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OPERATOR ORDERING BY ILL-POSEDNESS IN HILBERT AND BANACH SPACES*

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Abstract. For operators representing ill-posed problems, an ordering by ill-posedness is proposed, where one operator is considered more ill-posed than another one if the former can be expressed as a concatenation of bounded operators involving the latter. This definition is motivated by a recent one introduced by Mathé and Hofmann [Adv. Oper. Theory, 10 (2025), Paper No. 36] that utilizes bounded and orthogonal operators, and we show the equivalence of our new definition with this one for the case of compact and non-compact linear operators in Hilbert spaces. We compare our ordering with other measures of ill-posedness such as the decay of the singular values, norm estimates, and range inclusions. Furthermore, as the new definition does not depend on the notion of orthogonal operators, it can be extended to the case of linear operators in Banach spaces, and it also provides ideas for applications to nonlinear problems in Hilbert spaces. In the latter context, certain nonlinearity conditions can be interpreted as ordering relations between a nonlinear operator and its linearization.

Key words. ill-posed problem, measures of ill-posedness, singular values, degree of ill-posedness, range inclusion, nonlinearity condition

AMS subject classifications. 47A52, 65J20, 47J06, 47B01, 47B02

1. Introduction.

When studying ill-posed and inverse problems, quite frequently one encounters the situation that one problem might be "more difficult" to solve than another, and, with respect to regularization theory, this problem is often called "more ill-posed" than the other. Usually, in the case of linear compact operators in Hilbert spaces, the decay rate of the singular values is used as the indicator for such discrimination.

In this article we define an abstract general concept of when one problem is more ill-posed than another by using an ordering relation based on connecting operators. This is motivated by a recent definition introduced by Mathé and Hofmann in [28], where bounded and orthogonal operators are used.

The main point of this article is to define a more general ordering than that in [28]. While the latter hinges on the notion of unitary or orthogonal operators, our new definition is based only on bounded operators, and thus it enables extensions to the Banach space case and even to the nonlinear case. One of the main theoretical results is that the ordering defined here is equivalent to that in [28], namely in both the compact and non-compact cases; in the compact case it is also equivalent to the more classical approach of comparing the decay rates of the singular values.

After defining the ordering in Section 2, we prove the mentioned equivalence in Section 3 in the compact case and in Section 6 for the non-compact case. Section 4 is devoted to alternative orderings and the relation to regularization. Following this, Section 5 recalls the Douglas range inclusion theorem in light of our suggested ordering approach. A generalization to the Banach space case is done in Section 7. Furthermore, in Section 8 we extend the ordering ideas to nonlinear operator equations in Hilbert spaces, interpreting certain nonlinearity conditions as an ordering relation between an operator and its linearization.

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2. Orderings by ill-posedness. Let A and A' be bounded linear operators between Banach spaces, $A : X \to Y$ and $A' : X' \to Y'$. We use the following definitions: $\mathcal{N}(A)$ denotes the nullspace of A and $\mathcal{R}(A)$ its range. $\mathcal{B}(X, Y)$ denotes the set of *bounded* operators between Banach spaces X and Y. If the spaces are clear from the context, then we simply write $\mathcal{B}$. $\mathcal{O}(X, Y)$ denotes the set of *isometric* operators between *Hilbert* spaces X and Y. If the spaces are clear from the context, then we simply write $\mathcal{O}$. Moreover, we denote by $\mathcal{PO}(X, Y)$ the set of *partial isometries*. For consistency of notation, we also use $\mathcal{I}$ for the set consisting of only the identity operator $\mathcal{I} = \{I\}$.

Note that an isometric operator U satisfies $U^T U = I$. If it is additionally surjective, $\mathcal{R}(U) = Y$, then we speak of an *orthogonal* (in real spaces) or *unitary* (in complex spaces) operator. An operator U that has a nullspace but is an isometry when restricted to $\mathcal{N}(U)^\perp$ is a *partial isometry*.

In the sequel, we denote by A^* the adjoint/dual operator of A both in the Hilbert and in the Banach space cases, while A' simply means a generic operator that is compared with A. (Later, for nonlinear problems, F' also denotes the Fréchet derivative of an operator F.)

An ordering of inverse problems by ill-posedness was proposed in [18], where a measure of ill-posedness was abstractly defined as an ordering of operators. In this article, motivated by [28], the following ordering is the main tool that we work with.

DEFINITION 2.1. *Let X, Y, X', and Y' be Banach spaces. Let $A : X \to Y$ and $A' : X' \to Y'$ be bounded linear operators. The operator A' is said to be* more ill-posed *than the operator A if there applies an ordering defined as follows:*

$$(2.1)\qquad \begin{aligned} & A' \leq_{\mathcal{B},\mathcal{B}} A \\ \iff \quad & \exists T \in \mathcal{B}(\overline{\mathcal{R}(A)}, Y') \quad \text{and} \quad \exists S \in \mathcal{B}(X', X) \quad \text{such that} \quad A' = TAS. \end{aligned}$$

If

$$A' \leq_{\mathcal{B},\mathcal{B}} A \quad \text{and} \quad A \leq_{\mathcal{B},\mathcal{B}} A',$$

then both operators are equivalent with respect to ill-posedness, *and we write*

$$A' \sim_{\mathcal{B},\mathcal{B}} A.$$

If either A' is more ill-posed than A or A is more ill-posed than A', then A and A' are said to be comparable, *otherwise* non-comparable. *If A' is more ill-posed than A, but A fails to be more ill-posed than A', then A' is said to be* strictly more ill-posed *than A. The operators T and S in (2.1) are referred to as* connecting operators *in the ordering.*

As we will see, in many cases this definition reflects most of the established notions of A' is "more ill-posed" than A.

3. Compact operators in Hilbert spaces.

3.1. General assertions. Let us specialize the definition to the simplest situation when all spaces are Hilbertian and the operators A and A' are compact. In Sections 3–5 we assume throughout that

$$X,\ Y,\ X',\ \text{and } Y' \text{ are Hilbert spaces}$$

representing the domain and image spaces of operators A and A', i.e., $A' : X' \to Y'$ and $A : X \to Y$.

PROPOSITION 3.1. *A simple consequence of Definition 2.1 is the adjoint invariance of the ordering:*

$$A' \leq_{\mathcal{B},\mathcal{B}} A \quad \iff \quad A'^* \leq_{\mathcal{B},\mathcal{B}} A^*.$$

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This is an immediate consequence of the definition: $A' \leq_{\mathcal{B},\mathcal{B}} A$ holds if and only if $A' = TAS$. Hence $A'^* = S^*A^*T^*$, which means by definition that $A'^* \leq_{\mathcal{B},\mathcal{B}} A^*$.

Under the stated assumption, we may consider seemingly (stronger) alternative definitions of an ordering that use isometric operators instead of bounded ones, as was done in [28].

DEFINITION 3.2. *Define the following orderings:*

$$\begin{aligned} A' \leq_{\mathcal{O},\mathcal{B}} A \quad &\Longleftrightarrow \quad \exists\, O \in \mathcal{O}(\overline{\mathcal{R}(A)}, Y'),\ S \in \mathcal{B}(X', X) \quad \textit{with} \quad A' = OAS,\\ A' \leq_{\mathcal{B},\mathcal{O}} A \quad &\Longleftrightarrow \quad \exists\, T \in \mathcal{B}(Y, Y'),\ U \in \mathcal{O}(X', \mathcal{N}(A)^{\perp}) \quad \textit{with} \quad A' = TAU. \end{aligned}$$

If the operators in $\mathcal{O}$ can be chosen as the identity, then we write $\leq_{\mathcal{I},\mathcal{B}}$ or $\leq_{\mathcal{B},\mathcal{I}}$. Analogously, if $\mathcal{O}$ is out of the set of partial isometries, we write $\leq_{\mathcal{PO},\mathcal{B}}$ or $\leq_{\mathcal{B},\mathcal{PO}}$.

Moreover, in the case that A and A' are both compact, we may define an ordering by the singular values

$$A' \leq_{\sigma} A \quad \Longleftrightarrow \quad \forall\, n \in \mathbb{N}:\ \sigma_n(A') \leq \sigma_n(A),$$

and similarly

$$A' \lessapprox_{\sigma,C} A \quad \Longleftrightarrow \quad \exists\, C > 0 \quad \textit{such that} \quad \forall\, n \in \mathbb{N}:\ \sigma_n(A') \leq C\sigma_n(A).$$

Clearly, $A' \leq_{\mathcal{O},\mathcal{B}} A$ or $A' \leq_{\mathcal{B},\mathcal{O}} A$ imply that $A' \leq_{\mathcal{B},\mathcal{B}} A$. But we will show below that all three orderings are in fact equivalent. For this we need the results of the following lemma, and we refer in this context also to [28, Prop. 3].

LEMMA 3.3. *Let $A : X \to Y$ and $A' : X' \to Y'$ be compact operators acting between Hilbert spaces.*

- *In the case that $\dim(\mathcal{R}(A')) = \dim(\mathcal{R}(A))$ (finite or not), we have*

$$A' \lessapprox_{\sigma,C} A \quad \Longrightarrow \quad A' \leq_{\mathcal{O},\mathcal{B}} A,$$

with a connecting operator $O : \overline{\mathcal{R}(A)} \to \overline{\mathcal{R}(A')} \in \mathcal{O}$ unitary. Also, we have

$$A' \lessapprox_{\sigma,C} A \quad \Longrightarrow \quad A' \leq_{\mathcal{B},\mathcal{O}} A,$$

with a connecting operator $O : \mathcal{N}(A')^{\perp} \to \mathcal{N}(A)^{\perp} \in \mathcal{O}$ unitary.

- *In the case that $\mathcal{R}(A')$ is finite-dimensional and $\dim(\mathcal{R}(A')) < \dim(\mathcal{R}(A))$, we have*

$$A' \lessapprox_{\sigma,C} A \quad \Longrightarrow \quad A' = OAS,$$

where S is bounded and $\mathcal{O} : \overline{\mathcal{R}(A)} \to \overline{\mathcal{R}(A')}$ is a partial isometry.

Proof. Recall the singular-value decomposition of compact operators:

$$A' = \sum_{i \in I'} \sigma_i'(\cdot, \phi_i')\psi_i' \quad \text{and} \quad A = \sum_{i \in I} \sigma_i(\cdot, \phi_i)\psi_i.$$

Here, $\sigma_i, \sigma_i' > 0$ are the positive singular values of A and A', respectively. The functions ψ_i and ψ_i' are orthogonal bases of $\overline{R(A)}$ and $\overline{R(A')}$, respectively, and ϕ_i and ϕ_i' are orthogonal bases of $\mathcal{N}(A)^{\perp}$ and $\mathcal{N}(A')^{\perp}$, respectively. The index sets I and I' are either countably infinite or finite depending on the dimension of the ranges, or, equivalently, on the number of non-zero singular values. In any case by $A' \lessapprox_{\sigma,C} A$ we can assume that $I' \subset I$.

Define the following operators:

$$Sf := \sum_{i\in I'} \frac{\sigma_i'}{\sigma_i}(f,\phi_i')\phi_i,$$
$$Of := \sum_{i\in I'} (f,\psi_i)\psi_i'. \tag{3.1}$$

Then, inserting the definitions, this yields (the indices i, j, and k correspond to O, A, and S, respectively)

$$OASf = \sum_{i\in I'}\sum_{j\in I}\sum_{k\in I'} \sigma_j \frac{\sigma_k'}{\sigma_k}(f,\phi_k')(\phi_k,\phi_j)(\psi_j,\psi_i)\psi_i'.$$

By orthogonality $(\phi_k,\phi_j) = \delta_{k,j}$ and $(\psi_j,\psi_i) = \delta_{i,j}$, and thus two summations drop out:

$$OASf = \sum_{i\in I'} \sigma_i \frac{\sigma_i'}{\sigma_i}(f,\phi_i')\psi_i' = \sum_{i\in I'} \sigma_i'(f,\phi_i')\psi_i' = A'f.$$

By $A' \lessapprox_{\sigma,C} A$ it is easy to show that S is bounded by $\max_i |\sigma_i'/\sigma_i| \leq C$.

In the case that $\dim(\mathcal{R}(A')) = \dim(\mathcal{R}(A))$, we have that $I' = I$, and then it follows easily that O is an isometry, $\overline{R(A)} \to \overline{R(A')} \subset Y'$, and surjective, hence unitary. Since $A' \lessapprox_{\sigma,C} A$ clearly implies $A'^* \lessapprox_{\sigma,C} A^*$, applying the previous results gives $A'^* = OA^*S$, hence $A' = S^*AO^*$, where $O^* : \overline{R(A')} = \mathcal{N}(A')^\perp \to \overline{R(A)} = \mathcal{N}(A)^\perp$ is again unitary.

In the other case, $\dim(\mathcal{R}(A')) < \dim(\mathcal{R}(A))$, we have $I' < I$, and O is only a partial isometry since the elements ψ_i, $i \in I \setminus I'$, are in its nullspace. ∎

REMARK 3.4. The case that $\dim(\mathcal{R}(A')) < \dim(\mathcal{R}(A))$ cannot appear if A and A' are both ill-posed operators in the sense of Nashed [29]. Since we are mostly interested in the ill-posed situation, we do not consider it in detail.

A simple corollary of Lemma 3.3 is the following result in the case that the singular values of A and A' are comparable.

COROLLARY 3.5. *Assume that A and A' are compact operators between Hilbert spaces with equivalent decay rates of the singular values, i.e., $A \approx_\sigma A'$, or, in more detail,*

$$\exists\, c, C: \quad c\sigma_n(A') \leq \sigma_n(A) \leq C\sigma_n(A'), \quad \forall\, n \in \mathbb{N}. \tag{3.2}$$

Then there exists an isomorphism $T : \overline{R(A)} \to \overline{R(A')}$ (i.e., a bounded invertible linear map) and a unitary operator $U : N(A')^\perp \to N(A)^\perp$ such that

$$A' = TAU.$$

Also, there exists an isomorphism $S : N(A')^\perp \to N(A)^\perp$ and a unitary operator O : $\overline{R(A)} \to \overline{R(A')}$ such that

$$A' = OAS.$$

In particular, if both operators are injective, then S is an isomorphism $X' \to X$; and if both operators are surjective, then T is an isomorphism $Y \to Y'$.

Proof. The assumption (3.2) gives that $\dim(\mathcal{R}(A')) = \dim(\mathcal{R}(A))$; hence by Lemma 3.3 $A' = OAS$. From (3.2) it follows that $c \leq \sigma_i'/\sigma_i \leq C$, and thus the operator S in (3.1) is bounded. By interchanging σ', ϕ_i' and σ, ϕ_i in (3.1), we obtain the inverse $S^{-1} : N(A)^\perp \to N(A')^\perp$, which is bounded due to (3.2), and thus S is an isomorphism. The result with

$A' = TAU$ follows by applying the previous result to A'^* and A^*, yielding $A'^* = OA^*S$, and then taking adjoints. ∎

Now we are ready to state the first main result of the equivalence between Definition 2.1 and Definition 3.2.

THEOREM 3.6. *Let A and A' be compact operators between Hilbert spaces such that both of them are ill-posed, which means that* $\dim(\mathcal{R}(A)) = \dim(\mathcal{R}(A')) = \infty$.

Then the following statements are equivalent:

1. $A' \leq_{\mathcal{B},\mathcal{B}} A \iff$ *2.* $A' \leq_{\mathcal{O},\mathcal{B}} A \iff$ *3.* $A' \leq_{\mathcal{B},\mathcal{O}} A \iff$ *4.* $A' \lessapprox_{\sigma,C} A$.

Proof. Assume that item 1. holds. Then, $\sigma_n(A') \leq \|T\|\|S\|\sigma_n(A), \forall\, n \in \mathbb{N}$, and hence item 4. is also valid with $C = \|T\|\|S\|$. By Lemma 3.3, we have that items 2. and 3. hold, and this clearly implies item 1. Thus, all four statements are equivalent. ∎

As a corollary, we have a stability result of our ordering.

COROLLARY 3.7. *Let A_n and A'_n be sequences of compact operators that converge in norm to ill-posed limit operators as $A_n \to A$ and $A'_n \to A$ as $n \to \infty$, and where the ordering*

$$A'_n \leq_{\mathcal{B},\mathcal{B}} A_n, \quad \forall\, n \in \mathbb{N}$$

is satisfied. Then the limits preserve the ordering

$$A' \leq_{\mathcal{B},\mathcal{B}} A.$$

Proof. It follows directly that $\sigma_k(A_n) \leq \sigma_k(A'_n)$ for all k and n, and, since the singular values are continuous, we have $A' \lessapprox_{\sigma,C} A$. Consequently, the result follows by applying Theorem 3.6. ∎

REMARK 3.8. Let us point out an apparent paradox with our definition of ordering as "more ill-posed". Taking $A' = TAS$ for T an operator with finite-dimensional range, it follows that $A' \leq_{\mathcal{B},\mathcal{B}} A$ holds, but since A' has finite-dimensional range, it corresponds to a well-posed operator. In this sense any such well-posed operator with finite-dimensional range is "more ill-posed" than an ill-posed operator. The same problem does appear for other orderings like $\lessapprox_{\sigma,C}$ as well. The resolution of the "paradox" is that any such finite-dimensional operator contains in a neighborhood an ill-posed operator of arbitrary high degree of ill-posedness. Thus, as soon as we would like to have a certain stability property of the ordering, as in Corollary 3.7, such a paradox is unavoidable.

3.2. Representation of moderately ill-posed operators. An interesting consequence of the above results is a characterization of moderately ill-posed operators with the *degree of ill-posedness* $k \in \mathbb{N}$. By this, we mean a compact operator showing a power-type decay of the singular values with exponent $-k$. Basically, the canonical example of such an operator is k-times integration $I_k : L^2(0,1) \to L^2(0,1)$ with $I_k = J^k$, where the simple integration operator $J : L^2(0,1) \to L^2(0,1)$ is defined as $J : f \to \int_0^x f(t)\,dt$, which represents an injective ill-posed operator with dense range. This definition can be extended to non-integer k by using fractional integration [33]. The moderate ill-posedness of I_k with degree k is a direct consequence of the well-known decay rate $\sigma_n(I_k) \sim n^{-k}$ of the singular values of I_k (see, e.g., [22, 31, 33]). Now, by Corollary 3.5, every moderately ill-posed operator with such singular-value decay is isomorphic to the k-times integration operator.

THEOREM 3.9. *Let A be a compact linear operator mapping between infinite-dimensional Hilbert spaces, and assume that we have, for some $k \in \mathbb{N}$,*

$$\sigma_n(A) \sim \frac{1}{n^k}. \tag{3.3}$$

Then there are isomorphic linear operators $T : L^2(0,1) \to \overline{R(A)}$ *and* $S : N(A)^\perp \to L^2(0,1)$ *such that*

$$A = TJ^k S.$$

One of the operators S *or* T *can be chosen unitary.*

Thus, the theory of moderately ill-posed problem boils down to understanding differentiation.

This theorem can be extended to more general situations, keeping the main assumptions on A. With minor modifications, we may consider instead of (3.3) more general decay rates. For $\alpha > \beta$ and constants $c, C > 0$:

$$\frac{c}{n^\alpha} \le \sigma_n(A) \le \frac{C}{n^\beta}. \tag{3.4}$$

In this case we have

$$A = TJ^\beta S,$$

with one of the operators S or T being unitary and the other one, for example S, satisfying an error estimate:

$$c\|J^{\alpha-\beta}x\| \le \|Sx\| \le C\|x\|, \quad x \in N(A)^\perp.$$

This follows, for instance, by using S as in (3.1) from the proof of Lemma 3.3. This immediately yields $A = OJ^\beta S$, and it can then be verified that

$$c\frac{1}{n^{\alpha-\beta}} \le cn^\beta \sigma_n(A) \le \sigma_i(S) \le C.$$

Thus, $J^{\alpha-\beta} \lessapprox_{\sigma,C} S$ and using Lemma 3.3 then gives the estimate. Note that, instead of J, other operators might be used, for example Sobolev scale embeddings.

The estimate (3.4) implies that the *interval of ill-posedness* for a compact operator $A : X \to Y$ (see [21]), defined as

$$[\underline{\mu}, \overline{\mu}] := \left[\liminf_{n\to\infty} \frac{-\log(\sigma_n(A))}{\log(n)}, \limsup_{n\to\infty} \frac{-\log(\sigma_n(A))}{\log(n)}\right], \tag{3.5}$$

satisfies $[\underline{\mu}, \overline{\mu}] = [\beta, \alpha]$. Conversely, it has been shown in [21] that a finite interval of ill-posedness $[\underline{\mu}, \overline{\mu}]$ implies (3.4) with $\beta = \underline{\mu} - \epsilon$ and $\alpha = \overline{\mu} + \epsilon$ for all $\epsilon > 0$. Note that in this case the lower bound $\underline{\mu}$ characterizes the degree of ill-posedness.

4. Alternative orderings and the relation to regularization. Let us recall some alternative orderings that were defined in [18]. Consider operators A and A' having common domain space X, i.e., $A : X \to Y$ and $A' : X \to Y'$. Then, define the norm orderings as

$$\begin{aligned} A' \le_{\text{norm}} A \quad &\Longleftrightarrow \quad \|A'x\| \le \|Ax\|, \quad \forall x \in X, \\ A' \le_{\text{norm},C} A \quad &\Longleftrightarrow \quad \|A'x\| \le C\|Ax\|, \quad \forall x \in X. \end{aligned}$$

Let $M \subset X$ be a conical set. Define the *modulus of injectivity* as

$$j(A, M) := \inf_{x \in M,\, x \ne 0} \frac{\|Ax\|}{\|x\|}.$$

The modulus of injectivity is related to the *modulus of continuity*,

$$\omega(\delta, A, M) := \sup\{\|x\| : x \in M, \|Ax\| \leq \delta\}, \quad \forall\, \delta > 0,$$

by the identity [18]

$$j(A, M) = \frac{\delta}{\omega(\delta, A, M)}, \qquad \forall\, \delta > 0.$$

Thus, we may define the ordering by continuity (or injectivity) as follows. Let M_γ be a family of increasing conical sets with $\bigcup_\gamma M_\gamma = X$. Then,

$$A' \leq_{j,M_\gamma} A \quad \Longleftrightarrow \quad j(A', M_\gamma) \leq j(A, M_\gamma), \quad \forall\, \gamma.$$

It follows immediately that

$$A' \leq_{\text{norm}} A \quad \Longrightarrow \quad A' \leq_{j,M_\gamma} A \quad \text{for all families } M_\gamma \text{ with } \bigcup_\gamma M_\gamma = X.$$

Of particular interest in applications and for discretization, we let $M_\gamma = X_n$, where X_n is a strictly increasing sequence of finite-dimensional subspaces with $\dim(X_n) = n$ and $\bigcup_n X_n = X$. We have [18]

$$A' \leq_{\text{norm}} A \quad \Longrightarrow \quad A' \leq_{j,X_n} A \quad \Longrightarrow \quad A' \leq_\sigma A.$$

The opposite direction is obtained by replacing A by AO:

$$A' \leq_\sigma A \quad \Longrightarrow \quad \exists\, O \in \mathcal{O} : \forall\, (X_n)_n : j(A', X_n) \leq j(AO, X_n),$$

or

$$A' \leq_\sigma A \quad \Longleftrightarrow \quad \forall\, (X_n)_n : \exists\, Y_n : \, j(A', X_n) \leq j(A, Y_n).$$

The last identities are obtained by invoking Lemma 3.3 to get $A' = TAO$, yielding the norm ordering $\|A'x\| \leq \|AOx\|$.

One motivation for studying orderings of ill-posed operators comes from comparing approximation rates for (Tikhonov) regularization schemes. Assume that two linear ill-posed problems with the same exact solution $x^\dagger$ are modeled by the operators A and A', respectively. This means that we consider regularized solutions to $Ax^\dagger = y$ and $A'x^\dagger = y'$. For simplicity, we consider exact data $y = Ax^\dagger$ and $y' = Ax^\dagger$. Using Tikhonov regularization $x_{A,\alpha} := (A^*A + \alpha I)^{-1}A^*y$ and $x_{A',\alpha} := (A'^*A' + \alpha I)^{-1}A'^*y'$, we may compare the approximation errors between the regularized solutions and $x^\dagger$. It makes sense to define

$$A' \leq_{\text{Tik}} A \quad \Longleftrightarrow \quad \|x_{A',\alpha} - x^\dagger\| \geq \|x_{A,\alpha} - x^\dagger\|, \quad \forall\, x^\dagger, \, \forall\, \alpha > 0,$$

which means that A' is more ill-posed in this ordering if the approximation error is always larger than the "less" ill-posed problem with A.

The following relation for the above ordering was shown in [18]:

$$A' \leq_{\text{Tik}} A \quad \Longleftrightarrow \quad A'^*A' \leq_{\text{norm}} A^*A.$$

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5. Douglas range inclusion theorem. A quite useful tool for studying operator factorization is the Douglas range inclusion theorem [8], which was employed recently in [28] and which we recall here in the form of Theorem 5.1 for our concept. Assume that A and A' have a common image space Y.

THEOREM 5.1 (Douglas range inclusion theorem). *The following statements are equivalent:*

1. $\mathcal{R}(A') \subset \mathcal{R}(A)$.
2. $\exists\, C > 0 : \|A'^* y\| \leq C\|A^* y\|, \quad \forall\, y \in Y$.
3. $\exists\, S \in \mathcal{B} : A' = AS$.

The operator S can be chosen as

$$S = A^\dagger A'. \tag{5.1}$$

If we impose that $\mathcal{R}(S) \subset \mathcal{N}(A)^\perp$, then S is uniquely defined by (5.1). Moreover, under this condition, $S|_{\mathcal{N}(A')^\perp}$ is injective.

Proof. The proof of the equivalences appeared in [8]. The choice of S is as in (5.1), and it was shown that S has a closed graph and hence is continuous. We show the uniqueness of S.

Let S be an arbitrary operator with $A' = AS$ and S mapping into $\mathcal{N}(A)^\perp$. Then by using $A^\dagger$ and considering the Moore–Penrose equation of the form $A^\dagger A = P_{N(A)^\perp}$, with $P_{N(A)^\perp}$ the orthogonal projector onto $N(A)^\perp$, we get $A^\dagger A' = A^\dagger AS = P_{N(A)^\perp} S = S$. Thus S must have the structure (5.1). By applying A to (5.1), it follows easily that $\mathcal{N}(S) \subset N(A')$. ∎

We thus can introduce an ordering related to the Douglas theorem defined by a range inclusion as follows. For operators A' and A with common image spaces $Y = Y'$, we define

$$A' \leq_{\mathcal{R}} A \quad \Longleftrightarrow \quad \mathcal{R}(A') \subset \mathcal{R}(A).$$

See also [3] for range inclusions with index functions.

By Theorem 5.1 we obtain

$$A' \leq_{\mathcal{R}} A \quad \Longleftrightarrow \quad A'^* \leq_{\text{norm},C} A^* \quad \Longleftrightarrow \quad A' = AS \quad \Longrightarrow \quad A' \leq_{\mathcal{O},\mathcal{B}} A,$$

as well as

$$A'^* \leq_{\mathcal{R}} A^* \quad \Longleftrightarrow \quad A' \leq_{\text{norm},C} A \quad \Longleftrightarrow \quad A' = TA \quad \Longrightarrow \quad A' \leq_{\mathcal{B},\mathcal{O}} A.$$

The opposite direction is immediate:

$$A' \leq_{\mathcal{B},\mathcal{B}} A \quad \Longleftrightarrow \quad \exists\, Q \in \mathcal{O} : A' \leq_{\text{norm},C} AQ \quad \Longleftrightarrow \quad \mathcal{R}(A'^*) \subset \mathcal{R}(Q^* A^*).$$

Here, the first equivalence comes from Theorem 3.6, according to which $A' = TAO$, hence the norm ordering follows. Vice versa, the norm ordering implies by the Douglas range theorem the existence of a connecting operator $A' = TAO$. The last equivalence comes from item 2. in Theorem 5.1, yielding $\mathcal{R}(A'^*) \subset \mathcal{R}(Q^* A^*)$.

A schematic description of the different relations is given in Table 5.1.

TABLE 5.1
Relation between different orderings.

$\exists Q \in \mathcal{O} : A' \leq_{\text{norm},C} AQ$	$\Longleftrightarrow$	$A' \leq_{\mathcal{B},\mathcal{B}} A$ $A' \leq_{\mathcal{B},\mathcal{O}} A$ $A' \leq_{\mathcal{O},\mathcal{B}} A$	$\Longleftrightarrow$	$A' \lessapprox_{\sigma,C} A$
				$\Uparrow$
$\Uparrow$		$\Uparrow$		$\forall (X_n)_n : A' \leq_{j,X_n} A$
$A' \leq_{\text{norm},C} A$ $A'^* \leq_{\mathcal{R}} A^*$	$\Longleftrightarrow$	$A' \leq_{\mathcal{B},\mathcal{I}} A$	$\Nearrow$	
$\Uparrow$		$\Uparrow$		
$A'^* A' \leq_{\text{norm}} A^* A$	$\Longleftrightarrow$	$A' \leq_{\text{Tik}} A$		

6. Ordering in the non-compact case.

6.1. General assertions. The previous results for compact operators made heavy use of the singular-value decomposition. We next address the question as to how far they can be generalized to the non-compact case. It turns out that our main theorem, Theorem 3.6, is still valid.

First, we state the following well-known results [10, Prop. 2.18].

PROPOSITION 6.1. *Let A be bounded between Hilbert spaces. Then,*

$$\mathcal{R}(A^*) = \mathcal{R}(\sqrt{A^*A}).$$

A consequence is the polar decomposition; see [32, p. 323] and the remark therein.

PROPOSITION 6.2. *Let A be bounded. Then there exists a U which is a partial isometry such that*

$$A = U\sqrt{A^*A},$$

where

$$U : \overline{R(A^*)} \to \overline{R(A)}$$

is an isometry. If there is an isometry from $N(A)$ onto $N(A^*)$, then U is unitary.

We have the following lemma.

LEMMA 6.3. *Let $A, D \in \mathcal{B}$. Then there exists a partial isometry $Q : \overline{\mathcal{R}(A)} \to \overline{\mathcal{R}(DA)}$, and an $S \in \mathcal{B}$ with*

$$DA = QAS.$$

If

$$\overline{\mathcal{R}(A^*)} = \overline{\mathcal{R}(A^*D^*)}, \tag{6.1}$$

then Q is an isometry.

Proof. By the polar decomposition, we have an isometry U from $\overline{\mathcal{R}(A^*D^*)} \to \overline{\mathcal{R}(DA)}$ with

$$DA = U\sqrt{A^*D^*DA}.$$

Let $Z := \sqrt{A^*D^*DA}$. Thus,

$$\mathcal{R}(Z) = \mathcal{R}(A^*D^*) \subset \mathcal{R}(A^*) = \mathcal{R}(\sqrt{A^*A}).$$

By the Douglas range inclusion theorem, it follows that there exists an $S \in \mathcal{B}$ with

$$Z = \sqrt{A^*A}S.$$

Again, by the polar decomposition, we have an isometry $W : \overline{\mathcal{R}(A^*)} \to \overline{\mathcal{R}(A)}$ with $A = W\sqrt{A^*A}$. Now $W^*W = I$ when restricted to $\overline{\mathcal{R}(A^*)}$. Thus,

$$W^*A = \sqrt{A^*A} \quad \text{and} \quad DA = U\sqrt{A^*A}S = UW^*AS,$$

and UW^* is a partial isometry from $\overline{\mathcal{R}(A)} \to \overline{\mathcal{R}(DA)}$ as claimed.

Assume that (6.1) holds, and let $x \in \mathcal{N}(UW^*)$, i.e., $UW^*x = 0$. Then we have $\mathcal{R}(W^*) = \overline{\mathcal{R}(A^*)} = \overline{\mathcal{R}(A^*D^*)}$. Since $\mathcal{N}(U) = \mathcal{R}(A^*D^*)^\perp$, it follows that $W^*x = 0$. Since $\mathcal{N}(W^*) = \mathcal{R}(A)^\perp$, the result follows. ∎

The condition (6.1) can equally be written as

$$\mathcal{N}(A) = \mathcal{N}(DA).$$

Recall the set of partial isometries $\mathcal{PO}$ and its use in orderings in Definition 3.2. We have the following result.

THEOREM 6.4. *Let A and A' be bounded operators between Hilbert spaces. Then the following statements are equivalent:*

$$\textit{1. } A' \leq_{\mathcal{B},\mathcal{B}} A \quad \Longleftrightarrow \quad \textit{2. } A' \leq_{\mathcal{PO},\mathcal{B}} A \quad \Longleftrightarrow \quad \textit{3. } A' \leq_{\mathcal{B},\mathcal{PO}} A.$$

If the connecting operator T in item 1. is injective on $\overline{R(A)}$, then we have

$$A' \leq_{\mathcal{B},\mathcal{B}} A \quad \Longleftrightarrow \quad A' \leq_{\mathcal{O},\mathcal{B}} A.$$

If $\mathcal{R}(S)$ in item 1. is dense in $\mathcal{N}(A)^\perp$, then

$$A' \leq_{\mathcal{B},\mathcal{B}} A \quad \Longleftrightarrow \quad A' \leq_{\mathcal{B},\mathcal{O}} A.$$

Proof. Assume that item 1. holds. Then $A' = TAS$, and by Lemma 6.3 applied to TA we have $TAS = QA\tilde{S}S$, where Q is a partial isometry. Since $\tilde{S}S$ is bounded, item 2. follows. The same with A'^*, A^* in place of A', A yields item 3. The implication 3. $\Rightarrow$ 1. holds clearly.

If T is injective, then $\mathcal{N}(TA) = \mathcal{N}(A)$, and by Lemma 6.3, Q is an isometry. The same argument with adjoints and S yields the second statement. ∎

6.2. Composition of Hausdorff and Cesàro operators. In [27], we have considered compositions of the non-compact Hausdorff operator $H : L^2(0,1) \to \ell^2(\mathbb{N})$ with non-closed range defined as

$$[Hx]_j := \int_0^1 x(t)t^{j-1}\,dt \quad (j = 1, 2, \ldots,\ x \in L^2(0,1)),$$

possessing the corresponding adjoint operator $H^*: \ell^2(\mathbb{N}) \to L^2(0,1)$ of the form

$$[H^*y](t) := \sum_{j=1}^{\infty} y_j t^{j-1} \quad (0 \le t \le 1,\ y \in \ell^2(\mathbb{N})),$$

and the non-compact Cesàro operator $C: L^2(0,1) \to L^2(0,1)$ with non-closed range defined as

$$[Cx](s) := \frac{1}{s} \int_0^s x(t)\,dt \quad (0 \le s \le 1,\ x \in L^2(0,1)),$$

having $C^*: L^2(0,1) \to L^2(0,1)$ of the form

$$[C^*x](t) := \int_t^1 \frac{x(s)}{s}\,ds \quad (0 \le t \le 1,\ x \in L^2(0,1))$$

as adjoint operator. Generalizations of these operators in weighted spaces and their ill-posedness are discussed in [26].

There is a connection with the compact self-adjoint diagonal operator $D: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined as

$$[Dy]_j := \frac{y_j}{j} \quad (j = 1, 2, \dots,\ y \in \ell^2(\mathbb{N}))$$

of the form $DH = HC^*$ (see [27, Prop. 2]), but our focus here is on the adjoint version with the composition $H^*D: \ell^2(\mathbb{N}) \to L^2(0,1)$ leading to the equality

$$H^*D = CH^*. \tag{6.2}$$

Also here, the composition H^*D is a compact operator even though both factors C and H^* are non-compact operators.

We mention two facts in this context. On the one hand, equation (6.2) expresses some kind of *similarity* between the operators D and C, even if one cannot simply write $D = (H^*)^{-1}CH^*$, because $(H^*)^{-1}$ is an unbounded operator. For the factorization (6.2), relation (2.1) applies with $A := C$, $A' = H^*D$, $X = L^2(0,1)$, $X' = \ell^2(\mathbb{N})$, and $Y = Y' = L^2(0,1)$ as

$$H^*D \le_{\mathcal{I},\mathcal{B}} C,$$

with the identity operator I in $L^2(0,1)$ and the connecting operator in $\mathcal{B}(X', X)$ is $S := H^*$. In the sense of Definition 2.1, the compact operator H^*D is *strictly more ill-posed* than the non-compact Cesàro operator C because non-compact operators can never be more ill-posed than compact operators. This is a consequence of the factorization in (2.1), where a compact operator A on the right-hand side always implies compactness of A'.

For ill-posedness degree discussions of compositions for the Hausdorff operator, the Cesàro operator, and multiplication operators with the integration operator J, we refer the reader to [6, 12, 19] and [22, 23], respectively.

6.3. Non-compact multiplication operators mimicking compact operators. As outlined in [34], the Halmos spectral theorem can be helpful for measuring and comparing the ill-posedness of classes of injective, positive semi-definite, self-adjoint, and bounded linear operators with non-closed range by using orthogonal transformations leading to bounded linear multiplication operators M_λ mapping in the real Hilbert space $L^2([0,\infty))$ with non-closed range defined as

$$[M_\lambda, x](\omega) := \lambda(\omega)\, x(\omega) \quad (0 \le \omega < \infty,\ x \in L^2([0,\infty))), \tag{6.3}$$

for real multiplier functions $\lambda \in L^\infty([0,\infty))$. Note that, in general, the Halmos spectral theorem gives a multiplication operator M_λ on the space $L^2(\Omega,\mu)$ with a general semi-finite measure μ on a locally compact space Ω. For simplicity only, we consider the special case that μ is the Lebesgue measure on $\Omega = [0,\infty)$ leading to the Hilbert space $L^2([0,\infty))$ as basis space for M_λ as stated. Using the Lebesgue measure seems to be reasonable, and we refer to discussions and examples in [34] in this context.

Using multiplication operators, the point of this section is the observation that, in contrast to the compact case, cf. Theorem 3.6, the spectrum of, say, A^*A alone does not contain enough information to conclude about equivalences with respect to the ordering $\le_{\mathcal{B},\mathcal{B}}$. Specifically, we construct operators A' and A, where A is compact and A' not, which have the same spectrum but are not equivalent with respect to $\le_{\mathcal{B},\mathcal{B}}$.

For that purpose, we consider the diagonalized injective self-adjoint compact operator $A' : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ with non-increasingly ordered singular values σ_n^2 $(n = 1, 2, \ldots)$ tending to zeros as $n \to \infty$, which will be introduced as

$$[A'z]_n := \sigma_n^2\, z_n \quad (n = 1, 2, \ldots,\ z = (z_1, z_2, \ldots)^T \in \ell^2(\mathbb{N})).$$

Its mimicking multiplication operator counterpart is the operator

$$[Ax](\omega) := \sum_{n=1}^{\infty} \sigma_n^2 \chi_{[n-1,n)}(\omega) x(\omega) \quad (0 \le \omega < \infty,\ x \in L^2([0,\infty)),$$

defined as a sum of characteristic functions over sets of Lebesgue measure one.

The spectrum $\sigma(A)$ of the operator A equals the essential range of the multiplier function λ (see [14, Thm. 2.1(g)]), and this gives

$$\sigma(A) = \{0\} \cup \bigcup_{n=1}^{\infty} \{\sigma_n^2\} = \sigma(A').$$

One could conjecture then that A is also a compact operator, but this is not true, because all eigenvalues σ_n^2 of A possess infinite-dimensional eigenspaces; we refer to [34, Prop. 2] for the following proposition.

PROPOSITION 6.5. *All injective, positive semi-definite, self-adjoint, bounded linear multiplication operators M_λ of type* (6.3) *are non-compact.*

As a consequence

$$A \not\le_{\mathcal{B},\mathcal{B}} A'.$$

It is easy to see that there is a bounded linear operator $S : \ell^2(\mathbb{N}) \to L^2([0,\infty))$ defined as

$$[Sz](\omega) := \sum_{n=1}^{\infty} z_n \chi_{[n-1,n)}(\omega) \quad (\omega \in [0,\infty),\ z = (z_1, z_2, \ldots)^T \in \ell^2(\mathbb{N})),$$

such that, with $AS : \ell^2(\mathbb{N}) \to L^2([0,\infty))$,

$$SA' = AS.$$

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Again, we observe some kind of similarity between the operators A' and A, even if one cannot simply write $A' = S^{-1}AS$. But the situation here is different from the situation in Section 6.2, because $\|Sz\|^2_{L^2([0,\infty))} = \|z\|^2_{\ell^2(\mathbb{N})}$ for all $z \in \ell^2(\mathbb{N})$ and S has a closed range in $L^2([0,\infty))$ such that the Moore–Penrose pseudo-inverse operator $S^\dagger$ is bounded and $A' = S^\dagger AS$ holds true. Hence, we have as main conclusion

$$A' \leq_{\mathcal{B},\mathcal{B}} A, \quad A' \not\sim_{\mathcal{B},\mathcal{B}} A \quad \text{but} \quad \sigma(A') = \sigma(A).$$

Thus, in the non-compact case, a result analogous to Corollary 3.5 cannot hold.

7. Banach space case. In this section, we extend some results to the Banach space case. Since orthogonality does not make sense there, we only use the ordering $\leq_{\mathcal{B},\mathcal{B}}$. We denote by a superscript $*$ the topological dual operator.

THEOREM 7.1. *Let $A : X \to Y$ and $A' : X' \to Y'$ be bounded operators between Banach spaces. Then*

$$A' \leq_{\mathcal{B},\mathcal{B}} A \quad \Longleftrightarrow \quad A'^* \leq_{\mathcal{B},\mathcal{B}} A^*.$$

For compact operators mapping between Banach spaces, we can replace singular values by s-numbers.

DEFINITION 7.2. *Let A and A' be compact operators as above. Let s be an s-number* [30]. *Define the ordering*

$$A' \leq_{s,C} A \quad \Longleftrightarrow \quad \exists\, C > 0 \text{ such that } \forall\, n : s_n(A') \leq C s_n(A).$$

Directly from the axioms of s-numbers we have the following result.

LEMMA 7.3. *The following holds:*

$$A' \leq_{\mathcal{B},\mathcal{B}} A \quad \Longrightarrow \quad A' \leq_{s,C} A.$$

It is unknown to the authors if and under what conditions a reverse implication could hold true.

The Douglas range inclusion theorem does not hold in Banach spaces without additional assumptions. The following results are due to [1, 9].

THEOREM 7.4. *Let $A : X \to Y$ and $A' : X \to Y'$ be bounded linear operators mapping between the Banach spaces X, Y, and Y'.*

1. *The following statements are equivalent:*
   - $\mathcal{R}(A'^*) \subset \mathcal{R}(A^*)$.
   - $A' = TA, \quad T \in \mathcal{B}(\overline{\mathcal{R}(A)}, X)$.
   - $\exists\, C > 0 : \|A'x\| \leq C\|Ax\|, \quad \forall\, x \in X$.
2. *Let $Y' = Y$. Moreover, assume that*

$$\mathcal{R}(A') \subset \mathcal{R}(A)$$

and that $\mathcal{N}(A)$ has a closed complemented subspace $X = \mathcal{N}(A) \oplus W$. Then

$$A' = AS, \quad S \in \mathcal{B}(X, X).$$

As a consequence, we have the following implication.

THEOREM 7.5. *Let $A, A' : X \to Y$ and A be injective (or have finite-dimensional nullspace). Then*

$$A' \leq_{\mathcal{B},\mathcal{B}} A \quad \Longleftrightarrow \quad \exists\, T \in \mathcal{B} : A'^*T^* \leq_{\text{norm}} A^*.$$

Proof. The implication "⇒" follows easily with $A' = TAS$ from the boundedness of S^*. For the opposite direction "⇐", use Theorem 7.4, item 1. with $A' := A^*T^*$ and A^* in place of A. □

8. Nonlinear case studies.

8.1. General assertions. In this section, we discuss and extend the concept of ordering by ill-posedness for the case of nonlinear operator equations

$$F(x) = y, \tag{8.1}$$

modeling nonlinear ill-posed problems, where the *nonlinear* operator F with $F : D(F) \subset X \to Y$ is *weakly sequentially closed* with *non-compact, convex, and closed domain* $D(F)$, and X and Y are Hilbert spaces. Unless stated otherwise, we also assume that F has a Fréchet derivative as well.

We usually assume that F is *locally ill-posed* at a point $x^\dagger \in D(F)$ in the sense that there exists a closed ball $B_\rho(x^\dagger)$ around $x^\dagger$ with radius $\rho > 0$ and a sequence $(x_n)_{n=1}^\infty \in D(F) \cap B_\rho(x^\dagger)$ with

$$\lim_{n\to\infty} \|F(x_n) - F(x^\dagger)\|_Y = 0, \quad \text{but} \quad \|x_n - x^\dagger\|_X \not\to 0 \quad \text{as } n \to \infty.$$

For characterizing the strength and nature of local ill-posedness at $x^\dagger \in D(F)$, ideas have been outlined in [13, 16, 17, 20] to use the ill-posedness nature of linearizations to F at $x^\dagger$ and in particular the decay rate of singular values of the compact Fréchet derivative $F'[x^\dagger]$. This leads to the following definition.

DEFINITION 8.1 (Local degree of ill-posedness). *Let the* local degree of ill-posedness *of the nonlinear operator $F : D(F) \subset X \to Y$ at the point $x^\dagger \in D(F)$ be the degree of ill-posedness of the compact Fréchet derivative $F'[x^\dagger]$ at this point. Then the moderate and severe ill-posedness of $F'[x^\dagger]$ and the associated interval of ill-posedness (see (3.5)) transfer to the local nature of ill-posedness of the nonlinear operator F at $x^\dagger$.*

Generic examples of locally ill-posed operators are the completely continuous (compact) operators F that transform weak convergence sequences into norm convergence ones.

PROPOSITION 8.2. *If $F : D(F) \subset X \to Y$ is completely continuous, then the nonlinear equation (8.1) is locally ill-posed everywhere on the interior* $\operatorname{int}(D(F))$ *of the domain $D(F)$.*

Proof. Choose $x^\dagger \in \operatorname{int}(D(F))$ arbitrarily. Then, for a sufficiently small radius $\rho > 0$, we have that $B_\rho(x^\dagger) \subset D(F)$, and, for any orthonormal system $\{e_n\}_{n=1}^\infty$ in X, we have weak convergence as $e_n \rightharpoonup 0$ and $x^\dagger + \rho e_n \rightharpoonup x^\dagger$ for $n \to \infty$ with $x_n := x^\dagger + \rho e_n \in B_\rho(x^\dagger)$ and $\|x_n - x^\dagger\|_X = \rho > 0$. Then the assumed weak continuity yields the assertion of the proposition. ∎

Moreover, for completely continuous (compact) operators, the Fréchet derivative is compact as well, hence the linearization is also ill-posed. As Example 8.9 below will show, the converse assertion is not true. There exist non-compact nonlinear operators F such that the Fréchet derivatives $F'[\cdot]$ are compact linear operators. This illustrates the problem that, even for differentiable operators and with regard to ill-posedness, the nonlinear operator can behave completely differently than its linearization, and in particular Definition 8.1 might become useless in certain situations.

Another example illustrating this appeared in [11, A.1], where a nonlinear operator is everywhere ill-posed, but its linearization is well-posed on a dense set! This shows that, without additional conditions, linear concepts of ill-posedness, as we used in the previous sections, are not appropriate in the nonlinear case.

The additional condition that we need here is that all linearizations locally behave in a similar way.

DEFINITION 8.3. *Let F be a Fréchet-differentiable operator that is locally ill-posed at $x^\dagger \in D(F)$. We say that it is* stably ill-posed *if there exists a ball $\mathcal{B}_\rho(x^\dagger)$ around $x^\dagger$ with radius ρ such that*

$$F'(x) \sim_{\mathcal{B},\mathcal{B}} F'(x^\dagger) \quad \textit{for all } x \in \mathcal{B}_\rho(x^\dagger) \cap D(F).$$

As a consequence of Theorem 3.6, in the case that $F'(x)$ is compact for all $x \in \mathcal{B}_\rho(x^\dagger)$, a necessary and sufficient condition for stable ill-posedness is that there exist constants $\underline{c}$ and $\overline{c}$ such that

$$\underline{c}\sigma_n(F'[x]) \leq \sigma_n(F'[x^\dagger]) \leq \overline{c}\sigma_n(F'[x]), \quad \forall\, x \in B_\rho(x^\dagger) \cap D(F). \tag{8.2}$$

Conditions sufficient for (8.2) are given by certain nonlinearity conditions. Assume that, for a linear bounded operator $R_{x,\tilde{x}} \in \mathcal{B}(Y,Y)$ depending on $x, \tilde{x} \in B_\rho(x^\dagger)$ and for some exponent $0 < \kappa \leq 1$ with the constant $C_R > 0$, it holds that

$$F'[x] = R_{x,\tilde{x}} F'[\tilde{x}], \qquad \forall\, x, \tilde{x} \in B_\rho(x^\dagger) \cap D(F), \tag{8.3}$$

$$\|R_{x,\tilde{x}} - I\|_{\mathcal{B}(Y,Y)} \leq C_R \|x - \tilde{x}\|_X^\kappa, \quad \forall\, x, \tilde{x} \in B_\rho(x^\dagger) \cap D(F). \tag{8.4}$$

We refer to [24] for consequences and to [15] for applications. A similar, slightly more general, condition was also used in [2].

In the notation of the previous sections, condition (8.3) means that

$$F'[x] \sim_{\mathcal{B},\mathcal{I}} F'[\tilde{x}], \quad \forall\, x, \tilde{x} \in B_\rho(x^\dagger) \cap D(F),$$

where the operator $R_{x,\tilde{x}}$ constitutes the connecting operator in $\mathcal{B}$.

A direct consequence of the previous results (see Table 5.1) is the following.

PROPOSITION 8.4. *The condition (8.3) implies stable ill-posedness. Moreover, we have*

$$(8.3) \quad \Longleftrightarrow \quad (F'[x] \sim_{\text{norm},C} F'[\tilde{x}], \quad \forall\, x, \tilde{x} \in B_\rho(x^\dagger) \cap D(F)) \quad \Longrightarrow \quad (8.2).$$

We are now in the position to state different concepts of ordering in the nonlinear case.

DEFINITION 8.5. *Let F and G be nonlinear Fréchet-differentiable operators, and let $B_\rho(x^\dagger)$ be a ball around $x^\dagger$ with radius $\rho > 0$. Denote by $\mathcal{NB}$ the set of continuous possibly nonlinear operators defined on a ball B_ρ (with possibly additional properties stated when required).*

Define

$$F \leq_{\mathcal{NB},\mathcal{NB}} G \quad \Longleftrightarrow \quad \left\{ \begin{matrix} \Psi \in \mathcal{NB},\ \Phi \in \mathcal{NB}: \ \textit{with} \\ F(x) = \Psi \circ G \circ \Phi(x), \end{matrix} \right\} \quad \forall\, x \in B_\rho(x^\dagger) \cap D(F).$$

Moreover, for fixed $x^\dagger$ and for F and G defined on $\mathcal{B}_\rho(x^\dagger)$, we define the linearized ordering as

$$F \leq^{\text{Lin}}_{\mathcal{B},\mathcal{B}} G \quad \Longleftrightarrow \quad F'[x^\dagger] \leq_{\mathcal{B},\mathcal{B}} G'[x^\dagger]$$

and the uniform linearized ordering as

$$F \leq^{\text{UniLin}}_{\mathcal{B},\mathcal{B}} G \quad \Longleftrightarrow \quad F'[x] \leq_{\mathcal{B},\mathcal{B}} G'[x], \quad \forall\, x \in B_\rho(x^\dagger) \cap D(F).$$

Here the nonlinear ordering $\leq_{\mathcal{NB},\mathcal{NB}}$ reflects the idea that the information transfer in F from input to output is passed though G and hence any "information" lost in G will also be lost in F. Thus, F is considered more ill-posed.

Clearly, by the chain rule, we have

$$F \leq_{\mathcal{NB},\mathcal{NB}} G \quad \Longrightarrow \quad F \leq_{\mathcal{B},\mathcal{B}}^{\mathrm{UniLin}} G \quad \Longrightarrow \quad F \leq_{\mathcal{B},\mathcal{B}}^{\mathrm{Lin}} G.$$

Moreover, if F and G are stably ill-posed, then

$$F \leq_{\mathcal{B},\mathcal{B}}^{\mathrm{Lin}} \quad \Longrightarrow \quad F \leq_{\mathcal{B},\mathcal{B}}^{\mathrm{UniLin}} G.$$

As the nonlinear ordering is difficult to prove, the linearized orderings serve as practical computable conditions to verify it. Thus, it is of interest to consider conditions when the linearized orderings imply the nonlinear one.

In Section 8.4 we further state such conditions. For this we have to take into account the degree of nonlinearity.

8.2. Degree of nonlinearity. The nonlinearity conditions (8.2) or (8.3) constitute a class of restrictions on the nonlinearity. Yet another (weaker) class of conditions are the conditions of tangential cone type. A parametric class of such conditions were postulated in [20, Definition 1] by introducing the concept of a *local degree of nonlinearity* at $x^\dagger$ with the exponent triple $(\gamma_1, \gamma_2, \gamma_3) \in [0,1] \times [0,1] \times [0,2]$ as follows.

DEFINITION 8.6 (Local degree of nonlinearity). *We call the operator* $F : D(F) \subset X \to Y$ locally nonlinear at $x^\dagger \in D(F)$ of degree $(\gamma_1, \gamma_2, \gamma_3) \in [0,1] \times [0,1] \times [0,2]$ *if there is a radius* $\rho > 0$ *and a constant* $q > 0$ *such that the estimate*

$$\begin{aligned} &\|F(x) - F(x^\dagger) - F'[x^\dagger](x - x^\dagger)\|_Y \\ &\quad \leq q \|F'[x^\dagger](x - x^\dagger)\|_Y^{\gamma_1} \, \|F(x) - F(x^\dagger)\|_Y^{\gamma_2} \, \|x - x^\dagger\|_X^{\gamma_3}, \\ &\qquad \forall\, x \in B_\rho(x^\dagger) \cap D(F) \end{aligned} \tag{8.5}$$

holds true.

The inequality in (8.5) is not scaling-invariant in F (i.e., invariant when F is replaced by λF for any $\lambda \in \mathbb{R}$). This is, however, achieved when we restrict our considerations to the degree triple $(1-\gamma, \gamma, 0)$ for $\gamma \in [0,1]$ with constants $q = q_\gamma > 0$ (subsuming the γ_3 part into the constants) such that

$$\begin{aligned} &\|F(x) - F(x^\dagger) - F'[x^\dagger](x - x^\dagger)\|_Y \\ &\quad \leq q_\gamma \|F'[x^\dagger](x - x^\dagger)\|_Y^{1-\gamma} \, \|F(x) - F(x^\dagger)\|_Y^{\gamma}, \\ &\qquad \forall\, x \in B_\rho(x^\dagger) \cap D(F). \end{aligned} \tag{8.6}$$

Under the exceptional requirements that the constants q_γ must be less than one for $\gamma = 0$ and $\gamma = 1$, the conditions in (8.6) are basically equivalent for all $\gamma \in [0,1]$ in the sense that an inequality chain

$$\begin{aligned} &\underline{K} \|F'[x^\dagger](x - x^\dagger)\|_Y \leq \|F(x) - F(x^\dagger)\|_Y \leq \overline{K} \, \|F'[x^\dagger](x - x^\dagger)\|_Y, \\ &\forall\, x \in B_\rho(x^\dagger) \cap D(F) \end{aligned}$$

is valid for all such exponents γ with existing constants $0 < \underline{K} \leq \overline{K} < \infty$. Vice versa, such a chain implies (8.6) for all $\gamma \in [0,1]$ and appropriate constants $q_\gamma > 0$, as the following Lemma 8.7 shows. We refer also to [20], where the cases $\gamma = 0$ and $\gamma = 1$ have been discussed in this context.

We mention that the case $\gamma = 1$ in (8.6) corresponds to the well-known strong tangential cone condition of Hanke, Neubauer, and Scherzer, introduced in [15]. This condition plays a crucial role in the analysis of nonlinear iterative methods. Note also that, for this purpose, the constant q_1 has to be assumed small, say $q_1 < 1$.

LEMMA 8.7.

1. *Let* (8.6) *hold for some* $\gamma \in [0,1]$, *where, in the case* $\gamma = 1$, *we have to assume that* $q_1 < 1$. *Then there exists a constant* $\overline{K} > 0$ *such that*

$$\|F(x) - F(x^\dagger)\|_Y \le \overline{K}\|F'[x^\dagger](x - x^\dagger)\|_Y, \quad \forall\, x \in B_\rho(x^\dagger) \cap D(F). \tag{8.7}$$

2. *Let* (8.6) *hold for some* $\gamma \in [0,1]$, *where, in the case* $\gamma = 0$, *we have to assume that* $q_0 < 1$. *Then there also exists a constant* $\underline{K} > 0$ *such that*

$$\underline{K}\|F'[x^\dagger](x - x^\dagger)\|_Y \le \|F(x) - F(x^\dagger)\|_Y, \quad \forall\, x \in B_\rho(x^\dagger) \cap D(F). \tag{8.8}$$

Conversely, the conditions (8.7) *and* (8.8) *together imply* (8.6) *for any* $\gamma \in [0,1]$ *and associated constants* $0 < q_\gamma < \infty$ *(although for no* γ *necessarily connected with a smallness condition* $q_\gamma < 1$*).*

Proof. Assuming (8.6), the inequality (8.7) is valid by using first a triangle inequality and then Young's inequality

$$q_\gamma \left(\frac{|\Delta F|}{|F'|}\right)^\gamma \le \gamma \frac{|\Delta F|}{|F'|} + \frac{1-\gamma}{\gamma} q_\gamma^{\gamma/(1-\gamma)}$$

with $|\Delta F| = \|F(x) - F(x^\dagger)\|_Y$ and $|F'| = \|F'[x^\dagger](x - x^\dagger)\|_Y$.

The optimal constants $\overline{K} > 0$ can be found as the solutions of $z - 1 = q_\gamma z^\gamma$, $z \ge 0$, which exist for $\gamma \in [0,1)$ and for $\gamma = 1$ in the case that $q_1 < 1$.

The inequality (8.8) follows from (8.6) in an analog manner. The constants $\underline{K} > 0$ can be chosen as the inverse of the solution to $z - 1 = q_\gamma z^{1-\gamma}$.

The opposite direction, starting from (8.7) and (8.8), needs the triangle inequality again. Then, the inequality (8.6) is obtained for arbitrary $\gamma \in [0,1]$ with appropriate constants $q_\gamma > 0$ by estimating either $a \le \underline{K}^{-1} b$ or $b \le \overline{K} a$ with $a = \|F'[x^\dagger](x - x^\dagger)\|$ and $b = \|F(x) - F(x^\dagger)\|$ and then using that $\min\{a,b\} \le a^\gamma b^{1-\gamma}$. ◻

It is well known that the conditions stated above in (8.3) and (8.4) imply the tangential cone conditions (see, e.g., [15]).

PROPOSITION 8.8. *Let* (8.3) *and* (8.4) *hold. Then* (8.6) *holds in the form*

$$\|F(x) - F(x^\dagger) - F'[x^\dagger](x - x^\dagger)\|_Y \le \frac{C_R}{1+\kappa}\|F'[x^\dagger](x - x^\dagger)\|_Y \, \|x - x^\dagger\|_X^\kappa.$$

8.3. Autoconvolution as counterexample. The autoconvolution problem is an important practically relevant example, where many nonlinearity conditions fail and in particular where the definition of the local degree of ill-posedness is hardly meaningful. A comprehensive analysis of the following autoconvolution operator with respect to ill-posedness and the properties mentioned below was started with the seminal paper [13]. An interesting deficit study concerning usual nonlinearity conditions can be found in [5].

EXAMPLE 8.9. Let $X = Y = L^2(0,1)$ and with $D(F) = L^2(0,1)$ the *autoconvolution operator*

$$[F(x)(s) := \int_0^s x(s-t)\, x(t)\, dt \quad (0 \le s \le 1). \tag{8.9}$$

The associated nonlinear operator equation (8.1) with the autoconvolution operator from (8.9) is *locally ill-posed everywhere* on the whole space $L^2(0,1)$. This autoconvolution operator is a *non-compact* nonlinear operator with the Fréchet derivative $F'[x] \in \mathcal{B}(L^2(0,1), L^2(0,1))$ as

$$[F'[x]v](s) = 2\int_0^s x(s-t)\, v(t)\, dt \quad (0 \le s \le 1,\ v \in L^2(0,1)), \tag{8.10}$$

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which is a *compact* linear operator for all $x \in L^2(0,1)$ satisfying a Lipschitz condition

$$\|F'[x] - F'[x^\dagger]\|_{\mathcal{B}(L^2(0,1),L^2(0,1))} \le L\|x - x^\dagger\|_{L^2(0,1)}, \quad \forall\, x \in L^2(0,1)$$

with a global Lipschitz constant $L = 2$. This implies in detail that

$$\|F(x) - F(x^\dagger) - F'[x^\dagger](x - x^\dagger)\|_{L^2(0,1)} = \|F(x - x^\dagger)\|_{L^2(0,1)} \le \|x - x^\dagger\|^2_{L^2(0,1)}$$

for all $x \in L^2(0,1)$ and provides us with a *local degree of nonlinearity* $(0,0,2)$ at $x^\dagger$ everywhere. No tangential cone condition and no degree of nonlinearity $(\gamma_1, \gamma_2, \gamma_3)$ can be shown, where either γ_1 or γ_2 is positive for any $x^\dagger \in L^2(0,1)$. In particular, the nonlinearity condition (8.3) fails everywhere and for all $0 < \kappa \le 1$; see [5, Cor. 2.3].

Indeed, the condition (8.2) for stable ill-posedness cannot hold, and much more is true: For a fixed asymptotics (local degree of ill-posedness) of the singular values at $x^\dagger$, arbitrarily changing degrees may occur in any neighborhood of $x^\dagger$. This fact can be seen by inspection of the linear convolution operator (8.10) and has been outlined in detail in [13, Sec. 5]. This is an indicator for strong instability, and under such circumstances it does not make sense to use the Fréchet derivatives as measures for the strength of ill-posedness of F at $x^\dagger$.

8.4. Nonlinearity conditions and stable ill-posedness. We now study the consequences of the nonlinearity conditions above with respect to the relation between the nonlinear and the linear ill-posedness.

The main result in this section is that tangential cone conditions (8.6) *almost* imply a factorization into "nonlinear well-posed" and "linear ill-posed" problems, while the stronger conditions (8.3) and (8.4) remove the "almost".

The next result has already been stated in [20] for the case of injective $F'[x^\dagger]$ and (8.6) with $\gamma = 0$.

THEOREM 8.10. *Let $x^\dagger \in D(F)$ with $\rho > 0$ such that $B_\rho(x^\dagger) \subset D(F)$. Let (8.7) and (8.8) hold. Then F can be factorized as*

$$F(x) - F(x^\dagger) = N \circ (F'[x^\dagger](x - x^\dagger)), \quad \forall\, x \in B_\rho(x^\dagger), \tag{8.11}$$

where the nonlinear operator

$$N : F'[x^\dagger](B_\rho(0)) \to Y$$

has the property

$$\begin{aligned} &N(0) = 0, \\ &\underline{K}\|z\|_Y \le \|N(z)\|_Y \le \overline{K}\|z\|_Y, \quad \forall\, z \in F'[x^\dagger](B_\rho(0)). \end{aligned} \tag{8.12}$$

Conversely, if (8.11) holds with such an N satisfying (8.12), then (8.7) and (8.8) are satisfied.

Proof. For simplicity we set $A := F'[x^\dagger]$. It follows from (8.7) that $F(x + n) = F(x)$, when $n \in \mathcal{N}(A)$. Let $z \in F'[x^\dagger](x - x^\dagger)$ with $x - x^\dagger \in B_\rho(0)$. We may write $z = A(h + n)$, where $h + n = x - x^\dagger$ and $h \in \mathcal{N}(A)^\perp \cap B_\rho(0)$ and $n \in N(A)$. Note that $A^\dagger A = P_{\mathcal{N}(A)^\perp}$, i.e., the orthogonal projector onto $\mathcal{N}(A)^\perp$ such that $A^\dagger A(h + n) = h$. Let us define the operator N as

$$N(z) := F(x^\dagger + A^\dagger z) - F(x^\dagger), \quad \forall\, z \in F'[x^\dagger](B_\rho(0)).$$

Obviously, $N(0) = 0$ holds true. Moreover,

$$\begin{aligned} N(z) &= F(x^\dagger + A^\dagger z) - F(x^\dagger) = F(x^\dagger + A^\dagger A(h + n)) - F(x^\dagger) \\ &= F(x^\dagger + h) - F(x^\dagger) = F(x^\dagger + h + n) - F(x^\dagger), \end{aligned} \tag{8.13}$$

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since $n \in N(A)$. Thus, as we have defined $h + n = x - x^\dagger$ and $z = F'[x^\dagger](x - x^\dagger)$, the factorization (8.11) follows.

The properties (8.12) are directly obtained since

$$\|N(z)\|_Y = \|F(x) - F(x^\dagger)\|_Y, \quad \|z\| = \|F'[x^\dagger](x - x^\dagger)\|,$$

and taking into account the inequalities (8.7) and (8.8).

Conversely, if (8.11) holds with (8.12), then

$$\|F(x^\dagger + h) - F(x^\dagger)\|_Y = \|N \circ F'[x^\dagger]h\|_Y \sim \|F'[x^\dagger]h\|_Y,$$

i.e., (8.7) and (8.8) hold. □

The previous theorem indicates that the tangential cone conditions can *roughly* be stated as $F \sim_{\mathcal{NB},\mathcal{I}} F'[x^\dagger]$, i.e., the nonlinear operator is as ill-posed as its linearization. However, this interpretation is not completely correct since the mapping N is only defined on the non-closed set $\mathcal{R}(F'[x^\dagger])$ and, in particular, is *not necessarily a continuous* mapping. Thus, N does not satisfy the requirements in Definition 8.5.

However, by using the stronger version (8.3) and (8.4), we can prove that N is continuous and remove the "roughly" in the previous statement.

THEOREM 8.11. *Let F satisfy* (8.3) *and* (8.4)*. Then, for any $x^\dagger$, the factorization* (8.11) *holds with $N : \overline{F'[x^\dagger](\mathcal{B}_\rho(0))} \to Y$ a Lipschitz continuous operator that locally has a Lipschitz continuous inverse.*

Proof. Since (8.3) and (8.4) imply the tangential cone condition (see Proposition 8.8), the factorization exists. We only have to show continuity. Again set $A := F'[x^\dagger]$, and let $z_1, z_2 \in A(B_\rho(0))$ with $z_1 = A(x_1 - x^\dagger)$, $z_2 = A(x_2 - x^\dagger)$, and $x_1, x_2 \in B_\rho(x^\dagger)$. Then using $F'[x] \sim_{\text{norm}} F'[x^\dagger]$, we obtain from (8.13) that

$$\begin{aligned}
\|N(z_1) - N(z_2)\|_Y &= \|F(x_2) - F(x_1)\| \\
&= \left\| \int_0^1 F'[x_1 + t(x_2 - x_1)](x_2 - x_1)\, dt \right\| \\
&= \left\| \int_0^1 R_{x_1 + t(x_2 - x1), x_1}\, dt\, F'[x_1](x_2 - x_2) \right\|_Y \\
&\le C\|F'[x_1](x_2 - x_2)\| \\
&\le \|F'[x^\dagger](x_1 - x_2)\|_Y = \|z_1 - z_2\|_Y .
\end{aligned}$$

Thus, N is Lipschitz on $A(B_\rho(0))$ and in particular uniformly continuous. It follows that N can be extended to a Lipschitz continuous operator on $\overline{A(B_\rho(0))}$ (see, e.g., [4, p. 190]). Now according to [15, p. 28] conditions (8.3) and (8.4) imply the tangential cone property (8.6) also with $x^\dagger$ replaced there by an arbitrary element $x_2 \in B_\rho(x^\dagger) \cap D(F)$ (and a constant $q_\gamma < 1$ if ρ is sufficiently small). Thus, we have

$$\begin{aligned}
\|z_1 - z_2\|_Y = \|F'[x^\dagger](x_1 - x_2)\| &\le_{(8.3)} C\|F'[x_2](x_1 - x_2)\| \\
&\le_{(8.8)} C\|F(x_2) - F(x_1)\| = \|N(z_1) - N(z_2)\|_Y .
\end{aligned}$$

Consequently, N is injective, and the local inverse N^{-1} exists and is Lipschitz continuous. □

This means that (8.3) and (8.4) imply that

$$F \sim_{\mathcal{NB},\mathcal{I}} F'[x^\dagger].$$

Finally, under the conditions (8.3) for F and G, we obtain the result that the nonlinear ordering is equivalent to the linearized ordering.

THEOREM 8.12. *Assume that F and G are Fréchet-differentiable and defined on $\mathcal{B}_\rho(x^\dagger)$ and both satisfy* (8.3) *and* (8.4) *with certain operators R_1 and R_2. Then*

$$F \leq_{\mathcal{B},\mathcal{I}}^{\text{Lin}} G \quad \Longrightarrow \quad F \leq_{\mathcal{NB},\mathcal{I}} G.$$

Proof. We have that $F(x) = N_1 \circ F'[x^\dagger](x - x^\dagger)$ and $G(x) = N_2 \circ G'[x^\dagger](x - x^\dagger)$. Hence, with $T \in \mathcal{B}$,

$$N_1^{-1}\left(F(x^\dagger + h) - F(x^\dagger)\right) = F'[x^\dagger]h = TG'[x^\dagger]h = TN_2^{-1}(G(x^\dagger + h) - G(x^\dagger)).$$

Thus,

$$F(x^\dagger + h) - F(x^\dagger) = N_1(TN_2^{-1}(G(x^\dagger + h) - G(x^\dagger))),$$

which gives $F \leq_{\mathcal{NB},\mathcal{I}} G$. ∎

REMARK 8.13. The previous results have indicated a relation between nonlinearity conditions of tangential cone type and an operator factorization of F into a (left) nonlinear well-posed and a (right) linear ill-posed operator. We leave it to future work to investigate the analogous case of a factorization with the nonlinear well-posed operator on the right (in our notation $F \sim_{\mathcal{I},\mathcal{NB}} F'[x^\dagger]$). Possibly in this case alternative nonlinearity conditions, for instance, range invariance, Newton–Mysovskii, or affine invariance conditions (cf. [7, 25]) are relevant to characterize this situation. Finally, the case of factorization of a linear operator with nonlinear well-posed ones from left and right as in Definition 8.5 (i.e., $F \sim_{\mathcal{NB},\mathcal{NB}} F'[x^\dagger]$) could be an interesting piece of future research.

9. Conclusion.

We have studied a new definition of ordering by ill-posedness that generalizes various other orderings. The main result is the equivalence of the new definition with that in [19], in particular also for the non-compact case. We have compared several known orderings and have established some equivalences; cf. Table 5.1. Furthermore, we have extended the definition to the nonlinear case and investigated the relation between linearized orderings and nonlinearity conditions. From our perspective, we could in particular interpret nonlinearity conditions of tangential cone type as conditions for factorizing an operator into a nonlinear well-posed and a linear ill-posed one. Some open questions remain: for instance, if an equivalent characterization of the operator ordering by s-numbers as in Theorem 3.6 is possible in the Banach space case or if extensions of the results of Section 8 to nonlinear operator factorizations, as mentioned in Remark 8.13, hold true.

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