Electronic Transactions on Numerical Analysis.
Volume 63, pp. 424–451, 2025.

ISSN 1068–9613.
DOI: 10.1553/etna_vol63s424

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REVISITING THE NOTION OF APPROXIMATING CLASS OF SEQUENCES FOR HANDLING APPROXIMATED PDES ON MOVING OR UNBOUNDED DOMAINS*

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Abstract. In the current work we consider matrix sequences $\{B_{n,t}\}_n$, with matrices of increasing sizes, depending on n, and equipped with a parameter $t > 0$. For every fixed $t > 0$, we assume that each $\{B_{n,t}\}_n$ possesses a canonical spectral/singular values symbol f_t, defined on $D_t \subset \mathbb{R}^d$, which are sets of finite measure, for $d \geq 1$. Furthermore, we assume that $\{\{B_{n,t}\}_n : t > 0\}$ is an approximating class of sequences (a.c.s.) for $\{A_n\}_n$ and that $\bigcup_{t>0} D_t = D$ with $D_{t+1} \supset D_t$. Under such assumptions and via the notion of a.c.s, we prove results on the canonical distributions of $\{A_n\}_n$, whose symbol, when it exists, can be defined on the, possibly unbounded, domain D of finite or even infinite measure. We then extend the concept of a.c.s. to the case where the approximating sequence $\{B_{n,t}\}_n$ has possibly a different dimension than the one of $\{A_n\}_n$. This concept seems to be particularly natural when dealing, e.g., with the approximation both of a partial differential equation (PDE) and of its (possibly unbounded or moving) domain D, using an exhausting sequence of domains $\{D_t\}$. Examples coming from approximated PDEs with either moving or unbounded domains are presented in connection with the classical and the new notion of a.c.s., while numerical tests and a list of open questions conclude the present work.

Key words. discretization of PDEs, moving/unbounded domains, spectral distribution of matrix sequences, (generalized) approximating class of sequences, generalized locally Toeplitz (GLT) matrix sequences, GLT theory

AMS subject classifications. 15A18, 65N, 47B

1. Introduction.

Partial differential equations (PDEs) and more recently fractional differential equations (FDEs) represent standard tools employed for modeling real-world problems in applied sciences and engineering. In particular, the notion of FDE can be considered a generalization of that of an PDE, in which fractional-order derivatives are used describing anomalous diffusion processes stemming from concrete applications. The price to pay is the nonlocal nature of the underlying operators, which implies the dense character of the approximated equations and hence a higher computational cost.

In general, when considering either PDEs or FDEs, analytical solutions are not generally known in closed form, and when they are known via proper representation formulae, it happens that the related computation is costly. Indeed, while the analysis plays a fundamental role in establishing the well-posedness of a given PDE/FDE, numerical methods are crucial for an efficient computation of a numerical solution approximating the solution to the infinite-dimensional problem within a certain error.

*Received July 27, 2024. Accepted August 15, 2025. Published online on October 3, 2025. Recommended by Lothar Reichel.

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When using a linear numerical method and the given PDE/FDE is of linear type $Lu = g$ on some domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, we end up with a usually large linear system

$$A_n \mathbf{u}_n = \mathbf{g}_n, \tag{1.1}$$

where $\mathbf{g}_n$ incorporate the approximation of the right-hand side terms and the given boundary conditions and A_n is a square matrix of size d_n, $d_k < d_{k+1}$, $k \in \mathbb{N}$. In this way, as n tends to infinity, i.e., the matrix size d_n tends to infinity, the approximated solution $\mathbf{u}_n$ converges to the solution of the continuous problem $Lu = g$ in a given topology, depending on the continuous problem and on the given numerical method.

Looking at (1.1) collectively for every n, we observe that we are considering a whole sequence of linear systems with increasing matrix size d_n. Hence, it becomes useful to study the collective behavior of the sequence $\{A_n\}_n$ of coefficient matrices.

What is often observed in practice is that the sequence of matrices $\{A_n\}_n$ enjoys an asymptotic spectral distribution, which is somehow connected to the spectrum of the linear differential operator L associated either with the given PDE or with the given FDE. More in detail, for a large set of test functions F, usually for all continuous functions with bounded support or just continuous functions if the spectral norm of A_n is uniformly bounded by a constant independent of n, a weak-$*$ convergence stands. More specifically, for every test function F, we have

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\lambda_j(A_n)) = \frac{1}{\mu(D)} \int_D \frac{\sum_{i=1}^{p} F(\lambda_i(\mathbf{f}(\mathbf{y})))}{p} \,\mathrm{d}\mathbf{y}, \tag{1.2}$$

where $\lambda_j(A_n)$, $j = 1, \ldots, d_n$, are the eigenvalues of A_n, $\mu(\cdot)$ is the Lebesgue measure in $\mathbb{R}^d$, and $\lambda_i(\mathbf{f}(\mathbf{y}))$, $i = 1, \ldots, p$, are the eigenvalues of a certain matrix-valued function

$$\mathbf{f} : D \subset \mathbb{R}^d \to \mathbb{C}^{p\times p},$$

with $\mu(D) \in (0, \infty)$. The function $\mathbf{f}$ is referred to as the spectral symbol of the sequence of matrices $\{A_n\}_n$.

When the symbol is continuous (or more generally Riemann-integrable), relation (1.2) says that, for n large enough, the spectrum of A_n can be subdivided into p different subsets (or "branches") of approximately the same cardinality d_n/p, and the i-th branch is approximately a uniform sampling over D of the i-th eigenvalue function $\lambda_i(\mathbf{f}(\mathbf{y}))$, $i = 1, \ldots, p$. In particular, the number p coincides with the number of "branches" that compose the spectrum of A_n. For instance, if $d = 1$, $d_n = np$, and $D = [a, b]$, then up to $o(n)$ possible outliers, the eigenvalues of A_n are approximately equal to

$$\lambda_i\Big(\mathbf{f}\Big(a + j\,\frac{b-a}{n}\Big)\Big), \qquad j = 1, \ldots, n, \quad i = 1, \ldots, p.$$

If $d = 2$, $d_n = n^2 p$, and $D = [a_1, b_1] \times [a_2, b_2]$, then again up to $o(n^2)$ possible outliers, the eigenvalues of A_n are approximately equal to

$$\lambda_i\Big(\mathbf{f}\Big(a_1 + j_1\,\frac{b_1-a_1}{n},\, a_2 + j_2\,\frac{b_2-a_2}{n}\Big)\Big), \qquad j_1, j_2 = 1, \ldots, n, \quad i = 1, \ldots, p,$$

and so on in a d-dimensional setting, with $d > 2$ integer, $d_n = n^d p$, and $o(n^{d-1})$ possible outliers.

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A complementary notion is that of the singular values symbol and the singular value distribution. For this further notion, instead of (1.2), for every test function F, we have

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\sigma_j(A_n)) = \frac{1}{\mu(D)} \int_D \frac{\sum_{i=1}^{p} F(\sigma_i(\mathbf{f}(\mathbf{y})))}{p} \, d\mathbf{y},$$

where $\sigma_j(A_n)$, $j = 1, \ldots, d_n$, are the singular values of A_n, $\sigma_i(\mathbf{f}(\mathbf{y}))$, $i = 1, \ldots, p$, are the singular values of a certain matrix-valued function $\mathbf{f} : D \subset \mathbb{R}^d \to \mathbb{C}^{p\times p}$, with $\mu(D) \in (0, \infty)$, and where the informal meaning exactly mirrors that one described above for the spectral symbol. The function $\mathbf{f}$ is referred to as the singular values symbol of the sequence of matrices $\{A_n\}_n$.

It is then clear that the spectral symbol (singular values symbol) $\mathbf{f}$ provides a "compact" and quite accurate description of the spectrum (singular values) of the discretization matrices A_n. The identification and the study of the symbol are consequently two important steps in the analysis of A_n and, as a consequence, in the analysis and in the design of fast iterative solvers for the linear systems in (1.1), especially for large matrix sizes d_n.

We remind that this type of eigenvalue distribution results are studied since the beginning of the last century in the context of sequences of Toeplitz matrices generated by real-valued (or Hermitian-valued) functions, as the reader can verify in the papers [54, 56, 57] or in the books [10, 28, 29], taking into account related references therein. Indeed, starting from the early works [50, 51, 53], where also the notion of the approximating class of sequences appeared for the first time, and after it was formally defined in [49], wide extensions to the $*$-algebras of generalized locally Toeplitz (GLT) matrix sequences are considered in [5, 6, 27, 28, 29], where the construction of the GLT $*$-algebras are based on the idea of an approximating class of sequences. We remind that every d-level p-block GLT matrix sequence is equipped with a $p \times p$ matrix-valued GLT symbol defined on $[0, 1]^d \times [-\pi, \pi]^d$ [6], which is also the symbol in the sense of Definition 1.1: the domain $[0, 1]^d$, possibly after an affine change of variables, is replaced by the physical domain of the given PDE in the case of the reduced GLT matrix sequences; see [50, pp. 395–399], [51, Section 3.1.4], [4]. There are several related results and applications to the approximation via local numerical methods of (systems of) PDEs/FDEs also with nonsmooth variable coefficients and irregular bounded domains/manifolds; for the spectral analysis in the case of d-level p-block GLT asymptotic structures, see [5, 6, 22, 24, 25]; for GLT-based fast numerical solvers also on non-Cartesian domains, systems of PDEs, and variable coefficients, see [8, 19, 20, 21, 37, 38]. While in [26] general domains and trimmed geometries are considered, the review paper [31] contains an engineering perspective, and the work [1] includes the GLT analysis in the case of PDEs on manifolds.

As described in Section 1.2, in the current work we generalize the notion of an approximating class of sequences in order to overcome the limitation of a fixed bounded domain, thus including both moving and unbounded domains, whose use is briefly recalled at the beginning of Section 4. Hence, the present work builds the approximation theory foundation for defining new GLT $*$-algebras including approximations of PDEs, with either moving or unbounded domains, thus going beyond the recent theoretical setting treated in [4, 5, 6].

1.1. The spectral distribution and the approximating class of sequences. We introduce the notions of distribution for a matrix sequence and that of the approximating class of sequences, which is a fruitful concept in the theory of numerical approximation and asymptotic spectral analysis of matrix sequences. We start with the definitions and then recall some useful results connecting the various concepts. In the following definitions and results and in the rest of the paper, $\alpha \wedge \beta$ denotes the minimum between two real numbers α, β, and $\mu(\cdot)$ indicates the d-dimensional Lebesgue measure.

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DEFINITION 1.1. *Let $\{A_n\}_n$ be a matrix sequence, and let $f : D \to \mathbb{C}^{p_1 \times p_2}$ be a measurable matrix-valued function defined on the measurable (bounded) set $D \subset \mathbb{R}^d$, with $0 < \mu(D) < \infty$. We write that $\{A_n\}_n$ is distributed as f in the sense of singular values in D and we write $\{A_n\}_n \sim_\sigma (f, D)$, if*

$$\lim_{n\to\infty} \sum_{j=1}^{d_n \wedge d_n'} \frac{F(\sigma_j(A_n))}{d_n \wedge d_n'} = \frac{1}{\mu(D)} \int_D \sum_{k=1}^{p} \frac{F(\sigma_k(f(s)))}{p} ds, \qquad \forall F \in C_c(\mathbb{R}),$$

where $\sigma_1(f(s)), \ldots, \sigma_p(f(s))$ are the singular values of $f(s)$, $p = p_1 \wedge p_2$. Therefore, a matrix of size $d_n \times d_n'$ has $d_n \wedge d_n'$ singular values. We call f the singular values symbol of $\{A_n\}_n$.

With the same notation as before, assuming additionally that f is square Hermitian matrix-valued with $p = p_1 = p_2$ and imposing $d_n = d_n'$, we write that $\{A_n\}_n$ is distributed as f in the sense of eigenvalues in D and we write $\{A_n\}_n \sim_\lambda (f, D)$, if

$$\lim_{n\to\infty} \sum_{j=1}^{d_n} \frac{F(\lambda_j(A_n))}{d_n} = \frac{1}{\mu(D)} \int_D \sum_{k=1}^{p} \frac{F(\lambda_k(f(s)))}{p} ds, \qquad \forall F \in C_c(\mathbb{R}),$$

where $\lambda_1(f(s)), \ldots, \lambda_p(f(s))$ are the eigenvalues of $f(s)$. We call f the spectral symbol of $\{A_n\}_n$.

Definition 1.1 is very general, and one may think that such a generality is not needed in applications. However, the truth is the opposite, and all the above generality is required by the (numerical) applications, studied mainly via the GLT theory. For giving a global picture, despite the great variety of numerical techniques, the various parameters p, d, f in Definition 1.1 in the GLT analysis depend on a combination of the given continuous PDE problem and of the chosen approximation technique. For instance, the typical D is $\Omega \times [-\pi, \pi]^d$, if Ω is a domain in $\mathbb{R}^d$ of positive and finite Lebesgue measure, where the PDE under consideration is defined. As a consequence, the typical dimensionality equals $2d$ while, and this is natural, in the case of a submanifold of codimension c, we have $2(d - c)$, $1 \le c \le d-1$. The parameter p is usually equal to $m\alpha(d-c)$, where m is the size of the vector function g in the continuous equation $Lu = g$: in the case of a vector PDE, obviously we have $m > 1$. More interestingly, $\alpha(\cdot)$ depends very much on the approximation scheme, which is in general of local type, such as finite elements [12], discontinuous Galerkin [32], finite differences [52], finite volumes [60], isogeometric analysis [16] etc. More precisely, in the case of finite differences [50], finite volumes [21], isogeometric analysis of degree k and maximal regularity $k-1$ [20], finite elements of degree 1 [7], we have $\alpha(d) = 1$, while for isogeometric analysis of intermediate regularity l, with $l < k - 1$, we observe $\alpha(d) = (k - l)^d$ [31]. In the case of finite elements of higher order k, we have $\alpha(d) = k^d$ [30, 42], while when using the discontinuous Galerkin scheme, $\alpha(d) = (k + l)^d$ [25]. Notice that the latter two formulas are a special instance of $\alpha(d) = (k - l)^d$, since $l = 0$ for finite elements of high order and $l = -1$ in the DG setting, because of the global discontinuity.

We also remark that the exponential growth of type $\alpha(d) = (k - l)^d$ is a drawback from a spectral viewpoint, because only one branch of the spectrum is acoustic, that is, related to the spectrum of the continuous operator, while the remaining very numerous branches are optical, in the sense that they behave like a pathology introduced by the numerical method (see, e.g., [17] and the references therein). In addition, the optical branches represent a challenge for designing fast iterative solvers, due to the very involved structure of the spectrum (see the applications in [5, 6, 27, 28, 29, 31]). Hence, the case of $p > 1$ emerges naturally not only in the case of d-level p-block Toeplitz structures described in Section 3 but also when dealing either with high-order finite elements for PDEs or with any approximation of vector PDEs.

DEFINITION 1.2. *Let $\{A_n\}_n$ be a matrix sequence of size $d_n \times d'_n$, with d_n and d'_n monotonically increasing integer sequences, and let $\{\{B_{n,t}\}_n\}_t$ be a sequence of matrix sequences of the same size $d_n \times d'_n$. We say that $\{\{B_{n,t}\}_n\}_t$ is an approximating class of sequences (a.c.s.) for $\{A_n\}_n$ if the following condition is met: for every t there exists n_t such that, for $n > n_t$,*

$$A_n = B_{n,t} + R_{n,t} + N_{n,t}, \qquad \text{rank}\,(R_{n,t}) \leq c(t) d_n \wedge d'_n,$$
$$\|N_{n,t}\| \leq \omega(t),$$

where $n_t, c(t), \omega(t)$ depend only on t, and

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} \omega(t) = 0.$$

The following theorem constitutes a link between an a.c.s. and the spectral distribution.

THEOREM 1.1 ([49, 53]). *Let $\{A_n\}_n$ be a matrix sequence of size $d_n \times d'_n$, with d_n and d'_n monotonically increasing integer sequences, and let $\{\{B_{n,t}\}_n\}_t$ be an a.c.s. for $\{A_n\}_n$. Suppose that $\{B_{n,t}\}_n \sim_\sigma (f_t, D)$ and f_t converges in measure to f. Then $\{A_n\}_n \sim_\sigma (f, D)$. Furthermore, if $d_n = d'_n$, all the involved matrices are Hermitian, $\{B_{n,t}\}_n \sim_\lambda (f_t, D)$, and f_t converges in measure to f, then $\{A_n\}_n \sim_\lambda (f, D)$*

1.2. Novelty of the present contribution. The tool contained in Theorem 1.1 is quite powerful, and it was introduced in the seminal work by Tilli [53] on locally Toeplitz matrix sequences. Then, in [49], the terminology was provided and further results were given. An account of the general theory on generalized locally Toeplitz (GLT) matrix sequences, based on the a.c.s. notion, can be found in [5, 6, 28, 29, 31] and the references therein.

In these books and long research/exposition papers, one can find also examples of applications ranging from approximated integro-differential equations, approximated partial differential equations, approximated fractional differential equations with any kind of methods (finite elements, finite differences, isogeometric analysis, finite volumes, etc) and with very mild assumptions: only Riemann integrability in the case of variable coefficients, only Peano-Jordan measurability of the domain (see, e.g., [34] for the latter two notions), grids approximated by a given function applied on uniform grids. In particular the richness in the potential domains is obtained via the notion of reduced GLT matrix sequences; see [50, pp. 395–399] for a detailed example, [51, Section 3.1.4] for an initial proposal and the related terminology, and the long dense paper [4] for a systematic treatment of the subject and for a complete theoretical development.

However, in all cases the idea is the immersion of the given Peano-Jordan measurable domain into a cube or rectangle of the appropriate number of dimensions, where this idea is related to the immersed methods [18, 35] and to the less recent idea of fictitious domains [33, 36]. The reader is referred also to [26, 39] for an asymptotical analysis and numerical methods regarding standard PDEs and FDEs using the immersion idea and the reduced GLT tools.

As a consequence, due to the immersion into the cube $[0, 1]^d$, possibly after affine changes of variables, the previous techniques are not applicable in the case of unbounded domains. In this direction, the new theorems in the present paper proved in Section 2 and the applications in Section 4 fill the gap and represent a big step for building an extended GLT theory including moving and unbounded domains, with either finite or infinite Lebesgue measure. Here, the basic a.c.s. and the new g.a.c.s. notions are the main tools which allow us to go beyond the GLT machinery. The new g.a.c.s. concept allows to deal with matrix sequences defined by different sequences of dimensions, and this may happen, e.g., in several approximation schemes for

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differential equations. The idea is already present in [38, Theorem 4.3, Corollary 4.4], and it is reminiscent of the extra-dimensional approach proposed by Tyrtyshnikov more than two decades ago (see again [38, beginning of Section 4.2] and [2] for a systematic treatment).

The paper is organized as follows. Section 2 is devoted to the introduciton of further notation and to state and derive the main approximation results. In particular, in Section 2.1, we introduce the new tool of generalized approximating class of sequences (g.a.c.s.), and we give an approximation result which highlights its usefulness. Section 3 contains the basic tools regarding Toeplitz matrix sequences, which are employed in the numerical experiments. In Section 4 we give a few applications for constant-coefficient PDEs, complemented by numerical experiments and related visualizations, which are then extended to the case of variable coefficients. Finally, Section 5 is devoted to concluding remarks and to mention a few open problems and future directions of research.

2. Main results.

We start this section by stating and proving a theorem that extends the practical purposes of the a.c.s. notion to the case of unbounded domains, possibly of infinite measure, using exhaustions by bounded sets of finite measure.

THEOREM 2.1. *Let $\{A_n\}_n$ be a matrix sequence with A_n of size $d_n \times d_n'$, with d_n and d_n' monotonically increasing integer sequences. Let D be a measurable set in $\mathbb{R}^d$, for some $d \geq 1$, possibly unbounded and of infinite measure, and let $f : D \to \mathbb{C}^{p_1 \times p_2}$, $p_1, p_2 \geq 1$, be a measurable function with $p = p_1 \wedge p_2$. Let D_t be an exhaustion of D, that is, it holds that $D_{t+1} \supset D_t$ for every $t > 0$ and $\bigcup_{t>0} D_t = D$. Assume that there exists $\{\{B_{n,t}\}_n\}_t$ such that $\{\{B_{n,t}\}_n : t > 0\}$ is an a.c.s. for $\{A_n\}_n$ with $\{B_{n,t}\}_n \sim_\sigma (f_t, D_t)$, i.e.,*

$$\lim_{n\to\infty} \sum_{j=1}^{d_n \wedge d_n'} \frac{F(\sigma_j(B_{n,t}))}{d_n \wedge d_n'} = \frac{1}{\mu(D_t)} \int_{D_t} \sum_{k=1}^{p} \frac{F(\sigma_k(f_t(s)))}{p} ds = \alpha_t(F)$$

for every $F \in C_c(\mathbb{R})$. Assume that, for every $F \in C_c(\mathbb{R})$,

$$\lim_{t\to\infty} \alpha_t(F) = \alpha(F).$$

Then

$$\lim_{n\to\infty} \sum_{j=1}^{d_n \wedge d_n'} \frac{F(\sigma_j(A_n))}{d_n \wedge d_n'} = \alpha(F). \tag{2.1}$$

If in addition we have

$$\lim_{t\to\infty} f_t^E = f \tag{2.2}$$

in D almost everywhere, with

$$f_t^E = \begin{cases} f_t(s) & \text{if } s \in D_t, \\ 0 & \text{otherwise}, \end{cases}$$

then we write $\{A_n\}_n \sim_{\sigma,\text{moving}} (f, D)$. Moreover, if $\mu(D) < \infty$, we have

$$\alpha(F) = \frac{1}{\mu(D)} \int_D \sum_{k=1}^{p} \frac{F(\sigma_k(f(s)))}{p} ds.$$

Proof. By the assumption $\{B_{n,t}\}_n \sim (f_t, D_t)$, we know that $\mu(D_t) > 0$, and hence, since $D \supset D_t$ for every $t > 0$, we deduce that $\mu(D) > 0$.

First, we note that relation (2.1) is true if and only if it is true for any F in C^1 with bounded support by simple functional approximation techniques such as the Weierstrass theorem plus a proper mollification technique (see [49, Lemma 2.2, p. 124, lines 6–9]). At this point, taking $F = F^{\uparrow} + F^{\downarrow}$, with $F^{\uparrow}(x) = \int_0^x (F')^+(t)dt$ and $F^{\downarrow}(x) = \int_0^x (F')^-(t)dt$, where $g^+ = \max(g, 0)$ and $g^- = \max(-g, 0)$, we deduce by linearity that (2.1) is true if and only if it holds for every function G continuous, differentiable (with G' nonnegative), being equal to 0 for every $x \le c_1$ and being constant (equal to $\|G\|_\infty$) for every $x \ge c_2$. Now, since $\{\{B_{n,t}\}_n : t > 0\}$ is an a.c.s. for $\{A_n\}_n$, we deduce that there exists $\alpha(t) \ge 0$ for which

$$\sigma_{j+\alpha(t)(d_n \wedge d_n')}(B_{n,t}) - \frac{1}{t} \le \sigma_j(A_n) \le \sigma_{j-\alpha(t)(d_n \wedge d_n')}(B_{n,t}) + \frac{1}{t}, \tag{2.3}$$

where $\lim_{t\to\infty} \alpha(t) = 0$, and the expression $j - \alpha(t)(d_n \wedge d_n')$ has to be set equal to 1 if $j - \alpha(t)(d_n \wedge d_n') \le 1$ and it has to be set equal to $d_n \wedge d_n'$ if $j - \alpha(t)(d_n \wedge d_n') \ge d_n \wedge d_n'$. Inequalities in (2.3), together with the features of G, lead to

$$\begin{aligned}\frac{1}{d_n \wedge d_n'} \sum_{j=1}^{d_n \wedge d_n'} G\left(\sigma_j(B_{n,t}) - \frac{1}{t}\right) - \|G\|_\infty \alpha(t) \\ &\le \frac{1}{d_n \wedge d_n'} \sum_{j=1}^{d_n \wedge d_n'} G(\sigma_j(A_n)) \\ &\le \frac{1}{d_n \wedge d_n'} \sum_{j=1}^{d_n \wedge d_n'} G\left(\sigma_j(B_{n,t}) + \frac{1}{t}\right) + \|G\|_\infty \alpha(t).\end{aligned}$$

Since

$$G\left(x + \frac{1}{t}\right) \le G(x) + \|G'\|_\infty \frac{1}{t} \qquad \text{and} \qquad G\left(x - \frac{1}{t}\right) \ge G(x) - \|G'\|_\infty \frac{1}{t},$$

we conclude that

$$\begin{aligned}\frac{1}{d_n \wedge d_n'} \sum_{j=1}^{d_n \wedge d_n'} G(\sigma_j(B_{n,t})) - \frac{1}{t}\|G'\|_\infty - \|G\|_\infty \alpha(t) \\ &\le \frac{1}{d_n \wedge d_n'} \sum_{j=1}^{d_n \wedge d_n'} G(\sigma_j(A_n)) \\ &\le \frac{1}{d_n \wedge d_n'} \sum_{j=1}^{d_n \wedge d_n'} G(\sigma_j(B_{n,t})) + \frac{1}{t}\|G'\|_\infty + \|G\|_\infty \alpha(t).\end{aligned}$$

Since

$$\lim_{t\to\infty} \frac{1}{t}\|G'\|_\infty \pm \alpha(t)\|G\|_\infty = 0,$$

taking any $\varepsilon > 0$, there exists $n_\varepsilon = n_{\varepsilon,t,G}$, such that for every $n \ge n_\varepsilon$ we have

$$\frac{1}{d_n \wedge d_n'} \sum_{j=1}^{d_n \wedge d_n'} G(\sigma_j(A_n)) \in \left[\alpha_t(G) - \frac{\varepsilon}{2}, \alpha_t(G) + \frac{\varepsilon}{2}\right] \subseteq \left[\alpha(G) - \varepsilon, \alpha(G) + \varepsilon\right].$$

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This concludes the first part of the theorem.

For the "in addition" part, we observe that under the assumption $\mu(D) < \infty$, we have $\lim_{t\to\infty} \mu(D_t) = \mu(D)$ and $\lim_{t\to\infty} \mu(D \setminus D_t) = 0$. Using equation (2.2), it follows that

$$\lim_{t\to\infty} \int_{D_t} \frac{\sum_{k=1}^{p} F(\sigma_k(f_t(s)))}{p} ds = \lim_{t\to\infty} \int_{D} \frac{\sum_{k=1}^{p} F(\sigma_k(f_t^E(s)))}{p} ds - F(0)\mu(D \setminus D_t)$$
$$= \int_{D} \frac{\sum_{k=1}^{p} F(\sigma_k(f(s)))}{p} ds,$$

ending the proof. $\square$

Under the hypothesis that the sequences are constituted by Hermitian matrices, we can extend the previous result to the case of eigenvalues, thus obtaining the theorem below. The proof is an eigenvalue version of the proof of Theorem 2.1: the steps are identical, and we leave the details to the interested reader.

THEOREM 2.2. *Let $\{A_n\}_n$ be a matrix sequence with A_n of order d_n such that $A_n = A_n^*$ with d_n a monotonically increasing integer sequence. Let D be a measurable set in $\mathbb{R}^d$, for some $d \geq 1$, possibly unbounded and of infinite measure, and let $f : D \to \mathbb{C}^{p\times p}$, $p \geq 1$, be a measurable function. Let D_t be an exhaustion of D, that is, $D_{t+1} \supset D_t$ for every $t > 0$ and $\bigcup_{t>0} D_t = D$. Assume that there exists $\{\{B_{n,t}\}_n\}_t$ such that $B_{n,t} = B_{n,t}^*$ for every n, t and that $\{\{B_{n,t}\}_n : t > 0\}$ is an a.c.s. for $\{A_n\}_n$ with $\{B_{n,t}\}_n \sim_\lambda (f_t, D_t)$, i.e.,*

$$\lim_{n\to\infty} \sum_{j=1}^{d_n} \frac{F(\lambda_j(B_{n,t}))}{d_n} = \frac{1}{\mu(D_t)} \int_{D_t} \sum_{k=1}^{p} \frac{F(\lambda_k(f_t(s)))}{p} ds = \alpha_t(F)$$

for every $F \in C_c(\mathbb{R})$. Assume that, for every $F \in C_c(\mathbb{R})$,

$$\lim_{t\to\infty} \alpha_t(F) = \alpha(F).$$

Then

$$\lim_{n\to\infty} \sum_{j=1}^{d_n} \frac{F(\lambda_j(A_n))}{d_n} = \alpha(F).$$

If in addition we have

$$\lim_{t\to\infty} f_t^E = f$$

in D almost everywhere, with

$$f_t^E = \begin{cases} f_t(s) & \text{if } s \in D_t, \\ 0 & \text{otherwise}, \end{cases}$$

then we write $\{A_n\}_n \sim_{\lambda,\text{moving}} (f, D)$. Moreover, if $\mu(D) < \infty$, we have

$$\alpha(F) = \frac{1}{\mu(D)} \int_{D} \sum_{k=1}^{p} \frac{F(\lambda_k(f(s)))}{p} ds.$$

Proof. We follow the same derivations as in the proof of Theorem 2.1, where the interlacing results concerning the eigenvalues have to be used in place of those for the singular values. $\square$

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2.1. The generalized approximating class of sequences. In this section, we introduce a new tool inspired by the a.c.s. theory and by the extra-dimensional approach (see [38, Section 4.1 and Section 4.2]), but one that is more flexible and able to handle all the parameters needed in order to study discretizations on unbounded domains of finite measure or moving domains. As for the classical a.c.s. tool, also the notion of generalized a.c.s. carries information about the spectral and singular value distribution, which is what we prove in Theorem 2.4. Hereafter, we focus on domains with finite measure, which will be the ones treated in the numerical tests.

DEFINITION 2.1. *Let $\{A_n\}_n$ be a matrix sequence of size $d_n \times d'_n$, with d_n and d'_n monotonically increasing integer sequences, and let $\{\{B_{n,t}\}_n\}_t$ be a sequence of matrix sequences of size $d_{n,t} \times d'_{n,t}$. Denoting by $\oplus$ the standard direct sum of matrices, we say that $\{\{B_{n,t}\}_n\}_t$ is a generalized approximating class of sequences (g.a.c.s.) for $\{A_n\}_n$ if, for every t, there exists n_t such that, for $n > n_t$,*

$$A_n = U_{n,t}\left(B_{n,t} \oplus 0_{n,t}\right) V_{n,t} + R_{n,t} + N_{n,t},$$

where $0_{n,t}$ is the null matrix of size $(d_n - d_{n,t}) \times (d'_n - d_{n,t})$, $U_{n,t}$ and $V_{n,t}$ are two unitary matrices of order $d_n \times d_n$ and $d'_n \times d'_n$, respectively, and $R_{n,t}$, $N_{n,t}$ are matrices of the same size of A_n, satisfying

$$\begin{aligned} \operatorname{rank}\left(R_{n,t}\right) &\leq c(t) d_n \wedge d'_n, \\ \|N_{n,t}\| &\leq \omega(t), \\ d_n \wedge d'_n - d_{n,t} \wedge d'_{n,t} =: m_{n,t} &\leq m(t) d_n \wedge d'_n, \\ \lim_{t\to\infty} c(t) = \lim_{t\to\infty} \omega(t) &= \lim_{t\to\infty} m(t) = 0. \end{aligned}$$

If we have a sequence $\{A_n\}_n$ of square Hermitian matrices, then we also ask $\{\{B_{n,t}\}_n\}_t$ to be square Hermitian and $V_{n,t} = U^*_{n,t}$ for all n and t.

REMARK 2.3. The notion of g.a.c.s. is intended as a generalization of the idea of a.c.s. being more natural when dealing with the approximation of infinite-dimensional operators over moving or unbounded domains. In particular, in the case of a discretization of a (fractional) differential operator over moving or unbounded domains, we usually end up with a sequence of domains (either a precompact exhaustion of the unbounded domain or the sequence of moving domains or a combination of both). Using the same approximation procedure (finite differences, finite elements, isogeometric analysis, finite volumes etc), it is natural to obtain matrices of different dimensions, and the g.a.c.s. is intended as a tool for dealing with this difficulty in the spirit of the extra-dimensional approach used in [38, Sections 4.1, 4.2].

THEOREM 2.4. *Let $\{A_n\}_n$ be a matrix sequence of size $d_n \times d'_n$, with d_n and d'_n monotonically increasing integer sequences. Let $\{\{B_{n,t}\}_n\}_t$ be a g.a.c.s. for $\{A_n\}_n$. If for all t, there exist (f_t, D_t) such that*

- $\{B_{n,t}\}_n \sim_\sigma (f_t, D_t)$,
- $D_t \subset D_{t+1}$, $\forall t$,
- $D := \bigcup_{t>0} D_t$ *of finite measure,*
- $\exists f : D \to \mathbb{C}^{p_1 \times p_2}$, $p_1, p_2 \geq 1$, *measurable such that* $f_t^E \to f$ *in measure,* $t \to +\infty$, *with* $p = p_1 \wedge p_2$ *and*

$$f_t^E = \begin{cases} f_t(s) & \text{if } s \in D_t, \\ 0 & \text{if } s \in D \setminus D_t, \end{cases}$$

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then $\{A_n\}_n \sim_\sigma (f, D)$, *i.e.,*

$$\lim_{n\to\infty} \sum_{j=1}^{d_n \wedge d'_n} \frac{F(\sigma_j(A_n))}{d_n \wedge d'_n} = \frac{1}{\mu(D)} \int_D \sum_{k=1}^{p} \frac{F(\sigma_k(f(s)))}{p} ds, \qquad \forall F \in C_c(\mathbb{R}). \tag{2.4}$$

Proof. First, we note that relation (2.4) is true if and only if it is true for any F in C^1 with bounded support by simple functional approximation techniques (see [49, Lemma 2.2, p. 124, lines 6–9]). At this point, taking $F = F^\uparrow + F^\downarrow$, with $F^\uparrow(x) = \int_{-\infty}^x (F')^+(t)dt$ and $F^\downarrow(x) = \int_{-\infty}^x (F')^-(t)dt$, where $g^+ = \max(g, 0)$ and $g^- = \max(-g, 0)$, we deduce by linearity that (2.4) is true if and only if it holds for every function G continuous, differentiable (with G' nonnegative), being equal to 0 for every $x \le c_1$ and being constant (equal to $\|G\|_\infty$) for every $x \ge c_2$.

Since, by hypothesis, $\{B_{n,t}\}_n \sim_\sigma (f_t, D_t)$, $D_t \subset D_{t+1}$, and $D := \bigcup_{t>0} D_t$ is of finite measure, we have

$$0 < \mu(D_t) \le \mu(D_{t+1}) \to \mu(D) < +\infty, \qquad t \to +\infty,$$

and, since $f_t^E \to f$ in measure as $t \to +\infty$, we also have

$$\sigma_k(f_t^E) \to \sigma_k(f) \quad \text{in measure}, \qquad \forall 1 \le k \le p.$$

Hence

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{\mu(D_t)} \int_{D_t} \sum_{k=1}^{p} \frac{G(\sigma_k(f_t(s)))}{p} ds \\ &\quad = \lim_{t\to\infty} \frac{\mu(D)}{\mu(D_t)} \frac{1}{\mu(D)} \left(\int_D \sum_{k=1}^{p} \frac{G(\sigma_k(f_t^E(s)))}{p} ds - G(0)\mu(D \setminus D_t) \right) \\ &\quad = \frac{1}{\mu(D)} \int_D \sum_{k=1}^{p} \frac{G(\sigma_k(f(s)))}{p} ds. \end{aligned} \tag{2.5}$$

Now, since for all $n > n_t$, $A_n = U_{n,t}\,(B_{n,t} \oplus 0_{n,t})\,V_{n,t} + R_{n,t} + N_{n,t}$, we set $C_{n,t} = U_{n,t}\,(B_{n,t} \oplus 0_{n,t})\,V_{n,t}$, and we notice that the singular values of $C_{n,t}$ are the singular values of $B_{n,t}$ with $m_{n,t} = d_n \wedge d'_n - d_{n,t} \wedge d'_{n,t}$ additional singular values equal to zero. By classic results on the interlacing of singular values under rank and norm corrections, we have

$$\begin{aligned} \sigma_j(A_n) = \sigma_j(C_{n,t} + R_{n,t} + N_{n,t}) &\le \sigma_j(C_{n,t} + R_{n,t}) + \omega(t) \\ &\le \sigma_{j+(d_n \wedge d'_n)c(t)}(C_{n,t}) + \omega(t), \end{aligned}$$

where $\sigma_j(C_{n,t}) := +\infty$ for $j > d_n \wedge d'_n$. On the basis of the initial discussion, we take G differentiable, monotone nondecreasing, positive and bounded (so $\|G\|_\infty = G(+\infty)$), and we have

$$\begin{aligned} \sum_{j=1}^{d_n \wedge d'_n} \frac{G(\sigma_j(A_n))}{d_n \wedge d'_n} &= \sum_{j=1}^{d_n \wedge d'_n} \frac{G(\sigma_j(C_{n,t} + R_{n,t} + N_{n,t}))}{d_n \wedge d'_n} \\ &\le \sum_{j=1}^{d_n \wedge d'_n} \frac{G(\sigma_{j+(d_n \wedge d'_n)c(t)}(C_{n,t}) + \omega(t))}{d_n \wedge d'_n} \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{d_n\wedge d_n'} \frac{G(\sigma_{j+(d_n\wedge d_n')c(t)}(C_{n,t}))}{d_n\wedge d_n'} + \omega(t)\,\|G'\|_\infty \\
&\leq \sum_{j=1+(d_n\wedge d_n')c(t)}^{d_n\wedge d_n'} \frac{G(\sigma_j(C_{n,t}))}{d_n\wedge d_n'} + c(t)\,\|G\|_\infty + \omega(t)\,\|G'\|_\infty \\
&\leq \sum_{j=1}^{d_n\wedge d_n'} \frac{G(\sigma_j(C_{n,t}))}{d_n\wedge d_n'} + c(t)\,\|G\|_\infty + \omega(t)\,\|G'\|_\infty .
\end{aligned}
$$

Taking into account the information about the singular values of $C_{n,t}$, it follows that

$$
\begin{aligned}
\sum_{j=1}^{d_n\wedge d_n'} \frac{G(\sigma_j(C_{n,t}))}{d_n\wedge d_n'} &= \frac{d_{n,t}\wedge d_{n,t}'}{d_n\wedge d_n'} \sum_{j=1}^{d_{n,t}\wedge d_{n,t}'} \frac{G(\sigma_j(B_{n,t}))}{d_{n,t}\wedge d_{n,t}'} + \frac{m_{n,t}}{d_n\wedge d_n'}G(0) \\
&\leq \sum_{j=1}^{d_{n,t}\wedge d_{n,t}'} \frac{G(\sigma_j(B_{n,t}))}{d_{n,t}\wedge d_{n,t}'} + m(t)\,\|G\|_\infty .
\end{aligned} \tag{2.7}
$$

Now, given $\epsilon > 0$, we choose t large enough such that $\omega(t), c(t), m(t) < \epsilon$, and, also by (2.5), we find

$$
\frac{1}{\mu(D_t)} \int_{D_t} \sum_{k=1}^{p} \frac{G(\sigma_k(f_t(s)))}{p} ds \leq \frac{1}{\mu(D)} \int_{D} \sum_{k=1}^{p} \frac{G(\sigma_k(f(s)))}{p} ds + \epsilon. \tag{2.8}
$$

Once t is fixed, using that $\{B_{n,t}\}_n \sim_\sigma (f_t, D_t)$, we can find $N_\epsilon > n_t$ such that for all $n > N_\epsilon$ we have

$$
\sum_{j=1}^{d_{n,t}\wedge d_{n,t}'} \frac{G(\sigma_j(B_{n,t}))}{d_{n,t}\wedge d_{n,t}'} \leq \frac{1}{\mu(D_t)} \int_{D_t} \sum_{k=1}^{p} \frac{G(\sigma_k(f_t(s)))}{p} ds + \epsilon . \tag{2.9}
$$

Finally, using the derivations in (2.6), (2.7), (2.8), and (2.9), for all $n > N_\epsilon$, we deduce

$$
\begin{aligned}
\sum_{j=1}^{d_n\wedge d_n'} \frac{G(\sigma_j(A_n))}{d_n\wedge d_n'} &\leq \sum_{j=1}^{d_n\wedge d_n'} \frac{G(\sigma_j(C_{n,t}))}{d_n\wedge d_n'} + c(t)\,\|G\|_\infty + \omega(t)\,\|G'\|_\infty \\
&\leq \sum_{j=1}^{d_{n,t}\wedge d_{n,t}'} \frac{G(\sigma_j(B_{n,t}))}{d_{n,t}\wedge d_{n,t}'} + (m(t)+c(t))\,\|G\|_\infty + \omega(t)\,\|G'\|_\infty \\
&\leq \frac{1}{\mu(D_t)} \int_{D_t} \sum_{k=1}^{p} \frac{G(\sigma_k(f_t(s)))}{p} ds + (1 + 2\,\|G\|_\infty + \|G'\|_\infty)\epsilon \\
&\leq \frac{1}{\mu(D)} \int_{D} \sum_{k=1}^{p} \frac{G(\sigma_k(f(s)))}{p} ds + (2 + 2\,\|G\|_\infty + \|G'\|_\infty)\epsilon .
\end{aligned}
$$

Since ϵ is arbitrary, we infer

$$
\limsup_{n\to\infty} \sum_{j=1}^{d_n\wedge d_n'} \frac{G(\sigma_j(A_n))}{d_n\wedge d_n'} \leq \frac{1}{\mu(D)} \int_{D} \sum_{k=1}^{p} \frac{G(\sigma_k(f(s)))}{p} ds .
$$

In a similar way we can also obtain that

$$\liminf_{n\to\infty} \sum_{j=1}^{d_n\wedge d_n'} \frac{G(\sigma_j(A_n))}{d_n \wedge d_n'} \geq \frac{1}{\mu(D)} \int_D \sum_{k=1}^{p} \frac{G(\sigma_k(f(s)))}{p} ds\,.$$

For the latter we just need to be a bit more careful in two of the steps: indeed, when using that G is positive, we add terms in the sum in (2.6), and we use that $d_{n,t} \wedge d'_{n,t} \leq d_n \wedge d'_n$ in (2.7). Nevertheless, those two additional terms of correction depend on G, $c(t)$, and $m(t)$ and can be controlled similarly to those coming from the $\limsup$ derivations. This ends the proof. ∎

Under the hypothesis that all the involved matrices are Hermitian, we can extend the previous result to the case of eigenvalues (with essentially the same proof), by obtaining the following theorem. Since the proof mimics that of Theorem 2.4 we leave it to the reader.

THEOREM 2.5. *Let $\{A_n\}_n$ be a sequence of Hermitian matrices of size d_n, with d_n a monotonically increasing integer sequence. Let $\{\{B_{n,t}\}_n\}_t$ be a g.a.c.s. for $\{A_n\}_n$ (in the Hermitian sense). If for all t there exist (f_t, D_t) such that*

- $\{B_{n,t}\}_n \sim_\lambda (f_t, D_t)$,
- $D_t \subset D_{t+1}$, $\forall t$,
- $D := \bigcup_{t>0} D_t$ *of finite measure,*
- $\exists f : D \to \mathbb{C}^{p\times p}$ *measurable such that $f_t^E \to f$ in measure, $t \to +\infty$, with*

$$f_t^E = \begin{cases} f_t(s) & \text{if } s \in D_t, \\ 0 & \text{if } s \in D \setminus D_t, \end{cases}$$

then $\{A_n\}_n \sim_\lambda (f, D)$.

REMARK 2.6. In the case of domains of infinite measure, we expect that a similar extension as the one inferred for an a.c.s. can be obtained. We do not go into detail at the moment as we want to focus mainly on unbounded domains with finite measure, leaving this possible extension for future developments. At any rate, as Theorem 2.1 and Theorem 2.2 show, the infinite measure setting can be handled: however, we still need to make a fine tuning of the new concepts in order to work in the best possible way.

REMARK 2.7. Following the work of Barbarino [3], the developed concepts and results can be recast in probabilistic terms as vague convergence of measures, where $\chi_E(\cdot)$ is the characteristic function of a measurable set E, each matrix A_n with dimension $d_n \times d'_n$ is associated to the atomic measure

$$\mu_{A_n} := \frac{1}{d_n \wedge d_n'} \sum_{k=1}^{d_n\wedge d_n'} \delta_{\sigma_k(A_n)},$$

δ_σ being the delta Dirac concentrated at σ, and each symbol f defined on a domain D with finite measure is associated to the probability measure

$$\mu_f(E) := \frac{\int_{s\in D} \chi_E(f(s))ds}{\mu(D)}.$$

In this context, $\{A_n\}_n \sim_\sigma f$ is equivalent to say that μ_{A_n} converge vaguely to μ_f. Moreover, the g.a.c.s. can be linked to the *modified optimal matching distance* between sequences of matrices defined in [3], where it has been proved how to induce the vague convergence for the measures associated to the matrix sequences and therefore for their respective symbols.

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As our main focus is to analyze how changing domains for the symbols affects the convergence of singular values and spectra of the sequences, we do not make use of the probabilistic notations and results, since the information on the domain for the symbol f would be lost if we deal with the associated measure μ_f. We finally stress that the domain is important in the present setting, since it retains information on the physical domain where the differential operator is defined, as it is transparent in Section 4.

3. Multi-index notation, Toeplitz and multilevel Toeplitz matrices. In this section we first introduce a multi-index notation that we use hereafter. Given an integer $d \geq 1$, a d-index $\boldsymbol{k}$ is an element of $\mathbb{Z}^d$, that is, $\boldsymbol{k} = (k_1, \ldots, k_d)$, with $k_r \in \mathbb{Z}$ for every $r = 1, \ldots, d$. Given two d-indices $\boldsymbol{i} = (i_1, \ldots, i_d)$, $\boldsymbol{j} = (j_1, \ldots, j_d)$, we write $\boldsymbol{i} \lhd \boldsymbol{j}$ if $i_r < j_r$ for the first $r \in \{1, 2, \ldots, d\}$ such that $i_r \neq j_r$. We say that $\boldsymbol{i} \unlhd \boldsymbol{j}$ if either $\boldsymbol{i} = \boldsymbol{j}$ or $\boldsymbol{i} \lhd \boldsymbol{j}$. The relation $\unlhd$ is a total order on $\mathbb{Z}^d$, usually addressed as standard lexicographic ordering. The relations $\rhd, \unrhd$ are defined accordingly.

Given two d-indices $\boldsymbol{i}, \boldsymbol{j}$, we write $\boldsymbol{i} < \boldsymbol{j}$ if $i_r < j_r$ for every $r = 1, \ldots, d$. The relations $\leq, >, \geq$ are defined accordingly.

We use bold letters for vectors and vector/matrix-valued functions. We indicate with $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$, the d-dimensional constant vectors $(0, 0, \ldots, 0)$, $(1, 1, \ldots, 1)$, $(2, 2, \ldots, 2)$, $\ldots$, respectively. Finally, given a d-index $\boldsymbol{n}$, we write $\boldsymbol{n} \to \infty$, meaning that $\min_{r=1,\ldots,d}\{n_r\} \to \infty$.

A Toeplitz matrix of order n is characterized by the fact that all the diagonals are constant: $(T_n)_{i,j} = t_{i-j}$, for $i, j = 1, ..., n$, and some coefficients t_k, $k = 1-n, ..., n-1$:

$$T_n = \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{1-n} \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & t_1 & t_0 \end{bmatrix}.$$

When every term t_k is a matrix of fixed size $p_1 \times p_2$, we say that T_n is of block Toeplitz type. The definition of d-level Toeplitz matrices is more involved, and it is based on the following recursive idea: a d-level Toeplitz matrix is a Toeplitz matrix where each coefficient t_k is a $(d-1)$-level Toeplitz matrix, and a 1-level Toeplitz matrix is just a standard Toeplitz matrix. Using standard multi-index notation, we can give a more detailed definition as follows: a d-level Toeplitz matrix is a matrix $T_{\mathbf{n}}$ such that

$$T_{\boldsymbol{n}} = (t_{\boldsymbol{i}-\boldsymbol{j}})_{\boldsymbol{i},\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{n}} \in \mathbb{C}^{(n_1 \cdots n_d) \times (n_1 \cdots n_d)},$$

with the multi-index $\boldsymbol{n}$ such that $\mathbf{0} < \boldsymbol{n} = (n_1, \ldots, n_d)$ and $t_{\boldsymbol{k}} \in \mathbb{C}$, $-(\boldsymbol{n} - \mathbf{1}) \unlhd \boldsymbol{k} \unlhd \boldsymbol{n} - \mathbf{1}$. If the basic element $\mathbf{t}_{\boldsymbol{k}}$ is a block of fixed size $p_1 \times p_2$, $\max\{p_1, p_2\} \geq 2$, we write that the matrix is a d-level block Toeplitz matrix, and we denote it by $T_{\boldsymbol{n},p_1,p_2}$, that is,

$$T_{\boldsymbol{n},p_1,p_2} = (\mathbf{t}_{\boldsymbol{i}-\boldsymbol{j}})_{\boldsymbol{i},\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{n}} \in \mathbb{C}^{(n_1 \cdots n_d p_1) \times (n_1 \cdots n_d p_2)}, \qquad \mathbf{t}_{\boldsymbol{k}} \in \mathbb{C}^{p_1 \times p_2}.$$

Given now a function $\boldsymbol{f} : [-\pi, \pi]^d \to \mathbb{C}^{p_1 \times p_2}$ in $L^1([-\pi, \pi]^d)$, we denote its Fourier coefficients by

$$\hat{\boldsymbol{f}}_{\boldsymbol{k}} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \boldsymbol{f}(\boldsymbol{\theta}) \mathrm{e}^{-\hat{\imath}\, \boldsymbol{k} \cdot \boldsymbol{\theta}} d\boldsymbol{\theta} \in \mathbb{C}^{p_1 \times p_2}, \qquad \boldsymbol{k} \in \mathbb{Z}^d, \quad \boldsymbol{k} \cdot \boldsymbol{\theta} = \sum_{r=1}^{d} k_r \theta_r,$$

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(the integrals are understood component-wise), and we associate to $\boldsymbol{f}$ the family of d-level block Toeplitz matrices

$$T_{\boldsymbol{n},p_1,p_2}(\boldsymbol{f}) := \left(\hat{\boldsymbol{f}}_{\boldsymbol{i}-\boldsymbol{j}}\right)_{\boldsymbol{i},\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}}, \qquad \boldsymbol{n} \in \mathbb{N}^d.$$

In this context, $\boldsymbol{f}$ is called the generating function of the sequence $\{T_{\boldsymbol{n},p_1,p_2}(\boldsymbol{f})\}_{\boldsymbol{n}}$. The following result links the definition of symbol function and generating function for multilevel block Toeplitz matrix sequences:

THEOREM 3.1 ([22, 54, 55]). *Let $\boldsymbol{f} : [-\pi,\pi]^d \to \mathbb{C}^{p_1\times p_2}$ be a function belonging to $L^1([-\pi,\pi]^d)$, $p_1, p_2, d \geq 1$. Then*

$$\{T_{\boldsymbol{n},p_1,p_2}(\boldsymbol{f})\}_{\boldsymbol{n}} \sim_\sigma \boldsymbol{f},$$

that is, the generating function of $\{T_{\boldsymbol{n},p_1,p_2}(\boldsymbol{f})\}_{\boldsymbol{n}}$ coincides with its singular values symbol. When $p_1 = p_2 = p$ and $\boldsymbol{f}$ is Hermitian-valued almost everywhere or belongs to the Tilli class, i.e., $\boldsymbol{f}$ is essentially bounded, the closure of its range has empty interior, and the range does not disconnect the complex field (see [55] when $p = 1$ and [22] when $p > 1$), we have

$$\{T_{\boldsymbol{n},p,p}(\boldsymbol{f})\}_{\boldsymbol{n}} \sim_\lambda \boldsymbol{f},$$

that is, the generating function of $\{T_{\boldsymbol{n},p,p}(\boldsymbol{f})\}_{\boldsymbol{n}}$ coincides with its spectral symbol. Here, when $p_1 = p_2 = p$, by the range of $\boldsymbol{f}$ we mean the union of the ranges of the p eigenvalue functions of $\boldsymbol{f}$.

4. Applications, examples, numerical evidence. In this section we show a few basic numerical examples for proving the validity of the theory developed in the previous section. We start with a kind of trivial one in a bounded setting and then explore a more involved one where the domain is unbounded with finite measure and hence, of course, of non-Cartesian type: in this case we consider both P_1- and P_2-finite elements and as operator both the constant-coefficient and the variable-coefficient Laplacian. Of course, from a practical viewpoint, our quite elementary numerical experiments should be regarded as a starting point toward the use of the present framework in real world models, including either PDEs with moving boundaries in monument degradation, cell biology, etc [11, 13, 14, 23, 58] or PDEs on unbounded domains when considering elasticity problems in volcanology [15].

4.1. Approximating by finite elements over an exhaustion of (0,1). We consider the following model problem:

$$\begin{aligned} -\Delta u &= f && \text{on } (0,1),\\ u(0) &= u(1) = 0. \end{aligned}$$

It is probably the most studied test example, and it is already known that its P_1-finite elements approximation [12] on a uniform grid with a proper scaling leads to the solution of a linear system whose coefficient matrix is $A_n = T_n(2-2\cos(\theta))$. As widely known, the related sequence $\{A_n\}_n$ has a spectral distribution given by the function $2-2\cos(\theta)$ in $[-\pi,\pi]$: it is indeed a special case of Theorem 3.1 since its symbol is real-valued. We can see this also by applying linear finite elements to the problem

$$\begin{aligned} -\Delta u &= f && \text{on } (0,1-1/t),\\ u(0) &= u(1-1/t) = 0. \end{aligned}$$

In this case the spectral distribution of the associated matrix sequence $\{B_{n,t}\}_n$ is given by the symbol function $(1-1/t)(2-2\cos(\theta))$ for every $t > 0$. Clearly, all the assumption of

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Theorem 2.2 are satisfied, hence the sequence $\{A_n\}_n$ has a spectral distribution with symbol given by $\lim_{t\to\infty}(1-1/t)(2-2\cos(\theta)) = 2-2\cos(\theta)$.

REMARK 4.1. We note that one of the main aspects that allows us to infer the spectral distribution of the sequence $\{A_n\}_n$ is the "convergence" of the points in the discretization defining $\{\{B_{n,t}\}_n\}_t$ (and of the related domain) to those defining $\{A_n\}_n$ (and of the limit domain) as $t \to \infty$. In fact, this is what makes $\{\{B_{n,t}\}_n\}_t$ an a.c.s. for $\{A_n\}_n$, and, by definition, $\{\{B_{n,t}\}_n\}_t$ is also a g.a.c.s. for $\{A_n\}_n$.

REMARK 4.2. From now on we consider two-dimensional examples, and a multi-index notation as in Section 3 is employed. In these examples, the used numerical schemes lead naturally to approximants that are composed by matrices having a different size with respect to the matrices of the original matrix sequence. Hence, differently from before (see Remark 4.1), we are forced to consider the new g.a.c.s notion and the related notations, that is, $\{A_{\mathbf{n}}\}_{\mathbf{n}}$, $\{\{B_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$, and $\{\{C_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$. As done in the proof of Theorem 2.4, since $\dim(A_{\mathbf{n}}) \neq \dim(B_{\mathbf{n},t})$, we consider $\{\{C_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$, where $C_{\mathbf{n},t} = B_{\mathbf{n},t} \oplus 0_m$ and $m = m(\mathbf{n}, t)$ is such that $\dim(C_{\mathbf{n},t}) = \dim(A_{\mathbf{n}})$. In the numerical experiments, we will verify that $\{\{B_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$ is a g.a.c.s. for $\{A_{\mathbf{n}}\}_{\mathbf{n}}$ and that $\{\{C_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$ is a standard a.c.s. for $\{A_{\mathbf{n}}\}_{\mathbf{n}}$. Both are feasible asymptotic approximations of the original matrix sequence according to the theory. However the a.c.s. is less natural because of the zero added eigenvalues, which look somehow artificial, even if the number of these zero eigenvalues tends to became relatively negligible as t tends to infinity.

4.2. Quadratic finite elements for the model problem on a two-dimensional unbounded set of finite measure. We consider the following problem:

$$
\begin{aligned}
-\Delta u &= v \quad \text{on } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{4.1}
$$

where $\Omega = \{(x, y) \in \mathbb{R} : x > 0, y > 0 \text{ and } y < g(x)\}$ with

$$
g(x) = \begin{cases} 1 & x < 1, \\ \frac{1}{x^2} & x \geq 1. \end{cases}
$$

It is clear that $\mu(\Omega) = \int_0^\infty g(x)dx = 2 < \infty$. For the discretization, we consider as spatial stepsize $h = \frac{1}{n+1}$ and the nodes $(x_i, y_j) = (ih, jh)$. We consider the maximum value of i, $\bar{i}$, for which there exists j such that $(x_{\bar{i}}, y_j) \in \Omega$. We consider all the points in the rectangle $(0, \bar{i}) \times (0, 1)$. Note that the index $\bar{i} = \lfloor\sqrt{n+1}\rfloor$.

By using these points, we draw rectangular triangles, so defining uniform structured meshes to which we apply quadratic P_2-finite elements, which leads to a two-level Toeplitz matrix $T_{\mathbf{n}}(f)$ with $\mathbf{n} = (n, n\lfloor\sqrt{n+1}\rfloor)$ and $f = f_{P_2} : [-\pi, \pi]^2 \longrightarrow \mathbb{C}^{4\times 4}$ (see [42]) with

$$
f_{P_2}(\theta_1, \theta_2) = \left[\begin{array}{cccc}
\alpha & -\beta(1+e^{\hat{\imath}\theta_1}) & -\beta(1+e^{\hat{\imath}\theta_2}) & 0 \\
-\beta(1+e^{-\hat{\imath}\theta_1}) & \alpha & 0 & -\beta(1+e^{\hat{\imath}\theta_2}) \\
\hline
-\beta(1+e^{-\hat{\imath}\theta_2}) & 0 & \alpha & -\beta(1+e^{\hat{\imath}\theta_1}) \\
0 & -\beta(1+e^{-\hat{\imath}\theta_2}) & -\beta(1+e^{-\hat{\imath}\theta_1}) & \gamma + \frac{\beta}{2}(\cos(\theta_1)+\cos(\theta_2))
\end{array}\right],
$$

where $\alpha = 16/3$, $\beta = 4/3$, and $\gamma = 4$.

Clearly, this sequence has a spectral distribution given by f on the domain $[-\pi,\pi]^2$ by Theorem 3.1 since its symbol is Hermitian-valued: indeed given the even nature of the symbol, the domain can be reduced to the subdomain $[0,\pi]^2$.

We now consider the points $(x_i, y_j) \in \Omega$ and the points $(x_i, y_j) \in \Omega_t = \Omega \cap B(0,t)^{\|\cdot\|_\infty}$. As $h \to 0$, we obtain uniform discretizations both for the set Ω and for the sets Ω_t, which constitute an exhaustion of the domain Ω. Finally, we can obtain a discrete version $A_{\mathbf{n}}$ of the Laplacian on Ω by simply cutting from the matrix $T_{\mathbf{n}}(f)$ all the rows and columns corresponding to indices (i,j) such that (x_i, y_j) are not in Ω.

We can do the same on Ω_t obtaining another sequence of matrices $\{\{B_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$. In general $\dim(A_{\mathbf{n}}) \neq \dim(B_{\mathbf{n},t})$. According to Remark 4.2, we then consider the sequence $\{\{C_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$, where $C_{\mathbf{n},t} = B_{\mathbf{n},t} \oplus 0_m$ and $m = m(\mathbf{n}, t)$ is such that $\dim(C_{\mathbf{n},t}) = \dim(A_{\mathbf{n}})$. We want to prove that

a) there exists f_t such that

$$\{B_{\mathbf{n},t}\}_{\mathbf{n}} \sim_{\lambda,\sigma} (f_t, \Omega_t \times [-\pi,\pi]^2), \quad \{C_{\mathbf{n},t}\}_{\mathbf{n}} \sim_{\lambda,\sigma} (f_t^E, \Omega \times [-\pi,\pi]^2).$$

b) $\{\{C_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$ is an a.c.s. for a matrix sequence $\{\tilde{A}_{\boldsymbol{n}}\}_{\boldsymbol{n}}$, where $\tilde{A}_{\boldsymbol{n}}$ is similar to $A_{\mathbf{n}}$ for every $\boldsymbol{n}$ using a unique permutation matrix and $\{\{B_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$ is a g.a.c.s. for $\{\tilde{A}_{\boldsymbol{n}}\}_{\mathbf{n}}$.

c) The limit $\lim_{t\to\infty} f_t^E$ exists in $\Omega \times [-\pi,\pi]^2$.

For item a) we can simply use the theory of reduced GLT in [4], obtaining that $\{C_{\mathbf{n},t}\}_{\mathbf{n}} \sim_{\lambda,\sigma} (f_t^E, \Omega \times [-\pi,\pi]^2)$, where $f_t(\mathbf{x}, \theta_1, \theta_2) = \chi_{\Omega_t}(\mathbf{x}) \cdot f$, with f depending on the used approximation of the considered PDE and $\mathbf{x} = (x,y)$.

For item b) we note that $\tilde{A}_{\mathbf{n}} = C_{\mathbf{n},t} + R_{\mathbf{n},t}$ with $\text{rank}(R_{\mathbf{n},t}) \leq (2 - \mu(\Omega_t))n^2 + c(t)n$, where $c(t) \to 0$ as $t \to \infty$. Since $\mu(\Omega_t) \to 2$ as $t \to \infty$, we find that $\{\{B_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$ is a g.a.c.s. for $\{\tilde{A}_{\mathbf{n}}\}_{\mathbf{n}}$ and $\{\{C_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$ is an a.c.s. for $\{\tilde{A}_{\mathbf{n}}\}_{\mathbf{n}}$.

For item c) we have

$$\lim_{t\to\infty} f_t^E = f,$$

for every $(\mathbf{x}, \theta_1, \theta_2) \in \Omega \times [-\pi,\pi]^2$. Using either Theorem 2.1 and Theorem 2.2 or Theorem 2.4 and Theorem 2.5, recalling Definition 1.1, we conclude that the sequence $\{A_{\mathbf{n}}\}_{\mathbf{n}}$ has a distribution both in the eigenvalue sense and in the spectral value sense given by f on $\Omega \times [-\pi,\pi]^2$, or, equivalently, on $\Omega \times [0,\pi]^2$.

REMARK 4.3. Equivalently, we can choose as $C_{\boldsymbol{n},t}$ the matrix defined by

$$D_{\boldsymbol{n}}(\chi_{\Omega_t}) A_{\boldsymbol{n}} D_{\boldsymbol{n}}(\chi_{\Omega_t}),$$

and it is possible to see that the rank of $A_{\boldsymbol{n}} - C_{\boldsymbol{n},t}$ satisfies the same upper estimate as above and $\{C_{\boldsymbol{n},t}\}_{\mathbf{n}}$ is still distributed as $(f_t^E, \Omega \times [-\pi,\pi]^2)$. Here, $D_{\boldsymbol{n}}(\chi_{\Omega_t})$ is the diagonal multilevel matrix obtained by sampling the argument in a proper equispaced grid; see, e.g., [29, Section 4.1.1, equations (4.1)–(4.2)]. Furthermore the set $[-\pi,\pi]^2$ can be replaced by $[0,\pi]^2$ in all cases in which the function is even in θ_1, θ_2, separately. As already observed in [50, pp. 375–378], it is worth recalling that the obtained spectral symbol depends on three actors: the operator order, the coefficients/physical domain, the approximation technique. For instance, in our setting the underlying operator has order two, and this can be read in the minimal eigenvalue of the symbol, which is asymptotic to $2 - 2\cos(\theta)$ in one variable and to $4 - 2\cos(\theta_1) - 2\cos(\theta_2)$ in two variables. In both cases the order of the zero is two, and this decides the conditioning of the resulting matrices which will grow as $N^{\frac{2}{d}}$, with N being the matrix size and d the dimensionality of the domain Ω. However, there are also other features that can be recovered. Here we mention three of them:

- The fact that the zero is at zero informs that the subspace related to low frequencies is the one associated with the small eigenvalues.
- An unbounded diffusion coefficient $a(x,y)$, e.g., with a unique pole at $(\hat{x},\hat{y})$ of order γ, will be seen in the maximal eigenvalues exploding as $N^{\frac{\gamma}{d}}$, and the related subspaces that can be easily identified as a function of the point $(\hat{x},\hat{y})$: in this case the overall conditioning will be asymptotic to $N^{\frac{(2+\gamma)}{d}}$.
- The approximation technique plays a role in the structure of the underlying matrices and hence in the complexity of the associated matrix-vector product: the more the method is precise, the larger is the bandwidth in a multilevel sense, while the number of levels is decided by the dimensionality of the physical domain Ω, and the presence of blocks and their size is exactly the gap between the degree of the polynomials used in the finite elements and the global continuity which is imposed (see also the discussion at the end of the introduction).

The remark below concerns the approximation by Q_2-finite elements, i.e., using rectangles as basic geometric elements instead of triangles. Notice that the curve $y = \frac{1}{x^2}$, for $x \geq 1$, defining the domain is better approximated using triangles instead of rectangles, which would give an unpleasant staircase.

REMARK 4.4. Consider the Q_2-finite elements approximation of the considered one-dimensional model problem. The resulting stiffness matrix is essentially of block Toeplitz type with blocks

$$K_0 = \frac{1}{3}\begin{bmatrix} 16 & -8 \\ -8 & 14 \end{bmatrix}, \qquad K_1 = \frac{1}{3}\begin{bmatrix} 0 & -8 \\ 0 & 1 \end{bmatrix}.$$

Hence, the symbol of the sequence of the stiffness matrices is the function

$$\mathbf{f}_2(\theta) = \frac{1}{3}\begin{bmatrix} 16 & -8-8\mathrm{e}^{\hat{\imath}\theta} \\ -8-8\mathrm{e}^{-\hat{\imath}\theta} & 14+2\cos\theta \end{bmatrix}, \qquad \theta \in [-\pi,\pi].$$

The eigenvalues of $\mathbf{f}_2(\theta)$, are

$$\lambda_1(\mathbf{f}_2(\theta)) = 5 + \frac{1}{3}\cos\theta + \frac{1}{3}\sqrt{129+126\cos\theta+\cos^2\theta},$$
$$\lambda_2(\mathbf{f}_2(\theta)) = 5 + \frac{1}{3}\cos\theta - \frac{1}{3}\sqrt{129+126\cos\theta+\cos^2\theta} = \frac{16}{3}\,\frac{2-2\cos\theta}{\lambda_1(\mathbf{f}_2(\theta))}.$$

Since the eigenvalue functions $\lambda_i(\mathbf{f}_2(\theta))$, $i = 1,2$, are even, it follows from Definition 1.1 that $\mathbf{f}_2(\theta)$ restricted to $[0,\pi]$ is still a symbol for $\{K_n^{(2)}\}_n$. As already recalled in the introduction, the latter is equivalent to the fact that a suitable ordering of the eigenvalues $\lambda_j(K_n^{(2)})$, $j = 1,\ldots,2n-1$, assigned in correspondence with an equispaced grid on $[0,\pi]$, approximately reconstructs the graphs of the eigenvalue functions $\lambda_i(\mathbf{f}_2(\theta))$, $i = 1,2$.

Setting $K_n^{(2)}$ the stiffness matrix in one dimension, this is observed in Figure 4.1, where we fix the equispaced grid $\frac{k\pi}{n+1}$, $k = 1,\ldots,n$, in $[0,\pi]$ and plot the eigenvalue functions $\lambda_i(\mathbf{f}_2(\theta))$, $i = 1,2$, as well as the pairs $\left(\frac{k\pi}{n+1}, \lambda_k(K_n^{(2)})\right)$, $k = 1,\ldots,n$, and $\left(\frac{(2n-k)\pi}{n+1}, \lambda_k(K_n^{(2)})\right)$, $k = n+1,\ldots,2n-1$, for $n = 40$. We clearly see from Figure 4.1 that the eigenvalues of $K_n^{(2)}$ can be split into two subsets (or branches) of approximately the same cardinality, and the i-th branch is approximately given by a uniform sampling over $[0,\pi]$ of the i-th eigenvalue function $\lambda_i(\mathbf{f}_2(\boldsymbol{\theta}))$, $i = 1,2$.

The spectral symbol in the two-dimensional case is then

$$f = f_{Q_2}(\theta_1,\theta_2) = \mathbf{f}_2(\theta_1)\mathbf{h}_2(\theta_2) + \mathbf{h}_2(\theta_1)\mathbf{f}_2(\theta_2)$$

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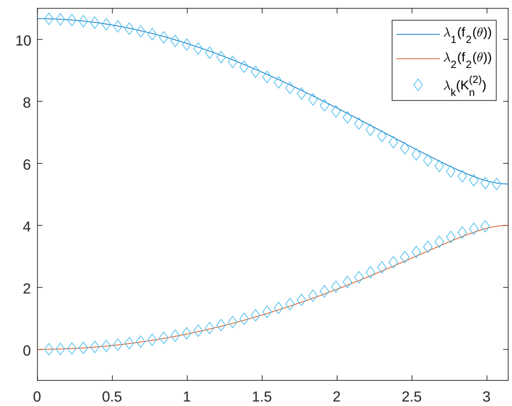

FIG. 4.1. *Graph over* $[0,\pi]$ *of the eigenvalue functions* $\lambda_i(\mathbf{f}_2(\theta))$, $i = 1,2$, *and of the pairs* $\left(\frac{k\pi}{n+1}, \lambda_k(K_n^{(2)})\right)$, $k = 1,\ldots,n$, *and* $\left(\frac{(2n-k)\pi}{n+1}, \lambda_k(K_n^{(2)})\right)$, $k = n+1,\ldots,2n-1$, *for* $n = 40$.

according to [30, Formula (5.1)] and with $\mathbf{f}_2$ as before and with $\mathbf{h}_2$ related to the mass matrix sequence as defined in [30]. Interestingly enough, the dimensionality of the two symbols f_{P_2}, f_{Q_2} is the same: two variables and the matrix size equal to four. Furthermore, in both cases three eigenvalues are strictly positive, and only one of them is asymptotic to the symbol of the standard discrete Laplacian by finite differences or P_1-finite elements, i.e., $4 - 2\cos(\theta_1) - 2\cos(\theta_2)$: compare [42] and [30].

Indeed, in the case of a general PDE with variable coefficients in a d-dimensional domains with standard finite elements, the number of variables is $2d$, which reduces to d if the coefficients of the differential operator are constants, and the dimensionality is k^d, where k is the degree of the used polynomials in the finite elements. For diminishing the presence of an exponential number of branches, as discussed in the introduction, the solution is isogeometric analysis [31], and only one branch is observed when maximal regularity is used [19, 20].

Finally, we consider a variable-coefficient version of the previous PDE in (4.1) expressed as follows:

$$\begin{aligned} \operatorname{div}(-a\nabla u) &= v \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.2}$$

where $a(x,y)$ is a positive non-degenerate variable coefficient on the domain $\Omega = \{(x,y) \in \mathbb{R} : x > 0, y > 0 \text{ and } y < g(x)\}$ with

$$g(x) = \begin{cases} 1 & x < 1, \\ \frac{1}{x^2} & x \geq 1. \end{cases}$$

Notice that for $a \equiv 1$ problem (4.2) reduces to (4.1). As in the constant-coefficient case, we opt for basic P_1-finite elements. According to the theory reported in [30, 42], the symbol is

$$f(\mathbf{x}, \theta_1, \theta_2) = (4 - 2\cos\theta_1 - 2\cos\theta_2)a(\mathbf{x}),$$

with four variables, $\mathbf{x} = (x,y)$ in the physical domain, (θ_1, θ_2) in the Fourier domain, and dimensionality 1 since the degree of the polynomials used in the finite elements is 1.

4.3. Numerical evidence. In this section we show numerical tests and visualizations corroborating the analysis conducted in the previous section. We focus on essentially a few discretization techniques for the approximation of (4.1) and (4.2), namely a standard centered finite differences scheme of order two, or equivalently P_1-finite elements, Q_1-finite elements, and P_2-finite elements.

With reference to Remark 4.3, we observed that the conditioning of the resulting matrices grows as $N^{\frac{2}{d}}$, with N being the matrix size and d being the dimensionality. This is due to the minimal eigenvalue, which converges to zero as $N^{-\frac{2}{d}}$ (see [9, 47, 48] for the pure Toeplitz setting and [41, 59] for the variable-coefficient case), and the related property is illustrated in Figures 4.2–4.4, where the univariate unique nondecreasing rearrangement of the symbol, for $d = 2$, has a positive bounded derivative so that the minimal eigenvalue tends to zero as N^{-1}. Things do not change, as expected, in the variable-coefficient setting since the diffusion coefficient $a(x, y)$ is positive and bounded: see Figure 4.5.

The P_2-finite elements case deserves further comments. Since the underlying symbol is 4×4 Hermitian-valued, we have four different branches, which can be represented in several ways. The univariate unique nondecreasing rearrangement of the symbol samplings is, in our opinion, the most effective way and even more when considering an error analysis with respect to the eigenvalue convergence. However, we may also plot the four branches separately, again by considering a nondecreasing rearrangement inside each branch (see Figure 4.6 top left) to analyze the features of each branch, or we may visualize them more properly as surfaces, thus stressing the matching with the two-dimensional sampling in $[0, \pi] \times [0, \pi]$ (see Figure 4.6 top right to bottom right). The same representation is considered in Figure 4.7 with respect to the $A_{\mathbf{n}}$-eigenvalues, showing a very good agreement with the functional samplings. We remark that a surface representation is important in terms of frequency subspaces, whose knowledge has a specific role when designing multigrid or multi-iterative solvers as in [19, 25, 46].

For the rest, there is nothing much to comment given the very strong agreement of the spectral behavior of the global matrix sequences and of the corresponding approximations $\{\{B_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$, $\{\{C_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$ as t grows. According to the notation in the proof of Theorem 2.4 and in Remark 4.2 (compare with Remark 4.1), $B_{\mathbf{n},t}$ has a smaller size than $A_{\mathbf{n}}$, $C_{\mathbf{n},t} = B_{\mathbf{n},t} \oplus 0_m$ has the same size as $A_{\mathbf{n}}$, $\{\{B_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$ is a g.a.c.s. for $\{A_{\mathbf{n}}\}_{\mathbf{n}}$, $\{\{C_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$ is a standard a.c.s. for $\{A_{\mathbf{n}}\}_{\mathbf{n}}$. Essentially, both $\{\{B_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$ and $\{\{C_{\mathbf{n},t}\}_{\mathbf{n}}\}_t$ carry the same information: however, the presence of the additional zero eigenvalues is visible in the plateau of zeros in Figures 1.7–1.8, in Figures 1.16–1.17, in Figures 2.6–2.7, in Figures 3.6–3.7 in the supplementary material[1], where the percentage of these additional zero eigenvalues goes to zero as t tends to infinity, since the ratio between the size of $B_{\mathbf{n},t}$ and that of $A_{\mathbf{n}}$ tends to 1 as t tends to infinity.

The very striking fact is that convergence is observed already for moderate values of t, giving evidence of the practical use of the used tools, i.e., the notion of a.c.s and that, which is indeed more natural, of the generalized a.c.s.: for the details on the approximating matrices, we refer to Figures 4.8–4.11, and compare with Figures 4.2–4.5 regarding the various matrix sequences $\{A_{\mathbf{n}}\}_{\mathbf{n}}$, respectively.

Furthermore, in Figures 4.12–4.13, the errors in the eigenvalue predictions are reported by considering the minimal distance of the eigenvalues from a suitable sampling of the symbol. More precisely, in the first column of Figure 4.12, the minimal distance is computed by considering a sampling of the symbol of approximately the same cardinality as the eigenvalues, thus highlighting the eigenvalue convergence as h decreases. In the second column the same analysis is performed by considering an high cardinality sampling of the symbol to better stress the quite good independence of the spectral approximation. The same quite good independence is observed in Figure 4.13 for different t-values as well.

[1] https://etna.ricam.oeaw.ac.at/volumes/2021-2030/vol63/addition/p424.php

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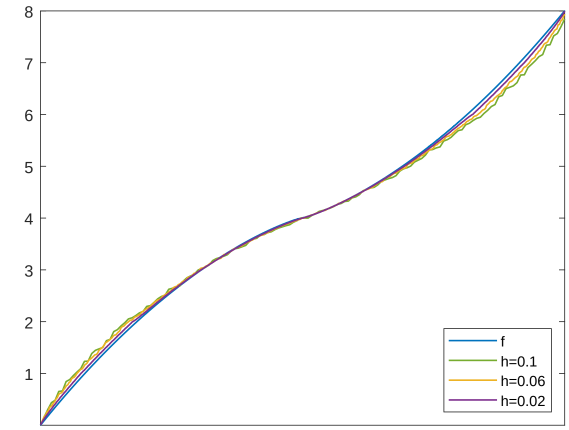

FIG. 4.2. *Eigenvalue distribution of* $\{A_{\mathbf{n}}\}_{\mathbf{n}}$ *for different* h*-values together with the sampling of* $f(\theta_1,\theta_2) = (2-2\cos\theta_1)+(2-2\cos\theta_2)$.

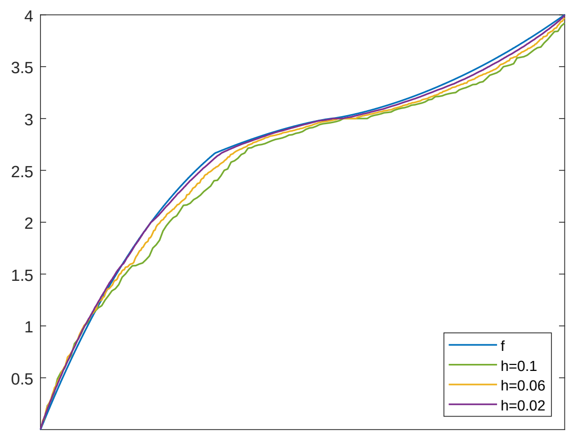

FIG. 4.3. *Eigenvalue distribution of* $\{A_{\mathbf{n}}\}_{\mathbf{n}}$ *for different* h*-values together with the sampling of* $f(\theta_1,\theta_2) = (8-2\cos\theta_1-2\cos\theta_2-4\cos\theta_1\cos\theta_2)/3$.

Finally, the pseudo-random behavior can be attributed to the nondecreasing (univariate) rearrangement of the spectral symbol of $\{A_{\mathbf{n}}\}_{\mathbf{n}}$. When maintaining the complete number of variables, a much smoother surface is expected. In addition, when looking at the figures related to the P_2-approximation, we observe 4 points where the rearranged symbol is not differentiable, and this corresponds to the 4 branches of the spectra since the symbol is 4×4 Hermitian-valued. Finally, when looking at Figures 4.5 and 4.11, we observe smoother curves and wider ranges, and this is due to the variable coefficient $a(x,y)$, since any of the four eigenvalue functions in the constant coefficient case is multiplied by $a(x,y)$. An exhaustive set of numerical evidence is reported in the supplementary material.

5. Conclusions. We have considered matrix sequences $\{B_{n,t}\}_n$, with matrices of increasing sizes, depending on n, and equipped with a parameter $t>0$. For every fixed $t>0$,

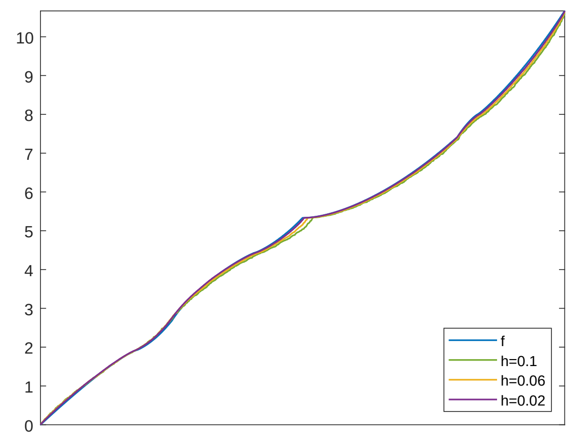

FIG. 4.4. *Eigenvalue distribution of* $\{A_{\mathbf{n}}\}_{\mathbf{n}}$ *for different* h*-values together with the sampling of* $f(\theta_1,\theta_2) = f_{P_2}(\theta_1,\theta_2)$.

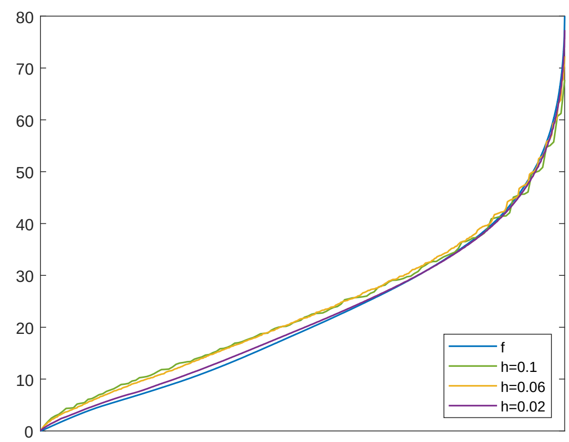

FIG. 4.5. *Eigenvalue distribution of* $\{A_{\mathbf{n}}(a)\}_{\mathbf{n}}$ *for different* h*-values together with the sampling of* $f(\theta_1,\theta_2,x,y) = [(2-2\cos\theta_1)+(2-2\cos\theta_2)]a(x,y)$.

we assume that each $\{B_{n,t}\}_n$ possesses a canonical spectral/singular values symbol f_t defined on $D_t \subset \mathbb{R}^d$ of finite measure, $d \geq 1$. Furthermore we assume that $\{\{B_{n,t}\}_n : t > 0\}$ is an a.c.s. for $\{A_n\}_n$ and that $\bigcup_{t>0} D_t = D$ with $D_{t+1} \supset D_t$. Under such assumptions and via the a.c.s. notion, we have proved general distribution results on the canonical distributions of $\{A_n\}_n$, whose symbol, when it exists, can be defined on the possibly unbounded domain D of finite or even infinite measure. In a second theoretical part, we have concentrated out attention on the case of unbounded domains of finite measure and have introduced a new concept, the g.a.c.s., which is suited particularly when moving or unbounded domains have to be treated.

Beside the notions of a.c.s and g.a.c.s, the main tool in concrete applications is the theory of GLT matrix sequences to which usually all the basic matrix sequences $\{B_{n,t}\}_n$ belong to for every $t > 0$, as shown in the examples stemming from the numerical approximation of

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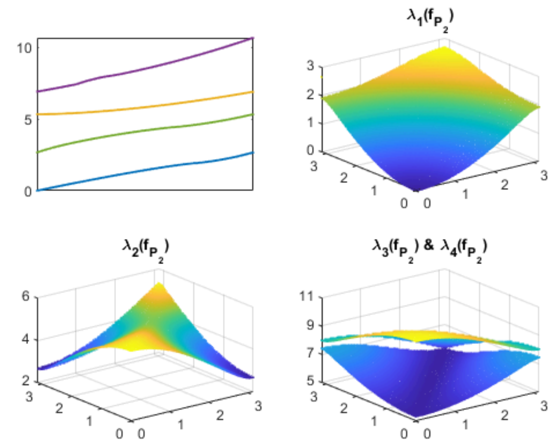

FIG. 4.6. *Eigenvalues of the* 4×4 *Hermitian-valued symbol* $f_{P_2}(\theta_1, \theta_2)$ *samplings* $\lambda_i(f_{P_2}(\theta_1, \theta_2))$*,* $i = 1, \ldots, 4$*, represented as ordered branches and as surfaces.*

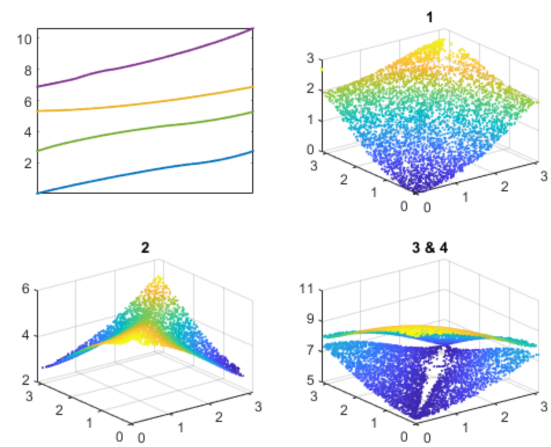

FIG. 4.7. *Matching of the eigenvalue distribution of* $\{A_{\mathbf{n}}\}_{\mathbf{n}}$ *with the eigenvalues of the* 4×4 *Hermitian-valued symbol* $f_{P_2}(\theta_1, \theta_2)$ *samplings* $\lambda_i(f_{P_2}(\theta_1, \theta_2))$*,* $i = 1, \ldots, 4$*, represented as ordered branches and as surfaces.*

PDEs with either moving or unbounded domains. Some numerical evidence has been given in order to corroborate the analysis.

As open questions we can mention the following main items:

1. In the present work we have focused on the case of unbounded domains with finite measure. However, as already mentioned, the problem of dealing with discretizations on domains of infinite measure remains interesting, both from the point of view of approximation and from the distributional point of view. In this context, also using the simpler a.c.s. notion, there is probably not a distributional symbol for the sequence as in the classical sense but rather a limit operator α that exploits some features of the discretized operator.

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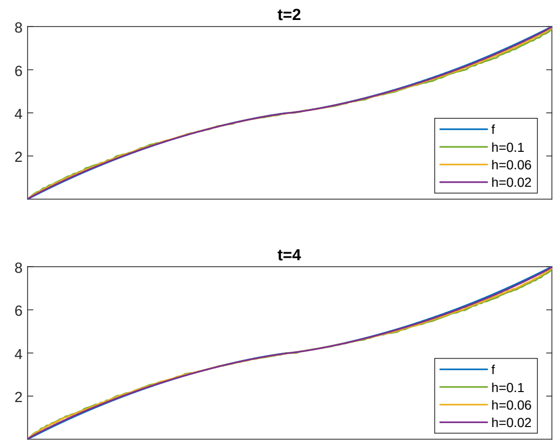

FIG. 4.8. *Eigenvalue distribution of* $\{B_{\mathbf{n},t}\}_{\mathbf{n}}$ *for different* h*-values together with the sampling of* $f(\theta_1, \theta_2) = (2 - 2\cos\theta_1) + (2 - 2\cos\theta_2)$*, and* $t = 2, 4$*.*

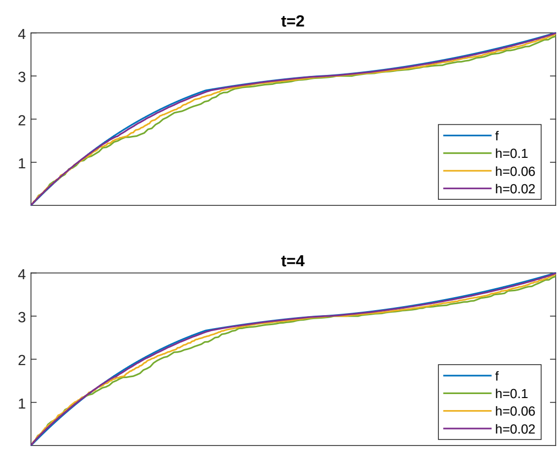

FIG. 4.9. *Eigenvalue distribution of* $\{B_{\mathbf{n},t}\}_{\mathbf{n}}$ *for different* h*-values together with the sampling of* $f(\theta_1, \theta_2) = (8 - 2\cos\theta_1 - 2\cos\theta_2 - 4\cos\theta_1\cos\theta_2)/3$*, and* $t = 2, 4$*.*

2. Examples regarding approximated FDEs on unbounded/moving domains represent a future challenge for the new tools to be investigated with care.
3. The a.c.s. notion has been used as one of the main tools at the foundation of the GLT theory. It is possible that a similar role could be played by the notion of g.a.c.s. for the construction of a new larger class of matrix sequences, with a structure of maximal $*$-algebra isometrically equivalent to measurable functions on a proper unbounded domain. This will be a subject of further studies in the future.
4. The study of the eigenvalue distribution in non-normal cases in which the matrix sequence cannot be viewed as a small perturbation of a Hermitian matrix sequence is still very intricate (see [40, 44, 45] and the use of potential theory in [43] and the references therein). This type of research is very challenging/difficult, and it is worth to be considered.

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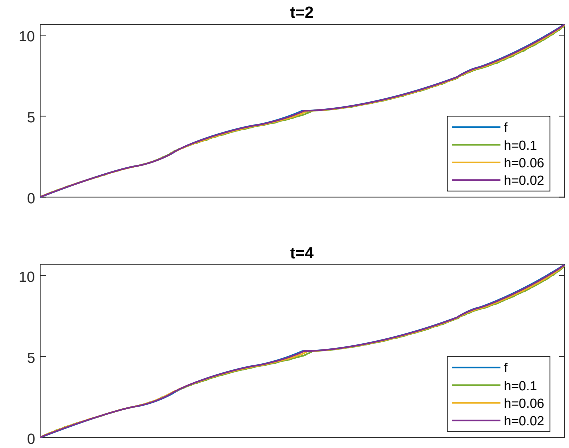

FIG. 4.10. *Eigenvalue distribution of* $\{B_{\mathbf{n},t}\}_{\mathbf{n}}$ *for different* h*-values together with the sampling of* $f(\theta_1, \theta_2) = f_{P_2}(\theta_1, \theta_2)$*, and* $t = 2, 4$*.*

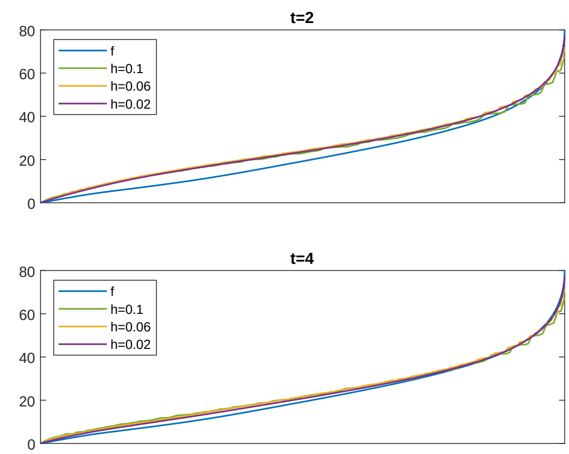

FIG. 4.11. *Eigenvalue distribution of* $\{B_{\mathbf{n},t}(a)\}_{\mathbf{n}}$ *for different* h*-values together with the sampling of* $f(\theta_1, \theta_2, x, y) = [(2-2\cos\theta_1)+(2-2\cos\theta_2)]a(x, y)$*,* $a(x, y) = (10+x^2+2y^2+\sin^2(x+y))/(1+x^2+y^2)$*, and* $t = 2, 4$*.*

Supplementary Material. Further information and numerical results can be found in the supplement to this paper at
https://etna.ricam.oeaw.ac.at/volumes/2021-2030/vol63/addition/p424.php.

Acknowledgments. All the authors are grateful for the support to “Gruppo Nazionale per il Calcolo Scientifico" (INdAM-GNCS). The work of Andrea Adriani is supported by MUR Excellence Department Project MatMod@TOV awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C23000330006. The work of Stefano Serra-Capizzano is partially supported by INdAM - GNCS Project “Analisi e applicazioni di matrici strutturate (a blocchi)" CUP E53C23001670001 and is funded via the European High-Performance Computing Joint Undertaking (JU) under grant agreement No 955701. The

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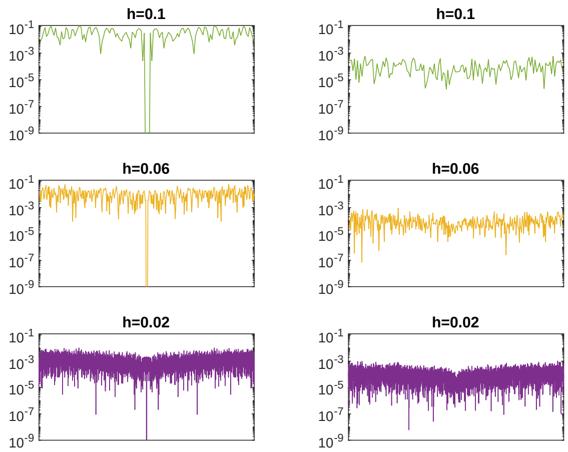

FIG. 4.12. *Minimal distance of eigenvalues of* $A_{\mathbf{n}}$ *from* $f(\theta_1, \theta_2) = (2 - 2\cos\theta_1) + (2 - 2\cos\theta_2)$ *for different* h*-values. First column with cardinality of* f *samplings comparable to eigenvalue cardinality, second column with high cardinality of* f *samplings.*

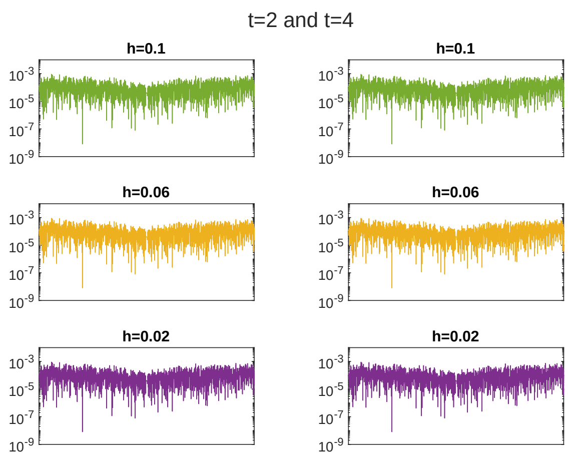

FIG. 4.13. *Minimal distance of eigenvalues of* $B_{\mathbf{n},t}$ *from* $f(\theta_1, \theta_2) = (2 - 2\cos\theta_1) + (2 - 2\cos\theta_2)$ *and* $t = 2, 4$ *for different* h*-values.*

JU receives support from the European Union's Horizon 2020 research and innovation programme and Belgium, France, Germany, Switzerland. Furthermore Stefano Serra-Capizzano is grateful for the support of the Laboratory of Theory, Economics and Systems—Department of Computer Science at Athens University of Economics and Business and to the "Como Lake center for AstroPhysics" of Insubria University.

Finally, the authors warmly thank the reviewers for their careful reading and the very useful suggestions: the paper substantially improved thanks to their help.

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