \sum_{j=0}^n \frac{j}{n} \binom{n}{j} X^j (1-X)^{n-j} \\ & \quad + \sum_{j=0}^n \left(\frac{j}{n} \right)^2 \binom{n}{j} X^j (1-X)^{n-j} \\ & = X^2 - 2X^2 \sum_{j=1}^n \binom{n-1}{j-1} X^{j-1} (1-X)^{(n-1)-(j-1)} \\ & \quad + \frac{X}{n} \sum_{j=1}^n j \binom{n-1}{j-1} X^{j-1} (1-X)^{(n-1)-(j-1)} \\ & = X^2 - 2X^2 + \frac{X(n-1)}{n} \sum_{j=1}^n \frac{j-1}{n-1} \binom{n-1}{j-1} X^{j-1} (1-X)^{(n-1)-(j-1)} + \frac{X}{n} \\ & = -X^2 + \frac{X^2(n-1)}{n} \sum_{j=2}^n \binom{n-2}{j-2} X^{j-2} (1-X)^{(n-2)-(j-2)} + \frac{X}{n} \\ & = -X^2 + \frac{X^2(n-1)}{n} + \frac{X}{n} = \frac{X(1-X)}{n} \leq \frac{1}{4n}, \end{aligned}$$

as $0 \leq X = \frac{x}{z} \leq 1$. Therefore,

$$(3.5) \quad \sum_{j \in K_2} \|(f(x : 0 : z) \ominus f(j : 0 : n))\|_{\mathbb{P}} \|P_{nj}(x : 0 : z)\|_{\mathbb{P}} \leq \frac{M}{2n\delta^2} \leq \frac{\epsilon}{2},$$

provided that $n > \frac{M}{\epsilon\delta^2}$. Hence, from (3.4), (3.5), and Lemma 3.8, we get

$$\|f(x : 0 : z) \ominus B_n[f](x : 0 : z)\|_{\mathbb{P}} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

if $n > \frac{M}{\epsilon\delta^2} = N_0$. Taking the supremum over all $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$, it follows that

$$\|f \ominus B_n[f]\|_{\mathbb{P}_{\infty}} \leq \epsilon,$$

for all $n > N_0$. This completes the proof. \square

REMARK 3.9. Any arbitrary closed projective interval is isomorphic to the projective interval $\mathbb{P}_{\mathbb{I} \times \{0\}}$. So, Theorem 3.6 can be extended to any arbitrary closed projective interval.

4. Non-affine real projective fractal functions. In Euclidean spaces, fractal functions form the basis of a constructive approximation theory for non-smooth functions. The non-affine fractal function (namely the α -fractal function), which is a perturbation of a classical function and in general non-differentiable, retains some of the properties of the germ function, like continuity and integrability. But, by suitably choosing the parameters of an IFS, it can also be made as regular as the classical one.

These developments delineate the fractal approximation theory, which serves as a link between the smoothness of a classical mathematical object and the pseudo-random nature of experimental data. In this section, we mainly introduce non-affine fractal interpolation schemes to construct non-affine fractal functions on the real projective plane. For a certain prescribed data set, by suitably defining an IFS on the real projective plane, we identify the corresponding fractal function as the fixed point of a Read-Bajraktarević operator.

4.1. Construction of non-affine fractal functions. Here, we provide the details of the construction of non-affine fractal functions. Let

$$\Delta_{\mathbb{P}} := \{(x_n : y_n : z_n) \in \mathbb{R}\mathbb{P}^2 \setminus \mathbb{H}_{e_3} : n = 0, 1, \dots, N\}$$

be a given set of points in $\mathbb{R}\mathbb{P}^2 \setminus \mathbb{H}_{e_3}$ satisfying the conditions

$$(x_0 : 0 : z_0) \prec (x_1 : 0 : z_1) \prec \dots \prec (x_N : 0 : z_N).$$

With a slight abuse of notation, we can write

$$\Delta_{\mathbb{P}} = \{(x_n : y_n : z_n) = (x_n : 0 : z_n) \oplus (0 : y_n : z_n) \in \mathbb{R}\mathbb{P}^2 \setminus \mathbb{H}_{e_3} : n = 0, 1, \dots, N\}.$$

Let

$$\mathbb{P}_{\mathbb{I} \times \{0\}} = \left\{ (x : 0 : z) \in \mathbb{H} : (x_0 : 0 : z_0) \preceq (x : 0 : z) \preceq (x_N : 0 : z_N) \right\}$$

be the closed projective interval containing the points $(x_n : 0 : z_n)$, for $n = 0, 1, \dots, N$, and let

$$\mathbb{P}_{\mathbb{I}_n \times \{0\}} = \left\{ (x : 0 : z) \in \mathbb{H} : (x_{n-1} : 0 : z_{n-1}) \preceq (x : 0 : z) \preceq (x_n : 0 : z_n) \right\}$$

be the corresponding projective subintervals. For $n = 1, 2, \dots, N$, let $L_n : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{P}_{\mathbb{I}_n \times \{0\}}$ be the contractive homeomorphisms given by

$$(4.1) \quad L_n(x : 0 : z) = (a_n x + b_n z : 0 : z)$$

such that

$$L_n(x_0 : 0 : z_0) = (x_{n-1} : 0 : z_{n-1}) \quad \text{and} \quad L_n(x_N : 0 : z_N) = (x_n : 0 : z_n).$$

Let $g : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{K}$ be a continuous function which passes through the data points. That is, $g(x_n : 0 : z_n) = (0 : y_n : z_n)$, for $n = 0, 1, \dots, N$.

Consider the continuous function $F_n : \mathbb{P}_{\mathbb{I} \times \{0\}} \oplus \mathbb{K} \rightarrow \mathbb{K}$,

$$(4.2) \quad F_n(x : y : z) = \alpha_n \odot (0 : y : z) \oplus g \circ L_n(x : 0 : z) \ominus (\alpha_n \odot b(x : 0 : z)),$$

where the α_n are scaling factors lying in $(-1, 1)$, $n = 0, 1, \dots, N$, and $b : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{K}$ is a continuous function such that

$$b(x_0 : 0 : z_0) = g(x_0 : 0 : z_0), \quad b(x_N : 0 : z_N) = g(x_N : 0 : z_N), \quad \text{with } b \neq g,$$

and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in (-1, 1)^N$ is the scale vector. Then, for $n = 1, 2, \dots, N$,

$$\begin{aligned} F_n(x_0 : y_0 : z_0) &= \alpha_n \odot (0 : y_0 : z_0) \oplus g \circ L_n(x_0 : 0 : z_0) \ominus (\alpha_n \odot b(x_0 : 0 : z_0)) \\ &= \alpha_n \odot (0 : y_0 : z_0) \oplus g(x_{n-1} : 0 : z_{n-1}) \ominus (\alpha_n \odot (0 : y_0 : z_0)) \\ &= (0 : y_{n-1} : z_{n-1}). \end{aligned}$$

Similarly,

$$F_n(x_N : y_N : z_N) = (0 : y_n : z_n).$$

That is, the functions F_n satisfy the so-called ‘join up’ conditions. Also,

$$\begin{aligned} &\|F_n((x : 0 : z) \oplus (0 : v : w)) \ominus F_n((x : 0 : z) \oplus (0 : v' : w'))\|_{\mathbb{P}} \\ &= \|\alpha_n \odot (0 : v : w) \ominus \alpha_n \odot (0 : v' : w')\|_{\mathbb{P}} = |\alpha_n| \|(0 : v : w) \ominus (0 : v' : w')\|_{\mathbb{P}} \\ &\leq \alpha_{\infty} \|(0 : v : w) \ominus (0 : v' : w')\|_{\mathbb{P}}, \end{aligned}$$

where $\alpha_{\infty} = \max\{|\alpha_n| \mid n = 1, 2, \dots, N\}$. If, for $n = 1, 2, \dots, N$, we have $|\alpha_n| < 1$, then $\alpha_{\infty} < 1$, and hence, the F_n are contractive with respect to the second variable. Now define the maps $W_n : \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} \rightarrow \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ by

$$(4.3) \quad W_n(x : y : z) = L_n(x : 0 : z) \oplus F_n(x : y : z).$$

Then, $\{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}; W_n : n = 1, 2, \dots, N\}$ is a RPIFS on the real projective plane $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$. By following the proof of Theorem 1 in [14], it can be observed that the RPIFS has an attractor G . Now, we prove that G is the graph of a fractal function from $\mathbb{P}_{\mathbb{I} \times \{0\}}$ to \mathbb{K} .

Let

$$\mathcal{G} := \{f \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}] : f(x_0 : 0 : z_0) = (0 : y_0 : z_0), f(x_N : 0 : z_N) = (0 : y_N : z_N)\}.$$

Then, \mathcal{G} is a closed subset of $(\mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}], \|\cdot\|_{\mathbb{P}_{\infty}})$. Hence, \mathcal{G} is complete. Define a *real projective Read-Bajraktarević operator* (RPRB) $\mathcal{T} : \mathcal{G} \rightarrow \mathcal{G}$ such that

$$(4.4) \quad \mathcal{T}f(x : 0 : z) = F_n(L_n^{-1}(x : 0 : z) \oplus f \circ L_n^{-1}(x : 0 : z)),$$

where $L_n^{-1} : \mathbb{P}_{\mathbb{I}_n \times \{0\}} \rightarrow \mathbb{P}_{\mathbb{I} \times \{0\}}$ is given by $L_n^{-1}(x : 0 : z) = (x - b_n z : 0 : a_n z)$, for $(x : 0 : z) \in \mathbb{P}_{\mathbb{I}_n \times \{0\}}$ and $n = 1, 2, \dots, N$. Now, from (4.2) and (4.4), \mathcal{T} can be written as

$$(4.5) \quad \mathcal{T}f(x : 0 : z) = g(x : 0 : z) \oplus \alpha_n \odot (f \ominus b) \circ L_n^{-1}(x : 0 : z).$$

THEOREM 4.1. *The RPRB-operator \mathcal{T} is well defined.*

Proof. Let $f \in \mathcal{G}$. Then,

$$\begin{aligned} \mathcal{T}f(x_0 : 0 : z_0) &= g(x_0 : 0 : z_0) \oplus \alpha_1 \odot (f \ominus b) \circ L_1^{-1}(x_0 : 0 : z_0) \\ &= (0 : y_0 : z_0) \oplus \alpha_n \odot (f \ominus b)(x_0 : 0 : z_0). \end{aligned}$$

Since $b(x_0 : 0 : z_0) = f(x_0 : 0 : z_0)$, it follows that $\mathcal{T}f(x_0 : 0 : z_0) = (0 : y_0 : z_0)$. Similarly, $\mathcal{T}f(x_N : 0 : z_N) = (0 : y_N : z_N)$.

Now $(x_n : 0 : z_n) \in \mathbb{P}_{\mathbb{I}_n \times \{0\}} \cap \mathbb{P}_{\mathbb{I}_{n+1} \times \{0\}}$. So, whenever $(x_n : 0 : z_n) \in \mathbb{P}_{\mathbb{I}_n \times \{0\}}$,

$$\begin{aligned} \mathcal{T}f(x_n : 0 : z_n) &= g(x_n : 0 : z_n) \oplus \alpha_n \odot (f \ominus b) \circ L_n^{-1}(x_n : 0 : z_n) \\ &= (0 : y_n : z_n) \oplus \alpha_n \odot (f \ominus b)(x_n : 0 : z_n). \end{aligned}$$

Since $b(x_n : 0 : z_n) = f(x_n : 0 : z_n)$, it follows that $\mathcal{T}f(x_n : 0 : z_n) = (0 : y_n : z_n)$. Also, whenever $(x_n : 0 : z_n) \in \mathbb{P}_{\mathbb{I}_{n+1} \times \{0\}}$,

$$\begin{aligned} \mathcal{T}f(x_n : 0 : z_n) &= g(x_n : 0 : z_n) \oplus \alpha_{n+1} \odot (f \ominus b) \circ L_{n+1}^{-1}(x_n : 0 : z_n) \\ &= (0 : y_n : z_n) \oplus \alpha_{n+1} \odot (f \ominus b)(x_n : 0 : z_n) \\ &= (0 : y_n : z_n). \end{aligned}$$

Hence \mathcal{T} is well defined. \square

THEOREM 4.2. *The RPRB-operator \mathcal{T} is contractive on $(\mathcal{G}, \|\cdot\|_{\mathbb{P}\infty})$.*

Proof. Let $f_1, f_2 \in \mathcal{G}$. Then, for $(x : 0 : z) \in \mathbb{P}_{\mathbb{I}_n \times \{0\}}$, from (4.5),

$$\begin{aligned} &\|(\mathcal{T}f_1)(x : 0 : z) \ominus (\mathcal{T}f_2)(x : 0 : z)\|_{\mathbb{P}} \\ &= \|\alpha_n \odot f_1 \circ L_n^{-1}(x : 0 : z) \ominus \alpha_n \odot f_2 \circ L_n^{-1}(x : 0 : z)\|_{\mathbb{P}} \\ &= |\alpha_n| \|f_1 \circ L_n^{-1}(x : 0 : z) \ominus f_2 \circ L_n^{-1}(x : 0 : z)\|_{\mathbb{P}}. \end{aligned}$$

Since $L_n^{-1}(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$,

$$\|(\mathcal{T}f_1)(x : 0 : z) \ominus (\mathcal{T}f_2)(x : 0 : z)\|_{\mathbb{P}} \leq |\alpha_n| \|f_1 \ominus f_2\|_{\mathbb{P}\infty} \leq \alpha_\infty \|f_1 \ominus f_2\|_{\mathbb{P}\infty},$$

where $\alpha_\infty = \max\{|\alpha_n| \mid n = 1, 2, \dots, N\} < 1$. Now, taking the supremum over all elements $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$, we get

$$\|\mathcal{T}f_1 \ominus \mathcal{T}f_2\|_{\mathbb{P}\infty} \leq \alpha_\infty \|f_1 \ominus f_2\|_{\mathbb{P}\infty}.$$

This shows that the RPRB-operator \mathcal{T} is contractive on $(\mathcal{G}, \|\cdot\|_{\mathbb{P}\infty})$. \square

Since $(\mathcal{G}, \|\cdot\|_{\mathbb{P}\infty})$ is complete, by the Banach fixed point theorem, \mathcal{T} has a unique fixed point g^α in \mathcal{G} . We call g^α the non-affine *real projective fractal function* (RPF). From (4.5), g^α satisfies the self-referential equation

$$(4.6) \quad g^\alpha(x : 0 : z) = g(x : 0 : z) \oplus (\alpha_n \odot (g^\alpha \ominus b) \circ L_n^{-1}(x : 0 : z)),$$

whenever $(x : 0 : z) \in \mathbb{P}_{\mathbb{I}_n \times \{0\}}$.

THEOREM 4.3. *The graph of the non-affine RPF g^α is the attractor of the RPIFS defined in (4.3). That is, $\text{graph}(g^\alpha) = G$.*

Proof. Let $\tilde{G} = \text{graph}(g^\alpha) := \{(x : 0 : z) \oplus g^\alpha(x : 0 : z) : (x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}\}$. Then,

$$(4.7) \quad \bigcup_{n=1}^N W_n(\tilde{G}) = \bigcup_{n=1}^N \{W_n((x : 0 : z) \oplus g^\alpha(x : 0 : z)) : (x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}\}.$$

Now, using (4.3), for $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$, we have

$$(4.8) \quad W_n((x : 0 : z) \oplus g^\alpha(x : 0 : z)) = L_n(x : 0 : z) \oplus F_n((x : 0 : z) \oplus g^\alpha(x : 0 : z)).$$

Also, since g^α is the fixed point of the RPRB operator \mathcal{T} , we obtain from (4.4),

$$(4.9) \quad g^\alpha(L_n(x : 0 : z)) = \mathcal{T}g^\alpha(L_n(x : 0 : z)) = F_n((x : 0 : z) \oplus g^\alpha(x : 0 : z)).$$

Using (4.8) and (4.9),

$$(4.10) \quad W_n((x : 0 : z) \oplus g^\alpha(x : 0 : z)) = L_n(x : 0 : z) \oplus g^\alpha(L_n(x : 0 : z)).$$

Therefore, using (4.7) and (4.10), it follows that

$$\begin{aligned} \bigcup_{n=1}^N W_n(\tilde{G}) &= \bigcup_{n=1}^N \{L_n(x : 0 : z) \oplus g^\alpha(L_n(x : 0 : z)) : (x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}\} \\ &= \bigcup_{n=1}^N \{(x : 0 : z) \oplus g^\alpha(x : 0 : z) : (x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}\} \\ &= \{(x : 0 : z) \oplus g^\alpha(x : 0 : z) : (x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}\} \\ &= \tilde{G}. \end{aligned}$$

Hence by uniqueness of the attractor, $\tilde{G} = G$. This completes the proof. \square

EXAMPLE 4.4. Here, we provide a graphical illustration of non-affine RPFs. For $N = 2$, consider the data set $\{(-1 : 1 : 1), (0 : 0 : 1), (1 : 1 : 1)\}$. Then, the projective interval is

$$\mathbb{P}_{\mathbb{I} \times \{0\}} := \{(x : 0 : z) \in \mathbb{H} : (-1 : 0 : 1) \preceq (x : 0 : z) \preceq (1 : 0 : 1)\},$$

and the corresponding projective subintervals are

$$\mathbb{P}_{\mathbb{I}_1 \times \{0\}} := \{(x : 0 : z) \in \mathbb{H} : (-1 : 0 : 1) \preceq (x : 0 : z) \preceq (0 : 0 : 1)\},$$

$$\mathbb{P}_{\mathbb{I}_2 \times \{0\}} := \{(x : 0 : z) \in \mathbb{H} : (0 : 0 : 1) \preceq (x : 0 : z) \preceq (1 : 0 : 1)\}.$$

To avoid difficulties in obtaining the figures, we consider the germ function $g : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{K}$ given by $g(x : 0 : z) = (0 : x^2 : z^2)$ and the base function $b : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{K}$ given by $b(x : 0 : z) = (0 : x^4 : z^4)$. Then, for the scale vector $(-0.5, 0.5)$, the corresponding RPIFS is generated by

$$L_1(x : 0 : z) = (0.5x - 0.5z : 0 : z),$$

$$L_2(x : 0 : z) = (0.5x + 0.5z : 0 : z)$$

and

$$F_1(x : y : z) = (0 : -0.5yz^3 + (0.5x - 0.5z)^2z^2 + 0.5x^4 : z^4),$$

$$F_2(x : y : z) = (0 : 0.5yz^3 + (0.5x + 0.5z)^2z^2 - 0.5x^4 : z^4).$$

The graphs of the classical function g and its corresponding non-affine RPF g^α are now displayed in Figure 4.1. Also, for the scale vector $(0.6, -0.2)$, the corresponding RPIFS is generated by the above $L_n(x : 0 : z)$, $n = 1, 2$, and

$$F_1(x : y : z) = (0 : 0.6yz^3 + (0.5x - 0.5z)^2z^2 - 0.6x^4 : z^4),$$

$$F_2(x : y : z) = (0 : -0.2yz^3 + (0.5x + 0.5z)^2z^2 + 0.2x^4 : z^4).$$

The graph of the corresponding non-affine RPF g^α is given in Figure 4.2.

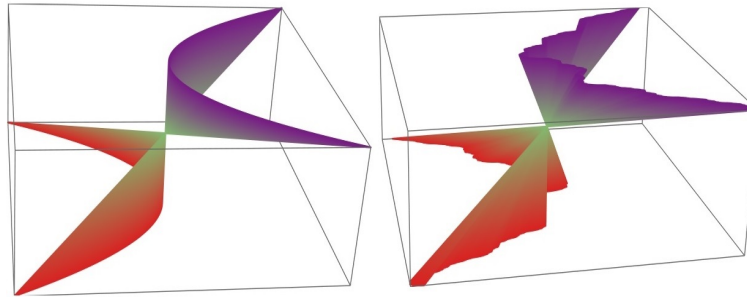


FIG. 4.1. Graphs of the classical function (left) and the corresponding non-affine RPF (right) with the scale vector $(-0.5, 0.5)$.

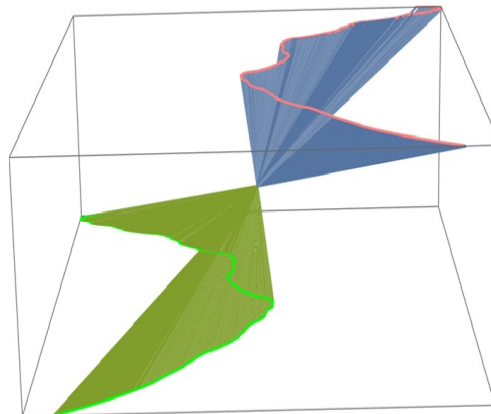


FIG. 4.2. Graphs of the corresponding non-affine RPF with the scale vector $(0.6, -0.2)$.

4.2. The projective fractal operator. In this context, we introduce a projective fractal operator and demonstrate its continuity, linearity, and boundedness. For a fixed partition $\Delta_{\mathbb{P}}$ and a continuous function b , define

$$\mathcal{P}_{\Delta_{\mathbb{P}}, b}^\alpha : \mathcal{G} \rightarrow \mathcal{G}$$

such that $\mathcal{P}_{\Delta_{\mathbb{P}}, b}^\alpha(g) = g^\alpha$. We call the operator $\mathcal{P}_{\Delta_{\mathbb{P}}, b}^\alpha$ the *projective fractal operator*. Although $\mathcal{P}_{\Delta_{\mathbb{P}}, b}^\alpha$ depends on $\Delta_{\mathbb{P}}$ and b , we omit the subindices and use the notation \mathcal{P}^α for

simplicity. Now, from (4.6), we get

$$\begin{aligned} \|g^\alpha(x : 0 : z) \ominus g(x : 0 : z)\|_{\mathbb{P}} &= \|\alpha_n \odot (g^\alpha \ominus b) \circ L_n^{-1}(x : 0 : z)\|_{\mathbb{P}} \\ &= |\alpha_n| \| (g^\alpha \ominus b) \circ L_n^{-1}(x : 0 : z) \|_{\mathbb{P}} \leq \alpha_\infty \|g^\alpha \ominus b\|_{\mathbb{P}\infty}. \end{aligned}$$

Therefore,

$$(4.11) \quad \|g^\alpha \ominus g\|_{\mathbb{P}\infty} \leq \alpha_\infty \|g^\alpha \ominus b\|_{\mathbb{P}\infty} \leq \alpha_\infty (\|g^\alpha \ominus g\|_{\mathbb{P}\infty} + \|g \ominus b\|_{\mathbb{P}\infty}).$$

Hence,

$$(4.12) \quad \|g^\alpha \ominus g\|_{\mathbb{P}\infty} \leq \frac{\alpha_\infty}{1 - \alpha_\infty} \|g \ominus b\|_{\mathbb{P}\infty}.$$

Now, if $\alpha_n = 0$, for $n = 1, 2, \dots, N$, then $\alpha_\infty = 0$. So in this case the fractal function g^α is nothing but the classical function g .

PROPOSITION 4.5. *For fixed $\Delta_{\mathbb{P}}$ and b , \mathcal{P}^α satisfies the Lipschitz condition*

$$\|\mathcal{P}^\alpha(g) \ominus \mathcal{P}^\alpha(h)\|_{\mathbb{P}\infty} \leq \frac{1}{1 - \alpha_\infty} \|g \ominus h\|_{\mathbb{P}\infty}.$$

Proof. Let $\mathcal{P}_{\Delta_{\mathbb{P}}, b}^\alpha(g) = g^\alpha$ and $\mathcal{P}_{\Delta_{\mathbb{P}}, b}^\alpha(h) = h^\alpha$. From (4.6), we have for all $(x : 0 : z) \in \mathbb{P}_{\mathbb{I}_n \times \{0\}}$,

$$\begin{aligned} g^\alpha(x : 0 : z) &= g(x : 0 : z) \oplus \alpha_n \odot (g^\alpha \ominus b) \circ L_n^{-1}(x : 0 : z), \\ h^\alpha(x : 0 : z) &= h(x : 0 : z) \oplus \alpha_n \odot (h^\alpha \ominus b) \circ L_n^{-1}(x : 0 : z). \end{aligned}$$

Therefore,

$$(4.13) \quad \begin{aligned} g^\alpha(x : 0 : z) \ominus h^\alpha(x : 0 : z) \\ = (g \ominus h)(x : 0 : z) \oplus \alpha_n \odot (g^\alpha \ominus h^\alpha) \circ L_n^{-1}(x : 0 : z). \end{aligned}$$

Then,

$$\begin{aligned} \|g^\alpha(x : 0 : z) \ominus h^\alpha(x : 0 : z)\|_{\mathbb{P}} \\ \leq \|(g \ominus h)(x : 0 : z)\|_{\mathbb{P}} + |\alpha_n| \| (g^\alpha \ominus h^\alpha) \circ L_n^{-1}(x : 0 : z) \|_{\mathbb{P}} \\ \leq \|g \ominus h\|_{\mathbb{P}\infty} + \alpha_\infty \|g^\alpha \ominus h^\alpha\|_{\mathbb{P}\infty}. \end{aligned}$$

Taking the supremum over all $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$, we get

$$\|g^\alpha \ominus h^\alpha\|_{\mathbb{P}\infty} \leq \|g \ominus h\|_{\mathbb{P}\infty} + \alpha_\infty \|g^\alpha \ominus h^\alpha\|_{\mathbb{P}\infty}.$$

The result now follows immediately. \square

THEOREM 4.6. *\mathcal{P}^α is continuous on \mathcal{G} .*

Proof. This follows from Proposition 4.5. \square

PROPOSITION 4.7. *Let $\alpha_m = (\alpha_1^m, \alpha_2^m, \dots, \alpha_N^m) \in \mathbb{R}^N$ be such that $|\alpha_m|_\infty < 1$ and $\alpha_m \rightarrow 0$ as $m \rightarrow \infty$. Then, $\mathcal{P}^\alpha(g) \rightarrow g$ uniformly as $m \rightarrow \infty$.*

Proof. From (4.12),

$$\|\mathcal{P}^\alpha(g) \ominus g\|_{\mathbb{P}\infty} \leq \frac{|\alpha_m|_\infty}{1 - |\alpha_m|_\infty} \|g \ominus b\|_{\mathbb{P}\infty}.$$

As $m \rightarrow \infty$, the condition $|\alpha_m|_\infty \rightarrow 0$ implies that $\mathcal{P}^\alpha(g) \rightarrow g$ uniformly. \square

We say that a function $h : \mathbb{H} \rightarrow \mathbb{K}$ is monotonically increasing if the condition $(x_1 : 0 : z_1) \preceq (x_2 : 0 : z_2)$ implies $h(x_1 : 0 : z_1) \preceq h(x_2 : 0 : z_2)$. Now in (4.2), consider $b = g \circ s$, where $s : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{P}_{\mathbb{I} \times \{0\}}$ is a continuous, monotonically increasing function such that

$$s(x_0 : 0 : z_0) = (x_0 : 0 : z_0) \quad \text{and} \quad s(x_N : 0 : z_N) = (x_N : 0 : z_N).$$

Then, the corresponding operator \mathcal{P}^α is known as the *projective fractal linear operator*, and we denoted it by $\mathcal{F}^\alpha := \mathcal{F}_{\Delta_{\mathbb{P}}, g \circ s}^\alpha$.

THEOREM 4.8. *For fixed $\Delta_{\mathbb{P}}$ and s , the operator \mathcal{F}^α is a bounded linear operator.*

Proof. First, we verify the linearity of the operator \mathcal{F}^α . For all $\lambda, \mu \in \mathbb{R}$, using (4.13), we get

$$\begin{aligned} & \lambda \odot g^\alpha \oplus \mu \odot h^\alpha \\ &= (\lambda \odot g \oplus \mu \odot h) \oplus \alpha_n \odot ((\lambda \odot g^\alpha \oplus \mu \odot h^\alpha) \ominus (\lambda \odot g \oplus \mu \odot h) \circ s) \circ L_n^{-1}. \end{aligned}$$

Also, from (4.6), we get

$$\begin{aligned} & (\lambda \odot g \oplus \mu \odot h)^\alpha \\ &= (\lambda \odot g \oplus \mu \odot h) \oplus \alpha_n \odot ((\lambda \odot g \oplus \mu \odot h)^\alpha \ominus (\lambda \odot g \oplus \mu \odot h) \circ s) \circ L_n^{-1}. \end{aligned}$$

The uniqueness of the solution of the fixed point equation defining the RPFY yields

$$(\lambda \odot g \oplus \mu \odot h)^\alpha = \lambda \odot g^\alpha \oplus \mu \odot h^\alpha.$$

Therefore,

$$\mathcal{F}^\alpha(\lambda \odot g \oplus \mu \odot h) = \lambda \odot \mathcal{F}^\alpha(g) \oplus \mu \odot \mathcal{F}^\alpha(h).$$

This shows that \mathcal{F}^α is linear.

Next, we verify the boundedness of the operator \mathcal{F}^α . From (4.12), for $b = g \circ s$, we get

$$(4.14) \quad \|\mathcal{F}^\alpha(g) \ominus g\|_{\mathbb{P}_\infty} \leq \frac{2\alpha_\infty}{1 - \alpha_\infty} \|g\|_{\mathbb{P}_\infty}.$$

From (4.14), it is clear that

$$\|\mathcal{F}^\alpha(g)\|_{\mathbb{P}_\infty} \leq \frac{2\alpha_\infty}{1 - \alpha_\infty} \|g\|_{\mathbb{P}_\infty} + \|g\|_{\mathbb{P}_\infty} \leq \frac{1 + \alpha_\infty}{1 - \alpha_\infty} \|g\|_{\mathbb{P}_\infty},$$

and as a consequence,

$$\|\mathcal{F}^\alpha\| \leq \frac{1 + \alpha_\infty}{1 - \alpha_\infty},$$

where $\|\mathcal{F}^\alpha\|$ represents the operator norm of \mathcal{F}^α . Hence, \mathcal{F}^α is a bounded linear operator. \square

5. Projective fractal uniform approximation. Using the results of the earlier sections, we now mainly prove fractal versions of some approximation results on the real projective plane. Also, we establish several properties of the projective fractal operator.

From now onward, we consider the general case

$$(5.1) \quad b = \mathbb{L}(f),$$

where $\mathbb{L} : \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}] \rightarrow \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ is a linear bounded operator with respect to the uniform norm $\|\cdot\|_{\mathbb{P}_{\infty}}$ such that

$$\mathbb{L}(f)(x_0 : 0 : z_0) = f(x_0 : 0 : z_0) \quad \text{and} \quad \mathbb{L}(f)(x_N : 0 : z_N) = f(x_N : 0 : z_N).$$

DEFINITION 5.1. Let g^α be the non-affine RPF generated by the IFS (4.1), (4.2), and (5.1).

By the idiomatic expression (4.6), g^α satisfies the self-referential equation

$$(5.2) \quad g^\alpha(x : 0 : z) = g(x : 0 : z) \oplus (\alpha_n \odot (g^\alpha \ominus \mathbb{L}(g)) \circ L_n^{-1}(x : 0 : z)),$$

whenever $(x : 0 : z) \in \mathbb{P}_{\mathbb{I}_n \times \{0\}}$.

LEMMA 5.2. For all $g \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$,

$$\|g^\alpha \ominus g\|_{\mathbb{P}_{\infty}} \leq \frac{\alpha_\infty}{1 - \alpha_\infty} \|g \ominus \mathbb{L}(g)\|_{\mathbb{P}_{\infty}}.$$

Proof. This follows from (4.12). \square

THEOREM 5.3. For any $g \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$,

$$\|(\mathcal{F}^\alpha \ominus \text{Id})(g)\|_{\mathbb{P}_{\infty}} \leq \frac{\alpha_\infty \|\text{Id} \ominus \mathbb{L}\|}{1 - \alpha_\infty} \|g\|_{\mathbb{P}_{\infty}},$$

where $\text{Id} : \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}] \rightarrow \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ represents the identity operator.

Proof. By using Lemma 5.2, the proof follows immediately. \square

Recall that $\mathcal{P}_m[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ is the set of all projective polynomials $p_m : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{K}$ of degree less than or equal to m and $\mathcal{P}[\mathbb{P}_{\mathbb{I} \times \{0\}}] := \bigcup_{m=1}^{\infty} \mathcal{P}_m[\mathbb{P}_{\mathbb{I} \times \{0\}}]$.

REMARK 5.4. The set $\{(0 : 1 : 1), (0 : x : z), (0 : x^2 : z^2), (0 : x^3 : z^3), \dots\}$ constitutes a basis of $\mathcal{P}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$.

DEFINITION 5.5. The fractal analogue of a projective polynomial $p(x : 0 : z) \in \mathcal{P}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ is defined as $\mathcal{F}^\alpha(p) = p^\alpha$. Thus, we denote $\mathcal{P}_m^\alpha[\mathbb{P}_{\mathbb{I} \times \{0\}}] = \mathcal{F}^\alpha(\mathcal{P}_m[\mathbb{P}_{\mathbb{I} \times \{0\}}])$ and $\mathcal{P}^\alpha[\mathbb{P}_{\mathbb{I} \times \{0\}}] = \mathcal{F}^\alpha(\mathcal{P}[\mathbb{P}_{\mathbb{I} \times \{0\}}])$.

THEOREM 5.6 (Fractal version of the Stone–Weierstrass-type theorem). Consider a continuous function $f : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{K}$.

For a given $\epsilon > 0$, for any partition $\Delta_{\mathbb{P}}$ of the projective interval $\mathbb{P}_{\mathbb{I} \times \{0\}}$ with N points, and for a bounded linear operator $\mathbb{L} : \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}] \rightarrow \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$, $\mathbb{L} \neq \text{Id}$, satisfying

$$\mathbb{L}(f)(x_0 : 0 : z_0) = f(x_0 : 0 : z_0), \quad \mathbb{L}(f)(x_N : 0 : z_N) = f(x_N : 0 : z_N),$$

there exists a fractal projective polynomial $p_0^\alpha \in \mathcal{P}^\alpha[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ with $\alpha_\infty \neq 0$, generated by the RPIFS (4.1), (4.2), and (5.1) such that $\|p_0^\alpha \ominus f\|_{\mathbb{P}_{\infty}} < \epsilon$.

Proof. Let $\epsilon > 0$. Then, by Theorem 3.6, there exists a projective polynomial $p_0 \in \mathcal{P}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ such that

$$(5.3) \quad \|p_0 \ominus f\|_{\mathbb{P}_{\infty}} < \frac{\epsilon}{2}.$$

Let $\mathbb{L} : \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}] \rightarrow \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ be a linear bounded operator with $\mathbb{L} \neq \text{Id}$, satisfying

$$\mathbb{L}(f)(x_0 : 0 : z_0) = f(x_0 : 0 : z_0), \quad \mathbb{L}(f)(x_N : 0 : z_N) = f(x_N : 0 : z_N).$$

For a partition $\Delta_{\mathbb{P}} : (x_0 : 0 : z_0) \prec (x_1 : 0 : z_1) \prec \cdots \prec (x_N : 0 : z_N)$ of the projective interval $\mathbb{P}_{\mathbb{I} \times \{0\}} = \left\{ (x : 0 : z) \in \mathbb{H} : (x_0 : 0 : z_0) \preceq (x : 0 : z) \preceq (x_N : 0 : z_N) \right\}$, we consider the scale vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in (-1, 1)^N$, with $\alpha_\infty \neq 0$, such that

$$(5.4) \quad \alpha_\infty < \frac{\frac{\epsilon}{2}}{\frac{\epsilon}{2} + \|\text{Id} \ominus \mathbb{L}\| \|p_0\|_{\mathbb{P}_\infty}}.$$

With these assumptions, for the projective polynomial p_0 , let p_0^α be the corresponding fractal projective polynomial, that is, $\mathcal{F}^\alpha(p_0) = p_0^\alpha$. Then,

$$\begin{aligned} \|p_0^\alpha \ominus f\|_{\mathbb{P}_\infty} &= \|p_0^\alpha \ominus p_0 \oplus p_0 \ominus f\|_{\mathbb{P}_\infty} \leq \|p_0^\alpha \ominus p_0\|_{\mathbb{P}_\infty} + \|p_0 \ominus f\|_{\mathbb{P}_\infty} \\ &\leq \frac{\alpha_\infty \|\text{Id} \ominus \mathbb{L}\|}{1 - \alpha_\infty} \|p_0\|_{\mathbb{P}_\infty} + \|p_0 \ominus f\|_{\mathbb{P}_\infty} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

The first step follows from the triangle inequality, the second step from Theorem 5.3, and the last step from the inequalities (5.3) and (5.4). \square

REMARK 5.7. The space $\mathcal{P}^\alpha[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ is dense in $\mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$.

The following theorem provides several properties of the projective fractal operator \mathcal{F}^α .

THEOREM 5.8. *Let \mathbb{L} be a linear bounded operator with respect to the uniform norm.*

Then,

(a) *the operator \mathcal{F}^α is bounded, and the following inequalities hold:*

$$\begin{aligned} \|\text{Id} \ominus \mathcal{F}^\alpha\| &\leq \frac{\alpha_\infty}{1 - \alpha_\infty} \|\text{Id} \ominus \mathbb{L}\|, \\ \|\mathcal{F}^\alpha\| &\leq 1 + \frac{\alpha_\infty}{1 - \alpha_\infty} \|\text{Id} \ominus \mathbb{L}\|. \end{aligned}$$

(b) *$\mathcal{F}^\alpha : \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}] \rightarrow \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ is linear and injective.*

(c) *If $\alpha = 0$, then $\mathcal{F}^\alpha = \text{Id}$.*

(d) *If $\alpha \neq 0$, then the fixed point of \mathcal{F}^α agrees with the fixed point of \mathbb{L} .*

(e) *If $\alpha \neq 0$, then $\mathcal{F}^\alpha = \text{Id}$ if and only if $\mathbb{L} = \text{Id}$.*

(f) *If $1 \in \sigma(\mathbb{L})$ (the spectrum of \mathbb{L}), then $1 \leq \|\mathcal{F}^\alpha\|$.*

Proof. For item (a), the first inequality follows immediately from Theorem 5.3. For any $g \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$,

$$\|\mathcal{F}^\alpha(g)\|_{\mathbb{P}_\infty} \leq \|\mathcal{F}^\alpha(g) \ominus g\|_{\mathbb{P}_\infty} + \|g\|_{\mathbb{P}_\infty} \leq \left(1 + \frac{\alpha_\infty}{1 - \alpha_\infty} \|\text{Id} \ominus \mathbb{L}\| \right) \|g\|_{\mathbb{P}_\infty}.$$

Therefore,

$$\|\mathcal{F}^\alpha\| \leq 1 + \frac{\alpha_\infty}{1 - \alpha_\infty} \|\text{Id} \ominus \mathbb{L}\|.$$

Regarding item (b), we have already verified the linearity of the operator \mathcal{F}^α . Let $g \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ be such that $\mathcal{F}^\alpha(g) = 0$. Then, from (5.2), for all $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$,

$$g(x : 0 : z) \oplus (\alpha_n \odot \mathbb{L}(g) \circ L_n^{-1}(x : 0 : z)) = (0 : 0 : 1).$$

Now, if $h(x : 0 : z) = (0 : 0 : 1)$, for all $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$, then

$$\begin{aligned} h(x : 0 : z) \oplus (\alpha_n \odot \mathbb{L}(h) \circ L_n^{-1}(x : 0 : z)) \\ = (0 : 0 : 1) \oplus (\alpha_n \odot (0 : 0 : 1)) = (0 : 0 : 1). \end{aligned}$$

By the uniqueness of the solution, we have $g(x : 0 : z) = (0 : 0 : 1)$ for all elements $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$. This shows that \mathcal{F}^α is injective.

For item (c), if $\alpha = 0$, then $\mathcal{F}^\alpha(g)(x : 0 : z) = g^\alpha(x : 0 : z) = g(x : 0 : z)$ for all $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$. Therefore, $\mathcal{F}^\alpha(g) = g$, and this implies $\mathcal{F}^\alpha = \text{Id}$.

For item (d), let $\alpha \neq 0$ and g be a fixed point of \mathbb{L} . According to (4.11),

$$\|g^\alpha \ominus g\|_{\mathbb{P}^\infty} \leq \alpha_\infty \|g^\alpha \ominus g\|_{\mathbb{P}^\infty} + \|g \ominus \mathbb{L}(g)\|_{\mathbb{P}^\infty}.$$

Therefore,

$$\|g^\alpha \ominus g\|_{\mathbb{P}^\infty} \leq \alpha_\infty \|g^\alpha \ominus g\|_{\mathbb{P}^\infty},$$

as $0 < \alpha_\infty < 1$. So, $\|g^\alpha \ominus g\|_{\mathbb{P}^\infty} = 0$. Hence, $\mathcal{F}^\alpha(g) = g$.

Conversely, let g be the fixed point of \mathcal{F}^α , then $g^\alpha = g$. As $\alpha \neq 0$, there is an $n \in \{1, 2, \dots, N\}$ such that $\alpha_n \neq 0$. For this n and for all $(x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$, from (5.2), we have

$$g(x : 0 : z) = g(x : 0 : z) \oplus (\alpha_n \odot (g \ominus \mathbb{L}(g)) \circ L_n^{-1}(x : 0 : z)).$$

Then,

$$\alpha_n \odot (g \ominus \mathbb{L}(g)) \circ L_n^{-1}(x : 0 : z) = (0 : 0 : 1).$$

This proves that $\mathbb{L}(g) = g$.

Item (e) is an immediate consequence of item (d).

For item (f), as 1 is the spectrum of \mathbb{L} , there is a non-null function f such that $\mathbb{L}(f) = f$. Then, by item (d), we have $\mathcal{F}^\alpha(f) = f$, which implies

$$1 \leq \|\mathcal{F}^\alpha\|. \quad \square$$

COROLLARY 5.9. *The system*

$$\{(0 : 1 : 1)^\alpha, (0 : x : z)^\alpha, (0 : x^2 : z^2)^\alpha, \dots, (0 : x^m : z^m)^\alpha\}$$

is linearly independent for any arbitrary scale vector α . In fact, the system forms a basis for $\mathcal{P}_m^\alpha[\mathbb{P}_{\mathbb{I} \times \{0\}}]$.

Proof. The result follows from the linearity and injectivity of \mathcal{F}^α . \square

Let $\mathcal{L}(\mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}])$ denotes the set of all invertible bounded linear operators on $\mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$. Then the following theorem holds:

THEOREM 5.10. *If $\alpha_\infty \|\mathbb{L}\| < 1$, then $\mathcal{F}^\alpha \in \mathcal{L}(\mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}])$, and*

$$\begin{aligned} \frac{1 - \alpha_\infty \|\mathbb{L}\|}{1 + \alpha_\infty} \|g\|_{\mathbb{P}^\infty} \leq \|\mathcal{F}^\alpha(g)\|_{\mathbb{P}^\infty} \leq \frac{1 + \alpha_\infty \|\mathbb{L}\|}{1 - \alpha_\infty} \|g\|_{\mathbb{P}^\infty} \quad \text{and} \\ \frac{1 - \alpha_\infty}{1 + \alpha_\infty \|\mathbb{L}\|} \|g\|_{\mathbb{P}^\infty} \leq \|(\mathcal{F}^\alpha)^{-1}(g)\|_{\mathbb{P}^\infty} \leq \frac{1 + \alpha_\infty}{1 - \alpha_\infty \|\mathbb{L}\|} \|g\|_{\mathbb{P}^\infty} \quad \text{for all } g \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]. \end{aligned}$$

Proof. The operator \mathcal{F}^α satisfies the equation

$$\mathcal{F}^\alpha(g)(x : 0 : z) = g(x : 0 : z) \oplus \alpha_n (\mathcal{F}^\alpha(g) \ominus L(g)) \circ L_n^{-1}(x : 0 : z)$$

for all $g \in \mathcal{C}[\mathbb{P}_{\mathbb{I}} \times \{0\}]$ and for all $(x : 0 : z) \in \mathbb{P}_{\mathbb{I}_n \times \{0\}}$. Then,

$$\|\mathcal{F}^\alpha(g) \ominus g\|_{\mathbb{P}_\infty} \leq \alpha_\infty \|\mathcal{F}^\alpha(g) \ominus L(g)\|_{\mathbb{P}_\infty} \leq \alpha_\infty \|L\| \|g\|_{\mathbb{P}_\infty} + \alpha_\infty \|\mathcal{F}^\alpha(g)\|_{\mathbb{P}_\infty}.$$

Using [9, Lemma 1], it follows that if $\alpha_\infty \|\mathbb{L}\| < 1$, then $\mathcal{F}^\alpha \in \mathcal{L}(\mathcal{C}[\mathbb{P}_{\mathbb{I}} \times \{0\}])$, and \mathcal{F}^α satisfies the above required inequalities. \square

REMARK 5.11. \mathcal{F}^α is a topological isomorphism.

COROLLARY 5.12. *If $\alpha_\infty \|\mathbb{L}\| < 1$, then*

$$\|g\|_{\mathbb{P}_\infty} \leq \left(\frac{1 + \alpha_\infty}{1 - \alpha_\infty \|\mathbb{L}\|} \right) \|g^\alpha\|_{\mathbb{P}_\infty}.$$

Proof. This is an immediate consequence of Theorem 5.10. \square

6. Dual non-affine real projective fractal functions. Duality of an object represents a symmetry in the sense that the dual of a point is a line and the dual of a line is a point. Barnsley et al. [6] provides conditions for the existence of the attractor of the hyperplane IFS on the dual space of the real projective space. In [13], the authors define the dual RPIFS on the dual space of the real projective plane and pose an open problem about the existence of a fractal function corresponding to a dual RPIFS. In this section, we first define the dual RPIFS, but for non-affine settings, and prove that the attractor of the dual RPIFS (corresponding to a RPIFS) is also the graph of a fractal function. To prove the main results in this section, recall the real projective metric $d_{\mathbb{P}}$ on $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ defined in Section 3. The hyperplane orthogonal to $p \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$ is expressed as

$$p^\perp = \{q \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} : q \perp p\}.$$

DEFINITION 6.1. *Let $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ be the collection of all hyperplanes of $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$, or equivalently $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}} = \{p^\perp : p \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}\}$.*

Then, the space $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ is said to be the dual space of $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$. The addition on $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ is induced from the addition on $\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$. That is, $p^\perp \oplus q^\perp = (p \oplus q)^\perp$. The dual space is endowed with a norm $\|\cdot\|_{\widehat{\mathbb{P}}}$ defined by

$$\|(x : y : z)^\perp\|_{\widehat{\mathbb{P}}} := \|(x : y : z)\|_{\mathbb{P}}, \quad \text{for all } (x : y : z)^\perp \in \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}.$$

Hence, the metric is given by

$$\widehat{d}_{\mathbb{P}}(p^\perp, q^\perp) := d_{\mathbb{P}}(p, q), \quad \text{for all } p^\perp, q^\perp \in \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}.$$

The map $Q : \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} \rightarrow \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ given by $Q(p) = p^\perp$ is called the duality map.

REMARK 6.2. The metric spaces $(\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}, d_{\mathbb{P}})$ and $(\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}, \widehat{d}_{\mathbb{P}})$ are isometrically isomorphic, and Q is the corresponding isometry. Hence, $(\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}, \widehat{d}_{\mathbb{P}})$ is a complete metric space. Since Q is continuous, it can in the usual way be extended to a map $\mathbb{Q} : \mathcal{H}(\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}) \rightarrow \mathcal{H}(\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}})$ such that for $A \in \mathcal{H}(\mathbb{RP}^2 \setminus \mathbb{H}_{e_3})$,

$$\mathbb{Q}(A) = \{Q(a) : a \in A\}.$$

Let $\widehat{\mathbb{H}} = Q(\mathbb{H})$ and $\widehat{\mathbb{K}} = Q(\mathbb{K})$. Now for $x \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3}$, x can be written as $x = x_1 \oplus x_2$, where $x_1 \in \mathbb{H}$ and $x_2 \in \mathbb{K}$. Then, from the definition of the addition on $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$, it follows that $x^\perp = (x_1 \oplus x_2)^\perp = x_1^\perp \oplus x_2^\perp$. So, $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ can be expressed as

$$\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}} = \widehat{\mathbb{H}} \oplus \widehat{\mathbb{K}}.$$

Here, we use the same notion \oplus for the addition.

Now consider the data set $\left\{ P_n \in \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}} : n = 0, 1, \dots, N \right\}$ on $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$, where $P_n = (x_n : y_n : z_n)^\perp$, for $n = 0, 1, \dots, N$, is given in (4.1). P_n can be written as

$$P_n = H_n \oplus K_n,$$

where $H_n = (x_n : 0 : z_n)^\perp \in \widehat{\mathbb{H}}$ and $K_n = (0 : y_n : z_n)^\perp \in \widehat{\mathbb{K}}$. Define

$$\widehat{L}_n : \widehat{\mathbb{P}_{1 \times \{0\}}} \rightarrow \widehat{\mathbb{P}_{1_n \times \{0\}}}, \quad \widehat{F}_n : \widehat{\mathbb{P}_{1 \times \{0\}}} \oplus \widehat{\mathbb{K}} \rightarrow \widehat{\mathbb{K}}$$

such that

$$\widehat{L}_n((x : 0 : z)^\perp) = (L_n(x : 0 : z))^\perp \quad \text{and} \quad \widehat{F}_n((x : y : z)^\perp) = (F_n(x : y : z))^\perp,$$

where L_n and F_n are given by (4.1), (4.2) and $\widehat{\mathbb{P}_{1 \times \{0\}}} = Q(\mathbb{P}_{1 \times \{0\}})$, $\widehat{\mathbb{P}_{1_n \times \{0\}}} = Q(\mathbb{P}_{1_n \times \{0\}})$. Define $\widehat{W}_n : \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}} \rightarrow \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$ such that

$$\widehat{W}_n((x : y : z)^\perp) = \widehat{L}_n((x : 0 : z)^\perp) \oplus \widehat{F}_n((x : y : z)^\perp).$$

The collection $\left\{ \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}; \widehat{W}_n : n = 1, 2, \dots, N \right\}$ forms an IFS on $\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}$, which we call the *dual RPIFS*. Using (4.3), \widehat{W}_n can be written as

$$\begin{aligned} \widehat{W}_n((x : y : z)^\perp) &= (L_n(x : 0 : z))^\perp \oplus (F_n(x : y : z))^\perp \\ &= (L_n(x : 0 : z) \oplus F_n(x : y : z))^\perp = (W_n(x : y : z))^\perp. \end{aligned}$$

Therefore,

$$\widehat{W}_n(P_0) = (W_n(x_0 : y_0 : z_0))^\perp = (x_{n-1} : y_{n-1} : z_{n-1})^\perp = P_{n-1}.$$

Similarly, $\widehat{W}_n(P_N) = P_n$. So, the \widehat{W}_n , for $n = 1, 2, \dots, N$, satisfy the join-up conditions corresponding to the data set $\left\{ P_n \in \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}} : n = 0, 1, \dots, N \right\}$. Also,

$$\begin{aligned} \widehat{d}_{\mathbb{P}} \left(\widehat{W}_n((x : y : z)^\perp), \widehat{W}_n((x' : y' : z')^\perp) \right) &= \widehat{d}_{\mathbb{P}} \left((W_n(x : y : z))^\perp, (W_n(x' : y' : z'))^\perp \right) \\ &= d_{\mathbb{P}}(W_n(x : y : z), W_n(x' : y' : z')). \end{aligned}$$

If $\alpha_n \in (-1, 1)$, then $\left\{ \widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}; \widehat{W}_n : n = 1, 2, \dots, N \right\}$ forms a contractive dual RPIFS on the complete metric space $\left(\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}}, \widehat{d}_{\mathbb{P}} \right)$. Hence it has a unique attractor \widehat{G} on $\mathcal{H} \left(\widehat{\mathbb{RP}^2 \setminus \mathbb{H}_{e_3}} \right)$. Now, we prove the following:

THEOREM 6.3. *There exists a continuous function $\widehat{g} : \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}} \rightarrow \widehat{\mathbb{K}}$ such that $\text{graph}(\widehat{g}) = \widehat{G}$. Moreover, $\widehat{G} = \{p^\perp : p \in G\}$, where G is the attractor of the RPIFS defined in (4.3).*

In order to prove the above theorem, we first need to establish some prerequisites. Consider the space $\mathcal{C}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}] := \left\{ \widehat{f} : \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}} \rightarrow \widehat{\mathbb{K}} \text{ such that } \widehat{f} \text{ is continuous} \right\}$, with a norm $\|\cdot\|_{\widehat{\mathbb{P}}_\infty}$ given by

$$(6.1) \quad \|\widehat{f}\|_{\widehat{\mathbb{P}}_\infty} = \sup \left\{ \|\widehat{f}(H)\|_{\widehat{\mathbb{P}}} : H \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}} \right\}.$$

In order to show that the above norm is well defined, we first prove the following lemma:

LEMMA 6.4. *For any $\widehat{f} \in \mathcal{C}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}]$, there exists a function $f \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{P}_{\mathbb{I} \times \{0\}} & \xrightarrow{f} & \mathbb{K} \\ Q \downarrow & & \downarrow Q \\ \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}} & \xrightarrow{\widehat{f}} & \widehat{\mathbb{K}} \end{array}$$

In particular, $\widehat{f}(p^\perp) = (f(p))^\perp$ for all $p^\perp \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}$.

Proof. Let $\widehat{f} \in \mathcal{C}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}]$. Define $f : \mathbb{P}_{\mathbb{I} \times \{0\}} \rightarrow \mathbb{K}$ such that

$$(6.2) \quad f = Q^{-1} \circ \widehat{f} \circ Q.$$

Since Q is an isometry and \widehat{f} is a continuous function, f is continuous. Also, from (6.2), we get $Q \circ f = \widehat{f} \circ Q$, and for $p^\perp \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}$,

$$\widehat{f}(p^\perp) = Q \circ f \circ Q^{-1}(p^\perp) = Q(f(p)) = (f(p))^\perp. \quad \square$$

Now, for $H \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}$, there is an element $p \in \mathbb{P}_{\mathbb{I} \times \{0\}}$ such that $H = p^\perp$. Therefore, from Lemma 6.4, for $\widehat{f} \in \mathcal{C}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}]$, there is a $f \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ such that

$$\widehat{f}(H) = \widehat{f}(p^\perp) = (f(p))^\perp.$$

Therefore, $\|\widehat{f}(H)\|_{\widehat{\mathbb{P}}} = \|(f(p))^\perp\|_{\widehat{\mathbb{P}}} = \|f(p)\|_{\mathbb{P}}$. This shows that the norm given in (6.1) is well defined.

PROPOSITION 6.5. $\left(\mathcal{C}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}], \|\cdot\|_{\widehat{\mathbb{P}}_\infty} \right)$ is isometrically isomorphic to $\left(\mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}], \|\cdot\|_{\mathbb{P}_\infty} \right)$. Hence, $\mathcal{C}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}]$ is complete with respect to the norm $\|\cdot\|_{\widehat{\mathbb{P}}_\infty}$.

Proof. Let us define a map $S : \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}] \rightarrow \mathcal{C}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}]$ such that for all $p^\perp \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}$

$$S(f)(p^\perp) = (f(p))^\perp.$$

Lemma 6.4 shows that S is a bijective map and also that, for $p^\perp \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}$,

$$\|S(f)(p^\perp)\|_{\widehat{\mathbb{P}}} = \|(f(p))^\perp\|_{\widehat{\mathbb{P}}} = \|f(p)\|_{\mathbb{P}}.$$

Therefore, for all $f \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$, $\|S(f)\|_{\widehat{\mathbb{P}}_\infty} = \|f\|_{\mathbb{P}_\infty}$. This completes the proof. \square

Proof of Theorem 6.3. Let

$$\mathcal{C}_{0,N}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}] = \left\{ \widehat{f} \in \mathcal{C}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}] : \widehat{f}(H_0) = K_0 \text{ and } \widehat{f}(H_N) = K_N \right\}.$$

Then, $\mathcal{C}_{0,N}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}]$ is a closed subset of $\mathcal{C}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}]$ and hence complete. Let us define an operator $\widehat{T} : \mathcal{C}_{0,N}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}] \rightarrow \mathcal{C}_{0,N}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}]$ such that for $\widehat{f} \in \mathcal{C}_{0,N}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}]$,

$$(6.3) \quad \widehat{T}(\widehat{f})(X) = \widehat{F}_n \left((\widehat{L}_n)^{-1}(X) \oplus \widehat{f} \circ (\widehat{L}_n)^{-1}(X) \right),$$

whenever $X \in \widehat{\mathbb{P}}_{\mathbb{I}_n \times \{0\}}$. Here, we claim that $(\widehat{L}_n)^{-1} = \widehat{L}_n^{-1}$. Let $H \in \widehat{\mathbb{P}}_{\mathbb{I}_n \times \{0\}}$ such that

$$(6.4) \quad (\widehat{L}_n)^{-1}(H) = H' \quad \text{for some } H' \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}.$$

Then, $H = \widehat{L}_n(H')$. Now, $H = (h_1 : 0 : h_3)^\perp$ and $H' = (h'_1 : 0 : h'_3)^\perp$ for some $(h_1 : 0 : h_3) \in \mathbb{P}_{\mathbb{I}_n \times \{0\}}$ and $(h'_1 : 0 : h'_3) \in \mathbb{P}_{\mathbb{I} \times \{0\}}$. Therefore,

$$\begin{aligned} (h_1 : 0 : h_3)^\perp &= \widehat{L}_n((h'_1 : 0 : h'_3)^\perp), \\ (h_1 : 0 : h_3)^\perp &= (L_n(h'_1 : 0 : h'_3))^\perp, \\ Q(h_1 : 0 : h_3) &= Q(L_n(h'_1 : 0 : h'_3)). \end{aligned}$$

As Q is injective, it follows that $(h_1 : 0 : h_3) = L_n(h'_1 : 0 : h'_3)$. This implies that $L_n^{-1}(h_1 : 0 : h_3) = (h'_1 : 0 : h'_3)$. Therefore,

$$\begin{aligned} (L_n^{-1}(h_1 : 0 : h_3))^\perp &= (h'_1 : 0 : h'_3)^\perp, \\ \widehat{L}_n^{-1}((h_1 : 0 : h_3)^\perp) &= (h'_1 : 0 : h'_3)^\perp, \\ (6.5) \quad \widehat{L}_n^{-1}(H) &= H'. \end{aligned}$$

Hence, from (6.4) and (6.5), $(\widehat{L}_n)^{-1}(H) = \widehat{L}_n^{-1}(H)$ for all $H \in \widehat{\mathbb{P}}_{\mathbb{I}_n \times \{0\}}$. Now, setting $X = (x : 0 : z)^\perp \in \widehat{\mathbb{P}}_{\mathbb{I}_n \times \{0\}}$. Then, (6.3) can be rewritten as

$$\begin{aligned} \widehat{T}(\widehat{f})(X) &= \widehat{F}_n \left(\widehat{L}_n^{-1}((x : 0 : z)^\perp) \oplus \widehat{f} \circ \widehat{L}_n^{-1}((x : 0 : z)^\perp) \right) \\ &= \widehat{F}_n \left((L_n^{-1}(x : 0 : z))^\perp \oplus \widehat{f} \left((L_n^{-1}(x : 0 : z))^\perp \right) \right). \end{aligned}$$

From Lemma 6.4, for $\widehat{f} \in \mathcal{C}_{0,N}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}]$, there is a function $f \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ such that $\widehat{f}((x : 0 : z)^\perp) = (f(x : 0 : z))^\perp$. Therefore,

$$\begin{aligned} \widehat{T}(\widehat{f})(X) &= \widehat{F}_n \left((L_n^{-1}(x : 0 : z))^\perp \oplus (f(L_n^{-1}(x : 0 : z)))^\perp \right) \\ &= \widehat{F}_n \left(\left(L_n^{-1}(x : 0 : z) \oplus f(L_n^{-1}(x : 0 : z)) \right)^\perp \right) \\ &= \left(F_n(L_n^{-1}(x : 0 : z) \oplus f \circ L_n^{-1}(x : 0 : z)) \right)^\perp = (\mathcal{T}(f)(x : 0 : z))^\perp, \end{aligned}$$

where \mathcal{T} is the RPRB-operator defined in (4.4). Hence, from above it can easily be seen that \widehat{T} is well defined, continuous, and also a contraction map with respect to the norm $\|\cdot\|_{\widehat{\mathbb{P}}_\infty}$. Therefore, \widehat{T} has a unique fixed point \widehat{g}^α in $\mathcal{C}_{0,N}[\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}]$. We refer to this as the *dual non-affine RPPF*.

Let $\widehat{M} = \text{graph}(\widehat{g}^\alpha) = \left\{ H \oplus \widehat{g}^\alpha(H) : H \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}} \right\}$. Then,

$$\begin{aligned}
 \widehat{W}_n(\widehat{M}) &= \left\{ \widehat{W}_n(H \oplus \widehat{g}^\alpha(H)) : H \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}} \right\} \\
 (6.6) \qquad &= \left\{ \widehat{L}_n(H) \oplus \widehat{F}_n((H \oplus \widehat{g}^\alpha(H))) : H \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}} \right\}.
 \end{aligned}$$

As \widehat{g}^α is the fixed point of \widehat{T} and $(\widehat{L}_n)^{-1} = \widehat{L}_n^{-1}$, it follows that

$$\begin{aligned}
 \widehat{g}^\alpha(\widehat{L}_n(H)) &= \widehat{T}(\widehat{g}^\alpha)(\widehat{L}_n(H)) = \widehat{F}_n(\widehat{L}_n^{-1}\widehat{L}_n(H) \oplus \widehat{g}^\alpha(\widehat{L}_n^{-1}\widehat{L}_n(H))) \\
 (6.7) \qquad &= \widehat{F}_n((H \oplus \widehat{g}^\alpha(H))).
 \end{aligned}$$

Therefore, from (6.6) and (6.7), we have

$$\widehat{W}_n(\widehat{M}) = \left\{ \widehat{L}_n(H) \oplus \widehat{g}^\alpha(\widehat{L}_n(H)) : H \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}} \right\}.$$

Taking the union over $n = 1, 2, \dots, N$, we get $\bigcup_{n=1}^N \widehat{W}_n(\widehat{M}) = \widehat{M}$. Hence, by the uniqueness of the attractor, $\widehat{G} = \widehat{M}$. Also, from Lemma 6.4, there exists a function $f \in \mathcal{C}[\mathbb{P}_{\mathbb{I} \times \{0\}}]$ such that $\widehat{g}^\alpha(p^\perp) = (f(p))^\perp$ for all $p^\perp \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}$. Therefore,

$$\begin{aligned}
 \widehat{G} &= \left\{ p^\perp \oplus \widehat{g}^\alpha(p^\perp) : p^\perp \in \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}} \right\} = \left\{ p^\perp \oplus (f(p))^\perp : p^\perp \in Q(\mathbb{P}_{\mathbb{I} \times \{0\}}) \right\} \\
 &= \left\{ (p \oplus f(p))^\perp : Q^{-1}(p^\perp) \in \mathbb{P}_{\mathbb{I} \times \{0\}} \right\} = \left\{ q^\perp : q = p \oplus f(p) \text{ and } p \in \mathbb{P}_{\mathbb{I} \times \{0\}} \right\} \\
 &= \left\{ q^\perp : q \in G \right\}.
 \end{aligned}$$

This completes the proof. \square

EXAMPLE 6.6. Consider the data points, the projective intervals, the classical function g , and the base function b as given in Example 4.4. Here, the points of the data set $\{P_0, P_1, P_2\}$ in $\widehat{\mathbb{RP}^2} \setminus \widehat{\mathbb{H}}_{e_3}$ are given by

$$\begin{aligned}
 P_0 &= (-1 : 1 : 1)^\perp = \{(x : y : z) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} : -x + y + z = 0\}, \\
 P_1 &= (0 : 0 : 1)^\perp = \{(x : y : z) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} : z = 0\}, \qquad \text{and} \\
 P_2 &= (1 : 1 : 1)^\perp = \{(x : y : z) \in \mathbb{RP}^2 \setminus \mathbb{H}_{e_3} : x + y + z = 0\}.
 \end{aligned}$$

The contraction homeomorphisms $\widehat{L}_n : \widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}} \rightarrow \widehat{\mathbb{P}}_{\mathbb{I}_n \times \{0\}}$, for $n = 1, 2$, are given by

$$\begin{aligned}
 \widehat{L}_1((x : 0 : z)^\perp) &= (L_1(x : 0 : z))^\perp = (0.5x - 0.5z : 0 : z)^\perp, \\
 \widehat{L}_2((x : 0 : z)^\perp) &= (L_2(x : 0 : z))^\perp = (0.5x + 0.5z : 0 : z)^\perp,
 \end{aligned}$$

where the interval $\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}}$ is given by

$$\widehat{\mathbb{P}}_{\mathbb{I} \times \{0\}} = \{p^\perp : p \in \mathbb{P}_{\mathbb{I} \times \{0\}}\} = \{(u : v : w) : xu + zw = 0 \text{ and } (x : 0 : z) \in \mathbb{P}_{\mathbb{I} \times \{0\}}\}$$

$$= \{(u : v : w) : xu + zw = 0 \text{ and } -z \leq x \leq z\}$$

and the subintervals $\widehat{\mathbb{P}}_{\mathbb{I}_n \times \{0\}}$, for $n = 1, 2$, are given by

$$\begin{aligned} \widehat{\mathbb{P}}_{\mathbb{I}_1 \times \{0\}} &= \{(u : v : w) : xu + zw = 0 \text{ and } -z \leq x \leq 0\}, \\ \widehat{\mathbb{P}}_{\mathbb{I}_2 \times \{0\}} &= \{(u : v : w) : xu + zw = 0 \text{ and } 0 \leq x \leq z\}. \end{aligned}$$

Also, the maps $\widehat{F}_n : \widehat{\mathbb{P}}_{\mathbb{I}_n \times \{0\}} \oplus \widehat{\mathbb{K}} \rightarrow \widehat{\mathbb{K}}$, for $n = 1, 2$, are given by

$$\begin{aligned} \widehat{F}_1((x : y : z)^\perp) &= (F_1(x : y : z))^\perp = (0 : -0.5yz^3 + (0.5x - 0.5z)^2z^2 + 0.5x^4 : z^4)^\perp, \\ \widehat{F}_2((x : y : z)^\perp) &= (F_2(x : y : z))^\perp = (0 : 0.5yz^3 + (0.5x + 0.5z)^2z^2 - 0.5x^4 : z^4)^\perp. \end{aligned}$$

The graphs of the classical dual function \widehat{g} and its corresponding dual non-affine RPF \widehat{g}^α are displayed in Figure 6.1.

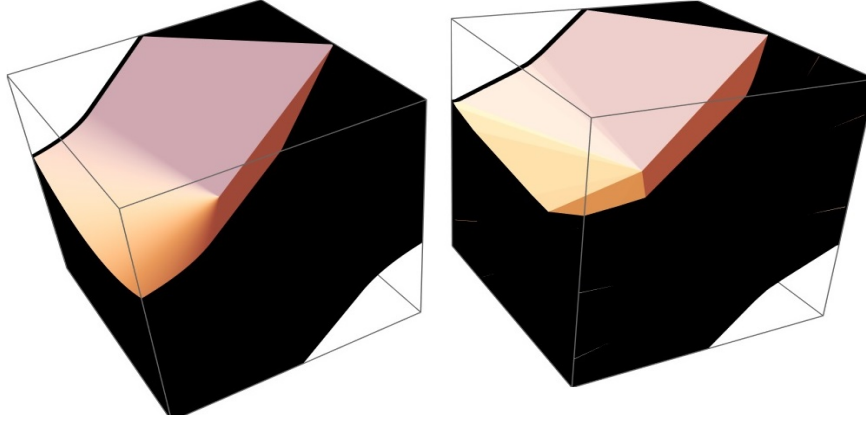


FIG. 6.1. Graphs of the classical dual function (left) and the corresponding dual non-affine RPF (right).

7. Conclusion. In this article, we introduce non-affine fractal functions on the real projective plane and prove the existence of a fractal function corresponding to a dual RPIFS on the dual of the real projective plane. First, we prove some classical results of approximation theory on the real projective plane. Then, by suitably designing a RPIFS, we construct a non-affine fractal function on it. After that, we define a projective fractal operator associated with the constructed non-affine fractal function and characterize some properties of it. A parallel study of fractal approximation theory on the real projective plane is initiated. Finally, we establish that the attractor of a dual RPIFS corresponding to an RPIFS is also the graph of a fractal function on the dual projective plane.

In future work, by choosing variable scaling functions, more flexible non-affine RPFs could be defined. The fractal dimensions of such fractal functions need further investigation. More complicated fractals, namely, bivariate fractal functions, hidden variable fractal function, fractal surfaces, etc. on the real projective plane are still unexplored and need further studies. Moreover, one may study fractal transformations and fractal image compression on the real projective plane, which may be useful for camera modeling.

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