Electronic Transactions on Numerical Analysis.
Volume 63, pp. 357–367, 2025.

ISSN 1068–9613.
DOI: 10.1553/etna_vol63s357

ETNA
Kent State University and
Johann Radon Institute (RICAM)

ASYMPTOTIC ANALYSIS OF THE NORMAL INVERSE GAUSSIAN CUMULATIVE DISTRIBUTION*

NICO M. TEMME†

Abstract. Using a recently derived integral representation in terms of elementary functions, we give new asymptotic expansions of the normal inverse Gaussian cumulative distribution function. One of its asymptotic representations is stated in terms of the normal Gaussian distribution or complementary error function.

Key words. normal inverse Gaussian distribution, asymptotic analysis, error function

AMS subject classifications. 41A60, 33B20, 62E20, 65D20

1. Introduction. The normal inverse Gaussian distribution is a four-parameter distribution $(\alpha, \beta, \mu, \delta)$ with argument x, which has been introduced by Barndorff-Nielsen [1, 2, 3]. In a recent preprint [7] the commonly used representation of the cumulative distribution function is given by

$$F(x;\alpha,\beta,\mu,\delta) = \frac{\alpha\delta e^{\delta\gamma}}{\pi}\int_{-\infty}^{x} \frac{K_1\left(\alpha\sqrt{\delta^2+(t-\mu)^2}\right)}{\sqrt{\delta^2+(t-\mu)^2}} e^{\beta(t-\mu)}\,dt,$$

where $\gamma = \sqrt{\alpha^2-\beta^2}$ and $K_1(z)$ denotes the modified Bessel function. The cited paper gives new convergent series and derives asymptotic expansions with the aim of developing a software package to compute the cumulative distribution function based on the normal inverse Gaussian distribution.

In this paper, we give more asymptotic expansions of $F(x;\alpha,\beta,\mu,\delta)$ after writing this function in a standard form to apply Laplace's method. Thereby, a modification is considered to handle the case that a pole is near a saddle point. This yields an expansion in which the complementary error function controls this phenomenon.

The starting point of our approach is an integral representation in terms of elementary functions of the complementary function, which is defined by

$$G(x;\alpha,\beta,\mu,\delta) = 1 - F(x;\alpha,\beta,\mu,\delta). \tag{1.1}$$

Especially for numerical computations, it is important to have a stable representation for both functions and to first compute the smaller one of the two. As worked out in this paper, the transition value x_0 with respect to x is given by

$$x_0 = \mu + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}}, \tag{1.2}$$

and, following from the asymptotic approximation, it holds $0 \le F(x;\alpha,\beta,\mu,\delta) \lessapprox \frac{1}{2}$ when $x \lessapprox x_0$.

*Received June 6, 2025. Accepted July 23, 2025. Published online on September 22, 2025. Recommended by Andre Weideman.

†Valkenierstraat 25, 1825BD Alkmaar, The Netherlands (nic@temme.net).

The following integral representation for the G-function has recently been derived in [7, Equation (2.8)]):

$$G(x;\alpha,\beta,\mu,\delta)=\frac{e^{\delta\gamma}}{\pi}\int_0^\infty \frac{re^{-\xi\left(\sqrt{r^2+\alpha^2}-\beta\right)}}{\sqrt{r^2+\alpha^2}\left(\sqrt{r^2+\alpha^2}-\beta\right)}\sin(\delta r)\,dr, \tag{1.3}$$

where

$$\xi=x-\mu\ge 0,\quad \delta>0,\quad \alpha>0,\quad -\alpha<\beta<\alpha,\quad \gamma=\sqrt{\alpha^2-\beta^2}.$$

As observed in [7], for $\xi<0$, we can use the relation

$$\begin{aligned}F(x;\alpha,\beta,\mu,\delta)&=1-F(-x;\alpha,-\beta,-\mu,\delta)=G(-x;\alpha,-\beta,-\mu,\delta)\\ &=\frac{e^{\delta\gamma}}{\pi}\int_0^\infty \frac{re^{\xi\left(\sqrt{r^2+\alpha^2}+\beta\right)}}{\sqrt{r^2+\alpha^2}\left(\sqrt{r^2+\alpha^2}+\beta\right)}\sin(\delta r)\,dr.\end{aligned} \tag{1.4}$$

In our new asymptotic results, the key term in the approximations is the complementary error function, defined by

$$\operatorname{erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\,dt,\quad z\in\mathbb{C},$$

leading to a representation that is not available in [7]. As we have explained previously, for example in [4, 5, 6] and also in [9, Chapter 21 and Part 7], such a representation can yield a powerful asymptotic approximation, also with respect to the so-called uniformity parameters. Moreover, it provides an excellent starting point for inverting cumulative distribution functions with respect to one of the parameters. This topic is intended for future research.

In Section 2 we use several transformations of the integral in (1.3) and obtain in Section 3 a representation suitable for asymptotic analysis. In Section 4 we give the asymptotic expansions, together with a figure and a table to explain the role of the transition value x_0. In Appendix A we present a short Maple code for the evaluation of the coefficients used in certain asymptotic expansions.

2. Transformations of the integral. The integrand in (1.3) is an even function of r, and so we substitute $r=\alpha\sinh(t)$. Then we obtain

$$G(x;\alpha,\beta,\mu,\delta)=\frac{e^{\delta\gamma+\beta\xi}}{2\pi}\int_{-\infty}^\infty e^{-\xi\alpha\cosh(t)}\frac{\sinh(t)\sin\left(\alpha\delta\sinh(t)\right)}{\cosh(t)-\cos(\tau)}\,dt, \tag{2.1}$$

where τ is given by

$$\frac{\beta}{\alpha}=\cos(\tau),\quad 0<\tau<\pi.$$

We use $\sin(z)=\frac{1}{i}\left(e^z-\cos(z)\right)$ with $z=\alpha\delta\sinh(t)$—observe that the cosine term will give an odd integrand—and write (2.1) as

$$G(x;\alpha,\beta,\mu,\delta)=\frac{e^{\delta\gamma+\beta\xi}}{2\pi i}\int_{-\infty}^\infty e^{-\alpha\omega\phi(t)}\frac{\sinh(t)}{\cosh(t)-\cos(\tau)}\,dt,$$

where

$$\phi(t)=\frac{\xi}{\omega}\cosh(t)-i\frac{\delta}{\omega}\sinh(t),\quad \omega=\sqrt{\xi^2+\delta^2}.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

We introduce $\nu \in (0, \frac{1}{2}\pi)$ by writing

$$\nu = \arctan\frac{\delta}{\xi} \quad \Longrightarrow \quad \xi = \omega\cos(\nu) \quad \text{and} \quad \delta = \omega\sin(\nu).$$

It follows that the function $\phi(t)$ can be expressed as

$$\phi(t) = \cosh(t)\cos(\nu) - i\sinh(t)\sin(\nu) = \cosh(t - i\nu).$$

Using this in the integral representation (2.1) yields

$$G(x;\alpha,\beta,\mu,\delta) = \frac{e^{\delta\gamma+\beta\xi}}{2\pi i}\int_{-\infty}^{\infty} e^{-\alpha\omega\cosh(t-i\nu)}\frac{\sinh(t)}{\cosh(t)-\cos(\tau)}\,dt. \tag{2.2}$$

This integral converges when $\cos(\nu) \geq 0$, but since for the representation of $G(x;\alpha,\beta,\mu,\delta)$ in (1.3) it is needed that $\delta = \omega\sin(\nu) > 0$, we therefore assume $\nu \in \left(0, \frac{1}{2}\pi\right)$.

We want to integrate the integral in (2.2) along the horizontal path with $\Im t = \nu$, and we need information about the poles of the integrand. Since it holds $-1 < \cos(\tau) < 1$, the poles closest to the origin are $t_\pm = \pm i\tau$, so we can shift the path of integration in (2.2) to the path $\Im t = \nu$. When $\nu > \tau$, we cross the pole at $i\tau$ and calculate the residue.

After shifting the path, we integrate along the horizontal line $\Im t = \nu$ using the substitution $t = s + i\nu$, and for $\nu > \tau$ we evaluate the relation

$$-\alpha\omega\cosh(i\tau - i\nu) = -\alpha\omega\cos(\tau-\nu) = -\beta\xi - \delta\gamma.$$

We obtain the representations

$$\begin{aligned} G(x;\alpha,\beta,\mu,\delta) &= \frac{e^{\delta\gamma+\beta\xi}}{2\pi i}\int_{-\infty}^{\infty} e^{-\alpha\omega\cosh(s)}\frac{\sinh(s+i\nu)}{\cosh(s+i\nu)-\cos(\tau)}\,ds, \qquad \tau > \nu, \\ G(x;\alpha,\beta,\mu,\delta) &= 1 - \frac{e^{\delta\gamma+\beta\xi}}{2\pi i}\int_{-\infty}^{\infty} e^{-\alpha\omega\cosh(s)}\frac{\sinh(s+i\nu)}{\cos(\tau)-\cosh(s+i\nu)}\,ds, \quad \tau < \nu. \end{aligned} \tag{2.3}$$

Moreover, the case $\tau < \nu$ gives

$$F(x;\alpha,\beta,\mu,\delta) = \frac{e^{\delta\gamma+\beta\xi}}{2\pi i}\int_{-\infty}^{\infty} e^{-\alpha\omega\cosh(s)}\frac{\sinh(s+i\nu)}{\cos(\tau)-\cosh(s+i\nu)}\,ds, \quad \tau < \nu. \tag{2.4}$$

For convergence of the integral in (1.3) we assumed $\xi \geq 0$, or $x \geq \mu$, but in the above three integrals this is no longer needed, and we can also let $\xi \leq 0$ or $x \leq \mu$.

These integrals have a saddle point at the origin, and the real axis is the path of steepest descent. There are poles at the s-values corresponding to the poles $t_\pm = \pm i\tau$, and by the transformation $t = s + i\nu$, the poles in the s-variable are given by

$$s_\pm = -i(\nu \mp \tau), \quad \nu \in (0,\pi), \quad \tau \in (0,\pi). \tag{2.5}$$

We see that the pole s_+ coincides with the saddle point at the origin when ξ takes the value ξ_0 that follows from

$$\tau = \nu \quad \Longrightarrow \quad \arctan\frac{\delta}{\xi_0} = \arccos\frac{\beta}{\alpha} \quad \Longrightarrow \quad \xi_0 = \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}} = \frac{\delta\beta}{\gamma}. \tag{2.6}$$

This yields, for $x = \mu + \xi$, the transition value $x_0 = \xi_0 + \mu$ announced in (1.2). The transition value $x_0 = \xi_0 + \mu$ coincides with the mean of the normal Gaussian distribution.

We have the following cases:

$$\begin{array}{lllll} \tau > \nu & \Longrightarrow & x > x_0, & \Im s_+ > 0, & \Im s_- < 0, \\ \tau = \nu & \Longrightarrow & x = x_0, & s_+ = 0, & \Im s_- < 0, \\ \tau < \nu & \Longrightarrow & x < x_0, & \Im s_+ < 0, & \Im s_- < 0. \end{array} \tag{2.7}$$

When one of these poles is close to the origin, we need special asymptotic methods to deal with it.

3. Further preparations for the asymptotic analysis. We continue with (2.4), assuming that $\nu > \tau$. Note that this condition will be relaxed after deriving the asymptotic expansions in Section 4. We write the real and imaginary parts of the integrand using

$$\begin{aligned} \sinh(s+i\nu) &= \sinh(s)\cos(\nu) + i\cosh(s)\sin(\nu), \\ \cosh(s+i\nu) &= \cosh(s)\cos(\nu) + i\sinh(s)\sin(\nu). \end{aligned}$$

We obtain

$$\begin{aligned} \frac{\sinh(s+i\nu)}{\cos(\tau)-\cosh(s+i\nu)} &= \frac{\sinh(s)\left(\cos(\tau)\cos(\nu)-\cosh(s)\right)}{\left(\cosh(s)-\cos(\tau)\cos(\nu)\right)^2-\sin^2(\tau)\sin^2(\nu)} \\ &\quad + i\frac{\sin(\nu)\left(\cos(\tau)\cosh(s)-\cos(\nu)\right)}{\left(\cosh(s)-\cos(\tau)\cos(\nu)\right)^2-\sin^2(\tau)\sin^2(\nu)}. \end{aligned}$$

We observe that the real part is odd with respect to s and the imaginary part is even. So we can write

$$F(x;\alpha,\beta,\mu,\delta) = \frac{e^{\delta\gamma+\beta\xi}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-\alpha\omega\cosh(s)}\sin(\nu)\left(\cos(\tau)\cosh(s)-\cos(\nu)\right)}{\left(\cosh(s)-\cos(\tau)\cos(\nu)\right)^2-\sin^2(\tau)\sin^2(\nu)}\,ds.$$

Next, we use $\cosh(s) = 1 + 2\sinh^2(s/2)$ and substitute

$$\sigma = \sinh(s/2), \qquad \frac{ds}{d\sigma} = \frac{2}{\sqrt{1+\sigma^2}}.$$

This gives

$$F(x;\alpha,\beta,\mu,\delta) = \frac{e^{\delta\gamma+\beta\xi-\alpha\omega}}{4\pi}\int_{-\infty}^{\infty} e^{-2\alpha\omega\sigma^2} f(\sigma)\,d\sigma, \quad \nu > \tau, \tag{3.1}$$

where

$$\begin{aligned} f(\sigma) &= \frac{\sin(\nu)}{\sqrt{1+\sigma^2}}\,\frac{\cos(\tau)(1+2\sigma^2)-\cos(\nu)}{\left(\sigma^2+\frac12-\frac12\cos(\tau)\cos(\nu)\right)^2-\frac14\sin^2(\tau)\sin^2(\nu)} \\ &= \frac{\sin(\nu)}{\sqrt{1+\sigma^2}}\,\frac{\cos(\tau)(1+2\sigma^2)-\cos(\nu)}{(\sigma^2-\sigma_+^2)(\sigma^2-\sigma_-^2)}. \end{aligned}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

The poles $\sigma_\pm$ follow from the poles $s_\pm$ via the relation $\sigma = \sinh(s/2)$. We have (see (2.5))

$$\sigma_\pm = \sinh\left(\tfrac{1}{2}s_\pm\right) = i\sin\left(\tfrac{1}{2}s_\pm\right) = -i\sin\left(\tfrac{1}{2}(\nu \mp \tau)\right). \tag{3.2}$$

By splitting into fractions, we obtain

$$\begin{aligned} f(\sigma) &= \frac{\sin(\nu)}{\sqrt{1+\sigma^2}\,(\sigma_-^2 - \sigma_+^2)}\Biggl(\frac{\cos(\nu) - \cos(\tau)(1+2\sigma_+^2)}{\sigma^2 - \sigma_+^2} \\ &\qquad\qquad - \frac{\cos(\nu) - \cos(\tau)(1+2\sigma_-^2)}{\sigma^2 - \sigma_-^2}\Biggr) \\ &= \frac{1}{\sqrt{1+\sigma^2}}\left(\frac{\sin(\nu-\tau)}{\sigma^2-\sigma_+^2} + \frac{\sin(\nu+\tau)}{\sigma^2-\sigma_-^2}\right). \end{aligned}$$

The argument of the exponential function in front of the integral in (3.1) can be written as

$$\delta\gamma + \beta\xi - \alpha\omega = -2\alpha\omega\sin^2\left(\tfrac{1}{2}(\nu-\tau)\right) = 2\alpha\omega\sigma_+^2.$$

After these steps, we summarize the results obtained so far in the following theorem:

THEOREM 3.1. *We can write* (3.1) *in the form*

$$\begin{aligned} F(x;\alpha,\beta,\mu,\delta) &= F^+(x;\alpha,\beta,\mu,\delta) + F^-(x;\alpha,\beta,\mu,\delta), \\ F^\pm(x;\alpha,\beta,\mu,\delta) &= \frac{e^{z\sigma_+^2}}{4\pi}\sin(\nu\mp\tau)U(\sigma_\pm,z), \\ U(\sigma_\pm,z) &= \int_{-\infty}^{\infty} e^{-z\sigma^2}\frac{d\sigma}{(\sigma^2-\sigma_\pm^2)\sqrt{1+\sigma^2}}, \end{aligned} \tag{3.3}$$

where, employing the notation used so far,

$$\begin{aligned} & z = 2\alpha\omega, \quad \xi = x-\mu, \quad x_0 = \mu + \xi_0, \quad \xi_0 = \frac{\beta\delta}{\gamma}, \\ & \sigma_+ = i\sin\left(\tfrac{1}{2}(\nu-\tau)\right), \quad \sigma_- = -i\sin\left(\tfrac{1}{2}(\nu+\tau)\right), \\ & \xi = x-\mu = \omega\cos(\nu), \quad \delta = \omega\sin(\nu), \\ & \nu = \arctan\frac{\delta}{\xi}, \quad \nu\in\left(0,\tfrac{1}{2}\pi\right), \\ & \omega = \sqrt{\xi^2+\delta^2}, \quad \cos(\tau) = \frac{\beta}{\alpha}, \quad \tau\in(0,\pi), \\ & \gamma = \sqrt{\alpha^2-\beta^2} = \alpha\sin(\tau). \end{aligned} \tag{3.4}$$

4. Asymptotic expansions. First, we give the asymptotic expansion of $U(\sigma_\pm, z)$ as defined in (3.3), with z a positive large parameter and $i\sigma_\pm \in (0,1)$. We initially assume that $i\sigma_\pm$ is not small, say $\frac{1}{2} \le i\sigma_\pm < 1$. In that case, an asymptotic expansion can be obtained by using Laplace's method; see [9, Chapter 3].

We expand

$$\frac{1}{(\sigma^2-\rho^2)\sqrt{1+\sigma^2}} = \sum_{k=0}^{\infty} u_k(\rho)\sigma^{2k}, \quad \rho = \sigma_\pm, \tag{4.1}$$

and obtain the asymptotic expansion

$$U(\rho, z) \sim \sqrt{\frac{\pi}{z}} \sum_{k=0}^{\infty} u_k(\rho) \frac{\left(\frac{1}{2}\right)_k}{z^k}, \quad z \to \infty, \quad \tfrac{1}{2} \le i\sigma_\pm < 1,$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol. The first coefficients are

$$u_0(\rho) = -\frac{1}{\rho^2}, \quad u_1(\rho) = \frac{\rho^2-2}{2\rho^4}, \quad u_2(\rho) = -\frac{3\rho^4-4\rho^2+8}{8\rho^6}.$$

Next, we want to obtain an expansion that is valid for small positive values of $|\sigma_\pm|$. As explained in [9, Chapter 21 and Part 7] and [4, 5, 6], we can use the complementary error function to obtain uniform approximations. We have the integral representations (for properties of the error functions, we refer to [8])

$$\begin{aligned} \operatorname{erfc}(z) &= \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2}\, dt, \quad z \in \mathbb{C}, \\ w(z) &= \frac{1}{\pi i} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{t-z} = e^{-z^2} \operatorname{erfc}(-iz), \quad \Im z > 0. \end{aligned}$$

In our analysis, we need the function

$$V(\rho, z) = \int_{-\infty}^{\infty} e^{-z\sigma^2} \frac{d\sigma}{\sigma^2-\rho^2} = \frac{\pi}{i\rho} w\left(\rho\sqrt{z}\right) = \frac{\pi}{i\sigma_\pm} e^{-z\sigma_\pm^2} \operatorname{erfc}\left(i\sigma_\pm \sqrt{z}\right),$$

where we used $\rho = \sigma_\pm$, with $\sigma_\pm$ defined in (3.2). Both $\sigma_\pm$ satisfy $\Im\sigma_\pm < 0$ (see also the third line of (2.7)).

Next, we consider $U(\sigma_\pm, z)$ defined in (3.3). We specify the role of the poles by writing

$$\frac{1}{(\sigma^2-\rho^2)\sqrt{1+\sigma^2}} = g(\sigma,\rho) + \frac{1}{(\sigma^2-\rho^2)\sqrt{1+\rho^2}}, \quad \rho = \sigma_\pm,$$

where

$$\begin{aligned} g(\sigma,\rho) &= \frac{1}{\sigma^2-\rho^2}\left(\frac{1}{\sqrt{1+\sigma^2}} - \frac{1}{\sqrt{1+\rho^2}}\right) \\ &= -\frac{1}{\sqrt{1+\sigma^2}\,\sqrt{1+\rho^2}\left(\sqrt{1+\sigma^2}+\sqrt{1+\rho^2}\right)}. \end{aligned} \tag{4.2}$$

Applying this to the function $U(\rho, z)$ defined in (3.3) gives

$$\begin{aligned}U(\rho, z) &= \int_{-\infty}^{\infty} e^{-z\sigma^2} \left(\frac{1}{(\sigma^2 - \rho^2)\sqrt{1+\rho^2}} + g(\sigma, \rho) \right) d\sigma \\ &= \frac{1}{\sqrt{1+\rho^2}} V(\rho, z) + \int_{-\infty}^{\infty} e^{-z\sigma^2} g(\sigma, \rho)\, d\sigma \\ &= -\frac{\pi i}{\rho\sqrt{1+\rho^2}} e^{-z\rho^2} \operatorname{erfc}\left(i\rho\sqrt{z}\right) + \int_{-\infty}^{\infty} e^{-z\sigma^2} g(\sigma, \rho)\, d\sigma.\end{aligned}$$

The function $g(\sigma, \rho)$ is analytic for $|\sigma| < 1$, thus, we can expand it as

$$g(\sigma, \rho) = g(0, \rho) \sum_{k=0}^{\infty} c_k(\rho) \sigma^{2k}, \quad g(0, \rho) = -\frac{1}{\sqrt{1+\rho^2}\left(1 + \sqrt{1+\rho^2}\right)} \tag{4.3}$$

to obtain the asymptotic expansion

$$U(\rho, z) \sim -\frac{\pi i}{\rho\sqrt{1+\rho^2}} e^{-z\rho^2} \operatorname{erfc}\left(i\rho\sqrt{z}\right) + g(0, \rho) \sqrt{\frac{\pi}{z}} \sum_{k=0}^{\infty} \frac{d_k(\rho)}{z^k},$$

where

$$d_k(\rho) = c_k(\rho) \left(\tfrac{1}{2}\right)_k, \quad k = 0, 1, 2, \ldots$$

From the second line in (4.2) we see that the computation of the coefficients c_k can be done by multiplying the Maclaurin series of $1/\sqrt{1+\sigma^2}$ and that of $1/\left(\sqrt{1+\sigma^2} + \sqrt{1+\rho^2}\right)$. For large values of k, both coefficients of these expansions are of small algebraic order of k, but the coefficients $d_k = c_k \left(\frac{1}{2}\right)_k = c_k \Gamma\left(k + \frac{1}{2}\right)/\Gamma\left(\frac{1}{2}\right)$ are of factorial order.

After these preparations, we obtain the asymptotic expansions for $F^{\pm}(x; \alpha, \beta, \mu, \delta)$ defined in (3.3):

$$\begin{aligned}&F^{\pm}(x; \alpha, \beta, \mu, \delta) \\ &\quad \sim \frac{e^{z\sigma_{\pm}^2} \sin(\nu \mp \tau)}{4\pi} \left(-\frac{\pi i e^{-z\sigma_{\pm}^2}}{\rho\sqrt{1+\rho^2}} \operatorname{erfc}\left(i\sigma_{\pm}\sqrt{z}\right) + g(0) \sqrt{\frac{\pi}{z}} \sum_{k=0}^{\infty} \frac{d_k(\sigma_{\pm})}{z^k} \right).\end{aligned}$$

Let us simplify a few of the expressions. We have (see equations (3.4) and(4.3))

$$\begin{aligned}\frac{\sin(\nu \mp \tau)}{\rho\sqrt{1+\rho^2}} &= \frac{\sin(\nu \mp \tau)}{-i \sin\left(\frac{1}{2}(\nu \mp \tau)\right) \cos\left(\frac{1}{2}(\nu \mp \tau)\right)} = 2i, \\ \sin(\nu \mp \tau) g(0, \sigma_{\pm}) &= -\frac{\sin(\nu \mp \tau)}{\cos\left(\frac{1}{2}(\nu \mp \tau)\right)\left(1 + \cos\left(\frac{1}{2}(\nu \mp \tau)\right)\right)} \\ &= -\frac{\sin\left(\frac{1}{2}(\nu \mp \tau)\right)}{\cos^2\left(\frac{1}{4}(\nu \mp \tau)\right)} = -2\tan\left(\tfrac{1}{4}(\nu \mp \tau)\right), \\ z(\sigma_+^2 - \sigma_-^2) &= -2\alpha\omega \sin(\nu)\sin(\tau) = -2\gamma\delta.\end{aligned}$$

In summary, we have the following theorem. The parameters ν, τ, $\sigma_{\pm}$, and z are given in (3.4).

THEOREM 4.1. *The functions $F^{\pm}(x;\alpha,\beta,\mu,\delta)$ composing the normal inverse Gaussian cumulative distribution $F(x;\alpha,\beta,\mu,\delta)$ as written in (3.3) of Theorem 3.1 have the asymptotic expansions*

$$F^{\pm}(x;\alpha,\beta,\mu,\delta) \sim \frac{e^{z\sigma_+^2}}{4\pi}\left(2\pi e^{-z\sigma_{\pm}^2}\operatorname{erfc}\left(i\sigma_{\pm}\sqrt{z}\right) - 2\tan\left(\tfrac{1}{4}(\nu\mp\tau)\right)\sqrt{\frac{\pi}{z}}\sum_{k=0}^{\infty}\frac{d_k(\sigma_{\pm})}{z^k}\right),$$

or

$$\begin{aligned} F^{+}(x;\alpha,\beta,\mu,\delta) &\sim \tfrac{1}{2}\operatorname{erfc}(\zeta_+) - e^{z\sigma_+^2}\frac{\tan\left(\frac{1}{4}(\nu-\tau)\right)}{2\sqrt{\pi z}}\sum_{k=0}^{\infty}\frac{d_k(\sigma_+)}{z^k}, \\ F^{-}(x;\alpha,\beta,\mu,\delta) &\sim \tfrac{1}{2}e^{-2\gamma\delta}\operatorname{erfc}(\zeta_-) - e^{z\sigma_+^2}\frac{\tan\left(\frac{1}{4}(\nu+\tau)\right)}{2\sqrt{\pi z}}\sum_{k=0}^{\infty}\frac{d_k(\sigma_-)}{z^k}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \zeta_+ &= i\sigma_+\sqrt{z} = \sin\left(\frac{1}{2}(\nu-\tau)\right)\sqrt{z} = \operatorname{sign}(x_0-x)\sqrt{\alpha\omega-\beta\xi-\gamma\delta}, \\ \zeta_- &= i\sigma_-\sqrt{z} = \sin\left(\frac{1}{2}(\nu+\tau)\right)\sqrt{z} = \sqrt{\alpha\omega-\beta\xi+\gamma\delta}. \end{aligned} \tag{4.5}$$

The expansions are valid for large values of z and $|\sigma_{\pm}| \le \frac{1}{2}$.

For the complementary function $G(x;\alpha,\beta,\mu,\delta)$ defined in (1.1) we have the next corollary:

COROLLARY 4.2. *The asymptotic expansion of the function $G(x;\alpha,\beta,\mu,\delta)$ follows from*

$$G(x;\alpha,\beta,\mu,\delta) = G^{+}(x;\alpha,\beta,\mu,\delta) - F^{-}(x;\alpha,\beta,\mu,\delta)$$

with

$$G^{+}(x;\alpha,\beta,\mu,\delta) \sim \tfrac{1}{2}\operatorname{erfc}(-\zeta_+) + e^{z\sigma_+^2}\frac{\tan\left(\frac{1}{4}(\nu-\tau)\right)}{2\sqrt{\pi z}}\sum_{k=0}^{\infty}\frac{d_k(\sigma_+)}{z^k}.$$

We make a few observations.

1. The square root forms of $\zeta_{\pm}$ in (4.5) follow from the relations in (3.4). The expression $\alpha\omega-\beta\xi-\gamma\delta$ for ζ_+ is a convex function of ξ with a double zero at ξ_0, the transition point. This follows from calculating the first terms in the Taylor expansion of ζ_+ as a function of ξ at ξ_0.
2. The argument ζ_+ of $\operatorname{erfc}(\zeta_+)$ in the first line of (4.4) vanishes when $\nu=\tau$, that is, when $x=x_0$; see the earlier discussion at equation (2.6). In the first paragraph of Section 3 we assumed that $\nu>\tau$ (or $x<x_0$) since we have started with (2.4). This corresponds to $\zeta_+>0$ in $\operatorname{erfc}(\zeta_+)$.
3. However, $\operatorname{erfc}(\zeta_+)$ allows us to take $\nu<\tau$ (or $x>x_0$), and the transition from $\nu>\tau$ to $\nu<\tau$ in $\operatorname{erfc}(\zeta_+)$ is smooth and analytic. In addition, other elements in the series expansion of the function $F^{+}(x;\alpha,\beta,\mu,\delta)$ remain well-defined, and $F^{+}(x;\alpha,\beta,\mu,\delta)$ changes from values smaller than $\frac{1}{2}$ ($x\le x_0$ or $\nu>\tau$) to values larger than $\frac{1}{2}$ ($x\ge x_0$ or $\nu<\tau$).
4. When $\nu<\tau$ we can repeat the analysis with the representation of the complementary function $G(x;\alpha,\beta,\mu,\delta)$, starting with the first line of (2.3). The result will be the same as in Corollary 4.2.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

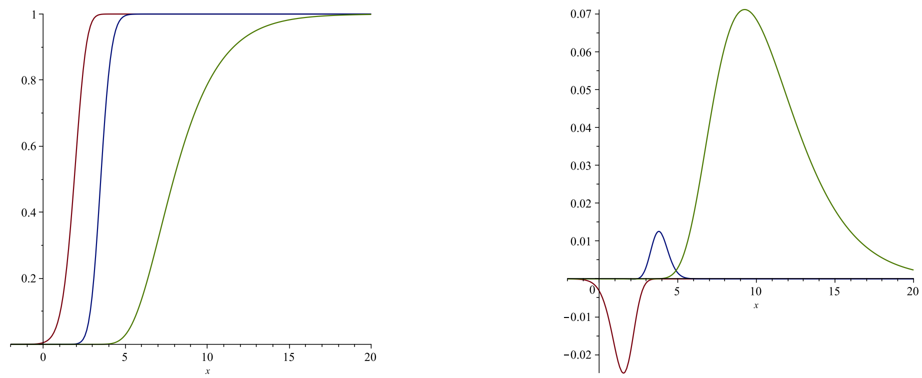

FIG. 4.1. *Graphs obtained using the asymptotic approximations, for $\alpha = 8$, $\mu = 3$, $\delta = 2$, $0 \le x \le 20$, and three values of β: left $\beta = -4$, middle $\beta = 2$, right $\beta = 7.5$. In the left figure we display graphs of the function $F(x;\alpha,\beta,\mu,\delta)$; in the right one we display $F^-(x;\alpha,\beta,\mu,\delta)$ for the same parameters.*

5. A notable point is that the asymptotic approximations can be used for negative values of $\xi = x - \mu$, while the starting representation in (2.1) is only valid for $\xi \ge 0$. For $\xi < 0$, see also the representation in (1.4).

The first coefficients $d_k(\sigma_\pm)$ are given by

$$\begin{aligned} &d_0(\sigma_\pm) = 1, \quad d_1(\sigma_\pm) = -\frac{w+2}{4(w+1)}, \quad d_2(\sigma_\pm) = \frac{3(3w^2+9w+8)}{32(w+1)^2}, \\ &d_3(\sigma_\pm) = -\frac{15(5w^3+20w^2+29w+16)}{128(w+1)^3}, \\ &d_4(\sigma_\pm) = \frac{105(35w^4+175w^3+345w^2+325w+128)}{2048(w+1)^4}. \end{aligned}$$

Here, $w = \sqrt{1+\sigma_\pm^2}$. Hence, $w = \cos\left(\frac{1}{2}(\nu \mp \tau)\right)$. In Appendix A we describe a recursive method for the symbolic evaluation of these coefficients, together with a short Maple code.

In Figure 4.1 we display graphs obtained using the asymptotic approximations in (4.4), for $\alpha = 8$, $\mu = 3$, $\delta = 2$, $0 \le x \le 20$, and three values of β: left $\beta = -4$, middle $\beta = 2$, right $\beta = 7.5$. In the figure on the left, we see three graphs of the function $F(x;\alpha,\beta,\mu,\delta)$, and on the right, we see three graphs of $F^-(x;\alpha,\beta,\mu,\delta)$ for the same parameters. We observe that the main contributions come from $F^+(x;\alpha,\beta,\mu,\delta)$, but for the numerical computations those of $F^-(x;\alpha,\beta,\mu,\delta)$ cannot be neglected.

The corresponding transition values x_0, the function values of $F(x_0;8,\beta,3,2)$, and the values of the large parameter $z = 2\alpha\omega$ are given in Table 4. In addition, we display the absolute errors in the computation of the asymptotic expansions relative to more accurate computations of the function F. The computations of the asymptotic approximations are performed with Maple, Digits $= 8$, with terms up to and including $k = 5$ in the asymptotic expansions in (4.4). The given absolute errors illustrate the quality of the asymptotic approximation with only 6 terms. In a future paper, we plan to perform more extensive tests to assess the accuracy of the asymptotic expansions for different regions of the parameters.

TABLE 4.1
For the values of β used in Figure 4.1, we give the transition values x_0, the function values $F(x_0;\alpha,\beta,\mu,\delta)$ for $\alpha=8$, $\mu=3$, $\delta=2$, the large parameter $z=2\alpha\omega$, and the absolute errors in the values of the F-function for these parameters.

β	x_0	F	z	absolute errors
-4.0	1.845299462	0.473833601	36.95041722	3.1×10^{-08}
2.0	3.516397780	0.512385772	33.04945788	1.7×10^{-09}
7.5	8.388159062	0.575900502	91.95791466	4.7×10^{-10}

Acknowledgements. The author thanks the reviewers for their constructive comments and helpful suggestions. One reviewer verified the paper in detail, noted in particular that the transition value x_0 coincides with the mean of the normal Gaussian distribution (see (2.6)), noted the typo in the first line of (2.4), and provided absolute errors for Table 4. The other reviewer suggested, among other things, including a Maple code for the coefficients $d_k(\rho)$ of the asymptotic expansions (see the new appendix) and provided a 31-page transcript of a ChatGPT session showing the verification of all steps in the paper, including transformations of variables, shifts of integration paths into the complex plane across the poles, and so on.

The author thanks Amparo Gil, Guillermo Navas-Palencia, and Javier Segura for their advice at the start of this project. The author acknowledges financial support from the project PID2021- 127252NB-I00 funded by MCIN/AEI/10.13039/501100011033/ FEDER, UE.

Appendix A. We can obtain the coefficients c_k of the expansion (4.3) of the function $g(\sigma,\rho)$ given in (4.2) using Maple's procedure *taylor*. To state an algorithm without this procedure and without Maple's procedure *pochhammer*, we use a recursive method for the symbolic evaluation of the coefficients c_k. Finally, we provide a Maple procedure *dkproc* for this method. A similar code can be written for the evaluation of the coefficients $u_k(\rho)$ of the Maclaurin series in (4.1).

We use the second line of equation (4.2) and write the expansion in (4.3) in the form

$$-g(0,\rho)\left(w(1+\sigma^2)+w^2\sqrt{1+\sigma^2}\right)\sum_{k=0}^{\infty}c_k(\rho)\sigma^{2k}=1,\quad w=\sqrt{1+\rho^2},$$

or

$$-g(0,\rho)\sum_{k=0}^{\infty}b_k(\rho)\sigma^{2k}\sum_{k=0}^{\infty}c_k(\rho)\sigma^{2k}=1,$$

where

$$c_0=1,\quad b_0=w+w^2,\quad b_1=w+\tfrac12 w^2,\quad b_k=w^2(-1)^k\frac{(-\frac12)_k}{k!},\quad k\ge 2.$$

The next step is to multiply the two series:

$$-g(0,\rho)\sum_{k=0}^{\infty}a_k(\rho)\sigma^{2k}=1,\quad a_k=\sum_{j=0}^{k}b_j(\rho)c_{k-j}(\rho).$$

All coefficients $a_k(\rho)$ should vanish, except $a_0(\rho)=b_0(\rho)c_0(\rho)$, and by using (4.3) we obtain the equation $-g(0,\rho)b_0(\rho)c_0(\rho)=1$, which yields $c_0(\rho)=1$. For the other $c_k(\rho)$, we obtain

ETNA
Kent State University and
Johann Radon Institute (RICAM)

the recursive relation

$$c_k(\rho) = -\frac{1}{b_0(\rho)}\sum_{j=1}^{k} b_j(\rho)c_{k-j}(\rho), \quad k = 1, 2, 3, \ldots$$

With these coefficients, we can obtain $d_k(\rho) = \left(\frac{1}{2}\right)_k c_k(\rho)$.

In the following Maple code, we describe an algorithm to compute $d_k(\rho)$.

```
restart;
dkproc:= proc(kmax, dk) local bk, ck, w, j, k, s, p;
# To compute d_k, see (4.24); dk is an output parameter.
  bk[0]:= w+w^2; bk[1]:= w+w^2/2; bk[2]:= -w^2/8;
  ck[0]:= 1; dk[0]:= 1;  ck[1]:= normal(-bk[1]/bk[0]);
  p:= 1/2; dk[1]:= p*ck[1];
  # p = (1/2)_1; a Pochhammer value to start a recursion.
  for k from 2 to kmax do
    # bk[k] is a binomial coefficient, generated by recursion.
    bk[k+1]:= -(k-1/2)*bk[k]/(k+1);
    s:= 0;
    for j from 1 to k do s:= normal(s + bk[j]*ck[k-j]) od;
    ck[k]:= normal(-s/bk[0]);  p:= p*(k-1/2);
    #This makes p = (1/2)_k;
     dk[k]:= p*ck[k];
  od;
  kmax
end;
# For example:
kmax:= 5; dkproc(kmax, dk);
for k from 0 to kmax do print(k,dk[k]) od;
```

REFERENCES

[1] O. BARNDORFF-NIELSEN, *Exponentially decreasing distributions for the logarithm of particle size*, Proc. Roy. Soc. Edinburgh Sect. A, 353 (1977), pp. 401–419.
[2] ———, *Hyperbolic distributions and distributions on hyperbolae*, Scand. J. Statist., 5 (1978), pp. 151–157.
[3] ———, *Normal inverse Gaussian distributions and stochastic volatility modelling*, Scand. J. Statist., 24 (1997), pp. 1–13.
[4] A GIL, J SEGURA, AND N. M. TEMME, *The asymptotic and numerical inversion of the Marcum Q-function*, Stud. Appl. Math., 133 (2014), pp. 257–278.
[5] ———, *On the computation and inversion of the cumulative noncentral beta distribution function*, Appl. Math. Comput., 361 (2019), pp. 74–86.
[6] ———, *A new asymptotic representation and inversion method for the Student's t distribution*, Integral Transforms Spec. Funct., 33 (2022), pp. 597–608.
[7] G. NAVAS-PALENCIA, *On the computation of the cumulative distribution function of the normal inverse Gaussian distribution*, Numer. Algorithms, Published online 2025. https://doi.org/10.1007/s11075-025-02202-w
[8] N. M. TEMME, *Error functions, Dawson's and Fresnel integrals*, in NIST Handbook of Mathematical Functions, Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert and Charles W. Clark, eds., U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC, 2010, pp. 159–171.
[9] ———, *Asymptotic Methods for Integrals*, World Scientific, Hackensack, 2015.