Electronic Transactions on Numerical Analysis.
Volume 63, pp. 401–423, 2025.

ISSN 1068–9613.
DOI: 10.1553/etna_vol63s401

ETNA
Kent State University and
Johann Radon Institute (RICAM)

STRUCTURED BACKWARD ERRORS OF SPARSE GENERALIZED SADDLE POINT PROBLEMS WITH HERMITIAN BLOCK MATRICES*

SK. SAFIQUE AHMAD† AND PINKI KHATUN†

Abstract. In this paper, we derive the structured backward error (BE) for a class of generalized saddle point problems (GSPPs) with perturbations preserving the sparsity pattern and the Hermitian structures of the block matrices. Additionally, we construct the optimal backward perturbation matrices for which the structured BE is achieved. Our analysis also examines the structured BE in cases where the sparsity pattern is not maintained. Through numerical experiments, we demonstrate the reliability of the obtained structured BEs and the corresponding optimal backward perturbations. Finally, the computed structured BEs are used to assess the strong backward stability of some numerical methods used to solve the GSPP.

Key words. Hermitian matrices, backward error, perturbation analysis, saddle point problems, sparsity

AMS subject classifications. 15A12, 65F20, 65F35, 65F99

1. Introduction.

The concept of backward error (BE) analysis, proposed in [29], plays a crucial role in the field of numerical linear algebra. It has several key applications: for example, it can be used to identify a nearby perturbed problem with minimal norm perturbation, ensuring that the approximate/computed solution of the original problem aligns with the exact solution of the perturbed problem; it can provide, through the product with condition numbers, an upper bound for the forward error; it is often employed as a stopping criterion for iterative algorithms. For a given problem, if the computed BE of an approximate solution is within the unit round-off error, the corresponding numerical algorithm is considered backward stable; see [15]. The notion of structured BE is introduced when the problem possesses some special structure, and the BE is analyzed with structure-preserving constraints imposed on the perturbation matrices. Furthermore, a numerical algorithm is classified as strongly backward stable if the computed structured BE remains within the unit round-off error; see [7, 8].

This paper considers the following 2×2 block linear system:

$$\mathfrak{B}\boldsymbol{x} \triangleq \begin{bmatrix} E & F^* \\ H & G \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} q \\ r \end{bmatrix} \triangleq \boldsymbol{f}, \tag{1.1}$$

where $E \in \mathbb{C}^{n\times n}, F, H \in \mathbb{C}^{m\times n}, G \in \mathbb{C}^{m\times m}$, $q \in \mathbb{C}^n$, and $r \in \mathbb{C}^m$. Hereafter, B^* represents the conjugate transpose of B. This 2×2 block linear system encompasses several important cases: the Hermitian saddle point problem (SPP) ($E = E^*$, $F = H$, $G = 0$), the non-Hermitian SPP ($F = H, G = G^*$), and the real standard SPP ($E \in \mathbb{R}^{n\times n}$, $F, H \in \mathbb{R}^{m\times n}$, $G \in \mathbb{R}^{m\times m}$, $q \in \mathbb{R}^n$, and $r \in \mathbb{R}^m$); see [5, 6]. Hereafter, 0 denotes the zero matrix of appropriate size. We refer to (1.1) as the generalized SPP (GSPP). The GSPP has broad applications across various scientific and engineering fields, including computational fluid dynamics [9, 12], optimal control [23], weighted and equality-constrained least-squares estimation [14]. For fundamental properties as well as for a comprehensive survey and applications of GSPPs, we refer to [6].

*Received November 1, 2024. Accepted July 13, 2025. Published online on September 30, 2025. Recommended by Maya Neytcheva.
†Corresponding author: Sk. S. Ahmad. Department of Mathematics, Indian Institute of Technology Indore, Simrol, Indore 453552, Madhya Pradesh, India (safique@iiti.ac.in), (pinki996.pk@gmail.com).

ETNA
Kent State University and
Johann Radon Institute (RICAM)

The development of various iterative methods for solving the GSPP (1.1) has become a focal point for many researchers, as reflected in recent studies [11, 16, 26, 30] and the references therein. However, the computed solution may still contain some errors and can potentially lead to insignificant results. Therefore, it is crucial to assess how closely the computed solution approximates the solution to the original problem.

Recently, many studies have been carried out on a structured BE analysis and the condition numbers for real SPPs; see [3, 4, 10, 20, 22, 27, 31, 34]. The existing literature primarily considers the block matrices E, F, G, and H real, with $(\cdot)^*$ in (1.1) intended as a real transpose and therefore denoted by $(\cdot)^T$. A brief overview of the literature is as follows: by considering $E = E^T \in \mathbb{R}^{n\times n}, F = H \in \mathbb{R}^{m\times n}$, and $G = 0$, the structured BE for the GSPP was derived in [27]. By employing Sun's methods, the BE for the GSPP when $E = I_n \in \mathbb{R}^{n\times n}$, $F = H \in \mathbb{R}^{m\times n}$, and $G = 0 \in \mathbb{R}^{m\times m}$ was investigated in [18], while the structured BE for the case $E \neq E^T \in \mathbb{R}^{n\times n}, F = H \in \mathbb{R}^{m\times n}$, and $G = 0 \in \mathbb{R}^{m\times m}$ was studied in [31]. Here, I_n denotes the $n \times n$ identity matrix. Further, when $G \neq 0$, structured BEs for the GSPP have been studied for the following matrix structures:

(a) $E = E^T \in \mathbb{R}^{n\times n}, F = H \in \mathbb{R}^{m\times n}$, and $G = G^T \in \mathbb{R}^{m\times m}$ in [10, 22, 34];
(b) $E = E^T \in \mathbb{R}^{n\times n}$, $F = H \in \mathbb{R}^{m\times n}$, and $G \in \mathbb{R}^{m\times m}$ in [34];
(c) $E \in \mathbb{R}^{n\times n}$, $F = H \in \mathbb{R}^{m\times n}$, and $G \in \mathbb{R}^{m\times m}$ in [10, 20];
(d) $E = E^T \in \mathbb{R}^{n\times n}$, $F \neq H \in \mathbb{R}^{m\times n}$, and $G \in \mathbb{R}^{m\times m}$ in [10, 22];
(e) $E = E^T \in \mathbb{R}^{n\times n}$, $F \neq H \in \mathbb{R}^{m\times n}$, and $G = G^T \in \mathbb{R}^{m\times m}$ in [34];
(f) $E \in \mathbb{R}^{n\times n}$, $F \neq H \in \mathbb{R}^{m\times n}$, and $G = G^T \in \mathbb{R}^{m\times m}$ in [10].

Recent studies have also focused on structured BEs for SPPs with three-by-three block structures; see [19, 21]. These investigations consider cases where the block matrices are real, with diagonal blocks being either symmetric or nonsymmetric. Nevertheless, the existing literature reveals several shortcomings: (1) it does not explore a structured BE analysis when the block matrices in (1.1) are complex, specifically when the block matrices possess Hermitian structure; (2) it does not provide the optimal backward perturbations needed to achieve the structured BE.

In many practical applications, such as the discretization of the Stokes equation in [12] and PDE-constrained optimization problems, the coefficient matrix of the GSPP includes a large number of zeros, i.e., it possesses a sparse structure. Preserving this sparsity is crucial for computational efficiency and for maintaining the problem's structure. Recent works on the structured BE analysis for eigenvalue problems have highlighted the importance of incorporating sparsity preservation in the perturbation analysis; see [1, 2, 33]. Therefore, performing perturbation analysis that preserves the sparsity pattern is essential and requires the construction of optimal sparse perturbation matrices to ensure accuracy and efficiency in solving the GSPP. The existing literature on structured BEs for GSPPs also does not preserve the sparsity of the coefficient matrix.

To overcome these drawbacks, in this paper, by preserving the sparsity pattern of $\mathfrak{B}$, we investigate the structured BEs in the following cases:

(i) $E \in \mathbb{HC}^{n\times n},\ F = H \in \mathbb{C}^{m\times n},\ G \in \mathbb{C}^{m\times m}$,
(ii) $E \in \mathbb{C}^{n\times n},\ F = H \in \mathbb{C}^{m\times n},\ G \in \mathbb{HC}^{m\times m}$,
(iii) $E \in \mathbb{HC}^{n\times n},\ F \neq H \in \mathbb{C}^{m\times n},\ G \in \mathbb{HC}^{m\times m}$,

where $\mathbb{HC}^{n\times n}$ represents the collection of all Hermitian matrices.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

The main highlights of this work are listed below:

- We investigate the structured BE for the GSPP (1.1) for the cases (i)–(iii) by retaining the sparsity pattern of $\mathfrak{B}$.
- We provide compact formulas for the structured BE in each of the three cases and derive the formulas for the optimal backward perturbations. Moreover, we derive the structured BEs for the GSPP when the sparsity of $\mathfrak{B}$ is not considered.
- We present numerical examples to validate the obtained results and evaluate the strong backward stability of some numerical methods used to solve the GSPP (1.1).

The remainder of the paper is organized as follows: Section 2 presents essential basic notations, preliminaries, and a few important results. In Section 3, we derive the structured BEs for the cases (i)–(iii). Section 4 provides numerical examples to validate the obtained formulas for structured BEs, while concluding remarks are presented in Section 5.

2. Notation and preliminaries.

2.1. Notation. Throughout the paper, let $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ real and complex matrices, respectively. We use $\mathbb{SR}^{n\times n}$, $\mathbb{SKR}^{n\times n}$, and $\mathbb{HC}^{n\times n}$ to represent the set of all $n \times n$ real symmetric, real skew-symmetric, and Hermitian matrices, respectively. For $X \in \mathbb{C}^{n\times m}$, $\mathfrak{R}(X)$, $\mathfrak{I}(X)$, X^T, X^*, and $X^\dagger$ represent the real part, imaginary part, transpose, conjugate transpose, and Moore-Penrose inverse of X, respectively. We use $0_{m\times n}$ and I_m to denote the $m \times n$ zero matrix and identity matrix, respectively. We only use 0 when the size is clear. We use e_i^m to denote the i^{th} column of the identity matrix I_m. The symbol $\mathbf{1}_{m\times m}$ represents the $m \times m$ matrix with all entries equal to 1. The notation $\|\cdot\|_2$ and $\|\cdot\|_{\mathrm{F}}$ represent the Euclidean norm and the Frobenius norm, respectively. The componentwise multiplication of the matrices of $X = [x_{ij}] \in \mathbb{R}^{m\times n}$ and $Z = [z_{ij}] \in \mathbb{R}^{m\times n}$ is defined as $X \odot Z := [x_{ij}z_{ij}] \in \mathbb{R}^{m\times n}$. For $Z \in \mathbb{R}^{m\times n}$, we define $\Theta_Z := \mathtt{sgn}(Z) = [\mathtt{sgn}(z_{ij})] \in \mathbb{R}^{m\times n}$, where

$$\mathtt{sgn}(z_{ij}) = \begin{cases} 1, & \text{for } z_{ij} \neq 0, \\ 0, & \text{for } z_{ij} = 0. \end{cases}$$

For $X = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n] \in \mathbb{R}^{m\times n}$, where $\mathbf{x}_i \in \mathbb{R}^m$, $i = 1, 2, \dots, n$, we define

$$\mathrm{vec}(X) := [\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_n^T]^T \in \mathbb{R}^{mn}.$$

For the matrices $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{p\times q}$, the Kronecker product [13] is defined as $X \otimes Y = [x_{ij}Y] \in \mathbb{R}^{mp\times nq}$. Given the matrix $A \in \mathbb{R}^{m\times n}$ and vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$, the following properties of the Kronecker product and the vec-operator hold [13, 17]:

$$\begin{aligned} Au &= (u^T \otimes I_m)\mathrm{vec}(A), \\ A^T v &= (I_n \otimes v^T)\mathrm{vec}(A). \end{aligned}$$

Given two positive weight vectors $\sigma_1 = [\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2]^T$ and $\sigma_2 = [\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2]^T$, we define the following weighted Frobenius norms:

$$\boldsymbol{\zeta}^{\sigma_1}(E, F, G, q, r) := \sqrt{\alpha_1^2\|E\|_{\mathrm{F}}^2 + \alpha_2^2\|F\|_{\mathrm{F}}^2 + \alpha_3^2\|G\|_{\mathrm{F}}^2 + \beta_1^2\|q\|_2^2 + \beta_2^2\|r\|_2^2},$$

$$\boldsymbol{\zeta}^{\sigma_2}(E, F, H, G, q, r) := \sqrt{\alpha_1^2\|E\|_{\mathrm{F}}^2 + \alpha_2^2\|F\|_{\mathrm{F}}^2 + \alpha_3^2\|H\|_{\mathrm{F}}^2 + \alpha_4^2\|G\|_{\mathrm{F}}^2 + \beta_1^2\|q\|_2^2 + \beta_2^2\|r\|_2^2}.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

2.2. Preliminaries. In this subsection, we first present the notion of unstructured and structured BEs and a few important definitions and lemmas.

DEFINITION 2.1 ([24]). *Let* $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ *be an approximate solution of the GSPP* (1.1). *Then, the normwise unstructured BE, denoted by* $\boldsymbol{\xi}(\widehat{\boldsymbol{x}})$, *is defined as*

$$\boldsymbol{\xi}(\widehat{\boldsymbol{x}}) := \min_{(\Delta\mathfrak{B},\ \Delta\boldsymbol{f})\in\mathcal{F}} \left\{ \left\| \begin{bmatrix} \frac{\|\Delta\mathfrak{B}\|_{\mathrm{F}}}{\|\mathfrak{B}\|_{\mathrm{F}}} & \frac{\|\Delta\boldsymbol{f}\|_2}{\|\boldsymbol{f}\|_2} \end{bmatrix} \right\|_2 \right\},$$

where

$$\mathcal{F} = \left\{ (\Delta\mathfrak{B},\ \Delta\boldsymbol{f}) \middle| (\mathfrak{B}+\Delta\mathfrak{B})\widehat{\boldsymbol{x}} = \boldsymbol{f} + \Delta\boldsymbol{f} \right\}.$$

In [24], the following explicit expression for the BE defined in Definition 2.1 was provided:

$$\boldsymbol{\xi}(\widehat{\boldsymbol{x}}) = \frac{\|\boldsymbol{f} - \mathfrak{B}\widehat{\boldsymbol{x}}\|_2}{\sqrt{\|\mathfrak{B}\|_{\mathrm{F}}^2\|\widehat{\boldsymbol{x}}\|_2^2 + \|\boldsymbol{f}\|_2^2}}. \tag{2.1}$$

When $\boldsymbol{\xi}(\widehat{\boldsymbol{x}})$ is sufficiently small, the approximate solution $\widehat{\boldsymbol{x}}$ becomes the exact solution to a slightly perturbed system $(\mathfrak{B}+\Delta\mathfrak{B})\widehat{\boldsymbol{x}} = \boldsymbol{f}+\Delta\boldsymbol{f}$, where both $\|\Delta\mathfrak{B}\|_F$ and $\|\Delta\boldsymbol{f}\|_2$ are relatively small. This implies that the corresponding numerical algorithm exhibits backward stability.

In the following definition, we introduce the concept of the structured BE for the GSPP in (1.1). Throughout the paper, we assume that the coefficient matrix $\mathfrak{B}$ in (1.1) is nonsingular.

DEFINITION 2.2. *Assume that* $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ *is a computed solution of the GSPP* (1.1). *Then, we define the normwise structured BEs* $\boldsymbol{\xi}^{\mathcal{G}_i}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$, $i = 1, 2, 3$, *as follows:*

$$\boldsymbol{\xi}^{\mathcal{G}_i}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = \min_{\left(\substack{\Delta E, \Delta F,\\ \Delta G, \Delta q, \Delta r}\right)\in\mathcal{G}_i} \boldsymbol{\zeta}^{\sigma_1}(\Delta E, \Delta F, \Delta G, \Delta q, \Delta r), \qquad \text{for } i = 1, 2,$$

$$\boldsymbol{\xi}^{\mathcal{G}_3}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = \min_{\left(\substack{\Delta E, \Delta F, \Delta H,\\ \Delta G, \Delta q, \Delta r}\right)\in\mathcal{G}_3} \boldsymbol{\zeta}^{\sigma_2}(\Delta E, \Delta F, \Delta H, \Delta G, \Delta q, \Delta r),$$

where

$$\begin{aligned} \mathcal{G}_1 = \Bigg\{ \begin{pmatrix} \Delta E, \Delta F, \\ \Delta G, \Delta q, \Delta r \end{pmatrix} \Bigg| \begin{bmatrix} E+\Delta E & (F+\Delta F)^* \\ F+\Delta F & G+\Delta G \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{u}} \\ \widehat{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} q+\Delta q \\ r+\Delta r \end{bmatrix}, \\ \Delta E \in \mathbb{HC}^{n\times n}, \Delta F \in \mathbb{C}^{m\times n}, \Delta G \in \mathbb{C}^{m\times m},\ \Delta q \in \mathbb{C}^n, \Delta r \in \mathbb{C}^m \Bigg\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \mathcal{G}_2 = \Bigg\{ \begin{pmatrix} \Delta E, \Delta F, \\ \Delta G, \Delta q, \Delta r \end{pmatrix} \Bigg| \begin{bmatrix} E+\Delta E & (F+\Delta F)^* \\ F+\Delta F & G+\Delta G \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{u}} \\ \widehat{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} q+\Delta q \\ r+\Delta r \end{bmatrix}, \\ \Delta E \in \mathbb{C}^{n\times n}, \Delta F \in \mathbb{C}^{m\times n}, \Delta G \in \mathbb{HC}^{m\times m},\ \Delta q \in \mathbb{C}^n, \Delta r \in \mathbb{C}^m \Bigg\}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathcal{G}_3 = \Bigg\{ \begin{pmatrix} \Delta E, \Delta F, \Delta H, \\ \Delta G, \Delta q, \Delta r \end{pmatrix} \Bigg| \begin{bmatrix} E+\Delta E & (F+\Delta F)^* \\ H+\Delta H & G+\Delta G \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{u}} \\ \widehat{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} q+\Delta q \\ r+\Delta r \end{bmatrix}, \\ \Delta E \in \mathbb{HC}^{n\times n}, \Delta F, \Delta H \in \mathbb{C}^{m\times n}, \Delta G \in \mathbb{HC}^{m\times m},\ \Delta q \in \mathbb{C}^n, \Delta r \in \mathbb{C}^m \Bigg\}. \end{aligned}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

By choosing $\alpha_1 = \frac{1}{\|E\|_{\mathrm{F}}}$, $\alpha_2 = \frac{1}{\|F\|_{\mathrm{F}}}$, $\alpha_3 = \frac{1}{\|G\|_{\mathrm{F}}}$, (or $\alpha_3 = \frac{1}{\|H\|_{\mathrm{F}}}$, $\alpha_4 = \frac{1}{\|G\|_{\mathrm{F}}}$ for $\boldsymbol{\xi}^{\mathcal{G}_3}(\widehat{\boldsymbol{u}}, \widehat{p})$), and $\beta_1 = \frac{1}{\|q\|_2}$, $\beta_2 = \frac{1}{\|r\|_2}$, we obtain relative structured BEs for the GSPP (1.1).

REMARK 2.3. We denote the optimal backward perturbations for the structured BEs by $\Delta E_{\mathtt{opt}}$, $\Delta F_{\mathtt{opt}}$, $\Delta H_{\mathtt{opt}}$, $\Delta G_{\mathtt{opt}}$, $\Delta q_{\mathtt{opt}}$, and $\Delta r_{\mathtt{opt}}$. Therefore, the following holds:

$$\begin{aligned}
&\boldsymbol{\xi}^{\mathcal{G}_i}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = \boldsymbol{\zeta}^{\sigma_1}(\Delta E_{\mathtt{opt}}, \Delta F_{\mathtt{opt}}, \Delta G_{\mathtt{opt}}, \Delta q_{\mathtt{opt}}, \Delta r_{\mathtt{opt}}), \qquad \text{for } i = 1, 2,\\
&\boldsymbol{\xi}^{\mathcal{G}_3}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = \boldsymbol{\zeta}^{\sigma_2}(\Delta E_{\mathtt{opt}}, \Delta F_{\mathtt{opt}}, \Delta H_{\mathtt{opt}}, \Delta G_{\mathtt{opt}}, \Delta q_{\mathtt{opt}}, \Delta r_{\mathtt{opt}}).
\end{aligned}$$

REMARK 2.4. Our focus is on examining the structured BE while preserving the sparsity pattern of the block matrices. To accomplish this, we substitute the perturbation matrices ΔE, ΔF, ΔH, and ΔG with $\Delta E \odot \Theta_E$, $\Delta F \odot \Theta_F$, $\Delta H \odot \Theta_H$, and $\Delta G \odot \Theta_G$, respectively. In this framework, the structured BEs are denoted as $\boldsymbol{\xi}^{\mathcal{G}_i}_{\mathbf{sps}}(\boldsymbol{u}, \boldsymbol{p})$, for $i = 1, 2, 3$, while the optimal perturbation matrices are denoted by $\Delta E^{\mathtt{sps}}_{\mathtt{opt}}$, $\Delta F^{\mathtt{sps}}_{\mathtt{opt}}$, $\Delta H^{\mathtt{sps}}_{\mathtt{opt}}$, $\Delta G^{\mathtt{sps}}_{\mathtt{opt}}$, $\Delta q_{\mathtt{opt}}$, and $\Delta r_{\mathtt{opt}}$.

Next, we discuss some important definitions and lemmas.

LEMMA 2.5 ([28]). *The linear system $Ax = b$, with $A \in \mathbb{R}^{n\times m}, b \in \mathbb{R}^n$, is consistent if and only if $AA^\dagger b = b$. Furthermore, when the system is consistent, the minimum-norm solution is given by $A^\dagger b$.*

DEFINITION 2.6. *Let $Z \in \mathbb{SR}^{m\times m}$. Then we define its generator vector as*

$$\mathrm{vec}_S(Z) := [\boldsymbol{z}_1^T, \boldsymbol{z}_2^T, \ldots, \boldsymbol{z}_m^T]^T \in \mathbb{R}^{\frac{m(m+1)}{2}},$$

where

$$\begin{aligned}
&\boldsymbol{z}_1 = [z_{11}, z_{21}, \ldots, z_{m1}]^T \in \mathbb{R}^m,\\
&\boldsymbol{z}_2 = [z_{22}, z_{32}, \ldots, z_{m2}]^T \in \mathbb{R}^{m-1}, \ldots, \boldsymbol{z}_{m-1} = [z_{(m-1)(m-1)}, z_{m(m-1)}]^T \in \mathbb{R}^2,\\
&\boldsymbol{z}_m = [z_{mm}] \in \mathbb{R}.
\end{aligned}$$

DEFINITION 2.7. *Let $Z \in \mathbb{SKR}^{m\times m}$. Then we define its generator vector as*

$$\mathrm{vec}_{SK}(Z) := [\boldsymbol{z}_1^T, \boldsymbol{z}_2^T, \ldots, \boldsymbol{z}_{m-1}^T]^T \in \mathbb{R}^{\frac{m(m-1)}{2}},$$

where

$$\begin{aligned}
&\boldsymbol{z}_1 = [z_{21}, \ldots, z_{m1}]^T \in \mathbb{R}^{m-1},\\
&\boldsymbol{z}_2 = [z_{32}, \ldots, z_{m2}]^T \in \mathbb{R}^{m-2}, \ldots, \boldsymbol{z}_{m-1} = [z_{(m-1)(m-1)}]^T \in \mathbb{R}.
\end{aligned}$$

The following two lemmas can also be found in [32].

LEMMA 2.8. *Let $M \in \mathbb{SR}^{m\times m}$. Then $\mathrm{vec}(M) = \mathcal{J}_S^m \mathrm{vec}_S(M)$, where*

$$\mathcal{J}_S^m = \begin{bmatrix} \mathcal{J}_S^{(1)} & \mathcal{J}_S^{(2)} & \cdots & \mathcal{J}_S^{(m)} \end{bmatrix} \in \mathbb{R}^{m^2 \times \frac{m(m+1)}{2}},$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

and the matrices $\mathcal{J}_S^{(i)} \in \mathbb{R}^{m^2 \times (m-i+1)}$ *are defined as*

$$\mathcal{J}_S^{(1)} = \begin{bmatrix} e_1^m & e_2^m & e_3^m & \cdots & e_{m-1}^m & e_m^m \\ 0 & e_1^m & 0 & \cdots & \cdots & 0 \\ 0 & 0 & e_1^m & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & e_1^m & 0 \\ 0 & 0 & 0 & \cdots & 0 & e_1^m \end{bmatrix},$$

$$\mathcal{J}_S^{(2)} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ e_2^m & e_3^m & \cdots & \cdots & e_m^m \\ 0 & e_2^m & 0 & \cdots & 0 \\ 0 & 0 & e_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & e_2^m \end{bmatrix}, \ldots, \mathcal{J}_S^{(m)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ e_m^m \end{bmatrix}.$$

LEMMA 2.9. *Let* $M \in \mathbb{SKR}^{m \times m}$. *Then* $\mathrm{vec}(M) = \mathcal{J}_{SK}^m \mathrm{vec}_{SK}(M)$, *where*

$$\mathcal{J}_{SK}^m = \begin{bmatrix} \mathcal{J}_{SK}^{(1)} & \mathcal{J}_{SK}^{(2)} & \cdots \mathcal{J}_{SK}^{(m-1)} \end{bmatrix} \in \mathbb{R}^{m^2 \times \frac{m(m-1)}{2}},$$

and the matrices $\mathcal{J}_{SK}^{(i)} \in \mathbb{R}^{m^2 \times (m-i)}$ *are defined as*

$$\mathcal{J}_{SK}^{(1)} = \begin{bmatrix} e_2^m & e_3^m & \cdots & e_{m-1}^m & e_m^m \\ -e_1^m & 0 & \cdots & \cdots & 0 \\ 0 & -e_1^m & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & -e_1^m & 0 \\ 0 & 0 & \cdots & 0 & -e_1^m \end{bmatrix},$$

$$\mathcal{J}_{SK}^{(2)} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ e_3^m & \cdots & \cdots & e_m^m \\ -e_2^m & 0 & \cdots & 0 \\ 0 & -e_2^m & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 0 & -e_2^m \end{bmatrix}, \ldots, \mathcal{J}_{SK}^{(m-1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ e_m^m \\ -e_{m-1}^m \end{bmatrix}.$$

Next, we present an important lemma that is crucial for computing structured BEs while preserving the sparsity pattern.

LEMMA 2.10. *Assume* $M \in \mathbb{R}^{m \times m}$ *and* $X \in \mathbb{HC}^{n \times n}$. *Then the following results hold:*

1. *If* $M \in \mathbb{SR}^{m \times m}$, *then*

$$\mathrm{vec}(M \odot \Theta_X) = \mathcal{J}_S^m \Phi_X \mathrm{vec}_S(M \odot \Theta_X), \tag{2.4}$$

where $\Phi_X = \mathrm{diag}(\mathrm{vec}_S(\Theta_X))$.

2. *If* $M \in \mathbb{SKR}^{m \times m}$, *then*

$$\mathrm{vec}(M \odot \Theta_X) = \mathcal{J}_{SK}^m \Psi_X \mathrm{vec}_{SK}(M \odot \Theta_X), \tag{2.5}$$

where $\Psi_X = \mathrm{diag}([\Theta_X(1, 2:m), \Theta_X(2, 3:m), \ldots, \Theta_X(m-1:m)]^T)$.

Proof. Let $M \in \mathbb{SR}^{m\times m}$ and $X \in \mathbb{HC}^{n\times n}$. By definition of the matrix Θ_X, we have $\Theta_X \in \mathbb{SR}^{m\times m}$ and consequently, $M \odot \Theta_X \in \mathbb{SR}^{m\times m}$. Then,

$$\mathrm{vec}(M \odot \Theta_X) = \mathcal{J}_S^m \mathrm{vec}_S(M \odot \Theta_X) = \mathcal{J}_S^m \Phi_X \mathrm{vec}_S(M \odot \Theta_X),$$

and (2.4) follows. Similarly, we can prove (2.5) when $M \in \mathbb{SKR}^{m\times m}$. ∎

REMARK 2.11. By considering $X = \mathbf{1}_{m\times m}$, Lemma 2.10 reduces to Lemmas 2.8 and 2.9, where $\mathbf{1}_{m\times m}$ denotes $m \times m$ matrix with all entries equal to 1.

REMARK 2.12. When $M, X \in \mathbb{C}^{m\times n}$, we have $\mathrm{vec}(M \odot \Theta_X) = \Sigma_X \mathrm{vec}(M \odot \Theta_X)$, where $\Sigma_X = \mathrm{diag}(\mathrm{vec}(\Theta_X))$.

To illustrate Lemma 2.10, we consider the following example:

EXAMPLE 2.13. Let $M \in \mathbb{SR}^{3\times 3}$ and $X \in \mathbb{HC}^{3\times 3}$ be given by

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \quad X = \begin{bmatrix} 7 & 8+i & 0 \\ 8-i & 9 & 10+2i \\ 0 & 10-2i & 0 \end{bmatrix}.$$

Then, $M \odot \Theta_X = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 0 \end{bmatrix}$, and we have

$$\begin{aligned}
\mathrm{vec}(M \odot \Theta_X) &= \begin{bmatrix} e_1^3 & e_2^3 & e_3^3 & 0 & 0 & 0 \\ 0 & e_1^3 & 0 & e_2^3 & e_3^3 & 0 \\ 0 & 0 & e_1^3 & 0 & e_2^3 & e_3^3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \\ 5 \\ 0 \end{bmatrix} = \mathcal{J}_S^3 \mathrm{vec}_S(M \odot \Theta_X) \\
&= \mathcal{J}_S^3 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \\ 5 \\ 0 \end{bmatrix} = \mathcal{J}_S^3 \Phi_X \mathrm{vec}_S(M \odot \Theta_X).
\end{aligned}$$

3. Computation of structured BEs.

In this section, we derive the closed-form expressions for $\boldsymbol{\xi}_{\mathsf{sps}}^{\mathcal{G}_i}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$, for $i = 1, 2, 3$, by preserving the sparsity of the coefficient matrix and the Hermitian structure of the matrices E and G, respectively. Moreover, we provide the optimal perturbations for which the structured BE is attained. The following lemma plays a crucial role in the computation of the structured BE.

LEMMA 3.1. *Consider the following* 2×2 *block linear system:*

$$\begin{bmatrix} E + \Delta E & (F + \Delta F)^* \\ H + \Delta H & G + \Delta G \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} q + \Delta q \\ r + \Delta r \end{bmatrix}. \tag{3.1}$$

Then (3.1) *can be reformulated as the systems of linear equations*

$$\begin{aligned}
\Re(\Delta E)\Re(\boldsymbol{u}) - \Im(\Delta E)\Im(\boldsymbol{u}) + \Re(\Delta F)^T\Re(\boldsymbol{p}) + \Im(\Delta F)^T\Im(\boldsymbol{p}) - \Re(\Delta q) &= \Re(Q), \\
\Re(\Delta E)\Im(\boldsymbol{u}) + \Im(\Delta E)\Re(\boldsymbol{u}) + \Re(\Delta F)^T\Im(\boldsymbol{p}) - \Im(\Delta F)^T\Re(\boldsymbol{p}) - \Im(\Delta q) &= \Im(Q),
\end{aligned}$$

and

$$\begin{aligned}
\Re(\Delta H)\Re(\boldsymbol{u}) - \Im(\Delta H)\Im(\boldsymbol{u}) + \Re(\Delta G)\Re(\boldsymbol{p}) - \Im(\Delta G)\Im(\boldsymbol{p}) - \Re(\Delta r) &= \Re(R), \\
\Re(\Delta H)\Im(\boldsymbol{u}) + \Im(\Delta H)\Re(\boldsymbol{u}) + \Re(\Delta G)\Im(\boldsymbol{p}) + \Im(\Delta G)\Re(\boldsymbol{p}) - \Im(\Delta r) &= \Im(R),
\end{aligned}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

where

$$Q = q - E\boldsymbol{u} - F^*\boldsymbol{p}, \quad R = r - H\boldsymbol{u} - G\boldsymbol{p}.$$

Proof. The 2×2 block linear system (3.1) can be equivalently expressed as

$$\Delta E\boldsymbol{u} + \Delta F^*\boldsymbol{p} - \Delta q = q - E\boldsymbol{u} - F^*\boldsymbol{p}, \tag{3.2}$$

$$\Delta H\boldsymbol{u} + \Delta G\boldsymbol{p} - \Delta r = r - H\boldsymbol{u} - G\boldsymbol{p}. \tag{3.3}$$

The proof follows by separating the real and imaginary parts from (3.2) and (3.3). ∎

3.1. Computation of the structured BE for the case (i). In this subsection, we derive an explicit expression for the structured BE $\boldsymbol{\xi}^{\mathcal{G}_1}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ by preserving the structure of $E \in \mathbb{HC}^{n\times n}$, $F \in \mathbb{C}^{m\times n}$, and $G \in \mathbb{C}^{m\times m}$ and by maintaining the sparsity pattern of the coefficient matrix $\mathfrak{B}$ within the perturbation matrices.

Prior to stating the main theorem of this section, we construct the following matrices: Let $\mathfrak{D}_{S,n} \in \mathbb{R}^{\frac{n(n+1)}{2}\times\frac{n(n+1)}{2}}$ and $\mathfrak{D}_{SK,n} \in \mathbb{R}^{\frac{n(n-1)}{2}\times\frac{n(n-1)}{2}}$ be the diagonal matrices with

$$\mathfrak{D}_{S,n}(j,j) := \begin{cases} 1, & \text{for } j = \frac{(2n-(i-2))(i-1)}{2} + 1,\ i = 1, 2, \ldots, n, \\ \sqrt{2}, & \text{otherwise}, \end{cases}$$

and

$$\mathfrak{D}_{SK,n}(j,j) := \sqrt{2}, \qquad \text{for } j = 1, 2, \ldots, \frac{n(n-1)}{2}.$$

Further, set

$$N_1 := \mathcal{J}^n_S \Phi_E \mathfrak{D}^{-1}_{S,n} \in \mathbb{R}^{n^2\times\frac{n(n+1)}{2}}, \quad N_2 := \mathcal{J}^n_{SK} \Psi_E \mathfrak{D}^{-1}_{SK,n} \in \mathbb{R}^{n^2\times\frac{n(n-1)}{2}}.$$

Let $\boldsymbol{s} = n^2 + 2mn + 2m^2$, $\mathbf{X}_1 \in \mathbb{R}^{2n\times\boldsymbol{s}}$, and $\mathbf{X}_2 \in \mathbb{R}^{2m\times\boldsymbol{s}}$ be defined as

$$\mathbf{X}_1 := [\widehat{\mathbf{X}}_1 \ \ 0_{2n\times 2m^2}], \quad \mathbf{X}_2 := [0_{2m\times n^2} \ \ \widehat{\mathbf{X}}_2],$$

where

$$\widehat{\mathbf{X}}_1 = \begin{bmatrix} \alpha_1^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_n)N_1 & -\alpha_1^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_n)N_2 & \alpha_2^{-1}(I_n \otimes \Re(\widehat{\boldsymbol{p}})^T)\Sigma_F & \alpha_2^{-1}(I_n \otimes \Im(\widehat{\boldsymbol{p}})^T)\Sigma_F \\ \alpha_1^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_n)N_1 & \alpha_1^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_n)N_2 & \alpha_2^{-1}(I_n \otimes \Im(\widehat{\boldsymbol{p}})^T)\Sigma_F & -\alpha_2^{-1}(I_n \otimes \Re(\widehat{\boldsymbol{p}})^T)\Sigma_F \end{bmatrix}$$

and

$$\widehat{\mathbf{X}}_2 = \begin{bmatrix} \alpha_2^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_F & -\alpha_2^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_F & \alpha_3^{-1}(\Re(\widehat{\boldsymbol{p}})^T \otimes I_m)\Sigma_G & -\alpha_3^{-1}(\Im(\widehat{\boldsymbol{p}})^T \otimes I_m)\Sigma_G \\ \alpha_2^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_F & \alpha_2^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_F & \alpha_3^{-1}(\Im(\widehat{\boldsymbol{p}})^T \otimes I_m)\Sigma_G & \alpha_3^{-1}(\Re(\widehat{\boldsymbol{p}})^T \otimes I_m)\Sigma_G \end{bmatrix}.$$

Set

$$\Delta\mathcal{Y} := \begin{bmatrix} \alpha_1 \mathfrak{D}_{S,n} \mathrm{vec}_S(\Re(\Delta E \odot \Theta_E)) \\ \alpha_1 \mathfrak{D}_{SK,n} \mathrm{vec}_{SK}(\Im(\Delta E \odot \Theta_E)) \\ \alpha_2 \mathrm{vec}(\Re(\Delta F \odot \Theta_F)) \\ \alpha_2 \mathrm{vec}(\Im(\Delta F \odot \Theta_F)) \\ \alpha_3 \mathrm{vec}(\Re(\Delta G \odot \Theta_G)) \\ \alpha_3 \mathrm{vec}(\Im(\Delta G \odot \Theta_G)) \end{bmatrix} \in \mathbb{R}^{\boldsymbol{s}}, \quad \Delta\mathcal{Z} := \begin{bmatrix} \beta_1\Re(\Delta q) \\ \beta_1\Im(\Delta q) \\ \beta_2\Re(\Delta r) \\ \beta_2\Im(\Delta r) \end{bmatrix} \in \mathbb{R}^{2(n+m)}. \tag{3.4}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Note that

$$\begin{aligned}\alpha_1^2\|E\|_{\mathrm{F}}^2 &= \alpha_1^2\|\Re(E)\|_{\mathrm{F}}^2 + \alpha_1^2\|\Im(E)\|_{\mathrm{F}}^2\\ &= \|\alpha_1\mathfrak{D}_{S,n}\mathrm{vec}_S(\Re(E))\|_2^2 + \|\alpha_1\mathfrak{D}_{SK,n}\mathrm{vec}_{SK}(\Im(E))\|_2^2\\ &= \left\|\begin{bmatrix}\alpha_1\mathfrak{D}_{S,n}\mathrm{vec}_S(\Re(E))\\ \alpha_1\mathfrak{D}_{SK,n}\mathrm{vec}_{SK}(\Im(E))\end{bmatrix}\right\|_2^2.\end{aligned}$$

The following theorem provides a compact representation of the structured BE $\boldsymbol{\xi}_{\mathsf{sps}}^{\mathcal{G}_1}(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}})$.

THEOREM 3.2. *Assume that $E \in \mathbb{HC}^{n\times n}$, $F \in \mathbb{C}^{m\times n}$, $G \in \mathbb{C}^{m\times m}$ and $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T,\widehat{\boldsymbol{p}}^T]^T$ is a computed solution of the GSPP (1.1). Then, the structured BE $\boldsymbol{\xi}_{\mathsf{sps}}^{\mathcal{G}_1}(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}})$ with preserving sparsity is given by*

$$\boldsymbol{\xi}_{\mathsf{sps}}^{\mathcal{G}_1}(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}}) = \left\|\begin{bmatrix}\mathbf{X}_1 & \mathcal{I}_1\\ \mathbf{X}_2 & \mathcal{I}_2\end{bmatrix}^T\left(\begin{bmatrix}\mathbf{X}_1 & \mathcal{I}_1\\ \mathbf{X}_2 & \mathcal{I}_2\end{bmatrix}\begin{bmatrix}\mathbf{X}_1 & \mathcal{I}_1\\ \mathbf{X}_2 & \mathcal{I}_2\end{bmatrix}^T\right)^{-1}\begin{bmatrix}\Re(Q)\\ \Im(Q)\\ \Re(R)\\ \Im(R)\end{bmatrix}\right\|_2,$$

where $Q = q - E\boldsymbol{u} - F^*\boldsymbol{p}$, $R = r - F\boldsymbol{u} - G\boldsymbol{p}$, and

$$\begin{aligned}\mathcal{I}_1 &= \begin{bmatrix}-\beta_1{}^{-1}I_{2n} & 0_{2n\times 2m}\end{bmatrix} \in \mathbb{R}^{2n\times 2(n+m)},\\ \mathcal{I}_2 &= \begin{bmatrix}0_{2m\times 2n} & -\beta_2{}^{-1}I_{2m}\end{bmatrix} \in \mathbb{R}^{2m\times 2(n+m)}.\end{aligned}$$

Proof. Let $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T,\widehat{\boldsymbol{p}}^T]^T$ be a computed solution of the GSPP (1.1) with $E \in \mathbb{HC}^{n\times n}$. Then, we need to find the perturbations $\Delta q \in \mathbb{C}^n$, $\Delta r \in \mathbb{C}^m$, and sparsity preserving perturbation matrices $\Delta E \in \mathbb{HC}^{n\times n}$, $\Delta F \in \mathbb{C}^{m\times n}$, and $\Delta G \in \mathbb{C}^{m\times m}$ so that (2.2) holds. Therefore, we replace ΔE, ΔF, and ΔG with $\Delta E \odot \Theta_E$, $\Delta F \odot \Theta_F$, and $\Delta G \odot \Theta_G$, respectively, such that the following holds:

$$\begin{bmatrix}E + (\Delta E\odot\Theta_E) & (F + (\Delta F\odot\Theta_F))^*\\ F + (\Delta F\odot\Theta_F) & G + (\Delta G\odot\Theta_G)\end{bmatrix}\begin{bmatrix}\widehat{\boldsymbol{u}}\\ \widehat{\boldsymbol{p}}\end{bmatrix} = \begin{bmatrix}q + \Delta q\\ r + \Delta r\end{bmatrix}.$$

By using Lemma 3.1 with $H = F$, we have

$$\text{(3.5)}\quad \begin{cases}\Re(\Delta E\odot\Theta_E)\Re(\widehat{\boldsymbol{u}}) - \Im(\Delta E\odot\Theta_E)\Im(\widehat{\boldsymbol{u}}) + \Re(\Delta F\odot\Theta_F)^T\Re(\widehat{\boldsymbol{p}})\\ \qquad + \Im(\Delta F\odot\Theta_F)^T\Im(\widehat{\boldsymbol{p}}) - \Re(\Delta q) = \Re(Q),\\ \Re(\Delta E\odot\Theta_E)\Im(\widehat{\boldsymbol{u}}) + \Im(\Delta E\odot\Theta_E)\Re(\widehat{\boldsymbol{u}}) + \Re(\Delta F\odot\Theta_F)^T\Im(\widehat{\boldsymbol{p}})\\ \qquad - \Im(\Delta F\odot\Theta_F)^T\Re(\widehat{\boldsymbol{p}}) - \Im(\Delta q) = \Im(Q),\end{cases}$$

and

$$\text{(3.6)}\quad \begin{cases}\Re(\Delta F\odot\Theta_F)\Re(\widehat{\boldsymbol{u}}) - \Im(\Delta F\odot\Theta_F)\Im(\widehat{\boldsymbol{u}}) + \Re(\Delta G\odot\Theta_G)\Re(\widehat{\boldsymbol{p}})\\ \qquad - \Im(\Delta G\odot\Theta_G)\Im(\widehat{\boldsymbol{p}}) - \Re(\Delta r) = \Re(R),\\ \Re(\Delta F\odot\Theta_F)\Im(\widehat{\boldsymbol{u}}) + \Im(\Delta F\odot\Theta_F)\Re(\widehat{\boldsymbol{u}}) + \Re(\Delta G\odot\Theta_G)\Im(\widehat{\boldsymbol{p}})\\ \qquad + \Im(\Delta G\odot\Theta_G)\Re(\widehat{\boldsymbol{p}}) - \Im(\Delta r) = \Im(R).\end{cases}$$

Now, using the properties of the vec-operator and the Kronecker product in (3.5), we obtain

$$\begin{cases} (\Re(\widehat{\boldsymbol{u}})^T \otimes I_n)\mathrm{vec}(\Re(\Delta E \odot \Theta_E)) - (\Im(\widehat{\boldsymbol{u}})^T \otimes I_n)\mathrm{vec}(\Im(\Delta E \odot \Theta_E)) \\ \qquad + (I_n \otimes \Re(\widehat{\boldsymbol{p}})^T)\mathrm{vec}(\Re(\Delta F \odot \Theta_F)) \\ \qquad + (I_n \otimes \Im(\widehat{\boldsymbol{p}})^T)\mathrm{vec}(\Im(\Delta F \odot \Theta_F)) - \Re(\Delta q) = \Re(Q), \\ (\Im(\widehat{\boldsymbol{u}})^T \otimes I_n)\mathrm{vec}(\Re(\Delta E \odot \Theta_E)) + (\Re(\widehat{\boldsymbol{u}})^T \otimes I_n)\mathrm{vec}(\Im(\Delta E \odot \Theta_E) \\ \qquad + (I_n \otimes \Im(\widehat{\boldsymbol{p}})^T)\mathrm{vec}(\Re(\Delta F \odot \Theta_F)) \\ \qquad - (I_n \otimes \Re(\widehat{\boldsymbol{p}})^T)\mathrm{vec}(\Im(\Delta F \odot \Theta_F)) - \Im(\Delta q) = \Im(Q). \end{cases} \tag{3.7}$$

As $\Delta E \in \mathbb{HC}^{n\times n}$, we have $\Re(\Delta E) \in \mathbb{SR}^{n\times n}$ and $\Im(\Delta E) \in \mathbb{SKR}^{n\times n}$. Furthermore, we have $\Delta E \odot \Theta_E \in \mathbb{HC}^{n\times n}$, $\Re(\Delta E \odot \Theta_E) \in \mathbb{SR}^{n\times n}$, and $\Im(\Delta E \odot \Theta_E) \in \mathbb{SKR}^{n\times n}$. Hence, applying Lemma 2.10 to (3.7), we get

$$\begin{cases} (\Re(\widehat{\boldsymbol{u}})^T \otimes I_n)\mathcal{J}_S^n \Phi_E \mathrm{vec}_S(\Re(\Delta E \odot \Theta_E)) \\ -(\Im(\widehat{\boldsymbol{u}})^T \otimes I_n)\mathcal{J}_{SK}^n \Psi_E \mathrm{vec}_{SK}(\Im(\Delta E \odot \Theta_E)) \\ +(I_n \otimes \Re(\widehat{\boldsymbol{p}})^T)\Sigma_F \mathrm{vec}(\Re(\Delta F \odot \Theta_F)) \\ +(I_n \otimes \Im(\widehat{\boldsymbol{p}})^T)\Sigma_F \mathrm{vec}(\Im(\Delta F \odot \Theta_F)) - \Re(\Delta q) = \Re(Q), \\ (\Im(\widehat{\boldsymbol{u}})^T \otimes I_n)\mathcal{J}_S^n \Phi_E \mathrm{vec}_S(\Re(\Delta E \odot \Theta_E)) \\ +(\Re(\widehat{\boldsymbol{u}})^T \otimes I_n)\mathcal{J}_{SK}^n \Psi_E \mathrm{vec}_{SK}(\Im(\Delta E \odot \Theta_E) \\ +(I_n \otimes \Im(\widehat{\boldsymbol{p}})^T)\Sigma_F \mathrm{vec}(\Re(\Delta F \odot \Theta_F)) \\ -(I_n \otimes \Re(\widehat{\boldsymbol{p}})^T)\Sigma_F \mathrm{vec}(\Im(\Delta F \odot \Theta_F)) - \Im(\Delta q) = \Im(Q). \end{cases} \tag{3.8}$$

Then, (3.8) can be reformulated as

$$\mathbf{X}_1 \Delta\mathcal{Y} + \begin{bmatrix} -\beta_1^{-1} I_{2n} & 0_{2m} \end{bmatrix} \Delta\mathcal{Z} = \begin{bmatrix} \Re(Q) \\ \Im(Q) \end{bmatrix}. \tag{3.9}$$

In a similar manner, from (3.6), we can deduce the following:

$$\mathbf{X}_2 \Delta\mathcal{Y} + \begin{bmatrix} 0_{2n} & -{\beta_2}^{-1} I_{2m} \end{bmatrix} \Delta\mathcal{Z} = \begin{bmatrix} \Re(R) \\ \Im(R) \end{bmatrix}. \tag{3.10}$$

Therefore, combining (3.9) and (3.10), we get

$$\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \Delta\mathcal{Y} + \begin{bmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \end{bmatrix} \Delta\mathcal{Z} = \begin{bmatrix} \Re(Q) \\ \Im(Q) \\ \Re(R) \\ \Im(R) \end{bmatrix} \iff \begin{bmatrix} \mathbf{X}_1 & \mathcal{I}_1 \\ \mathbf{X}_2 & \mathcal{I}_2 \end{bmatrix} \begin{bmatrix} \Delta\mathcal{Y} \\ \Delta\mathcal{Z} \end{bmatrix} = \begin{bmatrix} \Re(Q) \\ \Im(Q) \\ \Re(R) \\ \Im(R) \end{bmatrix}. \tag{3.11}$$

Since the matrix $\begin{bmatrix} \mathbf{X}_1 & \mathcal{I}_1 \\ \mathbf{X}_2 & \mathcal{I}_2 \end{bmatrix}$ has full row-rank, the consistency condition in Lemma 2.5 is satisfied, and the minimal-norm solution of (3.11) is expressed as:

$$\begin{aligned} \begin{bmatrix} \Delta\mathcal{Y} \\ \Delta\mathcal{Z} \end{bmatrix}_{\texttt{opt}} &= \begin{bmatrix} \mathbf{X}_1 & \mathcal{I}_1 \\ \mathbf{X}_2 & \mathcal{I}_2 \end{bmatrix}^{\dagger} \begin{bmatrix} \Re(Q) \\ \Im(Q) \\ \Re(R) \\ \Im(R) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{X}_1 & \mathcal{I}_1 \\ \mathbf{X}_2 & \mathcal{I}_2 \end{bmatrix}^T \left(\begin{bmatrix} \mathbf{X}_1 & \mathcal{I}_1 \\ \mathbf{X}_2 & \mathcal{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 & \mathcal{I}_1 \\ \mathbf{X}_2 & \mathcal{I}_2 \end{bmatrix}^T \right)^{-1} \begin{bmatrix} \Re(Q) \\ \Im(Q) \\ \Re(R) \\ \Im(R) \end{bmatrix}. \end{aligned} \tag{3.12}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

According to Definition 2.2, we have

$$\begin{aligned}
\boldsymbol{\xi}_{\mathsf{sps}}^{\mathcal{G}_1}(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}}) &= \min_{\left(\substack{\Delta E\odot E,\,\Delta F\odot F,\\ \Delta G\odot G,\,\Delta q,\,\Delta r}\right)\in\mathcal{G}_1} \boldsymbol{\zeta}^{\sigma_1}(\Delta E\odot E,\Delta F\odot F,\Delta G\odot G,\Delta q,\Delta r)\\
&= \min_{\left(\substack{\Delta E\odot E,\,\Delta F\odot F,\\ \Delta G\odot G,\,\Delta q,\,\Delta r}\right)\in\mathcal{G}_1} \left\| \begin{bmatrix} \alpha_1\mathrm{vec}(\Re(\Delta E\odot\Theta_E))\\ \alpha_1\mathrm{vec}(\Im(\Delta E\odot\Theta_E))\\ \alpha_2\mathrm{vec}(\Re(\Delta F\odot\Theta_F))\\ \alpha_2\mathrm{vec}(\Im(\Delta F\odot\Theta_F))\\ \alpha_3\mathrm{vec}(\Re(\Delta G\odot\Theta_G))\\ \alpha_3\mathrm{vec}(\Im(\Delta G\odot\Theta_G))\\ \beta_1\Re(\Delta q)\\ \beta_1\Im(\Delta q)\\ \beta_2\Re(\Delta r)\\ \beta_2\Im(\Delta r)\end{bmatrix}\right\|_2\\
&= \min_{\left(\substack{\Delta E\odot E,\,\Delta F\odot F,\\ \Delta G\odot G,\,\Delta q,\,\Delta r}\right)\in\mathcal{G}_1} \left\| \begin{bmatrix} \alpha_1\mathfrak{D}_{S,n}\mathrm{vec}_S(\Re(\Delta E\odot\Theta_E))\\ \alpha_1\mathfrak{D}_{SK,n}\mathrm{vec}_{SK}(\Im(\Delta E\odot\Theta_E))\\ \alpha_2\mathrm{vec}(\Re(\Delta F\odot\Theta_F))\\ \alpha_2\mathrm{vec}(\Im(\Delta F\odot\Theta_F))\\ \alpha_3\mathrm{vec}(\Re(\Delta G\odot\Theta_G))\\ \alpha_3\mathrm{vec}(\Im(\Delta G\odot\Theta_G))\\ \beta_1\Re(\Delta q)\\ \beta_1\Im(\Delta q)\\ \beta_2\Re(\Delta r)\\ \beta_2\Im(\Delta r)\end{bmatrix}\right\|_2\\
&= \min\left\{ \left\|\begin{bmatrix}\Delta\mathcal{Y}\\ \Delta\mathcal{Z}\end{bmatrix}\right\|_2 \;\middle|\; \begin{bmatrix}\mathbf{X}_1 & \mathcal{I}_1\\ \mathbf{X}_2 & \mathcal{I}_2\end{bmatrix}\begin{bmatrix}\Delta\mathcal{Y}\\ \Delta\mathcal{Z}\end{bmatrix} = \begin{bmatrix}\Re(Q)\\ \Im(Q)\\ \Re(R)\\ \Im(R)\end{bmatrix}\right\}\\
&= \left\|\begin{bmatrix}\Delta\mathcal{Y}\\ \Delta\mathcal{Z}\end{bmatrix}_{\mathsf{opt}}\right\|_2,
\end{aligned}$$

and the proof is complete. □

REMARK 3.3. Using (3.4) and (3.12), the optimal perturbation matrix $\Delta E_{\mathsf{opt}}^{\mathsf{sps}}$ is obtained as

$$\Delta E_{\mathsf{opt}}^{\mathsf{sps}} = \Re(\Delta E_{\mathsf{opt}}^{\mathsf{sps}}) + i\Im(\Delta E_{\mathsf{opt}}^{\mathsf{sps}}),$$

where

$$\mathrm{vec}_S(\Re(\Delta E_{\mathsf{opt}}^{\mathsf{sps}})) = \alpha_1^{-1}\mathfrak{D}_{S,n}^{-1}\begin{bmatrix}I_{\boldsymbol{n}_1} & 0_{\boldsymbol{n}_1\times(\boldsymbol{l}-\boldsymbol{n}_1)}\end{bmatrix}\begin{bmatrix}\Delta\mathcal{Y}\\ \Delta\mathcal{Z}\end{bmatrix}_{\mathsf{opt}},$$

$$\mathrm{vec}_{SK}(\Im(\Delta E_{\mathsf{opt}}^{\mathsf{sps}})) = \alpha_1^{-1}\mathfrak{D}_{SK,n}^{-1}\begin{bmatrix}0_{\boldsymbol{n}_2\times\boldsymbol{n}_1} & I_{\boldsymbol{n}_2} & 0_{\boldsymbol{n}_2\times(\boldsymbol{l}-n^2)}\end{bmatrix}\begin{bmatrix}\Delta\mathcal{Y}\\ \Delta\mathcal{Z}\end{bmatrix}_{\mathsf{opt}},$$

$\boldsymbol{n}_1 = \frac{n(n+1)}{2}$, $\boldsymbol{n}_2 = \frac{n(n-1)}{2}$, and $\boldsymbol{l} = \boldsymbol{s} + 2(m+n)$. Similarly, the optimal backward perturbation matrices $\Delta F_{\mathsf{opt}}^{\mathsf{sps}}$, $\Delta G_{\mathsf{opt}}^{\mathsf{sps}}$, Δr_{opt}, and Δq_{opt} are given by

$$\mathrm{vec}(\Delta F_{\mathsf{opt}}^{\mathsf{sps}}) = \alpha_2^{-1}\begin{bmatrix}0_{nm\times n^2} & I_{nm} & iI_{nm} & 0_{nm\times(\boldsymbol{l}-n^2-2nm)}\end{bmatrix}\begin{bmatrix}\Delta\mathcal{Y}\\ \Delta\mathcal{Z}\end{bmatrix}_{\mathsf{opt}},$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

$$\mathrm{vec}(\Delta G_{\mathtt{opt}}^{\mathtt{sps}}) = \alpha_3^{-1} \begin{bmatrix} 0_{m^2\times(n^2+2nm)} & I_{m^2} & iI_{m^2} & 0_{m^2\times 2(m+n)} \end{bmatrix} \begin{bmatrix} \Delta\mathcal{Y} \\ \Delta\mathcal{Z} \end{bmatrix}_{\mathtt{opt}},$$

$$\Delta q_{\mathtt{opt}} = \beta_1^{-1} \begin{bmatrix} 0_{n\times(\boldsymbol{l}-2(n+m))} & I_n & iI_n & 0_{n\times 2m} \end{bmatrix} \begin{bmatrix} \Delta\mathcal{Y} \\ \Delta\mathcal{Z} \end{bmatrix}_{\mathtt{opt}},$$

$$\Delta r_{\mathtt{opt}} = \beta_2^{-1} \begin{bmatrix} 0_{m\times(\boldsymbol{l}-2m)} & I_m & iI_m \end{bmatrix} \begin{bmatrix} \Delta\mathcal{Y} \\ \Delta\mathcal{Z} \end{bmatrix}_{\mathtt{opt}}.$$

We now compute a compact formula for the structured BE for case (i), in which the sparsity pattern of $\mathfrak{B}$ is not retained.

COROLLARY 3.4. *Assume that $E \in \mathbb{HC}^{n\times n}$, $F \in \mathbb{C}^{m\times n}$, $G \in \mathbb{C}^{m\times m}$ and that $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ is a computed solution of the GSPP (1.1). Then, the structured BE $\boldsymbol{\xi}^{\mathcal{G}_1}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ without preserving sparsity is expressed as*

$$\boldsymbol{\xi}^{\mathcal{G}_1}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = \left\| \begin{bmatrix} \mathbf{Y}_1 & \mathcal{I}_1 \\ \mathbf{Y}_2 & \mathcal{I}_2 \end{bmatrix}^T \left(\begin{bmatrix} \mathbf{Y}_1 & \mathcal{I}_1 \\ \mathbf{Y}_2 & \mathcal{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1 & \mathcal{I}_1 \\ \mathbf{Y}_2 & \mathcal{I}_2 \end{bmatrix}^T \right)^{-1} \begin{bmatrix} \Re(Q) \\ \Im(Q) \\ \Re(R) \\ \Im(R) \end{bmatrix} \right\|_2,$$

where

$$\mathbf{Y}_1 = [\widehat{\mathbf{Y}}_1 \;\; 0_{2n\times 2m^2}], \quad \mathbf{Y}_2 = [0_{2m^2\times n^2} \;\; \widehat{\mathbf{Y}}_2],$$

$$\widehat{\mathbf{Y}}_1 = \begin{bmatrix} \alpha_1^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_n)\mathcal{J}_S^n \mathfrak{D}_{S,n}^{-1} & -\alpha_1^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_n)\mathcal{J}_{SK}^n \mathfrak{D}_{SK,n}^{-1} & \alpha_2^{-1}(I_n \otimes \Re(\widehat{\boldsymbol{p}})^T) & \alpha_2^{-1}(I_n \otimes \Im(\widehat{\boldsymbol{p}})^T) \\ \alpha_1^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_n)\mathcal{J}_S^n \mathfrak{D}_{S,n}^{-1} & \alpha_1^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_n)\mathcal{J}_{SK}^n \mathfrak{D}_{SK,n}^{-1} & \alpha_2^{-1}(I_n \otimes \Im(\widehat{\boldsymbol{p}})^T) & -\alpha_2^{-1}(I_n \otimes \Re(\widehat{\boldsymbol{p}})^T) \end{bmatrix}$$

and

$$\widehat{\mathbf{Y}}_2 = \begin{bmatrix} \alpha_2^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_m) & -\alpha_2^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_m) & \alpha_3^{-1}(\Re(\widehat{\boldsymbol{p}})^T \otimes I_m) & -\alpha_3^{-1}(\Im(\widehat{\boldsymbol{p}})^T \otimes I_m) \\ \alpha_2^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_m) & \alpha_2^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_m) & \alpha_3^{-1}(\Im(\widehat{\boldsymbol{p}})^T \otimes I_m) & \alpha_3^{-1}(\Re(\widehat{\boldsymbol{p}})^T \otimes I_m) \end{bmatrix}.$$

Proof. As we are not preserving the sparsity structure of E, F and G, by setting $\Theta_E = \mathbf{1}_{n\times n}$, $\Theta_F = \mathbf{1}_{m\times n}$, and $\Theta_G = \mathbf{1}_{m\times m}$, the proof proceeds accordingly. ∎

In the following, we derive the structured BE for the GSPP (1.1) when $E \in \mathbb{HC}^{n\times n}$ and $G = 0_{m\times m}$.

COROLLARY 3.5. *Assume that $E \in \mathbb{HC}^{n\times n}$, $F \in \mathbb{C}^{m\times n}$, $G = 0_{m\times m}$ and that $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ is a computed solution of the GSPP (1.1). Then, the structured BE $\boldsymbol{\xi}_{\mathtt{sps}}^{\mathcal{G}_1}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ with preserving sparsity is given by*

$$\boldsymbol{\xi}_{\mathtt{sps}}^{\mathcal{G}_1}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = \left\| \begin{bmatrix} \widehat{\mathbf{X}}_1 & \mathcal{I}_1 \\ \mathbf{Z} & \mathcal{I}_2 \end{bmatrix}^T \left(\begin{bmatrix} \widehat{\mathbf{X}}_1 & \mathcal{I}_1 \\ \mathbf{Z} & \mathcal{I}_2 \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{X}}_1 & \mathcal{I}_1 \\ \mathbf{Z} & \mathcal{I}_2 \end{bmatrix}^T \right)^{-1} \begin{bmatrix} \Re(Q) \\ \Im(Q) \\ \Re(\widehat{R}) \\ \Im(\widehat{R}) \end{bmatrix} \right\|_2,$$

where $\mathbf{Z} = [0_{2m^2\times n^2} \;\; \widehat{\mathbf{Z}}]$, with

$$\widehat{\mathbf{Z}} = \begin{bmatrix} \alpha_2^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_F & -\alpha_2^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_F \\ \alpha_2^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_F & \alpha_2^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_F \end{bmatrix},$$

and $\widehat{R} = r - F\widehat{\boldsymbol{u}}$.

Proof. The proof follows by taking $\alpha_3 \to \infty$ and $G = 0_{m\times m}$ in Theorem 3.2. ∎

The structured BE for the GSPP (1.1) is analyzed in [34] under the assumption that the block matrices are real, i.e., $E = E^T \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{m\times n}$, $G \in \mathbb{R}^{m\times m}$, and the vectors $q \in \mathbb{R}^n$, $r \in \mathbb{R}^m$ are also real. From our results, we can also obtain the structured BE for this case, as given below.

COROLLARY 3.6. *Assume that $E \in \mathbb{SR}^{n\times n}$, $F \in \mathbb{R}^{m\times n}$, $G \in \mathbb{R}^{m\times m}$, $q \in \mathbb{R}^n$, $r \in \mathbb{R}^m$ and that $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ is a computed solution of the GSPP (1.1). Then, the structured BE $\boldsymbol{\xi}^{\mathcal{G}_1}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ with preserving sparsity is given by*

$$\boldsymbol{\xi}^{\mathcal{G}_1}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = \left\| \begin{bmatrix} \mathbf{X}_1^R & \widetilde{\mathcal{I}}_1 \\ \mathbf{X}_2^R & \widetilde{\mathcal{I}}_2 \end{bmatrix}^T \left(\begin{bmatrix} \mathbf{X}_1^R & \widetilde{\mathcal{I}}_1 \\ \mathbf{X}_2^R & \widetilde{\mathcal{I}}_2 \end{bmatrix} \begin{bmatrix} \mathbf{X}_1^R & \widetilde{\mathcal{I}}_1 \\ \mathbf{X}_2^R & \widetilde{\mathcal{I}}_2 \end{bmatrix}^T \right)^{-1} \begin{bmatrix} Q \\ R \end{bmatrix} \right\|_2,$$

where $Q = q - E\boldsymbol{u} - F^T\boldsymbol{p}$, $R = r - F\boldsymbol{u} - G\boldsymbol{p}$,

$$\begin{aligned}
\widetilde{\mathcal{I}}_1 &= \begin{bmatrix} -{\beta_1}^{-1} I_n & 0_{n\times m} \end{bmatrix} \in \mathbb{R}^{n\times(n+m)}, \\
\widetilde{\mathcal{I}}_2 &= \begin{bmatrix} 0_{m\times n} & -{\beta_2}^{-1} I_m \end{bmatrix} \in \mathbb{R}^{m\times(n+m)}, \\
\mathbf{X}_1^R &= \begin{bmatrix} \alpha_1^{-1}(\widehat{\boldsymbol{u}}^T \otimes I_n)N_1 & \alpha_2^{-1}(I_n \otimes \widehat{\boldsymbol{p}}^T)\Sigma_F & 0_{n\times m^2} \end{bmatrix}, \\
\mathbf{X}_2^R &= \begin{bmatrix} 0_{m\times \frac{n(n+1)}{2}} & \alpha_2^{-1}(\widehat{\boldsymbol{u}}^T \otimes I_m)\Sigma_F & \alpha_3^{-1}(\widehat{\boldsymbol{p}}^T \otimes I_m)\Sigma_G \end{bmatrix}.
\end{aligned}$$

Proof. Since $E \in \mathbb{SR}^{n\times n}$, $F \in \mathbb{R}^{m\times n}$, $G \in \mathbb{R}^{m\times m}$, $q \in \mathbb{R}^n$, and $r \in \mathbb{R}^m$, the computed solution $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ is real. Therefore, $\Im(\widehat{\boldsymbol{u}}) = 0$ and $\Im(\widehat{\boldsymbol{p}}) = 0$. Furthermore, $\Im(Q) = 0$ and $\Im(R) = 0$; substituting these values into Theorem 3.2 yields the desired structured BE. Hence, the proof is complete. ∎

By applying Remark 3.3, we can also obtain the optimal backward perturbation matrices for the structured BE. The above result highlights the generalized nature of the proposed framework. Furthermore, our analysis ensures the preservation of the sparsity pattern of the block matrices, unlike the results in [34], which neither maintain sparsity nor provide optimal backward perturbation matrices.

3.2. Computation of the structured BE for case (ii). In this subsection, we derive a compact expression for the structured BE $\boldsymbol{\xi}^{\mathcal{G}_2}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ by preserving the Hermitian structure of $G \in \mathbb{HC}^{m\times m}$ and preserving the sparsity of the coefficient matrix $\mathfrak{B}$. Before stating the main theorem, we construct the following matrices: Set

$$\begin{aligned}
S_1 &:= \mathcal{J}_S^m \Phi_G \mathfrak{D}_{S,m}^{-1} \in \mathbb{R}^{m^2\times \frac{m(m+1)}{2}}, & S_2 &:= \mathcal{J}_{SK}^m \Psi_G \mathfrak{D}_{SK,m}^{-1} \in \mathbb{R}^{m^2\times \frac{m(m-1)}{2}}, \\
\mathbf{K}_1 &:= [\widehat{\mathbf{K}}_1 \;\; 0_{2n\times m^2}], & \mathbf{K}_2 &:= [0_{2m\times 2n^2} \;\; \widehat{\mathbf{K}}_2],
\end{aligned}$$

where

$$\widehat{\mathbf{K}}_1 = \begin{bmatrix} \alpha_1^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_n)\Sigma_E & -\alpha_1^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_n)\Sigma_E & \alpha_2^{-1}(I_n \otimes \Re(\widehat{\boldsymbol{p}})^T)\Sigma_F & \alpha_2^{-1}(I_n \otimes \Im(\widehat{\boldsymbol{p}})^T)\Sigma_F \\ \alpha_1^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_n)\Sigma_E & \alpha_1^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_n)\Sigma_E & \alpha_2^{-1}(I_n \otimes \Im(\widehat{\boldsymbol{p}})^T)\Sigma_F & -\alpha_2^{-1}(I_n \otimes \Re(\widehat{\boldsymbol{p}})^T)\Sigma_F \end{bmatrix}$$

and

$$\widehat{\mathbf{K}}_2 = \begin{bmatrix} \alpha_2^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_F & -\alpha_2^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_F & \alpha_3^{-1}(\Re(\widehat{\boldsymbol{p}})^T \otimes I_m)S_1 & -\alpha_3^{-1}(\Im(\widehat{\boldsymbol{p}})^T \otimes I_m)S_2 \\ \alpha_2^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_F & \alpha_2^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_F & \alpha_3^{-1}(\Im(\widehat{\boldsymbol{p}})^T \otimes I_m)S_1 & \alpha_3^{-1}(\Re(\widehat{\boldsymbol{p}})^T \otimes I_m)S_2 \end{bmatrix}.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Define

$$\Delta\mathcal{Y} := \begin{bmatrix} \alpha_1 \mathrm{vec}(\Re(\Delta E \odot \Theta_E)) \\ \alpha_1 \mathrm{vec}_(\Im(\Delta E \odot \Theta_E)) \\ \alpha_2 \mathrm{vec}(\Re(\Delta F \odot \Theta_F)) \\ \alpha_2 \mathrm{vec}(\Im(\Delta F \odot \Theta_F)) \\ \alpha_3 \mathfrak{D}_{S,m} \mathrm{vec}_S(\Re(\Delta G \odot \Theta_G)) \\ \alpha_3 \mathfrak{D}_{SK,m} \mathrm{vec}_{SK,m}(\Im(\Delta G \odot \Theta_G)) \end{bmatrix} \in \mathbb{R}^{\boldsymbol{t}}, \quad \Delta\mathcal{Z} := \begin{bmatrix} \beta_1 \Re(\Delta q) \\ \beta_1 \Im(\Delta q) \\ \beta_2 \Re(\Delta r) \\ \beta_2 \Im(\Delta r) \end{bmatrix} \in \mathbb{R}^{2(n+m)},$$

where $\boldsymbol{t} = 2(n^2 + nm) + m^2$.

THEOREM 3.7. *Assume that $E \in \mathbb{C}^{n\times n}$, $F \in \mathbb{C}^{m\times n}$, $G \in \mathbb{HC}^{m\times m}$ and that $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ is a computed solution of the GSPP (1.1). Then, the structured BE $\boldsymbol{\xi}^{\mathcal{G}_2}_{\mathrm{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ with preserving sparsity is given by*

$$\boldsymbol{\xi}^{\mathcal{G}_2}_{\mathrm{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = \left\| \begin{bmatrix} \mathbf{K}_1 & \mathcal{I}_1 \\ \mathbf{K}_2 & \mathcal{I}_2 \end{bmatrix}^T \left(\begin{bmatrix} \mathbf{K}_1 & \mathcal{I}_1 \\ \mathbf{K}_2 & \mathcal{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{K}_1 & \mathcal{I}_1 \\ \mathbf{K}_2 & \mathcal{I}_2 \end{bmatrix}^T \right)^{-1} \begin{bmatrix} \Re(Q) \\ \Im(Q) \\ \Re(R) \\ \Im(R) \end{bmatrix} \right\|_2,$$

where $Q, R, \mathcal{I}_1$ and $\mathcal{I}_2$ are defined as in Theorem 3.2.

Proof. For the approximate solution $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$, we are required to find the perturbations $\Delta E \in \mathbb{C}^{n\times n}$, $\Delta F \in \mathbb{C}^{m\times n}$, and $\Delta G \in \mathbb{HC}^{m\times m}$ which retain the sparsity pattern of E, F, and G, respectively, and $\Delta q \in \mathbb{C}^n$, $\Delta r \in \mathbb{C}^m$ so that (2.3) holds. Thus, we replace $\Delta E, \Delta F$, and ΔG with $\Delta E \odot \Theta_E$, $\Delta F \odot \Theta_F$, and $\Delta G \odot \Theta_G$, respectively. Consequently, we have

$$\begin{bmatrix} E + (\Delta E \odot \Theta_E) & (F + (\Delta F \odot \Theta_F))^* \\ F + (\Delta F \odot \Theta_F) & G + (\Delta G \odot \Theta_G) \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{u}} \\ \widehat{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} q + \Delta q \\ r + \Delta r \end{bmatrix},$$

where $\Delta G \odot \Theta_G \in \mathbb{HC}^{m\times m}$. As $\Delta G \odot \Theta_G \in \mathbb{HC}^{m\times m}$, we get $\Re(\Delta G \odot \Theta_G) \in \mathbb{SR}^{m\times m}$ and $\Im(\Delta G \odot \Theta_G) \in \mathbb{SKR}^{m\times m}$. By applying Lemma 3.1 and following a similar approach as in Theorem 3.2, the proof is complete. ∎

REMARK 3.8. In a similar fashion to Remark 3.3, we can easily construct the optimal backward perturbations $\Delta E^{\mathrm{sps}}_{\mathrm{opt}}$, $\Delta F^{\mathrm{sps}}_{\mathrm{opt}}$, $\Delta G^{\mathrm{sps}}_{\mathrm{opt}}$, Δq_{opt}, and Δr_{opt}. Moreover, by considering $\Theta_E = \mathbf{1}_{n\times n}$, $\Theta_F = \mathbf{1}_{m\times n}$, and $\Theta_G = \mathbf{1}_{m\times m}$, as in Corollary 3.4, we can obtain the desired structured BE when the sparsity structure is not preserved.

Next, we present the structured BE for the GSPP (1.1) by considering $E \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{m\times n}$, $G = G^T \in \mathbb{R}^{m\times m}$, and the vectors $q \in \mathbb{R}^n$, $r \in \mathbb{R}^m$.

COROLLARY 3.9. *Assume that $E \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{m\times n}$, $G \in \mathbb{SR}^{m\times m}$, $q \in \mathbb{R}^n$, $r \in \mathbb{R}^m$ and that $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ is a computed solution of the GSPP (1.1). Then, the structured BE $\boldsymbol{\xi}^{\mathcal{G}_2}_{\mathrm{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ with preserving sparsity is given by*

$$\boldsymbol{\xi}^{\mathcal{G}_2}_{\mathrm{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = \left\| \begin{bmatrix} \mathbf{K}^R_1 & \widetilde{\mathcal{I}}_1 \\ \mathbf{K}^R_2 & \widetilde{\mathcal{I}}_2 \end{bmatrix}^T \left(\begin{bmatrix} \mathbf{K}^R_1 & \widetilde{\mathcal{I}}_1 \\ \mathbf{K}^R_2 & \widetilde{\mathcal{I}}_2 \end{bmatrix} \begin{bmatrix} \mathbf{K}^R_1 & \widetilde{\mathcal{I}}_1 \\ \mathbf{K}^R_2 & \widetilde{\mathcal{I}}_2 \end{bmatrix}^T \right)^{-1} \begin{bmatrix} Q \\ R \end{bmatrix} \right\|_2,$$

where $Q = q - E\boldsymbol{u} - F^T\boldsymbol{p}$, $R = r - F\boldsymbol{u} - G\boldsymbol{p}$,

$$\widetilde{\mathcal{I}}_1 = \begin{bmatrix} -{\beta_1}^{-1} I_n & 0_{n\times m} \end{bmatrix} \in \mathbb{R}^{n\times(n+m)},$$
$$\widetilde{\mathcal{I}}_2 = \begin{bmatrix} 0_{m\times n} & -{\beta_2}^{-1} I_m \end{bmatrix} \in \mathbb{R}^{m\times(n+m)},$$
$$\mathbf{K}^R_1 = \begin{bmatrix} \alpha_1^{-1}(\widehat{\boldsymbol{u}}^T \otimes I_n)\Sigma_E & \alpha_2^{-1}(I_n \otimes \widehat{\boldsymbol{p}}^T)\Sigma_F & 0_{n\times \frac{n(n+1)}{2}} \end{bmatrix},$$
$$\mathbf{K}^R_2 = \begin{bmatrix} 0_{m\times n^2} & \alpha_2^{-1}(\widehat{\boldsymbol{u}}^T \otimes I_m)\Sigma_F & \alpha_3^{-1}(\widehat{\boldsymbol{p}}^T \otimes I_m)S_1 \end{bmatrix}.$$

Proof. Given that $E \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{m\times n}$, $G \in \mathbb{SR}^{m\times m}$, $q \in \mathbb{R}^n$, and $r \in \mathbb{R}^m$, the computed solution $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ is real. As a result, $\Im(\widehat{\boldsymbol{u}}) = 0$ and $\Im(\widehat{\boldsymbol{p}}) = 0$. Additionally, we have $\Im(Q) = 0$ and $\Im(R) = 0$. Substituting these values into Theorem 3.7, we derive the desired structured BE, and the proof is complete. □

The structured BE for the case discussed in Corollary 3.9 has also been studied in [10, 20]. In contrast, our investigation preserves the sparsity pattern of the block matrices and the symmetry of the block $G \in \mathbb{R}^{m\times m}$, providing also optimal backward perturbation matrices.

3.3. Computation of the structured BE for case (iii). In this section, we derive the structured BE for the GSPP (1.1) when $E \in \mathbb{HC}^{n\times n}$, $F \neq H \in \mathbb{C}^{m\times n}$, and $G \in \mathbb{HC}^{m\times m}$. Let $\boldsymbol{k} = n^2 + 4mn + m^2$. Then we construct the matrices $\mathbf{M}_1 \in \mathbb{R}^{2n\times \boldsymbol{k}}$ and $\mathbf{M}_2 \in \mathbb{R}^{2m\times \boldsymbol{k}}$ as follows:

$$\mathbf{M}_1 := [\widehat{\mathbf{X}}_1 \;\; 0_{2n\times(m^2+2mn)}], \quad \mathbf{M}_2 := [0_{2m\times(n^2+2mn)} \;\; \widehat{\mathbf{M}}_2],$$

where

$$\widehat{\mathbf{M}}_2 = \begin{bmatrix} \alpha_2^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_H & -\alpha_2^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_H & \alpha_3^{-1}(\Re(\widehat{\boldsymbol{p}})^T \otimes I_m)S_1 & -\alpha_3^{-1}(\Im(\widehat{\boldsymbol{p}})^T \otimes I_m)S_2 \\ \alpha_2^{-1}(\Im(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_H & \alpha_2^{-1}(\Re(\widehat{\boldsymbol{u}})^T \otimes I_m)\Sigma_H & \alpha_3^{-1}(\Im(\widehat{\boldsymbol{p}})^T \otimes I_m)S_1 & \alpha_3^{-1}(\Re(\widehat{\boldsymbol{p}} \otimes I_m)^T)S_2 \end{bmatrix}.$$

Moreover, we define the following two vectors:

$$\Delta\mathcal{Y} := \begin{bmatrix} \alpha_1 \mathfrak{D}_{S,n}\mathrm{vec}_S(\Re(\Delta E \odot \Theta_E)) \\ \alpha_1 \mathfrak{D}_{SK,n}\mathrm{vec}_{SK}(\Im(\Delta E \odot \Theta_E)) \\ \alpha_2 \mathrm{vec}(\Re(\Delta F \odot \Theta_F)) \\ \alpha_2 \mathrm{vec}(\Im(\Delta F \odot \Theta_F)) \\ \alpha_3 \mathrm{vec}(\Re(\Delta H \odot \Theta_H)) \\ \alpha_3 \mathrm{vec}(\Im(\Delta H \odot \Theta_H)) \\ \alpha_4 \mathfrak{D}_{S,m}\mathrm{vec}_S(\Re(\Delta G \odot \Theta_G)) \\ \alpha_4 \mathfrak{D}_{S,m}\mathrm{vec}_{SK}(\Im(\Delta G \odot \Theta_G)) \end{bmatrix} \in \mathbb{R}^{\boldsymbol{k}}, \quad \Delta\mathcal{Z} := \begin{bmatrix} \beta_1\Re(\Delta q) \\ \beta_1\Im(\Delta q) \\ \beta_2\Re(\Delta r) \\ \beta_2\Im(\Delta r) \end{bmatrix} \in \mathbb{R}^{2(n+m)}.$$

In the next theorem, we present the structured BE $\boldsymbol{\xi}^{\mathcal{G}_3}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ by preserving the sparsity of the coefficient matrix $\mathfrak{B}$.

THEOREM 3.10. *Assume that $E \in \mathbb{HC}^{n\times n}$, $F, H \in \mathbb{C}^{m\times n}$, $G \in \mathbb{HC}^{m\times m}$ and that $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ is an approximate solution of the GSPP (1.1). Then, the structured BE $\boldsymbol{\xi}^{\mathcal{G}_3}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ with preserving sparsity is given by*

$$\boldsymbol{\xi}^{\mathcal{G}_3}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = \left\| \begin{bmatrix} \mathbf{M}_1 & \mathcal{I}_1 \\ \mathbf{M}_2 & \mathcal{I}_2 \end{bmatrix}^T \left(\begin{bmatrix} \mathbf{M}_1 & \mathcal{I}_1 \\ \mathbf{M}_2 & \mathcal{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{M}_1 & \mathcal{I}_1 \\ \mathbf{M}_2 & \mathcal{I}_2 \end{bmatrix}^T \right)^{-1} \begin{bmatrix} \Re(Q) \\ \Im(Q) \\ \Re(R) \\ \Im(R) \end{bmatrix} \right\|_2,$$

where

$$\mathcal{I}_1 = \begin{bmatrix} -\beta_1^{-1} I_{2n} & 0_{2n\times 2m} \end{bmatrix} \in \mathbb{R}^{2n\times 2(n+m)},$$
$$\mathcal{I}_2 = \begin{bmatrix} 0_{2m\times 2n} & -\beta_2^{-1} I_{2m} \end{bmatrix} \in \mathbb{R}^{2m\times 2(n+m)},$$

and Q and R defined as in Lemma 3.1.

Proof. For the computed solution $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$, we are required to find the perturbations $\Delta E \in \mathbb{HC}^{n\times n}$, $\Delta F, \Delta H \in \mathbb{C}^{m\times n}$, and $\Delta G \in \mathbb{HC}^{m\times m}$ which retain the sparsity of E, F, H, and G, respectively, and $\Delta q \in \mathbb{C}^n$, $\Delta r \in \mathbb{C}^m$ so that (2.3) holds. Thus, we replace E, F, H, and G with $\Delta E \odot \Theta_E$, $\Delta F \odot \Theta_F$, $\Delta H \odot \Theta_H$, and $\Delta G \odot \Theta_G$, respectively. Consequently, we obtain

$$\begin{bmatrix} E + (\Delta E \odot \Theta_E) & (F + (\Delta F \odot \Theta_F))^* \\ H + (\Delta H \odot \Theta_H) & G + (\Delta G \odot \Theta_G) \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{u}} \\ \widehat{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} q + \Delta q \\ r + \Delta r \end{bmatrix}, \tag{3.13}$$

where $\Delta E \odot \Theta_E \in \mathbb{HC}^{n \times n}$ and $\Delta G \odot \Theta_G \in \mathbb{HC}^{m \times m}$. By Lemma 3.1, equation (3.13) leads to

$$\begin{cases} \Re(\Delta E \odot \Theta_E)\Re(\boldsymbol{u}) - \Im(\Delta E \odot \Theta_E)\Im(\boldsymbol{u}) + \Re(\Delta F \odot \Theta_F)^T \Re(\boldsymbol{p}) \\ \qquad\qquad + \Im(\Delta F \odot \Theta_F)^T \Im(\boldsymbol{p}) - \Re(\Delta q) = \Re(Q), \\ \Re(\Delta E \odot \Theta_E)\Im(\boldsymbol{u}) + \Im(\Delta E \odot \Theta_E)\Re(\boldsymbol{u}) + \Re(\Delta F \odot \Theta_F)^T \Im(\boldsymbol{p}) \\ \qquad\qquad - \Im(\Delta F \odot \Theta_F)^T \Re(\boldsymbol{p}) - \Im(\Delta q) = \Im(Q), \end{cases}$$

and

$$\begin{cases} \Re(\Delta H \odot \Theta_H)\Re(\boldsymbol{u}) - \Im(\Delta H \odot \Theta_H)\Im(\boldsymbol{u}) + \Re(\Delta G \odot \Theta_G)\Re(\boldsymbol{p}) \\ \qquad\qquad - \Im(\Delta G \odot \Theta_G)\Im(\boldsymbol{p}) - \Re(\Delta r) = \Re(\widetilde{R}), \\ \Re(\Delta H \odot \Theta_H)\Im(\boldsymbol{u}) + \Im(\Delta H \odot \Theta_H)\Re(\boldsymbol{u}) + \Re(\Delta G \odot \Theta_G)\Im(\boldsymbol{p}) \\ \qquad\qquad + \Im(\Delta G \odot \Theta_G)\Re(\boldsymbol{p}) - \Im(\Delta r) = \Im(\widetilde{R}). \end{cases}$$

Given that $\Delta E \odot \Theta_E \in \mathbb{HC}^{n \times n}$ and $\Delta G \odot \Theta_G \in \mathbb{HC}^{m \times m}$, we find $\Re(\Delta E \odot \Theta_E) \in \mathbb{SR}^{n \times n}$, $\Im(\Delta E \odot \Theta_E) \in \mathbb{SKR}^{n \times n}$, $\Re(\Delta G \odot \Theta_G) \in \mathbb{SR}^{m \times m}$, and $\Im(\Delta G \odot \Theta_G) \in \mathbb{SKR}^{m \times m}$. Now, using a reasoning similar to the one in the proof of Theorem 3.2, we have

$$\begin{bmatrix} \mathbf{M}_1 & \mathcal{I}_1 \\ \mathbf{M}_2 & \mathcal{I}_2 \end{bmatrix} \begin{bmatrix} \Delta \mathcal{Y} \\ \Delta \mathcal{Z} \end{bmatrix} = \begin{bmatrix} \Re(Q) \\ \Im(Q) \\ \Re(R) \\ \Im(R) \end{bmatrix}. \tag{3.14}$$

Observe that $\begin{bmatrix} \mathbf{M}_1 & \mathcal{I}_1 \\ \mathbf{M}_2 & \mathcal{I}_2 \end{bmatrix}$ is a full-row-rank matrix. Hence, the linear system (3.14) is consistent, and its minimum-norm solution is given by

$$\begin{bmatrix} \Delta \mathcal{Y} \\ \Delta \mathcal{Z} \end{bmatrix}_{\texttt{opt}} = \begin{bmatrix} \mathbf{M}_1 & \mathcal{I}_1 \\ \mathbf{M}_2 & \mathcal{I}_2 \end{bmatrix}^T \left(\begin{bmatrix} \mathbf{M}_1 & \mathcal{I}_1 \\ \mathbf{M}_2 & \mathcal{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{M}_1 & \mathcal{I}_1 \\ \mathbf{M}_2 & \mathcal{I}_2 \end{bmatrix}^T \right)^{-1} \begin{bmatrix} \Re(Q) \\ \Im(Q) \\ \Re(\widetilde{R}) \\ \Im(\widetilde{R}) \end{bmatrix}. \tag{3.15}$$

The remainder of the proof follows from (3.15) and by applying a similar technique as in the proof of Theorem 3.2. □

ETNA
Kent State University and
Johann Radon Institute (RICAM)

REMARK 3.11. In a similar manner to Remark 3.3, the optimal backward perturbations $\Delta E^{\mathtt{sps}}_{\mathtt{opt}}$, $\Delta F^{\mathtt{sps}}_{\mathtt{opt}}$, $\Delta H^{\mathtt{sps}}_{\mathtt{opt}}$, $\Delta G^{\mathtt{sps}}_{\mathtt{opt}}$, $\Delta q_{\mathtt{opt}}$, and $\Delta r_{\mathtt{opt}}$ can be easily computed. Furthermore, by taking $\Theta_E = \mathbf{1}_{n\times n}$, $\Theta_F = \mathbf{1}_{m\times n} = \Theta_H$, and $\Theta_G = \mathbf{1}_{m\times m}$, as in Corollary 3.4, we can derive the structured BE under the condition that the sparsity structure is not maintained.

The structured BE for the GSPP (1.1) has been examined in [34] under the assumption that the block matrices are real, specifically, $E = E^T \in \mathbb{R}^{n\times n}$, $F, H \in \mathbb{R}^{m\times n}$, and $G = G^T \in \mathbb{R}^{m\times m}$, with the vectors $q \in \mathbb{R}^n$ and $r \in \mathbb{R}^m$. Based on our findings, we can also derive the structured BE for this scenario, which is presented below.

COROLLARY 3.12. *Assume that $E \in \mathbb{SR}^{n\times n}$, F, $H \in \mathbb{R}^{m\times n}$, $G \in \mathbb{SR}^{m\times m}$, $q \in \mathbb{R}^n$, $r \in \mathbb{R}^m$ and that $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ is a computed solution of the GSPP (1.1). Then, the structured BE $\boldsymbol{\xi}^{\mathcal{G}_3}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ with preserving sparsity is given by*

$$\boldsymbol{\xi}^{\mathcal{G}_3}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = \left\| \begin{bmatrix} \mathbf{M}_1^R & \widetilde{\mathcal{I}}_1 \\ \mathbf{M}_2^R & \widetilde{\mathcal{I}}_2 \end{bmatrix}^T \left(\begin{bmatrix} \mathbf{M}_1^R & \widetilde{\mathcal{I}}_1 \\ \mathbf{M}_2^R & \widetilde{\mathcal{I}}_2 \end{bmatrix} \begin{bmatrix} \mathbf{M}_1^R & \widetilde{\mathcal{I}}_1 \\ \mathbf{M}_2^R & \widetilde{\mathcal{I}}_2 \end{bmatrix}^T \right)^{-1} \begin{bmatrix} Q \\ R \end{bmatrix} \right\|_2,$$

where $Q = q - E\boldsymbol{u} - F^T\boldsymbol{p}$, $R = r - H\boldsymbol{u} - G\boldsymbol{p}$,

$$\begin{aligned}
\widetilde{\mathcal{I}}_1 &= \begin{bmatrix} -\beta_1{}^{-1} I_n & 0_{n\times m} \end{bmatrix} \in \mathbb{R}^{n\times(n+m)}, \\
\widetilde{\mathcal{I}}_2 &= \begin{bmatrix} 0_{m\times n} & -\beta_2{}^{-1} I_m \end{bmatrix} \in \mathbb{R}^{m\times(n+m)}, \\
\mathbf{M}_1^R &= \begin{bmatrix} \alpha_1^{-1}(\widehat{\boldsymbol{u}}^T \otimes I_n)N_1 & \alpha_2^{-1}(I_n \otimes \widehat{\boldsymbol{p}}^T)\Sigma_F & 0_{n\times \frac{m(m+1)}{2}} \end{bmatrix}, \\
\mathbf{M}_2^R &= \begin{bmatrix} 0_{m\times \frac{n(n+1)}{2}} & \alpha_2^{-1}(\widehat{\boldsymbol{u}}^T \otimes I_m)\Sigma_H & \alpha_3^{-1}(\widehat{\boldsymbol{p}}^T \otimes I_m)S_1 \end{bmatrix}.
\end{aligned}$$

Proof. Since $E \in \mathbb{SR}^{n\times n}$, F, $H \in \mathbb{R}^{m\times n}$, $G \in \mathbb{SR}^{m\times m}$, $q \in \mathbb{R}^n$, and $r \in \mathbb{R}^m$, the computed solution $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ is real. Consequently, $\Im(\widehat{\boldsymbol{u}}) = 0$ and $\Im(\widehat{\boldsymbol{p}}) = 0$. Moreover, we have $\Im(Q) = 0$ and $\Im(R) = 0$. Replacing these values in Theorem 3.10, we obtain the desired structured BE, and the proof is complete. □

Our analysis preserves the sparsity pattern of the block matrices, in contrast to the findings in [34], which do not retain the sparsity of the block matrices.

4. Numerical examples. In this section, we present a few numerical experiments to support our theoretical results. We compare structured BEs, both with and without retaining the sparsity structure, with the unstructured BE given in (2.1). Additionally, we construct the optimal perturbation matrices for achieving the structured BE. We report results illustrating the backward stability and strong backward stability of the numerical methods applied to the GSPP. For all the examples, we take $\alpha_1 = \frac{1}{\|E\|_{\mathrm{F}}}$, $\alpha_2 = \frac{1}{\|F\|_{\mathrm{F}}}$, $\alpha_3 = \frac{1}{\|G\|_{\mathrm{F}}}$ (or $\alpha_3 = \frac{1}{\|H\|_{\mathrm{F}}}$ and $\alpha_4 = \frac{1}{\|G\|_{\mathrm{F}}}$), and $\beta_1 = \frac{1}{\|q\|_2}$, $\beta_2 = \frac{1}{\|r\|_2}$.

All numerical experiments were performed using MATLAB R2024a on a system with an Intel(R) Core(TM) i7-10700 CPU running at 2.90 GHz, 16 GB of RAM.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

EXAMPLE 4.1. We consider the GSPP (1.1) with the following block matrices:

$$E = \begin{bmatrix} -0.7073 & -0.2258i & -0.3326+0.4370i & -0.3111-0.1089i & -0.2558i \\ 0.2258i & 1.6606 & 1.0022 & -0.0749 & 0.2357i \\ -0.3326-0.4370i & 1.0022 & 0 & -1.5009 & -0.1383-0.0928i \\ -0.3111+0.1089i & -0.0749 & -1.5009 & 0 & -0.1i \\ 0.2558i & -0.2357i & -0.1383+0.0928i & 0.1i & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} -0.0753+1.3412i & 0 & 0 & -1.3057 & 0 \\ -0.1974 & 0 & 2.9371 & 0.3806i & 0 \\ 0.2232+1.4354i & 0.7996i & 0.3985 & 0 & 1.6102 \\ 0.3862 & 0.0097 & 0 & 1.6286i & 0.1291i \end{bmatrix},$$

$$G = \begin{bmatrix} 1.5246-0.1337i & 0 & -0.6924 & -0.0408i \\ 0 & -0.9025 & 0 & 0.0704 \\ 0 & -0.6885+0.6028i & 0.7823i & 1.2309 \\ 0.2146i & 0 & 0 & -0.2746 \end{bmatrix},$$

$$q = \begin{bmatrix} -0.8098-0.3969i \\ -1.3853+0.5947i \\ 0.0909+0.2202i \\ -0.2140-0.7165i \\ 0.1509+0.0117i \end{bmatrix}, \quad r = \begin{bmatrix} -2.3554-0.9550i \\ 0.6201-0.7783i \\ 0.3106+1.5288i \\ -0.0908-1.8683i \end{bmatrix}.$$

Let $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ be a computed solution to the GSPP, where

$$\widehat{\boldsymbol{u}} = \begin{bmatrix} 0.9249+1.6011i \\ -0.5210+0.2407i \\ 0.0189+0.2151i \\ -1.5819+0.1480i \\ 0.5443+1.2113i \end{bmatrix} \quad \text{and} \quad \widehat{\boldsymbol{p}} = \begin{bmatrix} -1.2670-1.2768i \\ -0.7997+0.4628i \\ 0.4206-0.0082i \\ 1.1641-1.0531i \end{bmatrix}$$

with residual $\|\mathfrak{B}\widehat{\boldsymbol{x}} - \boldsymbol{f}\|_2 = 0.0012$.

The unstructured BE $\boldsymbol{\xi}(\widehat{\boldsymbol{x}})$ computed using the formula (2.1) is 3.9295×10^{-5}. Given that E is Hermitian, we calculate the structured BE with preserving the sparsity using Theorem 3.2 and without preserving the sparsity using Corollary 3.4. As a result we obtain

$$\boldsymbol{\xi}^{\mathcal{G}_1}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = 3.7327 \times 10^{-4} \quad \text{and}$$
$$\boldsymbol{\xi}^{\mathcal{G}_1}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = 3.2520 \times 10^{-4}.$$

We observe that the structured BEs in both cases are only one order larger than the unstructured BE. Moreover, the structure-preserving optimal perturbation matrices are given by

$$\Delta E_{\texttt{opt}}^{\texttt{sps}} = 10^{-5} \cdot \begin{bmatrix} 3.5894 & -3.0713 - 2.1972i & -4.2943 - 2.2135i & 5.4466 - 3.3205i & -2.1295 + 1.9597i \\ -3.0713 + 2.1972i & 1.2093 & 5.6451 - 1.0180i & 2.7731 + 3.7360i & -2.5866 + 2.5464i \\ -4.2943 + 2.2135i & 0.5645 + 1.0180i & 0 & 4.0067 + 2.8368i & -3.5135 + 1.8975i \\ 5.4466 + 3.3205i & 2.7731 - 3.7360i & 4.00675 - 2.8368i & 0 & 4.0822 - 2.4857i \\ -2.1295 - 1.9597i & -2.5866 - 2.5464i & -3.5135 - 1.8975i & 4.0822 + 2.4857i & 0 \end{bmatrix},$$

$$\Delta F_{\texttt{opt}}^{\texttt{sps}} = 10^{-5} \cdot \begin{bmatrix} -0.5850 + 5.2796i & 0 & 0 & -8.0796 + 2.8769i & 0 \\ -1.5373 + 1.8646i & 0 & 1.9608 - 4.3999i & 2.5302 + 7.2612i & 0 \\ -9.6826 - 8.4119i & -3.4175 + 4.2353i & -3.2788 + 1.0017i & 0 & -7.6773 - 2.9473i \\ -7.1701 - 5.4915i & -2.2259 + 7.7132i & 0 & -4.9771 - 3.4259i & -0.1621 + 3.5988i \end{bmatrix},$$

$$\Delta G_{\texttt{opt}}^{\texttt{sps}} = 10^{-5} \cdot \begin{bmatrix} -1.5374 + 2.9958i & -1.6890 - 0.3728i & 0.7551 - 0.2236i & 2.6735 + 1.2196i \\ 0 & 0.0349 + 2.83169i & 0 & 0.9484 - 4.7166i \\ 0 & -3.3959 + 4.2108i & 0.4254 - 2.4258i & 7.1382 - 5.7885i \\ -1.2947 + 3.6675i & 0 & 0 & 3.2493 + 0.0981i \end{bmatrix},$$

$$\Delta q_{\texttt{opt}} = 10^{-5} \cdot \begin{bmatrix} 2.8974 - 3.9154i \\ 3.7448 + 3.2211i \\ 4.9361 + 3.0047i \\ -1.1399 - 2.9206i \\ 2.8144 + 3.0114i \end{bmatrix}, \quad \Delta r_{\texttt{opt}} = 10^{-5} \cdot \begin{bmatrix} -1.7842 + 0.56641i \\ 1.5676 + 2.6335i \\ -0.8984 + 5.7852i \\ -1.9543 + 0.9252i \end{bmatrix}.$$

Observe that the computed perturbation matrices preserve the sparsity structure and $\Delta E_{\texttt{opt}}^{\texttt{sps}}$ is Hermitian. Moreover, we have

$$(\mathfrak{B} + \Delta \mathfrak{B}_{\texttt{opt}}^{\texttt{sps}})\widehat{\boldsymbol{x}} = \boldsymbol{f} + \Delta \boldsymbol{f}_{\texttt{opt}},$$

where

$$\Delta \mathfrak{B}_{\texttt{opt}}^{\texttt{sps}} = \begin{bmatrix} \Delta E_{\texttt{opt}}^{\texttt{sps}} & \Delta F_{\texttt{opt}}^{\texttt{sps}\,T} \\ \Delta F_{\texttt{opt}}^{\texttt{sps}} & \Delta G_{\texttt{opt}}^{\texttt{sps}} \end{bmatrix} \quad \text{and} \quad \Delta \boldsymbol{f}_{\texttt{opt}} = \begin{bmatrix} \Delta q_{\texttt{opt}} \\ \Delta r_{\texttt{opt}} \end{bmatrix}.$$

EXAMPLE 4.2. To discuss the strong backward stability of the GMRES algorithm [25] in solving the GSPP (1.1), we conduct a comparative analysis between unstructured and structured BEs in this example. The block matrices of the GSPP (1.1) are constructed as follows:

$$\begin{aligned} E &= \Re(E) + i\,\Im(E), \\ F &= \texttt{sprandn}(m, n, 0.5) + i\,\texttt{sprandn}(m, n, 0.5), \\ G &= \texttt{sprandn}(m, m, 0.5) + i\,\texttt{sprandn}(m, m, 0.5), \\ q &= \texttt{randn}(n, 1) + i\,\texttt{randn}(n, 1), \\ r &= \texttt{randn}(m, 1) + i\,\texttt{randn}(m, 1), \end{aligned}$$

where

$$\begin{aligned} \Re(E) &= E_1 + E_1^*, & \Im(E) &= E_2 + E_2^*, \\ E_1 &= \texttt{sprandn}(n, n, 0.4), & E_2 &= \texttt{sprandn}(n, n, 0.4). \end{aligned}$$

Here, $\texttt{sprandn}(m, n, \mu)$ denotes a sparse random matrix of size $m \times n$ with μmn nonzero entries, and $\texttt{randn(m,n)}$ denotes a random matrix of size $m \times n$. The matrix dimensions are set as $n = 3\mathbf{k}$ and $m = 2\mathbf{k}$.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

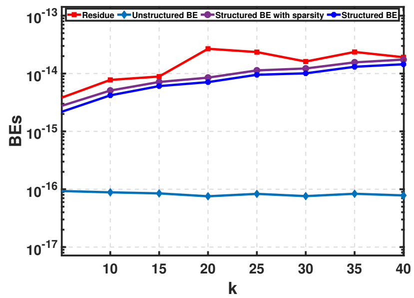

FIG. 4.1. *Relative residuals and BEs versus* $\mathbf{k}$.

We use the GMRES algorithm with a zero initial guess vector and a stopping tolerance of 10^{-8}. Let $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$ represent the computed solution of the GSPP. In Figure 4.1, for $\mathbf{k} = 5 : 5 : 40$, we plot the unstructured BE $\boldsymbol{\xi}(\widehat{\boldsymbol{x}})$ (labeled as "Unstructured BE"), the structured BE with preserving sparsity $\boldsymbol{\xi}^{\mathcal{G}_1}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ (labeled as "Structured BE with sparsity"), the structured BE without preserving sparsity $\boldsymbol{\xi}^{\mathcal{G}_1}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ (labeled as "Structured BE"), and the relative residual $\frac{\|\boldsymbol{f}-\mathfrak{B}\widehat{\boldsymbol{x}}\|_2}{\|\boldsymbol{f}\|_2}$ (labeled as "Residue").

From Figure 4.1, we observe that both structured BEs (with and without preserving sparsity) are one to two orders of magnitude larger than the unstructured BE. Despite this difference, the structured BEs consistently remain below $\mathcal{O}(10^{-14})$, indicating that they are significantly small. Hence, our analysis shows that the GMRES algorithm exhibits both backward stability and strong backward stability for these given test problems.

EXAMPLE 4.3. We consider the GSPP (1.1) with the block matrices $E \in \mathbb{C}^{4\times 4}$, $F \in \mathbb{C}^{3\times 4}$, $G \in \mathbb{HC}^{3\times 3}$, and the right-hand side vectors $q \in \mathbb{C}^4$, $r \in \mathbb{C}^3$ given by

$$E = \begin{bmatrix} 0.01i & 10^7(1+i) & 30(-1+i) & 0 \\ 100(1+i) & 0 & 0 & 10^5(-1+i) \\ 50(1+i) & 100(1+i) & 0 & 0 \\ 0 & 200(1-i) & 10^5(1+i) & 0.01(1+i) \end{bmatrix},$$

$$F = \begin{bmatrix} 10^{-5}(1+i) & 10^7(1+i) & 0 & 0 \\ 10^8(1-i) & 10^{-5}(1+i) & -10^{-6}(1+i) & 0 \\ 0 & 10^5(1+i) & 10^{-5}(1-i) & 10^6(1+i) \end{bmatrix},$$

$$G = \begin{bmatrix} 10^5 & 0 & 100+0.01i \\ 0 & 10^{-6} & 0 \\ 100-0.01i & 0 & -1 \end{bmatrix}, \ q = \begin{bmatrix} 10^4(1+i) \\ 10(1+i) \\ 0 \\ 10^{-6}(1+i) \end{bmatrix}, \ r = \begin{bmatrix} 0.01(1-i) \\ 0 \\ 0 \end{bmatrix}.$$

We use Gaussian elimination with partial pivoting (GEP) to solve the GSPP, and the computed approximate solution is $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{u}}^T, \widehat{\boldsymbol{p}}^T]^T$, where

$$\widehat{\boldsymbol{u}} = \begin{bmatrix} -0.0995-0.9904i \\ 0.0005+0.0003i \\ -99.0353+9.9509i \\ 0 \end{bmatrix} \in \mathbb{C}^4, \qquad \widehat{\boldsymbol{p}} = \begin{bmatrix} -0.01-0.099i \\ 0 \\ 0.9951+9.9035i \end{bmatrix} \in \mathbb{C}^3.$$

We compute the unstructured BE $\boldsymbol{\xi}(\widehat{\boldsymbol{x}})$ using formula (2.1), the structured BE by preserving sparsity $\boldsymbol{\xi}^{\mathcal{G}_2}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ using Theorem 3.7, and the structured BE without preserving sparsity $\boldsymbol{\xi}^{\mathcal{G}_1}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}})$ using Remark 3.8. The computed values are given by

$$\boldsymbol{\xi}(\widehat{\boldsymbol{x}}) = 9.3151 \times 10^{-20}, \ \boldsymbol{\xi}^{\mathcal{G}_2}_{\mathsf{sps}}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = 1.5494 \times 10^{-9}, \ \text{and} \ \boldsymbol{\xi}^{\mathcal{G}_2}(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{p}}) = 1.4964 \times 10^{-8}.$$

The unstructured backward error $\boldsymbol{\xi}(\widehat{\boldsymbol{x}})$ is of order $\mathcal{O}(10^{-28})$, whereas the structured BEs $\boldsymbol{\xi}^{\mathcal{G}_2}_{\mathsf{sps}}(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}})$ and $\boldsymbol{\xi}^{\mathcal{G}_2}(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}})$ are significantly larger. Thus, the computed solution is an exact solution to a nearby unstructured linear system but not to a nearby structure-preserving GSPP. Therefore, the GEP for solving this GSPP is backward stable but not strongly backward stable.

EXAMPLE 4.4. In this example, we consider the GSPP (1.1) with the real block matrices

$$E = \begin{bmatrix} I\otimes J + J\otimes I & 0 \\ 0 & I\otimes J + J\otimes I\end{bmatrix} \in \mathbb{R}^{2t^2\times 2t^2}, \quad F = \begin{bmatrix} I\otimes X & X\otimes I\end{bmatrix} \in \mathbb{R}^{t^2\times 2t^2},$$
$$H = \begin{bmatrix} Y\otimes X & X\otimes Y\end{bmatrix} \in \mathbb{R}^{t^2\times 2t^2}, \qquad G = 0,$$

where $J = \frac{1}{(t+1)^2}\,\mathrm{tridiag}(-1,2,-1) \in \mathbb{R}^{t\times t}$, $\quad X = \frac{1}{t+1}\,\mathrm{tridiag}(0,1,-1) \in \mathbb{R}^{t\times t}$, and $Y = \mathrm{diag}(1, t+1, \ldots, t^2-t+1) \in \mathbb{R}^{t\times t}$. Here, $\mathrm{tridiag}(a_1,a_2,a_3) \in \mathbb{R}^{t\times t}$ represents the tridiagonal matrix with the subdiagonal entry a_1, the diagonal entry a_2, and the superdiagonal entry a_3. The size of the coefficient matrix $\mathfrak{B}$ of GSPP is $(n+m) = 3t^2$. The right-hand side vector $\boldsymbol{f} \in \mathbb{R}^{n+m}$ is taken such that the exact solution of the GSPP is $[1,1,\ldots,1]^T \in \mathbb{R}^{n+m}$.

TABLE 4.1
Values of the structured and unstructured BEs of the approximate solution obtained using GMRES for Example 4.4.

t	$\boldsymbol{\xi}(\widehat{\boldsymbol{x}})$ [24]	$\boldsymbol{\xi}^{\mathcal{G}_3}_{\mathsf{sps}}(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}})$[1]	$\boldsymbol{\eta}(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}})$ [34]
4	3.2543e-16	4.9110e-15	2.4997e-14
5	2.3891e-15	5.4614e-14	7.4194e-14
6	8.8212e-16	3.1719e-14	3.7464e-14
7	6.5032e-16	3.1236e-14	8.6559e-14
8	6.0863e-16	3.7896e-14	4.0871e-14

[1]indicates our obtained structured BE.

We use the GMRES method to solve the GSPP. The initial guess vector is $0 \in \mathbb{R}^{n+m}$, and the stopping criterion is $\frac{\|\mathfrak{B}\boldsymbol{x}_k - \boldsymbol{f}\|_2}{\|\boldsymbol{f}\|_2} < 10^{-11}$, where $\boldsymbol{x}_k$ is the solution at the k^{th} iteration. We consider $t = 4,5,\ldots,8$ and compute the unstructured BE $\boldsymbol{\xi}(\widehat{\boldsymbol{x}})$ given in [24], the structured BE $\boldsymbol{\eta}(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}})$ given in [34], and the structured BE by preserving sparsity in Corollary 3.12 at the final iteration of the GMRES method.

From Table 4.1, we observe that $\boldsymbol{\eta}(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}})$ and $\boldsymbol{\xi}^{\mathcal{G}_3}_{\mathsf{sps}}(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{p}})$ are of order $\mathcal{O}(10^{-14})$ for all values of t, demonstrating the reliability of our structured BE formulas. Additionally, our structured BE formulation preserves the sparsity pattern of the coefficient matrix $\mathfrak{B}$. On the other hand, the unstructured backward error $\boldsymbol{\xi}(\widehat{\boldsymbol{x}})$ remains around $\mathcal{O}(10^{-16})$ for all values of t. This indicates that both the unstructured and structured BEs are significantly small, confirming that the GMRES method exhibits backward stability and strong backward stability for solving this GSPP.

5. Conclusions. In this paper, have investigated the structured BEs for the GSPP under the constraint that the block matrices E and G preserve the Hermitian structure. Furthermore, we have ensured that the perturbation matrices maintain the sparsity pattern of the coefficient matrix. We have derived optimal perturbation matrices that simultaneously preserve the Hermitian structure and the sparsity of the block matrices, identifying the closest perturbed structure-preserving GSPP. Thus, the approximate solution corresponds to the exact solution of the perturbed GSPP. We have conducted numerical experiments that validate the reliability and accuracy of our theoretical results. Finally, the derived structured BE formulas have been utilized to assess the strong backward stability of the numerical methods for solving GSPPs.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

Acknowledgments. During the course of this work, Pinki Khatun was supported by a fellowship from the Council of Scientific & Industrial Research (CSIR), New Delhi, India (File No. 09/1022(0098)/2020-EMR-I).

REFERENCES

[1] S. S. AHMAD AND P. KANHYA, *Structured perturbation analysis of sparse matrix pencils with s-specified eigenpairs*, Linear Algebra Appl., 602 (2020), pp. 93–119.
[2] ———, *Backward error analysis and inverse eigenvalue problems for Hankel and Symmetric-Toeplitz structures*, Appl. Math. Comput., 406 (2021), Paper No. 126288, 15 pages.
[3] S. S. AHMAD AND P. KHATUN, *Structured condition numbers for a linear function of the solution of the generalized saddle point problem*, Electron. Trans. Numer. Anal., 60 (2024), pp. 471–500. https://etna.ricam.oeaw.ac.at/vol.60.2024/pp471-500.dir/pp471-500.pdf
[4] ———, *Structured backward errors for special classes of saddle point problems with applications*, Linear Algebra Appl., 713 (2025), pp. 90–112.
[5] Z.-Z. BAI AND J.-Y. PAN, *Matrix Analysis and Computations*, SIAM, Philadelphia, 2021.
[6] M. BENZI, G. H. GOLUB, AND J. LIESEN, *Numerical solution of saddle point problems*, Acta Numer., 14 (2005), pp. 1–137.
[7] J. R. BUNCH, *The weak and strong stability of algorithms in numerical linear algebra*, Linear Algebra Appl., 88/89 (1987), pp. 49–66.
[8] J. R. BUNCH, J. W. DEMMEL, AND C. F. VAN LOAN, *The strong stability of algorithms for solving symmetric linear systems*, SIAM J. Matrix Anal. Appl., 10 (1989), pp. 494–499.
[9] Y. CAO, J.-L. DONG, AND Y.-M. WANG, *A relaxed deteriorated PSS preconditioner for nonsymmetric saddle point problems from the steady Navier-Stokes equation*, J. Comput. Appl. Math., 273 (2015), pp. 41–60.
[10] X. S. CHEN, W. LI, X. CHEN, AND J. LIU, *Structured backward errors for generalized saddle point systems*, Linear Algebra Appl., 436 (2012), pp. 3109–3119.
[11] G. EBADI, K. MEHRABI, AND P. S. STANIMIROVIĆ, *An Uzawa-DOS method for solving saddle-point problems*, Numer. Algorithms, 99 (2025), pp. 165—182.
[12] H. C. ELMAN, D. J. SILVESTER, AND A. J. WATHEN, *Finite Elements and Fast Iterative Solvers: with Applications in Incompressible Fluid Dynamics*, 2nd ed., Oxford University Press, Oxford, 2014.
[13] A. GRAHAM, *Kronecker Products and Matrix Calculus: with Applications*, Wiley, New York, 1981.
[14] M. GULLIKSSON, X.-Q. JIN, AND Y.-M. WEI, *Perturbation bounds for constrained and weighted least squares problems*, Linear Algebra Appl., 349 (2002), pp. 221–232.
[15] N. J. HIGHAM, *Accuracy and Stability of Numerical Algorithms*, 2nd ed., SIAM, Philadelphia, 2002.
[16] M.-Q. JIANG AND Y. CAO, *On local Hermitian and skew-Hermitian splitting iteration methods for generalized saddle point problems*, J. Comput. Appl. Math., 231 (2009), pp. 973–982.
[17] A. N. LANGVILLE AND W. J. STEWART, *The Kronecker product and stochastic automata networks*, J. Comput. Appl. Math., 167 (2004), pp. 429–447.
[18] X. LI AND X. LIU, *Structured backward errors for structured KKT systems*, J. Comput. Math., 22 (2004), pp. 605–610.
[19] P. LV, *Structured backward errors analysis for generalized saddle point problems arising from the incompressible Navier-Stokes equations*, AIMS Math., 8 (2023), pp. 30501–30510.
[20] W. MA, *On normwise structured backward errors for the generalized saddle point systems*, Calcolo, 54 (2017), pp. 503–514.
[21] W. MA, Y. FAN, AND X. XU, *Structured backward errors for block three-by-three saddle point systems*, Linear Multilinear Algebra, 72 (2024), pp. 3177-3197.
[22] L. MENG, Y. HE, AND S.-X. MIAO, *Structured backward errors for two kinds of generalized saddle point systems*, Linear Multilinear Algebra, 70 (2022), pp. 1345–1355.
[23] I. PERUGIA AND V. SIMONCINI, *Block-diagonal and indefinite symmetric preconditioners for mixed finite element formulations*, Numer. Linear Algebra Appl., 7 (2000), pp. 585–616.
[24] J. L. RIGAL AND J. GACHES, *On the compatibility of a given solution with the data of a linear system*, J. Assoc. Comput. Mach., 14 (1967), pp. 543–548.
[25] Y. SAAD AND M. H. SCHULTZ, *GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems*, SIAM J. Sci. Statist. Comput., 7 (1986), pp. 856–869.
[26] X.-H. SHAO AND H.-N. MENG, *A generalized variant of modified relaxed positive-semidefinite and skew-Hermitian splitting preconditioner for generalized saddle point problems*, Comput. Appl. Math., 41 (2022), Paper No. 351, 17 pages.
[27] J. G. SUN, *Structured backward errors for KKT systems*, Linear Algebra Appl., 288 (1999), pp. 75–88.
[28] D. S. WATKINS, *Fundamentals of Matrix Computations*, 3rd ed., Wiley, Hoboken, 2010.
[29] J. H. WILKINSON, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965.

[30] B. WU AND X.-B. GAO, *A modified parameterized shift-splitting preconditioner for saddle point problems*, Comput. Appl. Math., 40 (2021), Paper No. 1, 18 pages.
[31] H. XIANG AND Y. WEI, *On normwise structured backward errors for saddle point systems*, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 838–849.
[32] S.-F. YUAN AND Q.-W. WANG, *L-structured quaternion matrices and quaternion linear matrix equations*, Linear Multilinear Algebra, 64 (2016), pp. 321–339.
[33] K. ZHANG AND Y. SU, *Structured backward error analysis for sparse polynomial eigenvalue problems*, Appl. Math. Comput., 219 (2012), pp. 3073–3082.
[34] B. ZHENG AND P. LV, *Structured backward error analysis for generalized saddle point problems*, Adv. Comput. Math., 46 (2020), Paper No. 34, 27 pages.