Electronic Transactions on Numerical Analysis.
Volume 63, pp. 1–32, 2025.

ISSN 1068–9613.
DOI: 10.1553/etna_vol63s1

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STRUCTURE-PRESERVING DISCONTINUOUS GALERKIN APPROXIMATION OF A HYPERBOLIC-PARABOLIC SYSTEM*

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Abstract. We study the numerical approximation of a coupled hyperbolic-parabolic system by a family of discontinuous Galerkin (DG) space-time finite element methods. The model is rewritten as a first-order evolutionary problem that is treated by a unified abstract solution theory. For the discretization in space, generalizations of the distribution gradient and divergence operators on broken polynomial spaces are defined. Since their skew-selfadjointness is perturbed by boundary surface integrals, adjustments are introduced such that the skew-selfadjointness of the discrete counterpart of the total system's first-order differential operator in space is recovered. Well-posedness of the fully discrete problem and error estimates for the DG approximation in space and time are proved.

Key words. Coupled hyperbolic-parabolic problem, first-order system, Picard's theorem, discontinuous Galerkin space-time discretization, error estimates

AMS subject classifications. 65M15, 65M60, 35Q35

1. Introduction.

We study the numerical approximation by discontinuous Galerkin (DG) methods in space and time of solutions to the hyperbolic-parabolic system

$$\rho\partial_t^2 \boldsymbol{u} - \nabla\cdot(\boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) + \alpha\nabla p = \rho\boldsymbol{f}\,, \quad \text{in } \Omega\times(0,T]\,, \tag{1.1a}$$

$$c_0\partial_t p + \alpha\nabla\cdot\partial_t\boldsymbol{u} - \nabla\cdot(\boldsymbol{K}\nabla p) = g\,, \quad \text{in } \Omega\times(0,T]\,, \tag{1.1b}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0\,,\quad \partial_t\boldsymbol{u}(0) = \boldsymbol{u}_1\,,\quad p(0) = p_0\,, \quad \text{in } \Omega\,, \tag{1.1c}$$

$$\boldsymbol{u} = \boldsymbol{0}\,,\quad p = 0\,, \quad \text{on } \partial\Omega\times(0,T]\,. \tag{1.1d}$$

For this, we rewrite (1.1) as a first-order evolutionary problem in space and time on the open bounded domain $\Omega\subset\mathbb{R}^d$, with $d\in\{2,3\}$, and for the final time $T>0$. The system (1.1) is investigated as a prototype model problem for poro- and thermoelasticity; cf., e.g., [14, 15, 16, 37, 49]. In poroelasticity, equations (1.1a) and (1.1b) describe the conservation of momentum and mass. The unknowns are the effective solid phase displacement $\boldsymbol{u}$ and the effective fluid pressure p. The quantity $\boldsymbol{\varepsilon}(\boldsymbol{u}) := (\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top)/2$ denotes the symmetrized gradient or strain tensor. Further, ρ is the effective mass density, $\boldsymbol{C}$ is Gassmann's fourth-order effective elasticity tensor, α is Biot's pressure-storage coupling tensor, c_0 is the specific storage coefficient, and $\boldsymbol{K}$ is the permeability field. For simplicity, the positive quantities $\rho>0$, $\alpha>0$, and $c_0>0$ are assumed to be constant in space and time. Moreover, the tensors $\boldsymbol{C}$ and $\boldsymbol{K}$ are assumed to be symmetric and positive definite and independent of the space and time variables as well. In (1.1a), the effects of secondary consolidation (cf. [40]), described in certain models by the additional term $\lambda^*\delta_{ij}\varepsilon(\partial_t\boldsymbol{u})$ in the total stress, are not included here.

Beyond the classical applications of (1.1) in subsurface hydrology and geophysics, for instance in reservoir engineering, systems like (1.1) have recently attracted reseachers' interest in biomedical engineering; cf., e.g., [22, 25, 41]. In thermoelasticity, the system (1.1) describes the flow of heat through an elastic structure. In this context, p denotes the temperature, c_0

*Received November 1, 2023. Accepted November 7, 2024. Published online on January 15, 2025. Recommended by Simona Perotto.
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is the specific heat of the medium, and $\boldsymbol{K}$ is the conductivity. The homogeneous Dirichlet boundary conditions (1.1d) are studied here for simplicity and brevity.

By introducing the variable of unknowns $\boldsymbol{U} := (\boldsymbol{v}, \boldsymbol{\sigma}, p, \overline{\boldsymbol{q}})^\top$, with the quantities $\boldsymbol{v} := \partial_t \boldsymbol{u}$, $\boldsymbol{\sigma} := \boldsymbol{C}\boldsymbol{\varepsilon}$, and $\overline{\boldsymbol{q}} := -\boldsymbol{K}\nabla p + \alpha \boldsymbol{v}$, we transform the system (1.1) into an abstract evolutionary equation written as the sum of two unbounded first-order differential operators, one of them involving a first-order differential operator in time and the other one involving first-order differential operators in space. In the exponentially weighted-in-time Bochner space $H_\nu(\mathbb{R}; \boldsymbol{H})$ defined in (2.1), with some weight $\nu > 0$ and the Hilbert space $\boldsymbol{H} := L^2(\Omega)^{(d+1)^2}$, we then obtain the evolutionary equation for $\boldsymbol{U}$,

$$(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A})\boldsymbol{U} = \boldsymbol{F}\,. \tag{1.2}$$

In (1.2), $\boldsymbol{M}_0$ and $\boldsymbol{M}_1$ are bounded linear selfadjoint operators in $\boldsymbol{H}$, and $\boldsymbol{A}$ is an unbounded skew-selfadjoint operator in $\boldsymbol{H}$. The right-hand side function $\boldsymbol{F}$ in (1.2) depends on the source terms $\boldsymbol{f}$ and g of (1.1). For (1.2), a solution mechanism developed by R. Picard [42] can be applied. It is based on the monotonicity of both the sum of the mentioned unbounded operators together with its adjoint computed in the space-time Hilbert space. For the presentation of the solution theory we refer to [48, Theorem 6.2.1]. The well-posedness criterion for (1.2) that is summarized in Theorem 2.5 is elementary and general. It can be verified without dealing with the intricacies of more involved solution methods. This is an appreciable advantage of Picard's theorem [42]. A priori, there is no explicit initial condition implemented in the theory. For (1.2), an initial condition of the form $\lim_{t \searrow t_0} \boldsymbol{M}_0 \boldsymbol{U}(t) = \boldsymbol{M}_0 \boldsymbol{U}_0$, for some $t_0 \in \mathbb{R}$ and $\boldsymbol{U} \in D(\boldsymbol{A})$, can be implemented by a distributional right-hand side term $\boldsymbol{F} + \delta_{t_0} \boldsymbol{M}_0 \boldsymbol{U}_0$ for some $\boldsymbol{F} \in H_\nu(\mathbb{R}; \boldsymbol{H})$ supported on $[t_0, \infty)$ and the Dirac distribution δ_{t_0} at t_0. For details of this we refer to [43, Section 6.2.5] and [48, Chapter 9].

By introducing the four-field formulation for the unknown vector $\boldsymbol{U}$, the problem size is increased. However, in poroelasticity the explicit approximation of the flux variable $\boldsymbol{q} = -\boldsymbol{K}\nabla p$ is often desirable and of higher importance than the approximation of the fluid pressure itself. For instance, this holds if the reactive transport of species dissolved in the fluid is studied further. Simulations then demand for accurate approximations of the flux variable $\boldsymbol{q}$. A similar argument applies to the stress tensor $\boldsymbol{\sigma}$ if this variable is the goal quantity of physical interest in (1.1) or needs to be post-processed for elucidating phenomena modeled by (1.1). In implementations, the symmetry of the stress tensor $\boldsymbol{\sigma}$ can still be exploited to reduce the problem size.

In this work we propose and analyze fully discrete numerical approximation schemes that are built for the evolutionary equation (1.2). Their key feature is that they essentially preserve the abstract evolutionary form (1.2) and the operators' properties. However, due to the nonconforming discretization in space that is applied here, the skew-selfadjointness of the discrete counterpart $\boldsymbol{A}_h$ of $\boldsymbol{A}$ in (1.2) is perturbed by non-vanishing contributions arising from boundary face integrals. Therefore, a correction term is introduced on the discrete level to overcome this defect and to ensure that a discrete counterpart of the skew-selfadjointness, which is essentially used in the analyses, is satisfied. In the design of numerical methods, structure-preserving approaches ensuring that important properties of differential operators and solutions to the continuous problem are maintained on the fully discrete level are highly desirable and important to guarantee a physical realism of the numerical predictions. We focus on DG discretizations of the space and time variables. DG methods for the space discretization (cf., e.g., [26, 27, 46]) have shown their high flexibility and accuracy in approximating reliably solutions to partial differential equations, even solutions with complex structures or discontinuities, and in anisotropic or heterogeneous media. The application of DG schemes for the space discretization of (1.2) and the definition of the DG counterpart of $\boldsymbol{A}$ in (1.2)

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to preserve skew-selfadjointness represent the key innovation of this work over a series of previous ones [8, 32, 33, 34] based on Picard's theory. For the DG space discretization, the definition of the distribution gradient and divergence operator is extended to broken polynomial spaces by penalizing the jumps of the unknowns over interelement surfaces. By still adding some boundary correction due to the nonconformity of DG methods, the skew-selfadjointness of $\boldsymbol{A}$ is passed on to its discrete counterpart $\boldsymbol{A}_h$. This consistent definition and treatment of the DG gradient and DG divergence operators for the nonconforming approximation is essential for the overall approach and its analysis. It has not been studied yet. Finally, some penalization term is incorporated into the discrete scheme to ensure that a weak form of the Dirichlet boundary condition (1.1d) is satisfied.

For the discretization in time we use the DG method [50]. Variational time discretizations offer the appreciable advantage of the natural construction of families of schemes with higher-order members, even for complex coupled systems of equations. There exists a strong link to Runge–Kutta methods; cf. [3, 4]. DG time discretizations are known to be strongly A-stable. For elastodynamics and wave propagation they violate the energy conservation principle of solutions to the continuous problem. This might evoke effects of damping or dispersion. However, the convergence of the jump terms at the discrete time nodes is ensured; cf. [33, Theorem 2.3]. Continuous-in-time Galerkin (CG) methods (cf., e.g., [2, 6, 11, 12, 32] and the references therein) are known to be A-stable only, but they preserve the energy of solutions [12, Section 6]. These families are more difficult to analyze since they lead to Galerkin–Petrov methods with trial and test spaces differing from each other. For this reason and due to computational advantages gained for simulations of the second-order form (1.1), DG time discretizations are studied here. For studies of CG schemes we refer to, e.g., [6, 11, 12, 28, 32, 38] and the references therein. For a numerical study of DG and CG time discretizations of (1.1) we refer to [7].

In [34] and [32], one of the authors of this work studies with his coauthors numerical schemes based on DG and CG Galerkin methods in time and conforming Galerkin methods in space for evolutionary problems (1.2) of changing type. By decomposing Ω into three disjoint sets and defining $\boldsymbol{M}_0$ and $\boldsymbol{M}_1$ setwise, the system (1.2) degenerates to elliptic, parabolic, or hyperbolic type on these sets. Usually, degenerating problems are difficult to analyze. Due to the weak assumptions about the operators made in the theory of Picard [42], such type of problems can be embedded into this framework. The same applies to the concept of perfectly matched layers in wave propagation; cf., e.g., [13, 23]. They are used to truncate the entire space $\mathbb{R}^d$ or unbounded domains to bounded computational ones and mimic non-reflecting boundary conditions. The analysis of wave propagation with artificial absorbing layers and changing equations in either regions becomes feasible as well by the abstract solution theory.

In [24], space-time DG methods for weak solutions of hyperbolic linear first-order symmetric Friedrichs systems describing acoustic, elastic, or electromagnetic waves are proposed. For an introduction into the theory of first-order symmetric Friedrichs systems, we refer to [29, 30, 31] and [26, Chapter 7]. Similarly to this work, in [24] a first-order in space and time formulation of a second-order hyperbolic problem is used. In contrast to this work, no coupled system of mixed hyperbolic-parabolic type is considered there. In [24], the mathematical tools for proving well-posedness of the space-time DG discretization and error estimates are based on the theory of first-order Friedrichs systems. The theory strongly differs from Picard's theorem [42] that is used here. The differences of either approaches still require elucidation. In deriving space-time DG methods and proving error estimates, differences become apparent in the norms with respect to which convergence is proved. In [24], stability and convergence estimates are provided with respect to a mesh-dependent DG norm that includes the L^2-norm at the final time; cf. also [9].

Here, convergence of the fully discrete approximation $\boldsymbol{U}_{\tau,h}$ of (1.2) is proved in Theorem 4.1 with respect to the natural and induced norm of the exponentially weighted Bochner space $H_\nu(\mathbb{R};\boldsymbol{H})$, with the product space $\boldsymbol{H} = L^2(\Omega)^{(d+1)^2}$ equipped with the $\boldsymbol{L}^2$-norm. For the full discretization $\boldsymbol{U}_{\tau,h}$ of the solution $\boldsymbol{U}$ to (1.2), we show that

$$\sup_{t\in[0,T]} \langle \boldsymbol{M}_0(\boldsymbol{U}-\boldsymbol{U}_{\tau,h}))(t), (\boldsymbol{U}-\boldsymbol{U}_{\tau,h}))(t)\rangle_{\boldsymbol{H}} + \|\boldsymbol{U}-\boldsymbol{U}_{\tau,h}\|_\nu^2 \leq C(\tau^{2(k+1)} + h^{2r}), \tag{1.3}$$

where $\|\cdot\|_\nu$ is the exponentially weighted natural norm associated with $H_\nu(\mathbb{R};\boldsymbol{H})$. Further, k and r denote the piecewise polynomial degrees in time and space, respectively.

The paper is organized as follows. In Section 2 the evolutionary form (1.2) of (1.1) is derived, and its well-posedness is shown. The space-time discretization of (1.2) by the DG method is presented in Section 3. Its error analysis is addressed in Section 4. In Section 5, we illustrate our error estimate by numerical experiments for the scalar wave equation, studied for brevity and simplicity. In Section 6, we end with a summary and outlook.

2. Evolutionary formulation and its well-posedness. In this section we formally rewrite the coupled hyperbolic-parabolic problem (1.1) as an evolutionary problem (1.2) by introducing auxiliary variables. For the evolutionary problem we present a result of well-posedness that is based on Picard's theorem; cf. [42] and [48, Theorem 6.2.1]. Therein, the evolutionary problem is investigated on the whole time axis, for $t \in \mathbb{R}$, in the exponentially weighted Bochner space $H_\nu(\mathbb{R};H)$ introduced in Definition 2.1. Throughout, we use the usual notation for standard Sobolev spaces. Vector- and tensor-valued functions and their spaces are written in bold.

DEFINITION 2.1. *Let $\boldsymbol{H}$ be a real Hilbert space with associated norm $\|\cdot\|_{\boldsymbol{H}}$. For $\nu > 0$, we set*

$$H_\nu(\mathbb{R};\boldsymbol{H}) := \left\{ \boldsymbol{f} : \mathbb{R} \to \boldsymbol{H} : \int_{\mathbb{R}} \|\boldsymbol{f}(t)\|_{\boldsymbol{H}}^2 \, \mathrm{e}^{-2\nu t} \, \mathrm{d}t \right\}. \tag{2.1}$$

The space $H_\nu(\mathbb{R};\boldsymbol{H})$, equipped with the inner product

$$\langle \boldsymbol{f}, \boldsymbol{g}\rangle_\nu := \int_{\mathbb{R}} \langle \boldsymbol{f}(t), \boldsymbol{g}(t)\rangle_{\boldsymbol{H}} \, \mathrm{e}^{-2\nu t} \, \mathrm{d}t, \qquad \text{for } \boldsymbol{f},\boldsymbol{g} \in H_\nu(\mathbb{R};\boldsymbol{H}), \tag{2.2}$$

is a Hilbert space. The norm induced by the inner product (2.2) is denoted by $\|\cdot\|_\nu$. Moreover, we define ∂_t to be the closure of the operator

$$\begin{aligned} \partial_t : C_c^\infty(\mathbb{R};\boldsymbol{H}) \subset H_\nu(\mathbb{R};\boldsymbol{H}) &\to H_\nu(\mathbb{R};\boldsymbol{H}), \\ \boldsymbol{\phi} &\to \boldsymbol{\phi}', \end{aligned}$$

where $C_c^\infty(\mathbb{R};\boldsymbol{H})$ is the space of infinitely differentiable $\boldsymbol{H}$-valued functions on $\mathbb{R}$ with compact support. The domain of the time-derivative of order s, denoted by ∂_t^s, is the space $H_\nu^s(\mathbb{R};\boldsymbol{H})$. Before rewriting (1.1) in the form (1.2), we need to introduce differential operators with respect to the spatial variables.

DEFINITION 2.2. *Let $\Omega \subset \mathbb{R}^d$, for $d \in \mathbb{N}$, be an open non-empty set. Then we define*

$$L^2(\Omega)^{d\times d}_{\mathrm{sym}} := \left\{ (\phi_{ij})_{i,j=1,\dots,d} \in L^2(\Omega)^{d\times d} : \phi_{ij} = \phi_{ji} \;\; \forall\, i,j \in \{1,\dots,d\} \right\}.$$

DEFINITION 2.3. *Let $\Omega \subset \mathbb{R}^d$, for $d \in \mathbb{N}$, be an open non-empty set. We let*

$$\begin{aligned} \mathrm{grad}_0 : H_0^1(\Omega) \subset L^2(\Omega) &\to L^2(\Omega)^d, \\ \phi &\to (\partial_j \phi)_{j=1,\dots,d}, \end{aligned}$$

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and

$$\begin{aligned} \operatorname{Grad}_0 : H_0^1(\Omega)^d \subset L^2(\Omega)^d &\to L^2(\Omega)^{d\times d}_{\mathrm{sym}}\,, \\ (\phi_j)_{j=1,\dots,d} &\to \frac{1}{2}(\partial_l \phi_j + \partial_j \phi_l)_{j,l=1,\dots,d}\,. \end{aligned} \tag{2.3}$$

Moreover, we let

$$\begin{aligned} \operatorname{div} : D(\operatorname{div}) \subset L^2(\Omega)^d &\to L^2(\Omega)\,, \\ \operatorname{div} &:= -(\operatorname{grad}_0)^*\,, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \operatorname{Div} : D(\operatorname{Div}) \subset L^2(\Omega)^{d\times d}_{\mathrm{sym}} &\to L^2(\Omega)^d\,, \\ \operatorname{Div} &:= -(\operatorname{Grad}_0)^*\,. \end{aligned} \tag{2.5}$$

We note that $\operatorname{Grad}_0 \boldsymbol{u} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ for $\boldsymbol{u} \in H_0^1(\Omega)^d$. The operator div in (2.4) assigns each $\boldsymbol{L}^2$-vector field its distributional divergence with maximal domain, that is,

$$D(\operatorname{div}) = \left\{ \boldsymbol{\phi} \in L^2(\Omega)^d : \sum_{i=1}^{d} \partial_i \phi_i \in L^2(\Omega) \right\}.$$

Similarly, the operator Div in (2.5) assigns each $L^2(\Omega)^{d\times d}_{\mathrm{sym}}$ tensor field its distributional divergence with maximal domain, that is,

$$D(\operatorname{Div}) = \left\{ \boldsymbol{\phi} \in L^2(\Omega)^{d\times d}_{\mathrm{sym}} : \left(\sum_{i=1}^{d} \partial_i \phi_{ij} \right)_{j=1,\dots,d} \in L^2(\Omega)^d \right\}.$$

To rewrite (1.1) formally as a first-order evolutionary problem, we introduce the set of new unknowns

$$\boldsymbol{v} := \partial_t \boldsymbol{u}\,, \qquad \boldsymbol{\sigma} := \boldsymbol{C}\boldsymbol{\varepsilon}, \qquad \text{and} \qquad \boldsymbol{q} := -\boldsymbol{K}\nabla p\,. \tag{2.6}$$

Using (2.6) and differentiating the second definition in (2.6) with respect to the time variable, we recast (1.1a) and (1.1b) as the first-order-in-space-and-time system

$$\rho \partial_t \boldsymbol{v} - \operatorname{Div} \boldsymbol{\sigma} + \alpha \operatorname{grad}_0 p = \rho \boldsymbol{f}\,, \tag{2.7a}$$

$$\boldsymbol{S}\partial_t \boldsymbol{\sigma} - \operatorname{Grad}_0 \boldsymbol{v} = \boldsymbol{0}\,, \tag{2.7b}$$

$$c_0 \partial_t p + \alpha \operatorname{div} \boldsymbol{v} + \operatorname{div} \boldsymbol{q} = g\,, \tag{2.7c}$$

$$\boldsymbol{K}^{-1}\boldsymbol{q} + \operatorname{grad}_0 p = \boldsymbol{0}\,, \tag{2.7d}$$

where $\boldsymbol{S}$ denotes the positive definite, fourth-order compliance tensor of the inverse stress-strain relation of Hook's law of linear elasticity,

$$\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}\,.$$

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In matrix-vector notation the system (2.7) reads as

$$\begin{aligned}&\left(\partial_t \begin{bmatrix}\rho & \mathbf{0} & 0 & \mathbf{0}\\ 0 & \boldsymbol{S} & 0 & \mathbf{0}\\ 0 & \mathbf{0} & c_0 & \mathbf{0}\\ 0 & \mathbf{0} & 0 & \mathbf{0}\end{bmatrix} + \begin{bmatrix}0 & \mathbf{0} & 0 & \mathbf{0}\\ 0 & \mathbf{0} & 0 & \mathbf{0}\\ 0 & \mathbf{0} & 0 & \mathbf{0}\\ 0 & \mathbf{0} & 0 & \boldsymbol{K}^{-1}\end{bmatrix}\right.\\ &\qquad \left. + \begin{bmatrix}0 & -\operatorname{Div} & \alpha\,\operatorname{grad}_0 & 0\\ -\operatorname{Grad}_0 & 0 & 0 & 0\\ \alpha\,\operatorname{div} & 0 & 0 & \operatorname{div}\\ 0 & 0 & \operatorname{grad}_0 & 0\end{bmatrix}\right) \begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\sigma}\\ \boldsymbol{p}\\ \boldsymbol{q}\end{bmatrix} = \begin{bmatrix}\rho\boldsymbol{f}\\ \mathbf{0}\\ g\\ \mathbf{0}\end{bmatrix}.\end{aligned} \tag{2.8}$$

To further simplify the spatial differential operator in (2.8), we introduce the total flux variable

$$\overline{\boldsymbol{q}} := \boldsymbol{q} + \alpha\boldsymbol{v}$$

and then recast (2.8) as the evolutionary problem

$$\begin{aligned}&\left(\partial_t \begin{bmatrix}\rho & \mathbf{0} & 0 & \mathbf{0}\\ 0 & \boldsymbol{S} & 0 & \mathbf{0}\\ 0 & \mathbf{0} & c_0 & \mathbf{0}\\ 0 & \mathbf{0} & 0 & \mathbf{0}\end{bmatrix} + \begin{bmatrix}-\alpha^2\boldsymbol{K}^{-1} & \mathbf{0} & 0 & -\alpha\boldsymbol{K}^{-1}\\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0}\\ -\alpha\boldsymbol{K}^{-1} & \mathbf{0} & 0 & \boldsymbol{K}^{-1}\end{bmatrix}\right.\\ &\qquad \left. + \begin{bmatrix}0 & -\operatorname{Div} & 0 & 0\\ -\operatorname{Grad}_0 & 0 & 0 & 0\\ 0 & 0 & 0 & \operatorname{div}\\ 0 & 0 & \operatorname{grad}_0 & 0\end{bmatrix}\right) \begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\sigma}\\ p\\ \overline{\boldsymbol{q}}\end{bmatrix} = \begin{bmatrix}\rho\boldsymbol{f}\\ \mathbf{0}\\ g\\ \mathbf{0}\end{bmatrix}.\end{aligned}$$

Finally, we put

$$\boldsymbol{U} := (\boldsymbol{v}, \boldsymbol{\sigma}, p, \overline{\boldsymbol{q}})^\top \quad \text{and} \quad \boldsymbol{F} := (\rho\boldsymbol{f}, \mathbf{0}, g, \mathbf{0})^\top. \tag{2.9}$$

We define the operators

$$\begin{aligned}\boldsymbol{M}_0 &:= \begin{bmatrix}\rho & \mathbf{0} & 0 & \mathbf{0}\\ 0 & \boldsymbol{S} & 0 & \mathbf{0}\\ 0 & \mathbf{0} & c_0 & \mathbf{0}\\ 0 & \mathbf{0} & 0 & \mathbf{0}\end{bmatrix}, \qquad M_1 := \begin{bmatrix}-\alpha^2\boldsymbol{K}^{-1} & 0 & 0 & -\alpha\boldsymbol{K}^{-1}\\ \mathbf{0} & 0 & 0 & \mathbf{0}\\ \mathbf{0} & 0 & 0 & \mathbf{0}\\ -\alpha\boldsymbol{K}^{-1} & 0 & 0 & \boldsymbol{K}^{-1}\end{bmatrix},\\ \boldsymbol{A} &:= \begin{bmatrix}0 & -\operatorname{Div} & 0 & 0\\ -\operatorname{Grad}_0 & 0 & 0 & 0\\ 0 & 0 & 0 & \operatorname{div}\\ 0 & 0 & \operatorname{grad}_0 & 0\end{bmatrix}.\end{aligned} \tag{2.10}$$

Then we obtain the following evolutionary problem:

PROBLEM 2.4 (Evolutionary problem). *Let $\boldsymbol{H}$ denote the product space*

$$\boldsymbol{H} := L^2(\Omega)^d \times L^2(\Omega)^{d\times d}_{\text{sym}} \times L^2(\Omega) \times L^2(\Omega)^d, \tag{2.11}$$

equipped with the $\boldsymbol{L}^2$-inner product of $L^2(\Omega)^{(d+1)^2}$. Let $\boldsymbol{M}_0$, $\boldsymbol{M}_1 : \boldsymbol{H} \to \boldsymbol{H}$, and $\boldsymbol{A} : D(\boldsymbol{A}) \subset \boldsymbol{H} \to \boldsymbol{H}$, with

$$D(\boldsymbol{A}) := H^1_0(\Omega)^d \times D(\operatorname{Div}) \times H^1_0(\Omega) \times D(\operatorname{div}), \tag{2.12}$$

be defined by (2.10).

For a given $\boldsymbol{F} \in H_\nu(\mathbb{R}; \boldsymbol{H})$ according to (2.9), find $\boldsymbol{U} \in H_\nu(\mathbb{R}; \boldsymbol{H})$ such that

$$(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A})\boldsymbol{U} = \boldsymbol{F}\,, \tag{2.13}$$

where $\boldsymbol{U}$ is defined by (2.9) along with (2.6).

Well-posedness of (2.13) is ensured by the following abstract result; cf. [42] and [48, Theorem 6.2.1]:

THEOREM 2.5 (Well-posedness). *Let $\boldsymbol{H}$ denote a real Hilbert space. Let $\boldsymbol{M}_0$, $\boldsymbol{M}_1 : \boldsymbol{H} \to \boldsymbol{H}$ be bounded linear selfadjoint operators and $\boldsymbol{A} : D(\boldsymbol{A}) \subset \boldsymbol{H} \to \boldsymbol{H}$ skew-selfadjoint. Moreover, suppose that there exists some $\nu_0 > 0$ such that*

$$\exists\, \gamma > 0\ \forall \nu \geq \nu_0\,,\ \boldsymbol{x} \in \boldsymbol{H} : \quad \langle (\nu \boldsymbol{M}_0 + \boldsymbol{M}_1)\boldsymbol{x}, \boldsymbol{x}\rangle_{\boldsymbol{H}} \geq \gamma \langle \boldsymbol{x}, \boldsymbol{x}\rangle_{\boldsymbol{H}}\,. \tag{2.14}$$

Then, for each $\nu \geq \nu_0$ and each $\boldsymbol{F} \in H_\nu(\mathbb{R}; \boldsymbol{H})$, there exist a unique solution $\boldsymbol{U} \in H_\nu(\mathbb{R}; \boldsymbol{H})$ such that

$$\overline{(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A})}\boldsymbol{U} = \boldsymbol{F}\,, \tag{2.15}$$

where the closure is taken in $H_\nu(\mathbb{R}; \boldsymbol{H})$. Moreover, there holds the stability estimate

$$\|\boldsymbol{U}\|_\nu \leq \frac{1}{\gamma}\|\boldsymbol{F}\|_\nu\,. \tag{2.16}$$

If $\boldsymbol{F} \in H^s_\nu(\mathbb{R}; \boldsymbol{H})$ for some $s \in \mathbb{N}$, then the inclusion $\boldsymbol{U} \in H^s_\nu(\mathbb{R}; \boldsymbol{H})$ is satisfied, and the evolutionary equation is satisfied properly such that

$$(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A})\boldsymbol{U} = \boldsymbol{F}\,.$$

COROLLARY 2.6 (Well-posedness of Problem 2.4). *Problem 2.4 is well-posed. In particular, there exists a unique solution $\boldsymbol{U} \in H_\nu(\mathbb{R}; \boldsymbol{H})$ in the sense of (2.15).*

Proof. For Problem 2.4, the assumptions of Theorem 2.5 are fulfilled due to the conditions that the constants ρ, α, and c_0 in (2.10) are strictly positive and the compliance tensor $\boldsymbol{S}$ and the matrix $\boldsymbol{K}$ are symmetric and positive definite. The skew-selfadjointness of $\boldsymbol{A}$ directly follows from (2.4) and (2.5), respectively. Therefore, Theorem 2.5 proves the assertion of this corollary. ∎

REMARK 2.7.

- By a version of the Sobolev embedding theorem (cf. [43, Lemma 3.1.59]), we have that

$$H^1_\nu(\mathbb{R}; \boldsymbol{H}) \hookrightarrow C_\nu(\mathbb{R}; \boldsymbol{H})\,, \tag{2.17}$$

where

$$C_\nu(\mathbb{R}; \boldsymbol{H}) := \left\{ \boldsymbol{f} : \mathbb{R} \to \boldsymbol{H} : \boldsymbol{f} \text{ is continuous}\,,\ \sup_{t\in\mathbb{R}} \|\boldsymbol{f}(t)\|_{\boldsymbol{H}}\, \mathrm{e}^{-\nu t} < \infty \right\}.$$

- For $\boldsymbol{F} \in H^1_\nu(\mathbb{R}; \boldsymbol{H})$ there holds that $\boldsymbol{U} \in H^1_\nu(\mathbb{R}; \boldsymbol{H})$ and, consequently, that

$$\boldsymbol{A}\boldsymbol{U} = \boldsymbol{F} - \partial_t \boldsymbol{M}_0 \boldsymbol{U} - \boldsymbol{M}_1 \boldsymbol{U} \in H_\nu(\mathbb{R}; \boldsymbol{H})\,. \tag{2.18}$$

Therefore, we have that $\boldsymbol{U}(t) \in D(\boldsymbol{A})$ for almost every $t \in \mathbb{R}$. Moreover, for $\boldsymbol{F} \in H^2_\nu(\mathbb{R}; \boldsymbol{H})$ it follows that $\boldsymbol{U} \in H^2_\nu(\mathbb{R}; \boldsymbol{H})$, and, taking the time-derivative of the equation in (2.18), that $\boldsymbol{U} \in H^1_\nu(\mathbb{R}; D(\boldsymbol{A}))$. By the embedding result (2.17), it then follows that $\boldsymbol{U} \in C_\nu(\mathbb{R}; D(\boldsymbol{A}))$.

- The condition (2.14) of positive definiteness is assumed to hold uniformly in $\nu \geq \nu_0$. This ensures the causality of the solution operator $\boldsymbol{S}_\nu := \overline{(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A})}^{-1}$ to (2.15); cf. [48, Theorem 6.2.1]. Further, it holds that $\boldsymbol{S}_\nu \boldsymbol{F} = \boldsymbol{S}_\eta \boldsymbol{F}$, for $\nu, \eta \geq \nu_0$ and $\boldsymbol{F} \in H_\nu(\mathbb{R}; \boldsymbol{H}) \cap H_\eta(\mathbb{R}; \boldsymbol{H})$.
- Initial value problems for (2.13) are studied by a generalization of the solution theory to certain distributional right-hand sides; cf. [48, Theorem 9.4.3] and [43, Theorem 6.2.9]. Let $\boldsymbol{F} \in H_\nu(\mathbb{R}; \boldsymbol{H})$ be supported on $[t_0, \infty)$, for some $t_0 \in \mathbb{R}$, and $\boldsymbol{U}_0 \in D(\boldsymbol{A})$ be given. Then, the evolutionary equation

$$(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A})\boldsymbol{U} = \boldsymbol{F} + \delta_{t_0} \boldsymbol{M}_0 \boldsymbol{U}_0 \tag{2.19}$$

has a unique solution $\boldsymbol{U} \in H_\nu^{-1}(\mathbb{R}; \boldsymbol{H})$ satisfying $\boldsymbol{M}\boldsymbol{U}(t_0^+) = \boldsymbol{M}_0 \boldsymbol{U}_0$ in $H^{-1}(D(\boldsymbol{A}))$, where δ_{t_0} denotes the delta distribution at $t = t_0$. For $t \in (0, \infty)$, the evolutionary equation $(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A})\boldsymbol{U} = \boldsymbol{F}$ is satisfied in the sense of distributions for test functions $\boldsymbol{\varphi} \in \boldsymbol{H}_\nu^1(\mathbb{R}; \boldsymbol{H}) \cap H_\nu(\mathbb{R}; D(\boldsymbol{A}))$ supported on $[t_0, \infty)$. The distribution on the right-hand side of (2.19) can still be avoided by reformulating the initial value problem for (2.5) into the evolutionary equation

$$(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A})\boldsymbol{W} = \partial_t^{-1} \boldsymbol{F} + H_{t_0} \boldsymbol{U}_0$$

for $\boldsymbol{W} := \partial_t^{-1} \boldsymbol{U}$, where H_{t_0} denotes the Heaviside function with jump in $t = t_0$; cf. [32, Corollary 1.1] and [43, p. 446]. Then, it follows that

$$\boldsymbol{U} = \partial_t \overline{(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A})}^{-1} \partial_t^{-1} (\boldsymbol{F} + \delta_{t_0} \boldsymbol{U}_0)\,.$$

3. DG discretization and well-posedness. Here we derive a family of fully discrete schemes for Problem 2.4. Space and time discretization are based on DG approaches. Well-posedness of the discrete problem is shown. We assume that the weight ν in (2.1) is chosen such that the assumptions of Theorem 2.5 are satisfied.

3.1. Notation and auxiliaries. For the time discretization, we decompose $I = (0, T]$ into N subintervals $I_n = (t_{n-1}, t_n]$, $n = 1, \dots, N$, with $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$, such that $I = \bigcup_{n=1}^N I_n$. We put $\tau := \max_{n=1,\dots,N} \tau_n$, with $\tau_n = t_n - t_{n-1}$. Further, the set $\mathcal{M}_\tau := \{I_1, \dots, I_N\}$ is called the time mesh. For any $k \in \mathbb{N}_0$ and some Banach space $\boldsymbol{B}$, we let

$$\mathbb{P}_k(I_n; \boldsymbol{B}) := \left\{ \boldsymbol{w}_\tau \,:\, I_n \to \boldsymbol{B} \,:\, \boldsymbol{w}_\tau(t) = \sum_{j=0}^k \boldsymbol{W}^j t^j \;\forall t \in I_n\,,\; \boldsymbol{W}^j \in \boldsymbol{B}\;\forall j \right\}$$

denote the space of $\boldsymbol{B}$-valued polynomials of degree at most k defined on I_n. For a Hilbert space $\boldsymbol{H}$, the space $\mathbb{P}_k(I_n; \boldsymbol{H})$, equipped with the exponentially weighted inner product

$$\langle \boldsymbol{v}_\tau, \boldsymbol{w}_\tau \rangle_{\nu,n} := \int_{t_{n-1}}^{t_n} \langle \boldsymbol{v}_\tau(t), \boldsymbol{w}_\tau(t) \rangle_{\boldsymbol{H}}\, \mathrm{e}^{-2\nu(t - t_{n-1})}\, \mathrm{d}t\,, \tag{3.1}$$

is a Hilbert space. The semidiscretization in time of (2.13) by Galerkin methods is done in

$$\boldsymbol{Y}_{\tau,\nu}^k(\boldsymbol{B}) := \left\{ \boldsymbol{w}_\tau \in H_\nu(0, T; \boldsymbol{B}) : \boldsymbol{w}_{\tau|I_n} \in \mathbb{P}_k(I_n; \boldsymbol{B}), \forall I_n \in \mathcal{M}_\tau, \boldsymbol{w}_\tau(0) \in \boldsymbol{B} \right\}. \tag{3.2}$$

For any function $\boldsymbol{w} : \overline{I} \to \boldsymbol{B}$ that is piecewise sufficiently smooth with respect to the time mesh $\mathcal{M}_\tau$, for instance for $\boldsymbol{w} \in Y_\tau^k(\boldsymbol{B})$, we define the right-hand sided and left-hand sided limit at a mesh point t_n by

$$\begin{aligned} \boldsymbol{w}^+(t_n) &:= \lim_{t\to t_n+0} \boldsymbol{w}(t), \quad \text{for } n < N\,, \\ \boldsymbol{w}^-(t_n) &:= \lim_{t\to t_n-0} \boldsymbol{w}(t), \quad \text{for } n > 0\,. \end{aligned} \tag{3.3}$$

In the discrete scheme, a quadrature formula is applied for the evaluation of the time integrals. For the discontinuous-in-time finite element method, a natural choice is to consider the $(k+1)$-point right-sided Gauss–Radau quadrature formula on each time interval $I_n = (t_{n-1}, t_n]$. Here, we use a modification of the standard right-sided Gauss–Radau quadrature formula that is defined by

$$Q_{n,\nu}(w) := \frac{\tau_n}{2} \sum_{\mu=0}^{k} \hat{\omega}_\mu w(t_{n,\mu}) \approx \int_{I_n} \mathrm{e}^{-2\nu(t-t_{n-1})}\, w(t)\,\mathrm{d}t\,, \tag{3.4}$$

where $t_{n,\mu} = T_n(\hat{t}_\mu)$, for $\mu = 0,\dots,k$, are the quadrature points on I_n and $\hat{\omega}_\mu$ the corresponding weights. Here, $T_n(\hat{t}\,) := (t_{n-1}+t_n)/2 + (\tau_n/2)\hat{t}$ is the affine transformation from the reference interval $\hat{I} = (-1,1]$ to I_n, and $\hat{t}_\mu$, for $\mu = 0,\dots,k$, are the quadrature points of the weighted Gauss–Radau formula on $\hat{I}$ (cf. [44]) such that for all polynomials $p \in \mathbb{P}_{2k}(\hat{I};\mathbb{R})$ there holds

$$\int_{\hat{I}} \mathrm{e}^{-\nu\tau_n(\hat{t}+1)}\, p(\hat{t}\,)\,\mathrm{d}\hat{t} = \sum_{\mu=0}^{k} \hat{\omega}_\mu p(\hat{t}_\mu)\,.$$

Then, for polynomials $p \in \mathbb{P}_{2k}(I_n;\mathbb{R})$ we have that

$$Q_{n,\nu}(p) = \int_{I_n} \mathrm{e}^{-2\nu(t-t_{n-1})}\, p(t)\,\mathrm{d}t\,. \tag{3.5}$$

Finally, we introduce the time-mesh dependent quantities

$$Q_n[w]_\nu := Q_{n,\nu}(w)\,, \qquad Q_n[\boldsymbol{v},\boldsymbol{w}]_\nu := Q_{n,\nu}(\langle \boldsymbol{v},\boldsymbol{w}\rangle_{\boldsymbol{H}})\,, \tag{3.6a}$$

$$|w|^2_{\tau,\nu,n} := Q_n[w]_\nu\,, \qquad |w|^2_{\tau,\nu} := \sum_{n=1}^{N} Q_n[w]_\nu\,\mathrm{e}^{-2\nu t_{n-1}}\,, \tag{3.6b}$$

$$\|\boldsymbol{w}\|^2_{\tau,\nu,n} := Q_n[\boldsymbol{w},\boldsymbol{w}]_\nu\,, \qquad \|\boldsymbol{w}\|^2_{\tau,\nu} := \sum_{n=1}^{N} Q_n[\boldsymbol{w},\boldsymbol{w}]_\nu\,\mathrm{e}^{-2\nu t_{n-1}}\,, \tag{3.6c}$$

where the nonnegativity of w is assumed in (3.6b). Throughout, this will be satisfied below.

For the nodes $t_{n,\mu} \in (t_{n-1}, t_n]$, for $\mu = 0,\dots,k$ and $n = 1,\dots,N$, of the weighted Gauss–Radau formula (3.4), we define the global Lagrange interpolation operator $I_\tau : C([0,T];\boldsymbol{B}) \to Y_\tau^k(\boldsymbol{B})$ by

$$I_\tau \boldsymbol{f}(0) = \boldsymbol{f}(0)\,, \qquad I_\tau \boldsymbol{f}(t_{n,\mu}) = \boldsymbol{f}(t_{n,\mu})\,, \quad \mu = 0,\dots,k\,,\ n = 1,\dots,N\,. \tag{3.7}$$

For the Lagrange interpolation (3.7), on each I_n there holds that (cf. [36, Theorem 1])

$$\|\boldsymbol{w} - I_\tau \boldsymbol{w}\|_{C(\overline{I}_n;\boldsymbol{B})} \leq c\tau_n^{k+1} \|\partial_t^{k+1} w\|_{C(\overline{I}_n;\boldsymbol{B})}\,. \tag{3.8}$$

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Moreover, we need the Lagrange interpolation operator $\widehat{I}_\tau^{k+1} : C([0,T];\boldsymbol{B}) \to X_\tau^{k+1}(\boldsymbol{B})$ with respect to the Gauss–Radau quadrature points $t_{n,\mu}$, for $\mu = 0,\dots,k$, and t_{n-1}, for $n = 1,\dots,N$, which is defined by

$$\begin{aligned} \widehat{I}_\tau^{k+1}\boldsymbol{f}(t_{n,\mu}) &= \boldsymbol{f}(t_{n,\mu})\,, \quad \text{for } \mu = 0,\dots,k\,, \\ \widehat{I}_\tau^{k+1}\boldsymbol{f}(t_{n-1}) &= \boldsymbol{f}(t_{n-1})\,, \quad \text{for } n = 1,\dots,N\,. \end{aligned} \tag{3.9}$$

Then, for $\widehat{I}_\tau^{k+1}$ there holds that (cf. [36, Theorem 2])

$$\|\partial_t^s(\boldsymbol{w} - \widehat{I}_\tau^{k+1}\boldsymbol{w})\|_{C(\overline{I}_n;\boldsymbol{B})} \le c\tau_n^{k+2-s}\|\partial_t^{k+2}w\|_{C(\overline{I}_n;\boldsymbol{B})}\,, \qquad \text{for } s \in \{0,1\}\,. \tag{3.10}$$

For the space discretization, let the mesh $\mathcal{T}_h = \{K\}$ denote a decomposition of the polyhedron Ω into quadrilateral or hexahedral elements K with mesh size $h = \max\{h_K : K \in \mathcal{T}_h\}$, for $h_K := \operatorname{diam}(K)$. The mesh is assumed to be conforming (matching) and shape-regular; cf., e.g., [46]. The assumptions about $\mathcal{T}_h$ are sufficient to derive inverse and trace inequalities; cf. [26, Chapter 1]. Further, optimal polynomial approximation properties in the sense of [26, Definition 1.55] are satisfied; cf., e.g., [46, Theorem 2.6]. Simplicial triangulations can be considered analogously. For more general mesh concepts in the context of DG methods, we refer to [26, Section 1.4] or [27, Section 2.3.2].

For any $K \in \mathcal{T}_h$ we denote by n_K the outward unit normal to the faces (edges for $d = 2$) of K. Further, we let $\mathcal{E}_h$ be the union of the boundaries of all elements of $\mathcal{T}_h$. Let $\mathcal{E}_h^i = \mathcal{E}_h \backslash \partial\Omega$ be the set of interior faces (edges if $d = 2$) and $\mathcal{E}_h^\partial = \mathcal{E}_h \backslash \mathcal{E}_h^i$ denote the union of all boundary faces.

For any $r \in \mathbb{N}$, the space of continuous piecewise polynomial functions is denoted by

$$X_h^r := \{w_h \in C(\overline{\Omega}) \mid w_{h|K} \in W_r(K) \ \ \forall K \in \mathcal{T}_h\} \cap H_0^1(\Omega)\,,$$

where the local space $W_r(K)$ is defined by mapped versions of $\mathbb{Q}_r$; cf. [45, Section 3.2]. For any $r \in \mathbb{N}_0$, we denote the space of broken polynomials by

$$Y_h^r := \{w_h \in L^2(\Omega) \mid w_{h|K} \in W_r(K) \ \ \forall K \in \mathcal{T}_h\}\,.$$

For the spatial approximation of Problem 2.4 we consider using

$$\boldsymbol{H}_h \in \{\boldsymbol{H}_h^{\text{hy}}, \boldsymbol{H}_h^{\text{dg}}\}\,, \qquad \text{with } \boldsymbol{H}_h \subset \boldsymbol{H}\,, \tag{3.11}$$

where the finite element product spaces $\boldsymbol{H}_h^{\text{hy}}$ and $\boldsymbol{H}_h^{\text{dg}}$ are given by

$$\boldsymbol{H}_h^{\text{hy}} := (X_h^r)^d \times \big((Y_h^r)^{d\times d} \cap L^2(\Omega)_{\text{sym}}^{d\times d}\big) \times X_h^r \times (Y_h^r)^d\,, \tag{3.12a}$$

$$\boldsymbol{H}_h^{\text{dg}} := (Y_h^r)^d \times \big((Y_h^r)^{d\times d} \cap L^2(\Omega)_{\text{sym}}^{d\times d}\big) \times Y_h^r \times (Y_h^r)^d\,. \tag{3.12b}$$

Discretizations of Problem 2.4 in either spaces, $\boldsymbol{H}_h^{\text{hy}}$ and $\boldsymbol{H}_h^{\text{dg}}$, are studied simultaneously. The reason for considering also the hybrid space $\boldsymbol{H}_h^{\text{hy}}$ is that continuous and $\boldsymbol{H}_0^1(\Omega)$-conforming finite element methods lead to lower computational cost than discontinuous ones. $\boldsymbol{H}(\operatorname{div};\Omega)$-conforming approximations in the framework of Picard's theory have been studied in [34] for scalar-valued problems of changing type. These families of schemes can be applied analogously to the approximation of $\boldsymbol{\sigma}$ and $\overline{\boldsymbol{q}}$ in Problem 2.4. Since DG methods offer high flexibility combined with implementational advantages, DG methods are attractive and studied here.

In Section 4 we need the $\boldsymbol{L}^2$-orthogonal projection of functions $\boldsymbol{w} \in \boldsymbol{H}$ onto the broken polynomial space $\boldsymbol{H}_h^{\text{dg}}$ of (3.12b), which is very simple, even on more general meshes than studied here. For the $\boldsymbol{L}^2$-orthogonal projection $\Pi_h : \boldsymbol{H} \to \boldsymbol{H}_h^{\text{dg}}$, $\boldsymbol{v} \in \boldsymbol{H}$ and $\Pi_h \boldsymbol{v} \in \boldsymbol{H}_h^{\text{dg}}$, with

$$\langle \Pi_h \boldsymbol{v}, \boldsymbol{w}_h \rangle_{\boldsymbol{H}} = \langle \boldsymbol{v}, \boldsymbol{w}_h \rangle_{\boldsymbol{H}} \,, \quad \text{for all } \boldsymbol{w}_h \in \boldsymbol{H}_h^{\text{dg}} \,, \tag{3.13}$$

there holds for all $s \in \{0, \ldots, r+1\}$ and all $\boldsymbol{w} \in \boldsymbol{H}^s(K)$ that

$$|\boldsymbol{w} - \Pi_h \boldsymbol{w}|_{\boldsymbol{H}^m(K)} \leq C_{\text{app}} h_K^{s-m} |\boldsymbol{w}|_{\boldsymbol{H}^s(K)} \,, \quad \text{for } m \in \{0, \ldots, s\} \,, \tag{3.14}$$

where C_{app} is independent of both K and h; cf. [26, Lemma 1.58], [46, Theorem 2.6]. In (3.14), we denote by $|\cdot|_{\boldsymbol{H}^m(K)}$ the seminorm of the Sobolev space $\boldsymbol{H}^m(K)$. Also, the $\boldsymbol{L}^2$-orthogonal projection satisfies

$$\|\boldsymbol{w} - \Pi_h \boldsymbol{w}\|_{L^2(e)} \leq C'_{\text{app}} h_K^{s-1/2} |\boldsymbol{w}|_{\boldsymbol{H}^s(K)} \,, \qquad \text{for } s \geq 1 \,, \tag{3.15a}$$

$$\|\nabla(\boldsymbol{w} - \Pi_h \boldsymbol{w})_{|K} \cdot \boldsymbol{n}_e\|_{L^2(e)} \leq C''_{\text{app}} h_K^{s-3/2} |\boldsymbol{w}|_{\boldsymbol{H}^s(K)} \,, \qquad \text{for } s \geq 2 \,, \tag{3.15b}$$

where C'_{app} and C''_{app} are independent of both K and h and $\boldsymbol{n}_e$ is the unit outer normal vector to the face $e \in \partial K$; cf. [26, Lemma 1.59].

3.2. Gradient and divergence on broken function spaces. To define our DG discretization schemes we need to recall some general concepts for the definition of the gradient and divergence operator on broken function spaces with respect to the triangulation $\mathcal{T}_h$. For further details and concepts of broken function spaces, we refer to, e.g., [26]. On the triangulation $\mathcal{T}_h$, let $Y_h = Y_h(\mathcal{T}_h)$ and $\boldsymbol{Z}_h = \boldsymbol{Z}_h(\mathcal{T}_h)$ denote piecewise (broken) spaces of scalar- and vector-valued functions, respectively. On the set of (inner and outer) boundaries $\mathcal{E}_h$, let $\widehat{Y}_h = \widehat{Y}_h(\mathcal{E}_h)$ and $\widehat{\boldsymbol{Z}}_h = \widehat{\boldsymbol{Z}}_h(\mathcal{E}_h)$ be piecewise (broken) spaces of scalar- and vector-valued functions on $\mathcal{E}_h$, respectively. We put $\widetilde{Y}_h := Y_h \times \widehat{Y}_h$ and $\widetilde{\boldsymbol{Z}}_h := \boldsymbol{Z}_h \times \widehat{\boldsymbol{Z}}_h$. We denote the dual spaces of $\widetilde{Y}_h$ and $\widetilde{\boldsymbol{Z}}_h$ by $\widetilde{Y}_h^*$ and $\widetilde{\boldsymbol{Z}}_h^*$. In these spaces we define the following derivatives of the DG method; cf. [35]:

DEFINITION 3.1 (DG derivatives). *Let $\widetilde{y}_h = (y_h, \widehat{y}_h) \in \widetilde{Y}_h$ and $\widetilde{\boldsymbol{z}}_h = (\boldsymbol{z}_h, \widehat{\boldsymbol{z}}_h) \in \widetilde{\boldsymbol{Z}}_h$. The DG gradient* $\operatorname{grad}_{\text{dg}} : Y_h \to \widetilde{\boldsymbol{Z}}_h^*$ *and the DG divergence* $\operatorname{div}_{\text{dg}} : \boldsymbol{Z}_h \to \widetilde{Y}_h^*$ *are defined by*

$$\langle \operatorname{grad}_{\text{dg}} y_h, \widetilde{\boldsymbol{z}}_h \rangle := \langle \operatorname{grad}_h y_h, \boldsymbol{z}_h \rangle - \sum_{K \in \mathcal{T}_h} \langle y_h, \widehat{\boldsymbol{z}}_h \cdot \boldsymbol{n}_K \rangle_{\partial K} \,, \qquad \begin{array}{l} \forall y_h \in Y_h \,, \\ \forall \widetilde{\boldsymbol{z}}_h \in \widetilde{\boldsymbol{Z}}_h \,, \end{array} \tag{3.16a}$$

$$\langle \operatorname{div}_{\text{dg}} \boldsymbol{z}_h, \widetilde{y}_h \rangle := \langle \operatorname{div}_h \boldsymbol{z}_h, y_h \rangle - \sum_{K \in \mathcal{T}_h} \langle \boldsymbol{z}_h \cdot \boldsymbol{n}_K, \widehat{y}_h \rangle_{\partial K} \,, \qquad \begin{array}{l} \forall \boldsymbol{z}_h \in \boldsymbol{Z}_{\boldsymbol{h}} \,, \\ \forall \widetilde{y}_h \in \widetilde{Y}_h \,. \end{array} \tag{3.16b}$$

Here, $\langle \cdot, \cdot \rangle_S$ denotes the inner product of $L^2(S)$, where we drop the index S if $S = \Omega$. Further, grad_h and div_h are the broken gradient and divergence, respectively; cf. [26, Sections 1.2.5 and 1.2.6]. By $\boldsymbol{n}_K$ we denote the unit outer normal vector assigned to ∂K. We recall that on the usual Sobolev spaces, the broken gradient coincides with the distribution gradient; cf. [26, Lemma 1.22]. The same applies to the broken divergence; cf. [26, Section 1.2.6]. The dual operators of $\operatorname{grad}_{\text{dg}}$ and $\operatorname{div}_{\text{dg}}$ are denoted by $\operatorname{grad}_{\text{dg}}^* : \widetilde{\boldsymbol{Z}}_h \to Y_h^*$ and $\operatorname{div}_{\text{dg}} : \widetilde{Y}_h \to \boldsymbol{Z}_h^*$. Then, there holds that

$$\langle \operatorname{grad}_{\text{dg}}^* \widetilde{\boldsymbol{z}}_h, y_h \rangle = \langle \widetilde{\boldsymbol{z}}_h, \operatorname{grad}_{\text{dg}} y_h \rangle \,, \qquad \forall \widetilde{\boldsymbol{z}}_h \in \widetilde{\boldsymbol{Z}}_h \,, \forall y_h \in Y_h \,, \tag{3.17a}$$

$$\langle \operatorname{div}_{\text{dg}}^* \widetilde{y}_h, \boldsymbol{z}_h \rangle = \langle \widetilde{y}_h, \operatorname{div}_{\text{dg}} \boldsymbol{z}_h \rangle \,, \qquad \forall \widetilde{y}_h \in \widetilde{Y}_h \,, \forall \boldsymbol{z}_h \in \boldsymbol{Z}_h \,. \tag{3.17b}$$

The DG derivatives $\operatorname{grad}_{\mathrm{dg}}$ and $-\operatorname{div}_{\mathrm{dg}}$ are *conditionally dual* with each other; cf. [35]. To demonstrate this link, we deduce from (3.16) and (3.17) that

$$\begin{aligned}\langle -\operatorname{div}^*_{\mathrm{dg}}\, \widetilde{y}_h, \boldsymbol{z}_h\rangle &= -\langle \widetilde{y}_h, \operatorname{div}_{\mathrm{dg}} \boldsymbol{z}_h\rangle \\ &= \langle \operatorname{grad}_h y_h, \boldsymbol{z}_h\rangle + \sum_{K\in\mathcal{T}_h} \langle \widehat{y}_h - y_h, \boldsymbol{z}_h\cdot \boldsymbol{n}_K\rangle_{\partial K}\,,\end{aligned} \tag{3.18a}$$

$$\langle \operatorname{grad}_{\mathrm{dg}} y_h, \widetilde{\boldsymbol{z}}_h\rangle = \langle \operatorname{grad}_h y_h, \boldsymbol{z}_h\rangle - \sum_{K\in\mathcal{T}_h} \langle y_h, \widehat{\boldsymbol{z}}_h\cdot \boldsymbol{n}_K\rangle_{\partial K}, \tag{3.18b}$$

and

$$\begin{aligned}\langle \operatorname{grad}^*_{\mathrm{dg}}\, \widetilde{\boldsymbol{z}}_h, y_h\rangle &= \langle \widetilde{\boldsymbol{z}}_h, \operatorname{grad}_{\mathrm{dg}} y_h\rangle \\ &= -\langle \operatorname{div}_h \boldsymbol{z}_h, y_h\rangle + \sum_{K\in\mathcal{T}_h} \langle (\boldsymbol{z}_h - \widehat{\boldsymbol{z}}_h)\cdot \boldsymbol{n}_K, y_h\rangle_{\partial K}\,,\end{aligned} \tag{3.19a}$$

$$-\langle \operatorname{div}_{\mathrm{dg}} \boldsymbol{z}_h, \widetilde{y}_h\rangle = -\langle \operatorname{div}_h \boldsymbol{z}_h, y_h\rangle + \sum_{K\in\mathcal{T}_h} \langle \boldsymbol{z}_h\cdot \boldsymbol{n}_K, \widehat{y}_h\rangle_{\partial K}\,. \tag{3.19b}$$

The identities (3.18) and (3.19) directly imply the following conditional duality between $\operatorname{grad}_{\mathrm{dg}}$ and $\operatorname{div}_{\mathrm{dg}}$ under the assumption that $\widehat{y}_h = 0$ on $\mathcal{E}_h^\partial$; cf. also [35, Lemma 2.1]:

LEMMA 3.2. *Let $\widehat{y}_h = 0$ on $\mathcal{E}_h^\partial$. For the DG derivatives (3.16) there holds the duality*

$$\operatorname{div}_{\mathrm{dg}} = -(\operatorname{grad}_{\mathrm{dg}})^*\,,$$

if one of the following conditions is satisfied:

$$i)\quad \boldsymbol{z}_h\cdot \boldsymbol{n}_{K|\mathcal{E}_h} = \widehat{\boldsymbol{z}}_h\cdot \boldsymbol{n}_K\,; \tag{3.20a}$$

$$ii)\quad y_{h|\mathcal{E}_h} = \widehat{y}_h\,; \tag{3.20b}$$

$$iii)\quad \widehat{\boldsymbol{z}}_h\cdot \boldsymbol{n}_K = \frac{1}{2}(\boldsymbol{z}_h^+ + \boldsymbol{z}_h^-)\cdot \boldsymbol{n}_K \quad \text{and} \quad \widehat{y}_h = \frac{1}{2}(y_h^+ + y_h^-)\,. \tag{3.20c}$$

In (3.20c), we let $y_h^\pm := y_{h|\partial K^\pm}$ and $\boldsymbol{z}_h^\pm := \boldsymbol{z}_{h|\partial K^\pm}$ for two adjacent elements K^+ and K^- with common face $e \in \mathcal{E}_h^i$ and outer unit normal vector $\boldsymbol{n}_{K^\pm}$ to $\partial K^\pm$. We note that $\operatorname{grad}_{\mathrm{dg}} y_h = \operatorname{grad} y_h$ for $y_h \in H^1_0(\Omega)$ and $\operatorname{div}_{\mathrm{dg}} \boldsymbol{z}_h = \operatorname{div} \boldsymbol{z}_h$ for $\boldsymbol{z}_h \in \boldsymbol{H}(\operatorname{div},\Omega)$ since the second terms on the right-hand side of (3.16) yield that

$$\sum_{K\in\mathcal{T}_h} \langle y_h, \widehat{\boldsymbol{z}}_h\cdot \boldsymbol{n}_K\rangle_{\partial K} = \sum_{e\in\mathcal{E}_h^i} \langle y_h, \widehat{\boldsymbol{z}}_h\cdot (\boldsymbol{n}_{K^+} + \boldsymbol{n}_{K^-})\rangle_e = 0\,,$$

$$\sum_{K\in\mathcal{T}_h} \langle \boldsymbol{z}_h\cdot \boldsymbol{n}_K, \widehat{y}_h\rangle_{\partial K} = \sum_{e\in\mathcal{E}_h^i} \langle \boldsymbol{z}_h^+\cdot \boldsymbol{n}_{K^+} + \boldsymbol{z}_h^-\cdot \boldsymbol{n}_{K^-}, \widehat{y}_h\rangle_e = 0\,.$$

The matrix- and vector-valued operators Grad_0 and Div, introduced in (2.3) and (2.5), respectively, are defined on broken function spaces similarly to (3.16).

3.3. Gradient and divergence on broken polynomial spaces and the operator A_h. Now we specify the broken spaces $\widehat{Y}_h$ and $\widehat{\boldsymbol{Z}}_h$ of Definition 3.1 for the finite element spaces (3.11) and (3.12) that we consider for the approximation of Problem 2.4. In light of Lemma 3.2, we put

$$\widehat{y}_h := \frac{1}{2}(y_h^+ + y_h^-) \quad \text{and} \quad \widehat{\boldsymbol{z}}_h\cdot \boldsymbol{n}_e := \frac{1}{2}(\boldsymbol{z}_h^+ + \boldsymbol{z}_h^-)\cdot \boldsymbol{n}_e\,, \quad \text{for } e\in\mathcal{E}_h^i\,, \tag{3.21a}$$

$$\widehat{y}_h := y_h\,, \quad \text{and} \quad \widehat{\boldsymbol{z}}_h\cdot n_e := \boldsymbol{z}_h\cdot \boldsymbol{n}_e, \quad \text{for } e\in\mathcal{E}_h^\partial\,. \tag{3.21b}$$

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For (3.21), the definitions of the DG gradient $\operatorname{grad}_{\mathrm{dg}} : Y_h^r \to ((Y_h^r)^d)^*$ in (3.16a) and the DG divergence $\operatorname{div}_{\mathrm{dg}} : (Y_h^r)^d \to (Y_h^r)^*$ in (3.16b) of Definition 3.1 then read as follows:

DEFINITION 3.3 (DG derivatives for single-valued functions). *The DG gradient operator* $\operatorname{grad}_{\mathrm{dg}} : Y_h^r \to ((Y_h^r)^d)^*$ *and the DG divergence operator* $\operatorname{div}_{\mathrm{dg}} : (Y_h^r)^d \to (Y_h^r)^*$ *are defined by*

$$\langle \operatorname{grad}_{\mathrm{dg}} y_h, \boldsymbol{z}_h \rangle := \langle \operatorname{grad}_h y_h, \boldsymbol{z}_h \rangle - \sum_{e \in \mathcal{E}_h^i} \langle [\![y_h]\!], \{\!\{ \boldsymbol{z}_h \}\!\} \cdot \boldsymbol{n}_e \rangle_e - \sum_{e \in \mathcal{E}_h^\partial} \langle y_h, \boldsymbol{z}_h \cdot \boldsymbol{n}_e \rangle_e \,, \tag{3.22a}$$

$$\langle \operatorname{div}_{\mathrm{dg}} \boldsymbol{z}_h, y_h \rangle := \langle \operatorname{div}_h \boldsymbol{z}_h, y_h \rangle - \sum_{e \in \mathcal{E}_h^i} \langle [\![\boldsymbol{z}_h]\!] \cdot \boldsymbol{n}_e, \{\!\{ y_h \}\!\} \rangle_e - \sum_{e \in \mathcal{E}_h^\partial} \langle \boldsymbol{z}_h \cdot \boldsymbol{n}_e, y_h \rangle_e \,, \tag{3.22b}$$

for all $y_h \in Y_h^r$ *and* $\boldsymbol{z}_h \in (Y_h^r)^d$*, where standard notation (cf. [26]) is used for the averages and jumps*

$$\{\!\{ w \}\!\}_e := \frac{1}{2}(w^+ + w^-) \qquad \textit{and} \qquad [\![w]\!]_e := w^+ - w^- \,.$$

On the usual Sobolev spaces, the DG gradient and DG divergence of (3.22) coincide with the distribution gradient and divergence, respectively, since for functions in $H_0^1(\Omega)$, the jump terms $[\![y_h]\!]$ on $e \in \mathcal{E}_h^i$ and the traces on the boundary faces $e \in \mathcal{E}_h^\partial$ vanish in (3.22); cf. [26, Lemma 1.22 and 1.24]. Similarly, for functions in $\boldsymbol{H}(\operatorname{div};\Omega)$, the jumps $[\![\boldsymbol{z}_h]\!] \cdot \boldsymbol{n}_e$ for $e \in \mathcal{E}_h^i$ vanish as well; cf. [26, Lemma 1.22 and 1.24]. The assumption of Lemma 3.2 that $\widehat{y}_h = 0$ for $e \in \mathcal{E}_h^\partial$ is not fulfilled for the DG space (3.11) with (3.12b). This leads to a perturbation of the skew-selfadjointness of $\operatorname{grad}_{\mathrm{dg}}$ and $\operatorname{div}_{\mathrm{dg}}$ which is shown now.

LEMMA 3.4 (DG skew-selfadjointness). *For all* $y_h \in Y_h^r$ *and* $\boldsymbol{z}_h \in (Y_h^r)^d$ *there holds that*

$$\langle \operatorname{grad}_{\mathrm{dg}} y_h, \boldsymbol{z}_h \rangle + \sum_{e \in \mathcal{E}_h^\partial} \langle y_h, \boldsymbol{z}_h \cdot \boldsymbol{n}_e \rangle_e = \langle - \operatorname{div}_{\mathrm{dg}}^* y_h, \boldsymbol{z}_h \rangle \,, \tag{3.23a}$$

$$\langle \operatorname{div}_{\mathrm{dg}} \boldsymbol{z}_h, y_h \rangle + \sum_{e \in \mathcal{E}_h^\partial} \langle y_h, \boldsymbol{z}_h \cdot \boldsymbol{n}_e \rangle_e = \langle - \operatorname{grad}_{\mathrm{dg}}^* \boldsymbol{z}_h, y_h \rangle \tag{3.23b}$$

and

$$\langle \operatorname{grad}_{\mathrm{dg}} y_h, \boldsymbol{z}_h \rangle + \langle \operatorname{div}_{\mathrm{dg}} \boldsymbol{z}_h, y_h \rangle = - \sum_{e \in \mathcal{E}_h^\partial} \langle y_h, \boldsymbol{z}_h \cdot \boldsymbol{n}_e \rangle_e \,. \tag{3.24}$$

Proof. The identity (3.23) is a direct consequence of (3.18) and (3.19) along with the definition (3.21) of the broken spaces $\widehat{Y}_h$ and $\widehat{\boldsymbol{Z}}_{\boldsymbol{h}}$ on $\mathcal{E}_h$. From (3.23b) we then get that

$$\begin{aligned}
\langle \operatorname{grad}_{\mathrm{dg}} y_h, & \boldsymbol{z}_h \rangle + \langle \operatorname{div}_{\mathrm{dg}} \boldsymbol{z}_h, y_h \rangle \\
&= \langle \operatorname{grad}_{\mathrm{dg}}^* \boldsymbol{z}_h, y_h \rangle + \langle \operatorname{div}_{\mathrm{dg}} \boldsymbol{z}_h, y_h \rangle \\
&= -\langle \operatorname{div}_{\mathrm{dg}} \boldsymbol{z}_h, y_h \rangle - \sum_{e \in \mathcal{E}_h^\partial} \langle y_h, \boldsymbol{z}_h \cdot \boldsymbol{n}_e \rangle_e + \langle \operatorname{div}_{\mathrm{dg}} \boldsymbol{z}_h, y_h \rangle \\
&= - \sum_{e \in \mathcal{E}_h^\partial} \langle y_h, \boldsymbol{z}_h \cdot \boldsymbol{n}_e \rangle_e \,.
\end{aligned}$$

This proves (3.24). $\square$

For multi-valued functions, the DG derivatives $\operatorname{Grad}_{\mathrm{dg}} : (Y_h^r)^d \to \big((Y_h^r)^{d\times d} \cap L^2(\Omega)^{d\times d}_{\mathrm{sym}}\big)^*$ and $\operatorname{Div}_{\mathrm{dg}} : (Y_h^r)^{d\times d} \cap L^2(\Omega)^{d\times d}_{\mathrm{sym}} \to \big((Y_h^r)^d\big)^*$ are defined as follows:

DEFINITION 3.5 (DG derivatives for multi-valued functions). *For multi-valued functions, the DG gradient operator* $\operatorname{Grad}_{\mathrm{dg}} : (Y_h^r)^d \to \big((Y_h^r)^{d\times d} \cap L^2(\Omega)^{d\times d}_{\mathrm{sym}}\big)^*$ *and DG divergence operator* $\operatorname{Div}_{\mathrm{dg}} : (Y_h^r)^{d\times d} \cap L^2(\Omega)^{d\times d}_{\mathrm{sym}} \to \big((Y_h^r)^d\big)^*$ *are given by*

$$\begin{aligned}\langle \operatorname{Grad}_{\mathrm{dg}} \boldsymbol{y}_h, \boldsymbol{z}_h\rangle &:= \langle \operatorname{Grad}_h \boldsymbol{y}_h, \boldsymbol{z}_h\rangle \\ &\qquad - \sum_{e\in\mathcal{E}_h^i} \langle [\![\boldsymbol{y}_h]\!], \{\!\{\boldsymbol{z}_h\}\!\} \cdot \boldsymbol{n}_e\rangle_e - \sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{y}_h, \boldsymbol{z}_h \cdot \boldsymbol{n}_e\rangle_e\,,\end{aligned} \tag{3.25a}$$

$$\begin{aligned}\langle \operatorname{Div}_{\mathrm{dg}} \boldsymbol{z}_h, \boldsymbol{y}_h\rangle &:= \langle \operatorname{Div}_h \boldsymbol{z}_h, \boldsymbol{y}_h\rangle \\ &\qquad - \sum_{e\in\mathcal{E}_h^i} \langle [\![\boldsymbol{z}_h]\!] \cdot \boldsymbol{n}_e, \{\!\{\boldsymbol{y}_h\}\!\}\rangle_e - \sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{z}_h \cdot \boldsymbol{n}_e, \boldsymbol{y}_h\rangle_e\,,\end{aligned} \tag{3.25b}$$

for all $\boldsymbol{y}_h \in (Y_h^r)^d$ *and* $\boldsymbol{z}_h \in (Y_h^r)^{d\times d} \cap L^2(\Omega)^{d\times d}_{\mathrm{sym}}$.

In (3.25), the operators Grad_h and Div_h are the broken symmetrized gradient and the broken divergence that extend the distributional gradient in (2.3) and the divergence in (2.5) to broken polynomial spaces; cf. [26, Definition 1.21]. On the usual Sobolev spaces, the broken gradient $\operatorname{Grad}_{\mathrm{dg}}$ and divergence $\operatorname{Div}_{\mathrm{dg}}$ coincide with the distributional symmetrized gradient and divergence of (2.3) and (2.5), respectively. Similarly to Lemma 3.4, for DG spaces there holds for $\boldsymbol{y}_h \in (Y_h^r)^d$ and $\boldsymbol{z}_h \in (Y_h^r)^{d\times d} \cap L^2(\Omega)^{d\times d}_{\mathrm{sym}}$ that

$$\langle \operatorname{Grad}_{\mathrm{dg}} \boldsymbol{y}_h, \boldsymbol{z}_h\rangle + \sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{y}_h, \boldsymbol{z}_h \cdot \boldsymbol{n}_e\rangle_e = \langle -\operatorname{Div}^*_{\mathrm{dg}} \boldsymbol{y}_h, \boldsymbol{z}_h\rangle\,, \tag{3.26a}$$

$$\langle \operatorname{Div}_{\mathrm{dg}} \boldsymbol{z}_h, \boldsymbol{y}_h\rangle + \sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{y}_h, \boldsymbol{z}_h \cdot \boldsymbol{n}_e\rangle_e = \langle -\operatorname{Grad}^*_{\mathrm{dg}} \boldsymbol{z}_h, \boldsymbol{y}_h\rangle\,, \tag{3.26b}$$

and

$$\langle \operatorname{Grad}_{\mathrm{dg}} \boldsymbol{y}_h, \boldsymbol{z}_h\rangle + \langle \operatorname{Div}_{\mathrm{dg}} \boldsymbol{z}_h, \boldsymbol{y}_h\rangle = -\sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{y}_h, \boldsymbol{z}_h \cdot \boldsymbol{n}_e\rangle_e\,. \tag{3.27}$$

Now, we are able to define a discrete counterpart $\boldsymbol{A}_h : \boldsymbol{H}_h \to \boldsymbol{H}_h^*$ of the differential operator $\boldsymbol{A} : D(\boldsymbol{A}) \subset \boldsymbol{H} \to \boldsymbol{H}$ introduced in (2.10).

DEFINITION 3.6 (Discrete operator $\boldsymbol{A}_h$). *For the DG differential operators introduced in* (3.22) *and* (3.25)*, respectively, the operator* $\boldsymbol{A}_h : \boldsymbol{H}_h \to \boldsymbol{H}_h^*$ *is defined by*

$$\boldsymbol{A}_h := \begin{bmatrix} 0 & -\operatorname{Div}_{\mathrm{dg}} & 0 & 0 \\ -\operatorname{Grad}_{\mathrm{dg}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{div}_{\mathrm{dg}} \\ 0 & 0 & \operatorname{grad}_{\mathrm{dg}} & 0 \end{bmatrix}, \tag{3.28}$$

such that for $\boldsymbol{Y}, \boldsymbol{Z} \in \boldsymbol{H}_h$ *there holds*

$$\begin{aligned}\langle \boldsymbol{A}_h \boldsymbol{Y}, \boldsymbol{Z}\rangle &= -\langle \operatorname{Div}_{\mathrm{dg}} \boldsymbol{Y}_2, \boldsymbol{Z}_1\rangle - \langle \operatorname{Grad}_{\mathrm{dg}} \boldsymbol{Y}_1, \boldsymbol{Z}_2\rangle \\ &\qquad + \langle \operatorname{div}_{\mathrm{dg}} \boldsymbol{Y}_4, Z_3\rangle + \langle \operatorname{grad}_{\mathrm{dg}} Y_3, \boldsymbol{Z}_4\rangle\,.\end{aligned} \tag{3.29}$$

REMARK 3.7. From (3.24) and (3.27) we conclude that for $Y \in \boldsymbol{H}_h$ there holds

$$\langle \boldsymbol{A}_h \boldsymbol{Y}, \boldsymbol{Y}\rangle = \sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{Y}_1, \boldsymbol{Y}_2 \cdot \boldsymbol{n}_e\rangle_e - \sum_{e\in\mathcal{E}_h^\partial} \langle Y_3, \boldsymbol{Y}_4 \cdot \boldsymbol{n}_e\rangle_e\,. \tag{3.30}$$

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By (3.23) and (3.26), the operator $\boldsymbol{A}_h$ is not skew-selfadjoint on $\boldsymbol{H}_h^{\mathrm{dg}}$, as defined in (3.12b), due to perturbations by boundary face integrals. Consequently, the inner product $\langle \boldsymbol{A}_h \boldsymbol{Y}, \boldsymbol{Y}\rangle$ does no longer vanish as in the continuous case. However, the control of the latter term is essential for our analysis. Therefore, some correction term, defined in (3.32) below, will be introduced in the fully discrete scheme. Finally, we note that skew-selfadjointness is preserved for the hybrid space $\boldsymbol{H}_h^{\mathrm{hy}}$ of (3.12a).

3.4. Fully discrete problem with structure-preserving nonconforming approximation. For the discretization of Problem 2.4 in the space $\boldsymbol{H}_h$, defined in (3.11) and (3.12), we then consider the following family of fully discrete nonconforming approximation schemes:

PROBLEM 3.8 (Fully discrete problem). *Let $\boldsymbol{H}_h$ be given by (3.11) and (3.12). For the operators $\boldsymbol{M}_0$ and $\boldsymbol{M}_1$ of (2.10), $\boldsymbol{A}_h$ of (3.28), and given data $\boldsymbol{F} \in H^1_\nu(0,T;\boldsymbol{H})$ and $\boldsymbol{U}_{0,h} \in \boldsymbol{H}_h$, where $\boldsymbol{U}_{0,h} \in \boldsymbol{H}_h$ denotes an approximation of the initial value $\boldsymbol{U}_0 \in \boldsymbol{H}$ according to (2.9), find $\boldsymbol{U}_{\tau,h} \in Y_\tau^k(\boldsymbol{H}_h)$ with*

$$
\begin{aligned}
&Q_n[(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A}_h)\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h}]_\nu \\
&\qquad + Q_n[J_\partial(\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h}) + J_\gamma(\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h})]_\nu \\
&\qquad + \langle \boldsymbol{M}_0 [\![\boldsymbol{U}_{\tau,h}]\!]_{n-1}, \boldsymbol{V}_{\tau,h}^{+,n-1}\rangle_{\boldsymbol{H}} = Q_n[\boldsymbol{F}, \boldsymbol{V}_{\tau,h}]_\nu
\end{aligned}
\tag{3.31}
$$

for all $\boldsymbol{V}_{\tau,h} \in Y_\tau^k(\boldsymbol{H}_h)$ and $n \in \{1,2,\dots,N\}$, where

$$
J_\partial(\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h}) := -\sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{U}_{\tau,h}^{(2)} \cdot \boldsymbol{n}_e, \boldsymbol{V}_{\tau,h}^{(1)}\rangle_e + \sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{U}_{\tau,h}^{(4)} \cdot \boldsymbol{n}_e, \boldsymbol{V}_{\tau,h}^{(3)}\rangle_e\,, \tag{3.32}
$$

$$
J_\gamma(\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h}) := \sum_{e\in\mathcal{E}_h^\partial} \frac{1}{h}\Big(\gamma_1 \langle \boldsymbol{U}_{\tau,h}^{(1)}, \boldsymbol{V}_{\tau,h}^{(1)}\rangle_e + \gamma_2 \langle U_{\tau,h}^{(3)}, V_{\tau,h}^{(3)}\rangle_e\Big) \tag{3.33}
$$

with parameters $\gamma_i > 0$, for $i \in \{1,2\}$, and

$$
[\![\boldsymbol{U}_{\tau,h}]\!]_{n-1} := \begin{cases} \boldsymbol{U}_{\tau,h}^+(t_{n-1}) - \boldsymbol{U}_{\tau,h}^-(t_{n-1})\,, & \text{for } n \in \{2,\dots,N\}\,,\\ \boldsymbol{U}_{\tau,h}^+(t_{n-1}) - \boldsymbol{U}_{0,h}\,, & \text{for } n = 1\,, \end{cases}
$$

for $\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h} \in Y_\tau^k(\boldsymbol{H}_h)$, with $\boldsymbol{V}_{\tau,h}^{+,n-1} := \boldsymbol{V}_{\tau,h}^+(t_{n-1})$ being defined by (3.3).

REMARK 3.9.

- The algorithmic (or penalization) parameters $\gamma_i > 0$, for $i \in \{1,2\}$, in (3.33) have to be chosen sufficiently large; cf. [46]. The contribution J_γ, defined in (3.33), enforces a weak form of the homogeneous Dirichlet boundary conditions in (2.12). In the error estimation below, the term J_γ is further used in an essential way for absorption arguments and deriving convergence order estimates. Contributions to the upper bound of the discrete error are absorbed.
- In (3.31), the mathematical structure of the evolutionary problem (2.13) is essentially preserved, with the discrete operator $\boldsymbol{A}_h$ replacing $\boldsymbol{A}$. The perturbation of the skew-selfadjointness of $\boldsymbol{A}_h$, resulting from (3.23) and (3.26), is compensated in the analysis below by the additional (boundary) correction $Q_n[J_\partial(\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h})]_\nu$ in (3.31), along with the penalization induced by $Q_n[J_\gamma(\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h})]_\nu$.

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- A stronger penalization is obtained by further adding the term

$$
\begin{aligned}
&J_\delta(\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h}) \\
&:= \sum_{e\in\mathcal{E}_h^i} \frac{1}{h}\Big(\gamma_3\big\langle[\![\boldsymbol{U}_{\tau,h}^{(1)}]\!], [\![\boldsymbol{V}_{\tau,h}^{(1)}]\!]\big\rangle_e + \gamma_4\big\langle[\![U_{\tau,h}^{(3)}]\!], [\![V_{\tau,h}^{(3)}]\!]\big\rangle_e\Big) \\
&\qquad + \sum_{e\in\mathcal{E}_h^i} \frac{1}{h}\Big(\gamma_5\big\langle[\![\boldsymbol{U}_{\tau,h}^{(4)}\cdot \boldsymbol{n}_e]\!], [\![\boldsymbol{V}_{\tau,h}^{(2)}\cdot \boldsymbol{n}_e]\!]\big\rangle_e + \gamma_6\big\langle[\![\boldsymbol{U}_{\tau,h}^{(4)}\cdot \boldsymbol{n}_e]\!], [\![\boldsymbol{V}_{\tau,h}^{(3)}\cdot \boldsymbol{n}_e]\!]\big\rangle_e\Big)
\end{aligned}
$$

  to the left-hand side of (3.31), with constants $\gamma_i > 0$, for $i \in \{3,\dots,6\}$. The term J_δ penalizes the jumps of the variables over interior faces. For brevity, we do not include J_δ into our error analysis below since the presence of J_δ does not change the arguments and final results in an essential way. The nonnegativity of $J_\delta(\boldsymbol{V}_{\tau,h}, \boldsymbol{V}_{\tau,h})$ yields an error control for the jumps of the variables over interelement faces.
- Problem 3.8 yields a global-in-time formulation. For computations of space-time finite element discretizations, we propose using a temporal test basis that is supported on the subintervals I_n; cf. [5, 8]. Then, a time-marching process is obtained. For Problem 3.8, this amounts to assuming that the trajectory $\boldsymbol{U}_{\tau,h}$ has been computed before, for all $t \in [0, t_{n-1}]$, starting with an approximation $\boldsymbol{U}_{\tau,h}(t_0) := \boldsymbol{U}_{0,h}$ of $\boldsymbol{U}_0 \in D(\boldsymbol{A})$. On $I_n = (t_{n-1}, t_n]$, for given $\boldsymbol{U}_{\tau,h}(t_{n-1}) \in \boldsymbol{H}_h$, we consider then finding $\boldsymbol{U}_{\tau,h} \in \mathbb{P}_k(I_n, \boldsymbol{H}_h)$ such that (3.31) is satisfied for all $\boldsymbol{V}_{\tau,h} \in \mathbb{P}_k(I_n, \boldsymbol{H}_h)$.
- In (3.31), there holds that

$$
\begin{aligned}
\langle \boldsymbol{A}_h U_{\tau,h}, \boldsymbol{V}_{\tau,h}\rangle_H &= \left\langle \begin{bmatrix} -\operatorname{Div}_{\mathrm{dg}} \boldsymbol{U}_{\tau,h}^{(2)} \\ -\operatorname{Grad}_{\mathrm{dg}} \boldsymbol{U}_{\tau,h}^{(1)} \\ \operatorname{div}_{\mathrm{dg}} \boldsymbol{U}_{\tau,h}^{(4)} \\ \operatorname{grad}_{\mathrm{dg}} U_{\tau,h}^{(3)} \end{bmatrix}, \begin{bmatrix} \boldsymbol{V}_{\tau,h}^{(1)} \\ \boldsymbol{V}_{\tau,h}^{(2)} \\ V_{\tau,h}^{(3)} \\ \boldsymbol{V}_{\tau,h}^{(4)} \end{bmatrix} \right\rangle_H \\
&= \left\langle \begin{bmatrix} \boldsymbol{U}_{\tau,h}^{(2)} \\ -\operatorname{Grad}_{\mathrm{dg}} \boldsymbol{U}_{\tau,h}^{(1)} \\ \boldsymbol{U}_{\tau,h}^{(4)} \\ \operatorname{grad}_{\mathrm{dg}} U_{\tau,h}^{(3)} \end{bmatrix}, \begin{bmatrix} -\operatorname{Div}_{\mathrm{dg}}^* \boldsymbol{V}_{\tau,h}^{(1)} \\ \boldsymbol{V}_{\tau,h}^{(2)} \\ \operatorname{div}_{\mathrm{dg}}^* V_{\tau,h}^{(3)} \\ \boldsymbol{V}_{\tau,h}^{(4)} \end{bmatrix} \right\rangle_H .
\end{aligned}
\tag{3.34}
$$

  By (3.23a) and (3.26a), the operators $-\operatorname{Div}_{\mathrm{dg}}^*$ and $\operatorname{div}_{\mathrm{dg}}^*$ in (3.34) are transformed into the DG gradients $\operatorname{Grad}_{\mathrm{dg}}$ and $\operatorname{grad}_{\mathrm{dg}}$, respectively, applied to the test functions, and additional sums of boundary face integrals. This can be exploited in the assembly process and error analysis.

THEOREM 3.10 (Well-posedness of fully discrete problem). *There exists a unique solution $U_{\tau,h} \in Y_{\tau,h}^k(\boldsymbol{H}_h)$ of Problem 3.8.*

Proof. The proof follows the ideas of [34, Proof of Proposition 3.2]. To keep this work self-contained and due to adaptations of the proof required by the perturbation of the skew-selfadjointness, we briefly present it. Since Problem 3.8 is finite-dimensional, it suffices to prove uniqueness of solutions to (3.31) for $n \in \{1,\dots,N\}$. The existence of solutions then directly follows from their uniqueness. By means of the first of the items in Remark 3.9 and an induction argument, it suffices to prove the uniqueness of solutions to (3.31) on a fixed subinterval I_n.

For this, let $\widetilde{\boldsymbol{U}}_{\tau,h} \in \mathbb{P}_k(I_n, \boldsymbol{H}_h)$ and $\widehat{\boldsymbol{U}}_{\tau,h} \in \mathbb{P}_k(I_n, \boldsymbol{H}_h)$ be two solutions of (3.31). Then, their difference $\boldsymbol{U}_{\tau,h} := \widetilde{\boldsymbol{U}}_{\tau,h} - \widehat{\boldsymbol{U}}_{\tau,h}$ satisfies for all $\boldsymbol{V}_{\tau,h} \in \mathbb{P}_k(I_n, \boldsymbol{H}_h)$

$$
\begin{aligned}
Q_n[(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A}_h)\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h}]_\nu + Q_n[J_\partial(\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h}) + J_\gamma(\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h})]_\nu& \\
+ \langle M_0 \boldsymbol{U}^+_{\tau,h}(t_{n-1}) \,.\, \boldsymbol{V}^{+,n-1}_{\tau,h} \rangle_{\boldsymbol{H}} = 0\,.&
\end{aligned}
\tag{3.35}
$$

Next, we recall an argument of [34, Proof of Proposition 3.2]. We note that

$$
\begin{aligned}
\partial_t : \mathbb{P}_k(I_n, \boldsymbol{H}) &\to \mathbb{P}_k(I_n, \boldsymbol{H})\,,\\
\boldsymbol{w}_\tau &\mapsto \partial_t \boldsymbol{w}_\tau
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_{n-1} : \mathbb{P}_k(I_n, \boldsymbol{H}) &\to \boldsymbol{H}\,,\\
\boldsymbol{w}_\tau &\mapsto \delta_{n-1}\boldsymbol{w}_\tau := \boldsymbol{w}^+_\tau(t_{n-1})
\end{aligned}
$$

are bounded linear operators with respect to the norm of $\mathbb{P}_k(I_n, \boldsymbol{H})$ induced by the inner product (3.1). Consequently, the mapping

$$
\begin{aligned}
\mathbb{P}_k(I_n, \boldsymbol{H}) &\to \mathbb{R}\,,\\
\boldsymbol{w}_\tau &\mapsto \langle \boldsymbol{z}, \delta_{n-1}\boldsymbol{w}_\tau \rangle_{\boldsymbol{H}}
\end{aligned}
$$

is linear and bounded for each $\boldsymbol{z} \in \boldsymbol{H}$. Then, by the Riesz representation theorem, there exists a unique $\boldsymbol{\Psi}_\tau(\boldsymbol{z}) \in \mathbb{P}_k(I_n; \boldsymbol{H})$ such that

$$
\langle \boldsymbol{\Psi}_\tau(\boldsymbol{z}), \boldsymbol{w}_\tau \rangle_{\nu,n} = \langle \boldsymbol{z}, \delta_{n-1}\boldsymbol{w}_\tau \rangle_{\boldsymbol{H}}\,. \tag{3.36}
$$

The mapping $\boldsymbol{\Psi}_\tau : \boldsymbol{H} \to \mathbb{P}_k(I_n, \boldsymbol{H})$ is linear and bounded since for $\boldsymbol{z} \in \boldsymbol{H}$ there holds that

$$
\|\boldsymbol{\Psi}_\tau(\boldsymbol{z})\|^2_{\nu,n} = \langle \boldsymbol{\Psi}_\tau(\boldsymbol{z}), \boldsymbol{\Psi}_\tau(\boldsymbol{z}) \rangle_{\nu,n} = \langle \boldsymbol{z}, \delta_{n-1}\boldsymbol{\Psi}_\tau(z) \rangle_{\boldsymbol{H}} \le \|\boldsymbol{z}\|_{\boldsymbol{H}} \|\delta_{n-1}\| \|\boldsymbol{\Psi}_\tau(\boldsymbol{z})\|_{\nu,n}\,.
$$

Now, using integration by parts along with (3.36), we have or all $\boldsymbol{v}_\tau \in \mathbb{P}_k(I_n; \boldsymbol{H})$ that

$$
\begin{aligned}
&\langle \partial_t \boldsymbol{M}_0 \boldsymbol{v}_\tau, \boldsymbol{v}_\tau \rangle_{\nu,n}\\
&\quad = \frac{1}{2}\langle \partial_t \boldsymbol{M}_0 \boldsymbol{v}_\tau, \boldsymbol{v}_\tau \rangle_{\nu,n} + \frac{1}{2}\int_{t_{n-1}}^{t_n} \langle \boldsymbol{M}_0 \partial_t \boldsymbol{v}_\tau(t), \boldsymbol{v}_\tau(t) \rangle_{\boldsymbol{H}}\, \mathrm{e}^{-2\nu(t-t_{n-1})}\,\mathrm{d}t\\
&\quad = \frac{1}{2}\langle \partial_t \boldsymbol{M}_0 \boldsymbol{v}_\tau, \boldsymbol{v}_\tau \rangle_{\nu,n} - \frac{1}{2}\int_{t_{n-1}}^{t_n} \langle \boldsymbol{M}_0 \partial_t \boldsymbol{v}_\tau(t), \boldsymbol{v}_\tau(t) \rangle_{\boldsymbol{H}}\, \mathrm{e}^{-2\nu(t-t_{n-1})}\,\mathrm{d}t\\
&\qquad + \nu \int_{t_{n-1}}^{t_n} \langle \boldsymbol{M}_0 \boldsymbol{v}_\tau(t), \boldsymbol{v}_\tau(t) \rangle_{\boldsymbol{H}}\, \mathrm{e}^{-2\nu(t-t_{n-1})}\,\mathrm{d}t\\
&\qquad + \frac{1}{2}\langle \boldsymbol{M}_0 \boldsymbol{v}_\tau(t_n), \boldsymbol{v}_\tau(t_n) \rangle_{\boldsymbol{H}}\, \mathrm{e}^{-2\nu\tau_n} - \frac{1}{2}\langle \boldsymbol{v}^+_\tau(t_{n-1}), \boldsymbol{M}_0 \boldsymbol{v}^+_\tau(t_{n-1}) \rangle_{\boldsymbol{H}}\\
&\quad \ge \nu \langle \boldsymbol{M}_0 \boldsymbol{v}_\tau, \boldsymbol{v}_\tau \rangle_{\nu,n} - \frac{1}{2}\langle \boldsymbol{\Psi}_\tau(\boldsymbol{M}_0 \delta_{n-1}\boldsymbol{v}_\tau), \boldsymbol{v}_\tau \rangle_{\nu,n}\,.
\end{aligned}
\tag{3.37}
$$

Using (3.36), we rewrite (3.35) as

$$
\begin{aligned}
Q_n[(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A}_h)\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h}]_\nu + Q_n[J_\partial(\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h}) + J_\gamma(\boldsymbol{U}_{\tau,h}, \boldsymbol{V}_{\tau,h})]_\nu& \\
+ \langle \boldsymbol{\Psi}_\tau(\boldsymbol{M}_0 \delta_{n-1}\boldsymbol{U}_{\tau,h}), \boldsymbol{V}_{\tau,h} \rangle_{\boldsymbol{H}} = 0\,.&
\end{aligned}
\tag{3.38}
$$

In (3.38), we choose $\boldsymbol{V}_{\tau,h} = \boldsymbol{U}_{\tau,h}$. By (3.30) along with (3.32) we have for $\boldsymbol{Z} \in \boldsymbol{H}_h$ that

$$\begin{aligned} \langle \boldsymbol{A}_h \boldsymbol{Z}, \boldsymbol{Z}\rangle + J_\partial(\boldsymbol{Z},\boldsymbol{Z}) = \sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{Z}_1, \boldsymbol{Z}_2\cdot \boldsymbol{n}_e\rangle_e - \sum_{e\in\mathcal{E}_h^\partial} \langle Z_3, \boldsymbol{Z}_4\cdot \boldsymbol{n}_e\rangle_e \\ - \sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{Z}_2\cdot \boldsymbol{n}_e, \boldsymbol{Z}_1\rangle_e + \sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{Z}_4\cdot \boldsymbol{n}_e, Z_3\rangle_e = 0\,. \end{aligned} \tag{3.39}$$

From (3.38) we deduce by (3.37), (3.39), and the nonnegativity of $J_\gamma(\boldsymbol{U}_{\tau,h}, \boldsymbol{U}_{\tau,h})$, which follows from (3.33), that

$$\begin{aligned} 0 &= Q_n[(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A}_h)\boldsymbol{U}_{\tau,h}, \boldsymbol{U}_{\tau,h}]_\nu \\ &\quad + Q_n[J_\partial(\boldsymbol{U}_{\tau,h}, \boldsymbol{U}_{\tau,h}) + J_\gamma(\boldsymbol{U}_{\tau,h}, \boldsymbol{U}_{\tau,h})]_\nu + \langle \boldsymbol{\Psi}_\tau(\boldsymbol{M}_0\delta_{n-1}\boldsymbol{U}_{\tau,h}), \boldsymbol{U}_{\tau,h}\rangle_H \\ &\geq \langle \partial_t \boldsymbol{M}_0 \boldsymbol{U}_{\tau,h}, \boldsymbol{U}_{\tau,h}\rangle_{\nu,n} + \langle \boldsymbol{M}_1 \boldsymbol{U}_{\tau,h}, \boldsymbol{U}_{\tau,h}\rangle_{\nu,n} + \langle \boldsymbol{\Psi}_\tau(\boldsymbol{M}_0\delta_{n-1}\boldsymbol{U}_{\tau,h}), \boldsymbol{U}_{\tau,h}\rangle_{\nu,n} \\ &\geq \langle (\nu \boldsymbol{M}_0 + \boldsymbol{M}_1)\boldsymbol{U}_{\tau,h}, \boldsymbol{U}_{\tau,h}\rangle_{\nu,n} + \frac{1}{2}\langle \boldsymbol{\Psi}_\tau(\boldsymbol{M}_0\delta_{n-1}\boldsymbol{U}_{\tau,h}), \boldsymbol{U}_{\tau,h}\rangle_{\nu,n} \\ &\geq \gamma\langle \boldsymbol{U}_{\tau,h}, \boldsymbol{U}_{\tau,h}\rangle_{\nu,n}\,, \end{aligned} \tag{3.40}$$

where the nonnegativity of $\langle \boldsymbol{\Psi}_\tau(\boldsymbol{M}_0\delta_{n-1}\boldsymbol{U}_{\tau,h}), \boldsymbol{U}_{\tau,h}\rangle_{\nu,n}$ is ensured by

$$\langle \boldsymbol{\Psi}_\tau(\boldsymbol{M}_0\delta_{n-1}\boldsymbol{U}_{\tau,h}), \boldsymbol{U}_{\tau,h}\rangle_{\nu,n} = \langle \boldsymbol{M}_0 \boldsymbol{U}_{\tau,h}(t_{n-1}^+), \boldsymbol{U}_{\tau,h}(t_{n-1}^+)\rangle_{\boldsymbol{H}} \geq 0\,.$$

The latter inequality follows from the assumption (2.14). From (3.40) we directly conclude the uniqueness of solutions to (3.31) and, thereby, the assertion of this lemma. ∎

4. Error estimation for the structure-preserving nonconforming approximation. Here we prove an error estimate for the solution $\boldsymbol{U}_{\tau,h}$ of Problem 3.8. For brevity, the proof is done only for the full DG approximation in space, corresponding to the choice $\boldsymbol{H}_h = \boldsymbol{H}_h^{\text{dg}}$ in (3.11), with $r \in \mathbb{N}_0$ in (3.12b). The adaptation of the proof to the hybrid case $\boldsymbol{H}_h = \boldsymbol{H}_h^{\text{hy}}$ in (3.11) is straightforward.

THEOREM 4.1 (Error estimate for the fully discrete problem). *Let $\boldsymbol{H}$ be defined by (2.11). For the solution $\boldsymbol{U}$ of Problem 2.4, suppose that the regularity condition*

$$\boldsymbol{U} \in H_\nu^{k+3}(\mathbb{R}; \boldsymbol{H}) \cap H_\nu^2\big(\mathbb{R}; \boldsymbol{H}^{r+1}(\Omega)^{(d+1)^2}\big) \tag{4.1}$$

is satisfied. Let the discrete initial value $\boldsymbol{U}_{0,h} \in \boldsymbol{H}_h$ in Problem 3.8 be chosen such that $\|\boldsymbol{U}_0 - \boldsymbol{U}_{0,h}\| \leq ch^r$ holds.

Then, for the numerical solution $\boldsymbol{U}_{\tau,h}$ of Problem 3.8 there holds that

$$\begin{aligned} \sup_{t\in[0,T]} \langle \boldsymbol{M}_0(\boldsymbol{U}-\boldsymbol{U}_{\tau,h})(t), (\boldsymbol{U}-\boldsymbol{U}_{\tau,h})(t)\rangle + \mathrm{e}^{2\nu T}\,\|\boldsymbol{U}-\boldsymbol{U}_{\tau,h}\|_{\tau,\nu}^2 \\ \leq C(1+T)\,\mathrm{e}^{2\nu T}\big(\tau^{2(k+1)} + h^{2r}\big)\,. \end{aligned} \tag{4.2}$$

Proof. We split the error $\boldsymbol{U} - \boldsymbol{U}_{\tau,h}$ into the two parts

$$\boldsymbol{U} - \boldsymbol{U}_{\tau,h} = \boldsymbol{Z} + \boldsymbol{E}_{\tau,h} \qquad \text{with} \quad \boldsymbol{Z} := \boldsymbol{U} - I_\tau \Pi_h \boldsymbol{U}\,, \quad \boldsymbol{E}_{\tau,h} := I_\tau\Pi_h \boldsymbol{U} - \boldsymbol{U}_{\tau,h}\,, \tag{4.3}$$

where I_τ and Π_h are defined in (3.7) and (3.13), respectively. The errors $\boldsymbol{Z}$ and $\boldsymbol{E}_{\tau,h}$ are estimated in Lemma 4.2–4.5 below. By means of the triangle inequality, the splitting (4.3) along with these lemmas then proves (4.2). ∎

For the error $\boldsymbol{Z}$ in (4.3) there holds the following estimate:

LEMMA 4.2 (Estimation of the error $\boldsymbol{Z}$). *Let* $U \in H_\nu^{k+2}(\mathbb{R}; \boldsymbol{H}) \cap H_\nu^1(\mathbb{R}; \boldsymbol{H}^s(\Omega)^{(d+1)^2})$ *be satisfied for some* $s \in \{0, \dots, r+1\}$. *For the error* $\boldsymbol{Z} = \boldsymbol{U} - I_\tau \Pi_h \boldsymbol{U}$ *there holds that*

$$\sup_{t\in[0,T]} \langle \boldsymbol{M}_0 \boldsymbol{Z}(t), \boldsymbol{Z}(t)\rangle_{\boldsymbol{H}} + \mathrm{e}^{2\nu T} \|\boldsymbol{Z}\|_{\tau,\nu}^2 \le C(1+T)\,\mathrm{e}^{2\nu T}(\tau^{2(k+1)} + h^{2s})\,. \tag{4.4}$$

Proof. We split the error $\boldsymbol{Z}$ into the two parts

$$\boldsymbol{Z} = \boldsymbol{U} - I_\tau \Pi_h \boldsymbol{U} = (\boldsymbol{U} - \Pi_h \boldsymbol{U}) + (\Pi_h \boldsymbol{U} - I_\tau \Pi_h \boldsymbol{U})\,. \tag{4.5}$$

From (4.5) along with the commutativity of Π_h and I_τ and the boundedness of $\boldsymbol{M}_0$ and Π_h, we get that

$$\begin{aligned} \langle \boldsymbol{M}_0 \boldsymbol{Z}(t), \boldsymbol{Z}(t)\rangle_{\boldsymbol{H}} &\le C\|\boldsymbol{Z}(t)\|_{\boldsymbol{H}}^2 = C\|\boldsymbol{U}(t) - I_\tau \Pi_h \boldsymbol{U}(t)\|_H^2 \\ &\le C\Big(\|\boldsymbol{U}(t) - \Pi_h \boldsymbol{U}(t)\|_{\boldsymbol{H}}^2 + \|\boldsymbol{U}(t) - I_\tau \boldsymbol{U}(t)\|_{\boldsymbol{H}}^2\Big)\,. \end{aligned} \tag{4.6}$$

Using (3.8) and (3.14), we obtain from (4.6) that

$$\langle \boldsymbol{M}_0 \boldsymbol{Z}(t), \boldsymbol{Z}(t)\rangle_{\boldsymbol{H}} \le C\Big(\tau^{2(k+1)} \sup_{t\in I_n} \|\partial_t^{k+1} \boldsymbol{U}\|_{\boldsymbol{H}}^2 + h^{2s} \sup_{t\in I_n} \|\boldsymbol{U}\|_{\boldsymbol{H}^s(\Omega)}^2\Big) \tag{4.7}$$

for $t \in I_n$. We note that the norms on the right-hand side of (4.7) remain finite under the assumptions about $\boldsymbol{U}$; cf. (2.17). This shows the first of the estimates in (4.4). By the definition of $\|\cdot\|_{\tau,\nu}$ in (3.6c), the second of the estimates in (4.4) follows from (4.6) along with (3.8) and (3.14). ∎

For the error $\boldsymbol{E}_{\tau,h}$ in (4.3) the following estimate holds:

LEMMA 4.3 (Estimation of the error $\boldsymbol{E}_{\tau,h}$). *For the error* $\boldsymbol{E}_{\tau,h} = I_\tau \Pi_h \boldsymbol{U} - \boldsymbol{U}_{\tau,h}$ *there holds*

$$\begin{aligned} &\langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}^-(t_N), \boldsymbol{E}_{\tau,h}^-(t_N)\rangle_{\boldsymbol{H}} + \mathrm{e}^{2\nu T}(\|\boldsymbol{E}_{\tau,h}\|_{\tau,\nu}^2 + |J_\gamma(\boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h})|_{\tau,\nu}^2) \\ &\le C\,\mathrm{e}^{2\nu T}\Big(\langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}^-(t_0), \boldsymbol{E}_{\tau,h}^-(t_0)\rangle_{\boldsymbol{H}} + \|\partial_t \boldsymbol{M}_0(\boldsymbol{U} - \widehat{I}_\tau^{k+1}\Pi_h \boldsymbol{U})\|_{\tau,\nu}^2 \\ &\qquad + \|\boldsymbol{M}_1 \boldsymbol{Z}\|_{\tau,\nu}^2 + \|\boldsymbol{A}_h \boldsymbol{Z}\|_{\tau,\nu}^2 + |J_\gamma(\boldsymbol{Z},\boldsymbol{Z})|_{\tau,\nu}^2 + |J_\partial^{\boldsymbol{n}}(\boldsymbol{Z},\boldsymbol{Z})|_{\tau,\nu}^2 \\ &\qquad + T \max_{1\le n\le N} \Big\{\|\boldsymbol{M}_0(\Pi_h \boldsymbol{U}^+(t_{n-1}) - I_\tau \Pi_h \boldsymbol{U}^+(t_{n-1}))\|_H^2\, \mathrm{e}^{-2\nu t_{n-1}}\Big\}\Big), \end{aligned} \tag{4.8}$$

where, with $\boldsymbol{Z}$ *being defined in* (4.3),

$$J_\partial^{\boldsymbol{n}}(\boldsymbol{Z},\boldsymbol{Z}) := \sum_{e\in\mathcal{E}_h^\partial} h\left(\langle \boldsymbol{Z}_2\cdot \boldsymbol{n}_e, \boldsymbol{Z}_2\cdot \boldsymbol{n}_e\rangle_e + \langle \boldsymbol{Z}_4\cdot \boldsymbol{n}_e, \boldsymbol{Z}_4\cdot \boldsymbol{n}_e\rangle_e\right). \tag{4.9}$$

Proof. Essentially, the proof follows [34, Theorem 3.8] for the semidiscretization in time. In [34, Theorem 3.8] the skew-selfadjointness of the continuous operator is a key ingredient for proving the error estimate. To keep this work self-contained, we summarize in Appendix A the proof of (4.8) for the setting of Problem 4.3 and the perturbed skew-selfadjointness of $\boldsymbol{A}_h$, depicted by Lemma 3.4 and (3.26). ∎

For the error $\boldsymbol{E}_{\tau,h}$ in (4.3) we further have the following improved estimate:

LEMMA 4.4 (Improved estimation of the error $\boldsymbol{E}_{\tau,h}$). *For the error $\boldsymbol{E}_{\tau,h}$ in (4.3) there holds that*

$$
\begin{aligned}
\sup_{t\in[0,T]} & \langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}(t), \boldsymbol{E}_{\tau,h}(t)\rangle_H \\
&\le C\Big(\langle \boldsymbol{M}_0 \boldsymbol{E}^-_{\tau,h}(t_0), \boldsymbol{E}^-_{\tau,h}(t_0)\rangle_{\boldsymbol{H}} \\
&\qquad + \|\partial_t \boldsymbol{M}_0(\boldsymbol{U} - \widehat{I}_\tau^{k+1}\Pi_h \boldsymbol{U})\|^2_{\tau,\nu} + \|\boldsymbol{M}_1\boldsymbol{Z}\|^2_{\tau,\nu} + \|\boldsymbol{A}_h\boldsymbol{Z}\|^2_{\tau,\nu} \\
&\qquad + |J_\gamma(\boldsymbol{Z},\boldsymbol{Z})|^2_{\tau,\nu} + |J^{\boldsymbol{n}}_\partial(\boldsymbol{Z},\boldsymbol{Z})|^2_{\tau,\nu} \\
&\qquad + T\max_{1\le n\le N}\Big\{\|\boldsymbol{M}_0(\Pi_h\boldsymbol{U}^+(t_{n-1}) - I_\tau\Pi_h\boldsymbol{U}^+(t_{n-1}))\|^2_{\boldsymbol{H}}\, \mathrm{e}^{-2\nu t_{n-1}}\Big\}\Big).
\end{aligned} \tag{4.10}
$$

Proof. The proof follows the lines of [34, Theorem 3.12] for the semidiscretization in time. Again, to keep this work self-contained, we summarize the proof for the setting of Problem 3.8 in Appendix B. ◻

Next, we estimate the terms on the right-hand side of (4.8) and (4.10), respectively, one by one.

LEMMA 4.5. *For $s \in \{1,\dots,r+1\}$, let $\boldsymbol{U} \in H^{k+3}_\nu(\mathbb{R};\boldsymbol{H}) \cap H^2_\nu(\mathbb{R};\boldsymbol{H}^s(\Omega)^{(d+1)^2})$ be satisfied. With $\boldsymbol{Z} = \boldsymbol{U} - I_\tau\Pi_h\boldsymbol{U}$ there holds that*

$$
\|\partial_t \boldsymbol{M}_0(\boldsymbol{U} - \widehat{I}_\tau^{k+1}\Pi_h\boldsymbol{U})\|^2_{\tau,\nu} + \|\boldsymbol{M}_1\boldsymbol{Z}\|^2_{\tau,\nu} \le CT(\tau^{2(k+1)} + h^{2s}), \tag{4.11a}
$$

$$
\|\boldsymbol{A}_h\boldsymbol{Z}\|^2_{\tau,\nu} + |J_\gamma(\boldsymbol{Z},\boldsymbol{Z})|^2_{\tau,\nu} + |J^{\boldsymbol{n}}_\partial(\boldsymbol{Z},\boldsymbol{Z})|^2_{\tau,\nu} \le CTh^{2(s-1)}, \tag{4.11b}
$$

$$
\max_{1\le n\le N}\Big\{\|\boldsymbol{M}_0(\Pi_h\boldsymbol{U}^+(t_{n-1}) - I_\tau\Pi_h\boldsymbol{U}^+(t_{n-1}))\|^2_{\boldsymbol{H}}\,\mathrm{e}^{-2\nu t_{n-1}}\Big\}\Big) \le C\tau^{2(k+1)}. \tag{4.11c}
$$

Proof. Using a splitting as in (4.5) and commutation properties of the operators show that

$$
\begin{aligned}
\|\partial_t \boldsymbol{M}_0(\boldsymbol{U}(t) - \widehat{I}_\tau^{k+1}\Pi_h\boldsymbol{U}(t))\|_H &\le \|\boldsymbol{M}_0\Pi_h\partial_t(\boldsymbol{U}(t) - \widehat{I}_\tau^{k+1}\boldsymbol{U}(t))\|_{\boldsymbol{H}} \\
&\quad + \|\boldsymbol{M}_0(\partial_t\boldsymbol{U}(t) - \Pi_h\partial_t\boldsymbol{U}(t))\|_{\boldsymbol{H}}.
\end{aligned} \tag{4.12}
$$

By (3.10) and (3.14) along with the boundedness of M_0 and Π_h, we conclude from (4.12) that

$$
\|\partial_t \boldsymbol{M}_0(\boldsymbol{U}(t) - \widehat{I}_\tau^{k+1}\Pi_h\boldsymbol{U}(t))\|_H \le C\Big(\tau^{k+1}\sup_{t\in I_n}\|\partial_t^{k+2}\boldsymbol{U}\|_{\boldsymbol{H}} + h^s\sup_{t\in I_n}\|\partial_t\boldsymbol{U}\|_{\boldsymbol{H}^s(\Omega)}\Big).
$$

Recalling the definition of the norm $\|\cdot\|_{\tau,\nu}$ in (3.6c), this directly proves the first of the bounds in (4.11a). The estimate of $\|\boldsymbol{M}_1\boldsymbol{Z}\|_{\tau,\nu}$ in (4.11a) and inequality (4.11c) follow similarly.

It remains to prove (4.11b). By (3.9), the interpolation operator $\widehat{I}_\tau^{k+1}$ acts as the identity at the Gauss–Radau points $t_{n,\mu}$. Thus, for $\mu = 0,\dots,k$, we get that

$$
\|\boldsymbol{A}_h\boldsymbol{Z}(t_{n,\mu})\|_{\boldsymbol{H}} = \|\boldsymbol{A}_h(\boldsymbol{U} - \Pi_h\boldsymbol{U})(t_{n,\mu})\|_{\boldsymbol{H}}, \tag{4.13a}
$$

$$
|J_\gamma(\boldsymbol{Z}(t_{n,\mu}),\boldsymbol{Z}(t_{n,\mu}))|_{\tau,\nu} = |J_\gamma((\boldsymbol{U} - \Pi_h\boldsymbol{U})(t_{n,\mu}),(\boldsymbol{U} - \Pi_h\boldsymbol{U})(t_{n,\mu}))|_{\tau,\nu}, \tag{4.13b}
$$

$$
|J^{\boldsymbol{n}}_\partial(\boldsymbol{Z},\boldsymbol{Z})|_{\tau,\nu} = |J^{\boldsymbol{n}}_\partial((\boldsymbol{U} - \Pi_h\boldsymbol{U})(t_{n,\mu}),(\boldsymbol{U} - \Pi_h\boldsymbol{U})(t_{n,\mu}))|_{\tau,\nu}. \tag{4.13c}
$$

Setting $\boldsymbol{\Theta}_\mu^n := (\boldsymbol{U} - \Pi_h \boldsymbol{U})(t_{n,\mu})$, using (3.34), and recalling the duality relations (3.23a) and (3.26a), we get for the right-hand side of (4.13a) that

$$
\begin{aligned}
\langle \boldsymbol{A}_h \boldsymbol{\Theta}_\mu^n, \boldsymbol{Y}_h \rangle &= \langle \boldsymbol{\Theta}_{\mu,2}^n, -\operatorname{Div}_{\mathrm{dg}}^* \boldsymbol{Y}_{h,1} \rangle - \langle \operatorname{Grad}_{\mathrm{dg}} \boldsymbol{\Theta}_{\mu,1}^n, \boldsymbol{Y}_{h,2} \rangle \\
&\qquad + \langle \boldsymbol{\Theta}_{\mu,4}^n, \operatorname{div}_{\mathrm{dg}}^* Y_{h,3} \rangle + \langle \operatorname{grad}_{\mathrm{dg}} \Theta_{\mu,3}^n, \boldsymbol{Y}_{h,4} \rangle \\
&= \langle \boldsymbol{\Theta}_{\mu,2}^n, \operatorname{Grad}_{\mathrm{dg}} \boldsymbol{Y}_{h,1} \rangle + \sum_{e \in \mathcal{E}_h^\partial} \langle \boldsymbol{\Theta}_{\mu,2}^n \cdot \boldsymbol{n}_e, \boldsymbol{Y}_{h,1} \rangle_e \\
&\qquad - \langle \operatorname{Grad}_{\mathrm{dg}} \boldsymbol{\Theta}_{\mu,1}^n, \boldsymbol{Y}_{h,2} \rangle - \langle \boldsymbol{\Theta}_{\mu,4}^n, \operatorname{grad}_{\mathrm{dg}} Y_{h,3} \rangle \\
&\qquad - \sum_{e \in \mathcal{E}_h^\partial} \langle \boldsymbol{\Theta}_{\mu,4}^n \cdot \boldsymbol{n}_e, Y_{h,3} \rangle_e + \langle \operatorname{grad}_{\mathrm{dg}} \Theta_{\mu,3}^n, \boldsymbol{Y}_{h,4} \rangle \,,
\end{aligned} \tag{4.14}
$$

for all $\boldsymbol{Y}_h \in \boldsymbol{H}_h^{\mathrm{dg}}$. Next, we bound the right-hand side of (4.14) term by term. We start with the last term in (4.14). By (3.22a), the definition of $\operatorname{grad}_{\mathrm{dg}}$, and the Cauchy–Schwarz inequality, it follows that

$$
\begin{aligned}
&\langle \operatorname{grad}_{\mathrm{dg}} \Theta_{\mu,3}^n, \boldsymbol{Y}_{h,4} \rangle \\
&\quad = \langle \operatorname{grad}_h \Theta_{\mu,3}^n, \boldsymbol{Y}_{h,4} \rangle - \sum_{e \in \mathcal{E}_h^i} \langle [\![\Theta_{\mu,3}^n]\!], \{\!\{ \boldsymbol{Y}_{h,4} \}\!\} \cdot \boldsymbol{n}_e \rangle_e - \sum_{e \in \mathcal{E}_h^\partial} \langle \Theta_{\mu,3}^n, \boldsymbol{Y}_{h,4} \cdot \boldsymbol{n}_e \rangle_e \\
&\quad \leq \sum_{K \in \mathcal{T}_h} \| \nabla \Theta_{\mu,3}^n \|_{L^2(K)} \| \boldsymbol{Y}_{h,4} \|_{L^2(K)} + \sum_{e \in \mathcal{E}_h^i} \| [\![\Theta_{\mu,3}^n]\!] \|_{L^2(e)} \| \{\!\{ \boldsymbol{Y}_{h,4} \}\!\} \cdot \boldsymbol{n}_e \|_{L^2(e)} \\
&\qquad\qquad + \sum_{e \in \mathcal{E}_h^\partial} \| \Theta_{\mu,3}^n \|_{L^2(e)} \| \boldsymbol{Y}_{h,4} \cdot \boldsymbol{n}_e \|_{L^2(e)} \,.
\end{aligned} \tag{4.15}
$$

Using (3.14) with $m = 1$, (3.15a), and the inverse relation (cf. [26, Lemma 1.46])

$$
h_K^{1/2} \| w_h \|_{L^2(e)} \leq C_{\mathrm{inv}} \| w_h \|_{L^2(K)} \,, \qquad \text{for } e \subset K \,, \quad w \in \mathbb{Q}_r^d \,, \tag{4.16}
$$

we obtain from (4.15) that

$$
\langle \operatorname{grad}_{\mathrm{dg}} \Theta_{\mu,3}^n, \boldsymbol{Y}_{h,4} \rangle \leq C h^{s-1} \| \boldsymbol{U} \|_{\boldsymbol{H}^s(\Omega)} \Big(\sum_{K \in \mathcal{T}_h} \| \boldsymbol{Y}_{h,4} \|_{L^2(K)}^2 \Big)^{1/2} . \tag{4.17}
$$

For the fourth and fifth term on the right-hand side of (4.14) we have that

$$
\begin{aligned}
&\langle \boldsymbol{\Theta}_{\mu,4}^n, \operatorname{grad}_{\mathrm{dg}} Y_{h,3} \rangle + \sum_{e \in \mathcal{E}_h^\partial} \langle \boldsymbol{\Theta}_{\mu,4}^n \cdot \boldsymbol{n}_e, Y_{h,3} \rangle_e \\
&\qquad = \langle \boldsymbol{\Theta}_{\mu,4}^n, \operatorname{grad}_h Y_{h,3} \rangle - \sum_{e \in \mathcal{E}_h^i} \langle \{\!\{ \boldsymbol{\Theta}_{\mu,4}^n \}\!\} \cdot \boldsymbol{n}_e, [\![Y_{h,3}]\!] \rangle_e \,.
\end{aligned}
$$

From this, we find that

$$
\begin{aligned}
\langle \boldsymbol{\Theta}_{\mu,4}^n, \operatorname{grad}_{\mathrm{dg}} Y_{h,3} \rangle &\leq \sum_{K \in \mathcal{T}_h} \| \boldsymbol{\Theta}_{\mu,4}^n \|_{L^2(K)} \| \nabla Y_{h,3} \|_{L^2(K)} \\
&\qquad + C \sum_{e \in \mathcal{E}_h^i} \| \{\!\{ \boldsymbol{\Theta}_{\mu,4}^n \}\!\} \cdot \boldsymbol{n}_e \|_{L^2(e)} \| [\![Y_{h,3}]\!] \|_{L^2(e)} \,.
\end{aligned} \tag{4.18}
$$

Using (3.14) with $m = 0$, bounding $\|\nabla Y_{h,3}\|_{L^2(K)}$ by the the H^1-L^2 inverse inequality, and applying (3.15a) and (4.16), we deduce from (4.18) that

$$\langle \boldsymbol{\Theta}^n_{\mu,4}, \operatorname{grad}_{\mathrm{dg}} Y_{h,3}\rangle \leq C h^{s-1} \|\boldsymbol{U}\|_{\boldsymbol{H}^s(\Omega)} \Big(\sum_{K\in\mathcal{T}_h} \|Y_{h,3}\|^2_{L^2(K)}\Big)^{1/2} . \tag{4.19}$$

The first three terms on the right-hand side of (4.14) can be treated similarly. Since

$$\|\boldsymbol{A}_h \boldsymbol{W}\|_{\boldsymbol{H}} = \sup_{\boldsymbol{Y}_h \in \boldsymbol{H}_h^{\mathrm{dg}}\setminus\{\boldsymbol{0}\}} \frac{\langle \boldsymbol{A}_h \boldsymbol{W}, \boldsymbol{Y}_h\rangle_{\boldsymbol{H}}}{\|\boldsymbol{Y}_h\|_{\boldsymbol{H}}} , \qquad \text{for } \boldsymbol{W} \in D(\boldsymbol{A}) + \boldsymbol{H}_h^{\mathrm{dg}} ,$$

combining (4.14) with (4.17) and (4.19) and their counterparts for the first and the second term on the right-hand side of (4.14) proves for (4.13) that

$$\|\boldsymbol{A}_h(\boldsymbol{U}(t_{n,\mu}) - \Pi_h \boldsymbol{U}(t_{n,\mu}))\|^2_{\boldsymbol{H}} \leq C h^{2(s-1)} , \qquad \text{for } \mu = 1,\ldots,k . \tag{4.20}$$

Applying the temporal quadrature formula (3.4) to (4.20), summing up the resulting inequality from $n = 1$ to N and recalling the definition in (3.6c) yields the error bound for $\|\boldsymbol{A}_h \boldsymbol{Z}\|^2_{\tau,\nu}$ in (4.11b). The bound for $|J_\gamma(\boldsymbol{Z},\boldsymbol{Z})|^2_{\tau,\nu}$ follows similarly. Using (3.15a) we get that

$$J_\gamma(\boldsymbol{\Theta}^n_\mu, \boldsymbol{\Theta}^n_\mu) \leq \frac{C}{h} h^{2s-1} \left(\|\boldsymbol{U}^{(1)}\|^2_{\boldsymbol{H}^s(\Omega)} + \|U^{(3)}\|^2_{H^s(\Omega)} \right) . \tag{4.21}$$

Applying the temporal quadrature formula (3.4) to (4.21), summing up the resulting inequality from $n = 1$ to N and recalling the definition in (3.6b) yields the error bound for $|J_\gamma(\boldsymbol{Z},\boldsymbol{Z})|^2_{\tau,\nu}$ in (4.11b). The term $|J^{\boldsymbol{n}}_\partial(\boldsymbol{Z},\boldsymbol{Z})|^2_{\tau,\nu}$ in (4.9) is bounded by the same arguments. This completes the proof of (4.11). ∎

Theorem 4.1 proves convergence of the error measured in the time-mesh dependent norm $\|\cdot\|_{\tau,\nu}$ defined in (3.6c). Using a result of [33, Theorem 2.5], convergence with respect to the norm $\|\cdot\|_\nu$ induced by the inner product (2.2) of the continuous function space $H_\nu(\mathbb{R};\boldsymbol{H})$ can still be ensured.

COROLLARY 4.6. *Under the assumptions of Theorem 4.1, there holds that*

$$\|\boldsymbol{U} - \boldsymbol{U}_{\tau,h}\|^2_\nu \leq C(1+T)\,\mathrm{e}^{2\nu T}(\tau^{2(k+1)} + h^{2r}) . \tag{4.22}$$

Proof. We split the error into the parts

$$\|\boldsymbol{U} - \boldsymbol{U}_{\tau,h}\|_\nu \leq \|\boldsymbol{U} - I_\tau \boldsymbol{U}\|_\nu + \|I_\tau \boldsymbol{U} - \boldsymbol{U}_{\tau,h}\|_\nu . \tag{4.23}$$

For the first of the terms on the right-hand side of (4.23), we deduce from the interpolation error estimate proved in [33, Theorem 2.5] that

$$\|\boldsymbol{U} - I_\tau \boldsymbol{U}\|^2_\nu \leq C \tau^{2(k+1)} .$$

For the second of the terms on the right-hand side of (4.23), we get, by the exactness (3.5) of the quadrature in time for all $p \in \mathbb{P}_{2k}(I_n;\mathbb{R})$ along with (4.2), that

$$\begin{aligned} \|I_\tau \boldsymbol{U} - \boldsymbol{U}_{\tau,h}\|^2_\nu &= \|I_\tau \boldsymbol{U} - \boldsymbol{U}_{\tau,h}\|^2_{\tau,\nu} = \|\boldsymbol{U} - \boldsymbol{U}_{\tau,h}\|^2_{\tau,\nu} \\ &\leq C(1+T)\,\mathrm{e}^{2\nu T}(\tau^{2(k+1)} + T h^{2r}) . \end{aligned} \tag{4.24}$$

Together, (4.23) to (4.24) prove the assertion (4.22). ∎

By Theorem 4.1 and Corollary 4.6, the main result (1.3) of this work is thus proved. We still comment on the optimality of this error estimate.

REMARK 4.7. The error estimates (4.2) and (4.22) are of optimal order for the time discretization, but they are of suboptimal order for the space discretization with respect to the approximation properties of the discrete spaces defined in (3.11) and (3.12). The numerical evaluation of the first-order approach, presented in Section 5, argues for sharpness of the error bounds. In the error analysis, the loss of one order of convergence in space is due to the occurrence of the term $\|\boldsymbol{A}_h\boldsymbol{Z}\|_{\tau,\nu}$, for instance in the upper error bounds in (4.8) and (4.10), involving first-order derivatives of the interpolation error $\boldsymbol{Z}$ defined in (4.3). We did not succeed in proving sharper bounds for the quantities estimated from above by $\|\boldsymbol{A}_h\boldsymbol{Z}\|_{\tau,\nu}$.

The abstract solution theory of Picard [42] relies on weak assumptions on the operators of the evolutionary problem only (cf. Theorem 2.5), which is considered to be advantageous for problems that are of interest in practice. However, stability of the solution is then guaranteed in $\boldsymbol{L}^2$ only; cf. (2.16). As a consequence, the error estimates (4.2) and (4.22) ensure convergence in the $\boldsymbol{L}^2$-norm in space only, and absorption arguments, which are often useful in error analyses, are difficult to apply in this context. Due to the identity (A.3), which preserves the skew-selfadjointness of $\boldsymbol{A}_h$ on the discrete level by the addition of the correction term J_∂, no stricter error control than in the $\boldsymbol{L}^2$-norm is obtained by our analysis in the evolutionary framework. In this respect, the error analysis of this work differs strongly from standard error analyses of DG methods for first- and second-order (in space) problems. They are based on stability estimates for the spatial differential operator and lead to stricter error control in the DG energy norm, cf., e.g., [20, Section 5.2]. For such error analyses of DG methods, we refer in particular to [18, 20] and also to [26, 46]. At the current state, the feasibility of optimal-order error bounds for the space discretization thus remains an open problem and is left as a work for the future. A redesign of the interelement fluxes (penalization terms) might also enable improved error estimates.

For error estimates and numerical investigations of the system (1.1), based on approximations of the second-order in space formulation with the native unknowns $\boldsymbol{u}$, $\boldsymbol{v} = \partial_t\boldsymbol{u}$, and p, we also refer to [7, 8, 11].

5. Numerical experiments. Here we present some results of our performed numerical experiments to illustrate the error estimates (4.2) and (4.22). For simplicity and brevity, the scalar-valued wave equation is considered only. In first-order form, this equation reads as

$$\left(\partial_t \begin{bmatrix} I & \mathbf{0} \\ 0 & \boldsymbol{I} \end{bmatrix} + \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & \mathbf{0} \end{bmatrix}\right) \begin{bmatrix} u \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \partial_t^{-1} f \\ \mathbf{0} \end{bmatrix}, \tag{5.1}$$

where $I \in R^{1,1}$ and $\boldsymbol{I} \in \mathbb{R}^{d,d}$, with $d > 1$, denote the identity matrix. Applying the DG discretization of Section 3.4 to (5.1) and following the error analysis of Section 4, the error estimates (4.2) and (4.22) continue to hold for the discrete solution of (5.1). For this reason, we study the simplified model (5.1) only. Moreover, the weight ν in (2.1) and in the quadrature formula (3.4) is chosen as $\nu = 0$, for simplicity. More sophisticated numerical studies of the discrete scheme (3.31), also in three space dimensions, will be considered in a forthcoming work. They do not become feasible for (3.31) without an efficient linear solver and preconditioner that still need to be developed for (3.31). For this, we also refer to [8], where an iterative solver is presented and evaluated for the second-order in space formulation (1.1). Our implementation was done in an in-house frontend for the `deal.II` library [1].

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We study (5.1) for $\Omega = (0,1)^2$ and $I = (0,T]$, with $T = 1$, and the prescribed solution

$$u(x,y) = \frac{1}{4}\sin(2t\omega_1)\left(\sin\left(2\omega_2\left(x-\frac{1}{4}\right)\right)+1\right)\left(\sin\left(2\omega_2\left(y-\frac{1}{4}\right)\right)+1\right), \tag{5.2a}$$

(5.2b)
$$\boldsymbol{v}(x,y) = \begin{bmatrix} \frac{1}{4}\cos(2tw_1)\cos\left(2\omega_2\left(x-\frac{1}{4}\right)\right)\left(\sin\left(2\omega_2\left(y-\frac{1}{4}\right)\right)+1\right) \\ \frac{1}{4}\cos(2t\omega_1)\cos\left(2\omega_2\left(y-\frac{1}{4}\right)\right)\left(\sin\left(2\omega_2\left(x-\frac{1}{4}\right)\right)+1\right) \end{bmatrix},$$

with $\omega_1 = \omega_2 = \pi$. The norm of $L^\infty(I;L^2)$ is approximated by

$$\|w\|_{L^\infty(I;L^2)} \approx \max\{\|w_{|I_n}(t_{n,m})\| \ : \ m = 1,\dots,M\,,\ n = 1,\dots,N\}\,, \quad \text{with} \quad M = 100\,,$$

and the Gauss quadrature nodes $t_{n,m}$ of I_n. We investigate the space-time convergence of the scheme in Problem 3.8 applied to (5.1) in order to study the sharpness of (4.2) and (4.22). For this, the domain Ω is decomposed into a sequence of successively refined meshes of quadrilateral finite elements. The spatial and temporal mesh sizes are halved in each of the refinement steps. The step sizes of the coarsest mesh are $h_0 = 1/2\sqrt{2}$ and $\tau_0 = 0.2$. We choose the polynomial degree in time k and in space r as $k = r = 1$, $k = r = 2$, and $k = r = 3$ such that a solution $(u_{\tau,h}, \boldsymbol{v}_{\tau,h}) \in Y^k_{\tau,0}(H_h^{\text{dg}}) \times \left(Y^k_{\tau,0}(H_h^{\text{dg}})\right)^2$ is obtained; cf. (3.2) and (3.12b) for the definition of the discrete spaces. The calculated errors and corresponding experimental orders of convergence are summarized in Table 5.1. For $k = r = 1$ and $k = r = 3$, the computed errors nicely confirm the results of Theorem 4.1 and Corollary 4.6. In particular, the suboptimality of the convergence in space is illustrated. For $k = r = 2$, optimal-order approximation properties with respect to the space-time $L^2(L^2)$-norm are ensured for the DG scheme of Problem 3.8 applied to (5.1). This difference needs further elucidation in future work. It might be due to effects of superconvergence in space on the highly structured grids used in the convergence tests. For the analysis of superconvergence in the discrete time nodes of variational discretizations in time to the wave equation, we also refer to [12] and the discussion and references therein.

6. Summary and outlook. In this work we presented and analyzed the numerical approximation of a prototype hyperbolic-parabolic model of dynamic poro- or thermoelasticity that was rewritten as a first-order evolutionary system in space and time such that the abstract solution theory of Picard [42, 48] in exponentially weighted Bochner spaces became applicable. A family of DG schemes in space and time was studied where the innovation came through the DG discretization in space of the first-order formulation. By a consistent definition of the first-order spatial differential operators on broken polynomials spaces and the addition of boundary correction terms, the mathematical evolutionary structure was inherited by the discrete system from the continuous one. Well-posedness of the fully discrete problem and error estimates for its solution were proved. For the discretization in time, optimal order of convergence was ensured in the analysis. For the discretization in space, suboptimality was obtained only. Numerical experiments performed for a simplified hyperbolic model problem indicate sharpness of the presented error estimation. Numerical studies for more complex problems, involving strong heterogeneities and anisotropies, and the comparison with approximations based on the second-order-in-space problem formulation are also in our scope of interest. However, such numerical studies require efficient iterative solver tailored to the first-order system structure and remain as a work for the future.

Acknowledgement. Computational resources (HPC-cluster HSUper) have been provided by the project hpc.bw, funded by dtec.bw—Digitalization and Technology Research Center of the Bundeswehr. dtec.bw is funded by the European Union—NextGenerationEU.

TABLE 5.1
$L^2(L^2)$ and $L^\infty(L^2)$ errors and experimental orders of convergence (EOC) for (5.2).

τ	h	$\|u-u_{\tau,h}\|_{L^2(L^2)}$	EOC	$\|\boldsymbol{v}-\boldsymbol{v}_{\tau,h}\|_{L^2(\boldsymbol{L}^2)}$	EOC	$\|u-u_{\tau,h}\|_{L^\infty(L^2)}$	EOC	$\|\boldsymbol{v}-\boldsymbol{v}_{\tau,h}\|_{L^\infty(\boldsymbol{L}^2)}$	EOC
$\tau_0/2^0$	$h_0/2^0$	5.3194992212e-02	–	4.5002473555e-02	–	4.7295130546e-02	–	6.5759571453e-02	–
$\tau_0/2^1$	$h_0/2^1$	2.3727189278e-02	1.16	1.7391785239e-02	1.37	2.4542534222e-02	0.95	3.1747780152e-02	1.05
$\tau_0/2^2$	$h_0/2^2$	1.1501095599e-02	1.04	8.2761888911e-03	1.07	1.2881433392e-02	0.93	1.5750496027e-02	1.01
$\tau_0/2^3$	$h_0/2^3$	5.7478425626e-03	1.00	4.1880359747e-03	0.98	6.5079886919e-03	0.99	8.2527298893e-03	0.93
$\tau_0/2^4$	$h_0/2^4$	2.8804514083e-03	1.00	2.1087518836e-03	0.99	3.2684508287e-03	0.99	4.2300827636e-03	0.96
$\tau_0/2^5$	$h_0/2^5$	1.4431961280e-03	1.00	1.0587306355e-03	0.99	1.6558549975e-03	0.98	2.1319495371e-03	0.99

$k = r = 1$

τ	h	$\|u-u_{\tau,h}\|_{L^2(L^2)}$	EOC	$\|\boldsymbol{v}-\boldsymbol{v}_{\tau,h}\|_{L^2(\boldsymbol{L}^2)}$	EOC	$\|u-u_{\tau,h}\|_{L^\infty(L^2)}$	EOC	$\|\boldsymbol{v}-\boldsymbol{v}_{\tau,h}\|_{L^\infty(\boldsymbol{L}^2)}$	EOC
$\tau_0/2^0$	$h_0/2^0$	4.7374720709e-03	–	4.3371374886e-03	–	6.0899775976e-03	–	5.7444903854e-03	–
$\tau_0/2^1$	$h_0/2^1$	3.1848403311e-04	3.89	3.4698808171e-04	3.64	2.0092654075e-04	4.92	2.7491608725e-04	4.39
$\tau_0/2^2$	$h_0/2^2$	3.4399296400e-05	3.21	4.0798369154e-05	3.09	1.0462351708e-05	4.26	1.5873381996e-05	4.11
$\tau_0/2^3$	$h_0/2^3$	4.1449361940e-06	3.05	5.0350201039e-06	3.02	6.4029326291e-07	4.03	9.7645179146e-07	4.02
$\tau_0/2^4$	$h_0/2^4$	5.1338358092e-07	3.01	6.2747015866e-07	3.00	3.9831928910e-08	4.01	6.0803199465e-08	4.01

$k = r = 2$

τ	h	$\|u-u_{\tau,h}\|_{L^2(L^2)}$	EOC	$\|\boldsymbol{v}-\boldsymbol{v}_{\tau,h}\|_{L^2(\boldsymbol{L}^2)}$	EOC	$\|u-u_{\tau,h}\|_{L^\infty(L^2)}$	EOC	$\|\boldsymbol{v}-\boldsymbol{v}_{\tau,h}\|_{L^\infty(\boldsymbol{L}^2)}$	EOC
$\tau_0/2^0$	$h_0/2^0$	5.4568232110e-04	–	4.7831874231e-04	–	4.3042613140e-04	–	6.1497828029e-04	–
$\tau_0/2^1$	$h_0/2^1$	6.8748757804e-05	2.99	5.2596282338e-05	3.18	7.6944178401e-05	2.48	9.9985648321e-05	2.62
$\tau_0/2^2$	$h_0/2^2$	9.2647645044e-06	2.89	7.0737321746e-06	2.89	1.2948270810e-05	2.57	1.6227622934e-05	2.62
$\tau_0/2^3$	$h_0/2^3$	1.1623900524e-06	2.99	9.0027150466e-07	2.97	1.6899479588e-06	2.94	2.0259945045e-06	3.00
$\tau_0/2^4$	$h_0/2^4$	1.4429510473e-07	3.01	1.1253393808e-07	3.00	2.1124349485e-07	3.00	2.5324931306e-07	3.00

$k = r = 3$

Appendix A. Proof of Lemma 4.3.

To keep this work self-contained, we present the proof of Lemma 4.3.

Proof. Let $\boldsymbol{U}_{\tau,h} \in Y_\tau^k(\boldsymbol{H}_h^{\mathrm{dg}})$ be the solution of Problem 3.8 and $I_\tau \Pi_h \boldsymbol{U} \in Y_\tau^k(\boldsymbol{H}_h^{\mathrm{dg}})$ its approximation in $Y_\tau^k(\boldsymbol{H}_h^{\mathrm{dg}})$ by combined interpolation and projection. Under the regularity assumption (4.1), there holds for the solution $\boldsymbol{U} \in H_\nu(\mathbb{R}; D(\boldsymbol{A}))$ of Problem 2.4 that

$$(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A})\boldsymbol{U}(t) = \boldsymbol{F}(t)\,, \qquad \text{for } t \in [0,T]\,,$$

such that, for $n \in \{1,\dots,N\}$,

$$\begin{aligned}Q_n[(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A}_h)\boldsymbol{U}, \boldsymbol{V}_{\tau,h}]_\nu + Q_n[J_\partial(\boldsymbol{U}, \boldsymbol{V}_{\tau,h}) + J_\gamma(\boldsymbol{U}, \boldsymbol{V}_{\tau,h})]_\nu \\ + \langle \boldsymbol{M}_0 [\![\boldsymbol{U}]\!]_{n-1}, \boldsymbol{V}_{\tau,h}^{+,n-1}\rangle = Q_n[I_\tau \Pi_h \boldsymbol{F}, \boldsymbol{V}_{\tau,h}],\end{aligned} \tag{A.1}$$

for all $\boldsymbol{V}_{\tau,h} \in Y_\tau^k(\boldsymbol{H}_h^{\mathrm{dg}})$. In (A.1), $\boldsymbol{A}_h$ is the natural extension of Definition 3.6 to $D(\boldsymbol{A})$ with

$$\langle \mathrm{grad}_{\mathrm{dg}}\, y, \boldsymbol{v}_h\rangle := \langle \mathrm{grad}_0\, y, \boldsymbol{v}_h\rangle\,, \qquad \forall y \in H_0^1(\Omega)\,, \forall \boldsymbol{v}_h \in (Y_h^r)^d\,,$$

$$\langle \mathrm{div}_{\mathrm{dg}}\, \boldsymbol{z}, y_h\rangle := \langle \mathrm{div}\, \boldsymbol{z}, y_h\rangle - \sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{z}\cdot\boldsymbol{n}_e, y_h\rangle_e\,, \qquad \forall \boldsymbol{z} \in D(\mathrm{div})\,, \forall v_h \in Y_h^r\,,$$

$$\langle \mathrm{Grad}_{\mathrm{dg}}\, \boldsymbol{y}, \boldsymbol{v}_h\rangle := \langle \mathrm{Grad}_0\, \boldsymbol{y}, \boldsymbol{v}_h\rangle\,, \qquad \forall \boldsymbol{y} \in H_0^1(\Omega)^d\,, \ \forall \boldsymbol{v}_h \in (Y_h^r)^{d\times d} \cap L^2(\Omega)^{d\times d}_{\mathrm{sym}}\,,$$

$$\langle \mathrm{Div}_{\mathrm{dg}}\, \boldsymbol{z}, \boldsymbol{y}_h\rangle := \langle \mathrm{Div}\, \boldsymbol{z}, \boldsymbol{y}_h\rangle - \sum_{e\in\mathcal{E}_h^\partial} \langle z\cdot\boldsymbol{n}_e, y_h\rangle_e\,, \qquad \forall \boldsymbol{z} \in D(\mathrm{Div}) \cap L^2(\Omega)^{d\times d}_{\mathrm{sym}}\,, \ \forall v_h \in (Y_h^r)^d\,.$$

For (A.1), we note that by (3.29) and (3.32), there holds for $\boldsymbol{V}_h \in \boldsymbol{H}_h^{\mathrm{dg}}$ that

$$\begin{aligned}
&\langle \boldsymbol{A}_h \boldsymbol{U}, \boldsymbol{V}_h\rangle + J_\partial(\boldsymbol{U}, \boldsymbol{V}_h)\\
&\quad = -\langle \operatorname{Div}_{\mathrm{dg}} \boldsymbol{U}^2, \boldsymbol{V}_h^{(1)}\rangle - \langle \operatorname{Grad}_{\mathrm{dg}} \boldsymbol{U}^{(1)}, \boldsymbol{V}_h^{(2)}\rangle\\
&\qquad + \langle \operatorname{div}_{\mathrm{dg}} \boldsymbol{U}^{(4)}, V_h^{(3)}\rangle + \langle \operatorname{grad}_{\mathrm{dg}} U^{(3)}, \boldsymbol{V}_h^{(4)}\rangle\\
&\qquad - \sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{U}^{(2)}\cdot \boldsymbol{n}_e, V_{\tau,h}^{(1)}\rangle_e + \sum_{e\in\mathcal{E}_h^\partial} \langle \boldsymbol{U}^{(4)}\cdot \boldsymbol{n}_e, V_{\tau,h}^{(3)}\rangle_e\\
&\quad = -\langle \operatorname{Div} \boldsymbol{U}^2, \boldsymbol{V}_h^{(1)}\rangle - \langle \operatorname{Grad}_0 \boldsymbol{U}^{(1)}, \boldsymbol{V}_h^{(2)}\rangle\\
&\qquad + \langle \operatorname{div} \boldsymbol{U}^{(4)}, V_h^{(3)}\rangle + \langle \operatorname{grad}_0 U^{(3)}, \boldsymbol{V}_h^{(4)}\rangle\\
&\quad = \langle \boldsymbol{A}\boldsymbol{U}, \boldsymbol{V}_h\rangle.
\end{aligned}$$

Further, we have that $J_\gamma(\boldsymbol{U}, \boldsymbol{V}_{\tau,h}) = 0$ for $\boldsymbol{U} \in D(\boldsymbol{A})$. Under the assumption (4.1), $[\![\boldsymbol{U}]\!]_{n-1} = \boldsymbol{0}$ is satisfied. The identity $Q_n[\boldsymbol{F}, \boldsymbol{V}_{\tau,h}] = Q_n[I_\tau \Pi_h \boldsymbol{F}, \boldsymbol{V}_{\tau,h}]$ follows from (3.4), (3.7), and (3.13). Subtracting now (3.31) from (A.1) yields, with the splitting (4.3), the error equation

$$\begin{aligned}
&Q_n[(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A}_h)\boldsymbol{E}_{\tau,h}, \boldsymbol{V}_{\tau,h}]_\nu + Q_n[J_\partial(\boldsymbol{E}_{\tau,h}, \boldsymbol{V}_{\tau,h}) + J_\gamma(\boldsymbol{E}_{\tau,h}, \boldsymbol{V}_{\tau,h})]_\nu\\
&\qquad + \langle \boldsymbol{M}_0 [\![\boldsymbol{E}_{\tau,h}]\!]_{n-1}, \boldsymbol{V}_{\tau,h}^{+,n-1}\rangle\\
&\quad = -Q_n[(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A}_h)\boldsymbol{Z}, V_{\tau,h}]_\nu - Q_n[J_\partial(\boldsymbol{Z}, \boldsymbol{V}_{\tau,h})\\
&\qquad + J_\gamma(\boldsymbol{Z}, \boldsymbol{V}_{\tau,h})]_\nu - \langle M_0 [\![\boldsymbol{Z}]\!]_{n-1}, \boldsymbol{V}_{\tau,h}^{+,n-1}\rangle,
\end{aligned} \tag{A.2}$$

for all $\boldsymbol{V}_{\tau,h} \in Y_\tau^k(H_h^{\mathrm{dg}})$ and $n = 1, \dots, N$. Choosing $\boldsymbol{V}_{\tau,h} = \boldsymbol{E}_{\tau,h}$ and recalling that

$$\langle \boldsymbol{A}_h \boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h}\rangle + J_\partial(\boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h}) = 0\,, \tag{A.3}$$

by the arguments of (3.39), we get that

$$\begin{aligned}
&Q_n[(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1)\boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h}]_\nu\\
&\qquad + Q_n[J_\gamma(\boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h})]_\nu + \langle \boldsymbol{M}_0 [\![\boldsymbol{E}_{\tau,h}]\!]_{n-1}, \boldsymbol{E}_{\tau,h}^{+,n-1}\rangle\\
&\quad = -Q_n[(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A}_h)\boldsymbol{Z}, \boldsymbol{E}_{\tau,h}]_\nu - Q_n[J_\partial(\boldsymbol{Z}, \boldsymbol{E}_{\tau,h})\\
&\qquad + J_\gamma(\boldsymbol{Z}, \boldsymbol{E}_{\tau,h})]_\nu - \langle \boldsymbol{M}_0 [\![\boldsymbol{Z}]\!]_{n-1}, \boldsymbol{E}_{\tau,h}^{+,n-1}\rangle =: E_i^n,
\end{aligned} \tag{A.4}$$

for $n = 1, \dots N$ and $\boldsymbol{E}_{\tau,h}^{+,n-1} := \boldsymbol{E}_{\tau,h}^{+}(t_{n-1})$ by the definition in (3.3).

By [34, Lemma 3.5] for the left-hand side of (A.4), there holds that

$$\begin{aligned}
&Q_n[(\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1)\boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h}]_\nu\\
&\qquad + Q_n[J_\gamma(\boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h})]_\nu + \langle \boldsymbol{M}_0 [\![\boldsymbol{E}_{\tau,h}]\!]_{n-1}, \boldsymbol{E}_{\tau,h}^{+,n-1}\rangle\\
&\quad \geq \gamma \|\boldsymbol{E}_{\tau,h}\|_{\tau,\nu,n}^2 + |J_\gamma(\boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h})|_{\tau,\nu,n}^2\\
&\qquad + \frac{1}{2}\Big[\langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}^-(t_n), \boldsymbol{E}_{\tau,h}^-(t_n)\rangle\, \mathrm{e}^{-2\nu\tau_n} - \langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}^-(t_{n-1}), \boldsymbol{E}_{\tau,h}^-(t_{n-1})\rangle\\
&\qquad\quad + \langle \boldsymbol{M}_0 [\![\boldsymbol{E}_{\tau,h}]\!]_{n-1}, [\![\boldsymbol{E}_{\tau,h}]\!]_{n-1}\rangle\Big],
\end{aligned} \tag{A.5}$$

for $n = 1, \dots, N$, where $\boldsymbol{E}^-_{\tau,h}(t_0) = \Pi_h \boldsymbol{U}_0 - \boldsymbol{U}_{0,h}$. Multiplying (A.5) with the weight $\mathrm{e}^{-2\nu t_{n-1}}$, combining this with (A.4), summing up the resulting equation, and neglecting the positive jump terms yield that

$$\begin{aligned}(\text{A.6})\qquad &\langle \boldsymbol{M}_0 \boldsymbol{E}^-_{\tau,h}(t_N), \boldsymbol{E}^-_{\tau,h}(t_N)\rangle \,\mathrm{e}^{-2\nu T} + \gamma \|\boldsymbol{E}_{\tau,h}\|^2_{\tau,\nu} + |J_\gamma(\boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h})|^2_{\tau,\nu} \\ &\qquad \le C\Big(\langle \boldsymbol{M}_0 \boldsymbol{E}^-_{\tau,h}(t_0), \boldsymbol{E}^-_{\tau,h}(t_0)\rangle + \sum_{n=1}^N \mathrm{e}^{-2\nu t_{n-1}}\, E_i^n \Big)\,.\end{aligned}$$

To bound E_i^n in (A.6), we need auxiliary results. Recalling the exactness of the quadrature formula (3.4) for all $w \in P_{2k}(I_n;\mathbb{R})$, along with (3.1), and using integration by part, we get that

$$\begin{aligned} Q_n[\partial_t \boldsymbol{M}_0 I_\tau \Pi_h \boldsymbol{U}, \boldsymbol{E}_{\tau,h}]_\nu &= \langle \partial_t \boldsymbol{M}_0 I_\tau \Pi_h \boldsymbol{U}, \boldsymbol{E}_{\tau,h}\rangle_{\nu,n} \\ &= \underbrace{\langle \mathrm{e}^{-2\nu(t-t_{n-1})}\, \boldsymbol{M}_0 I_\tau \Pi_h \boldsymbol{U}, \boldsymbol{E}_{\tau,h}\rangle_{\boldsymbol{H}} \Big|_{t_{n-1}}^{t_n}}_{=:a} - \langle \boldsymbol{M}_0 I_\tau \Pi_h \boldsymbol{U}, \partial_t \boldsymbol{E}_{\tau,h}\rangle_{\nu,n} \\ &= a - Q_n[\boldsymbol{M}_0 I_\tau \Pi_h \boldsymbol{U}, \partial_t \boldsymbol{E}_{\tau,h}]_\nu = a - Q_n[\boldsymbol{M}_0 \widehat{I}_\tau^{k+1} \Pi_h \boldsymbol{U}, \partial_t \boldsymbol{E}_{\tau,h}]_\nu \\ &= a - \langle \boldsymbol{M}_0 \widehat{I}_\tau^{k+1} \Pi_h \boldsymbol{U}, \partial_t \boldsymbol{E}_{\tau,h}\rangle_{\nu,n} \\ &= a + \langle \partial_t \boldsymbol{M}_0 \widehat{I}_\tau^{k+1} \Pi_h \boldsymbol{U}, \boldsymbol{E}_{\tau,h}\rangle_{\nu,n} - \underbrace{\langle \mathrm{e}^{-2\nu(t-t_{n-1})}\, \boldsymbol{M}_0 \widehat{I}_\tau^{k+1} \Pi_h \boldsymbol{U}, \boldsymbol{E}_{\tau,h}\rangle_H \Big|_{t_{n-1}}^{t_n}}_{=:b} \\ &= a - b + Q_n[\partial_t \boldsymbol{M}_0 \widehat{I}_\tau^{k+1} \Pi_h \boldsymbol{U}, \boldsymbol{E}_{\tau,h}]_\nu\,. \end{aligned}$$

From this, along with the definition of I_τ and $\widehat{I}_\tau^{k+1}$ in (3.7) and (3.9), respectively, we conclude that

$$\begin{aligned}(\text{A.7})\qquad &Q_n[\partial_t \boldsymbol{M}_0 I_\tau \Pi_h \boldsymbol{U}, \boldsymbol{E}_{\tau,h}]_\nu + \langle \boldsymbol{M}_0 [\![I_\tau \Pi_h \boldsymbol{U}]\!]_{n-1}, \boldsymbol{E}^{+,n-1}_{\tau,h}\rangle_H \\ &\qquad = Q_n[\partial_t \boldsymbol{M}_0 \widehat{I}_\tau^{k+1} \Pi_h \boldsymbol{U}, \boldsymbol{E}_{\tau,h}]_\nu\,.\end{aligned}$$

By the inequalities of Cauchy–Schwarz and Cauchy–Young, there holds that

$$(\text{A.8})\qquad |J_\partial(\boldsymbol{Z}, \boldsymbol{E}_{\tau,h})| \le C J_\partial^n(\boldsymbol{Z}, \boldsymbol{Z}) + \beta J_\gamma(\boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h}),$$

for any $\beta > 0$; cf. (3.33) and (4.9). Then, for the errors E_i^n in (A.6), defined by (A.4), we obtain by (A.7) and (A.8) and the inequalities of Cauchy–Schwarz and Cauchy–Young that

$$\begin{aligned}(\text{A.9})\qquad &\sum_{n=1}^N \mathrm{e}^{-2\nu t_{n-1}}\, |E_i^n| \\ &\quad \le C\Big(\|\partial_t \boldsymbol{M}_0 (\boldsymbol{U} - \widehat{I}_\tau^{k+1} \Pi_h \boldsymbol{U})\|^2_{\tau,\nu} \\ &\qquad + \|\boldsymbol{M}_1 \boldsymbol{Z}\|^2_{\tau,\nu} + \|\boldsymbol{A}_h \boldsymbol{Z}\|^2_{\tau,\nu} + |J_\partial^n(\boldsymbol{Z}, \boldsymbol{Z})|^2_{\tau,\nu} + |J_\gamma(\boldsymbol{Z}, \boldsymbol{Z})|^2_{\tau,\nu} \\ &\qquad + T \max_{1\le n\le N} \Big\{ \|\boldsymbol{M}_0 (\Pi_h \boldsymbol{U}^+(t_{n-1}) - I_\tau \Pi_h \boldsymbol{U}^+(t_{n-1}))\|^2_{\boldsymbol{H}}\, \mathrm{e}^{-2\nu t_{n-1}} \Big\} \Big) \\ &\quad + \beta_1 \|\boldsymbol{E}_{\tau,h}\|^2_{\tau,\nu} + \beta_2 |J_\gamma(\boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h})|^2_{\tau,\nu},\end{aligned}$$

for any $\beta_1, \beta_2 > 0$. Finally, combining (A.6) with (A.9) and choosing β_1 and β_2 sufficiently small, proves the assertion (4.8). ∎

Appendix B. Proof of Lemma 4.4.

To keep this work self-contained, we present the proof of Lemma 4.4.

Proof. Using an idea of [2, Corollary 2.1], for $\boldsymbol{E}_{\tau,h} \in Y_\tau^k(\boldsymbol{H}_h^{\mathrm{dg}})$, we define the local interpolant

$$\widehat{\boldsymbol{E}}_{\tau,h} := I_\tau^n \boldsymbol{\Phi}\,, \qquad \text{with} \quad \boldsymbol{\Phi} := \frac{\tau_n}{t - t_{n-1}} \boldsymbol{E}_{\tau,h}\,, \quad \text{for } t \in I_n\,,\ n = 1, \dots, N\,,$$

where the local Lagrange interpolation operator $I_\tau^n : C(I_n; B) \to \mathbb{P}_k(I_n; B)$, for $n \in \{1, \dots, N\}$, satisfies

$$I_\tau^n \boldsymbol{f}(t_{n,\mu}) = \boldsymbol{f}(t_{n,\mu})\,, \qquad \text{for } \mu = 0, \dots, k\,,$$

for the quadrature nodes $t_{n,\mu} \in I_n$, for $\mu = 0, \dots, k$, of the (non-weighted) Gauss–Radau formula on I_n. Then, there holds that

$$\begin{aligned}\langle \boldsymbol{M}_0 \widehat{\boldsymbol{E}}_{\tau,h}(t_{n,\mu}), \widehat{\boldsymbol{E}}_{\tau,h}(t_{n,\mu})\rangle_{\boldsymbol{H}} &= \frac{\tau_n^2}{(t_{n,\mu} - t_{n-1})^2} \langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}(t_{n,\mu}), \boldsymbol{E}_{\tau,h}(t_{n,\mu})\rangle_H \\ &\geq \langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}(t_{n,\mu}), \boldsymbol{E}_{\tau,h}(t_{n,\mu})\rangle_{\boldsymbol{H}}\,.\end{aligned} \tag{B.1}$$

By [34, Lemma 3.10], based on [2, Lemma 2.1], along with (B.1), we obtain that

$$\begin{aligned}&Q_n[\partial_t \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}, 2\widehat{\boldsymbol{E}}_{\tau,h}]_\nu + \langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}^+(t_{n-1}), 2\widehat{\boldsymbol{E}}_{\tau,h}^+(t_{n-1})\rangle_{\boldsymbol{H}} \\ &\qquad \geq \frac{1}{\tau_n} Q_n[\boldsymbol{M}_0 \widehat{\boldsymbol{E}}_{\tau,h}, \widehat{\boldsymbol{E}}_{\tau,h}]_\nu \geq \frac{1}{\tau_n} Q_n[\boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}, E_{\tau,h}]_\nu\,.\end{aligned} \tag{B.2}$$

By the norm equivalence

$$\sup_{t \in [0,t]} |w(t))| \leq C_e \|w\|_{L^1((0,1);\mathbb{R})}\,, \qquad \text{for } w \in \mathbb{P}_k([0,1]; \mathbb{R})\,,$$

along with the transformation of $[t_{n-1}, t_n]$ to $[0,1]$, we have that

$$\begin{aligned}\sup_{t \in I_m} \langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}(t), \boldsymbol{E}_{\tau,h}(t)\rangle_{\boldsymbol{H}} &\leq \frac{C_e}{\tau_n} \mathrm{e}^{2\nu\tau_n}\, Q_n[\boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h}]_\nu \\ &\leq \frac{C}{\tau_n} Q_n[\boldsymbol{M}_0 \boldsymbol{E}_{\tau,h} \boldsymbol{E}_{\tau,h}]_\nu,\end{aligned} \tag{B.3}$$

with $C := C_e\, \mathrm{e}^{2\nu T} \geq \max_{n=1,\dots,N}\{\mathrm{e}^{2\nu\tau_n}\} C_e$. Further, by (3.30) and (3.32) we have that

$$\begin{aligned}&Q_n[\boldsymbol{A}_h \boldsymbol{E}_{\tau,h}, 2\widehat{\boldsymbol{E}}_{\tau,h}]_\nu + Q_n[J_\partial(\boldsymbol{E}_{\tau,h}, 2\widehat{\boldsymbol{E}}_{\tau,h})]_\nu \\ &\qquad = \frac{\tau_n}{2} \sum_{\mu=0}^k \hat{\omega}_\mu \frac{2\tau_n}{t_{n,\mu} - t_{n-1}} \Big(\langle \boldsymbol{A}_h \boldsymbol{E}_{\tau,h}(t_{n,\mu}), \boldsymbol{E}_{\tau,h}(t_{n,\mu})\rangle_{\boldsymbol{H}} \\ &\qquad\qquad + J_\partial(\boldsymbol{E}_{\tau,h}(t_{n,\mu}), \boldsymbol{E}_{\tau,h}(t_{n,\mu})) \Big) = 0\,.\end{aligned} \tag{B.4}$$

Combining (B.3) with (B.2) and then using (B.4), it follows that

$$
\begin{aligned}
&\sup_{t\in I_m} \langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}(t), \boldsymbol{E}_{\tau,h}(t)\rangle_H \\
&\qquad \le C\Big(Q_n[\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A}_h)\boldsymbol{E}_{\tau,h}, 2\widehat{\boldsymbol{E}}_{\tau,h}]_\nu + Q_n[J_\partial(\boldsymbol{E}_{\tau,h}, 2\widehat{\boldsymbol{E}}_{\tau,h}) \\
&\qquad\qquad + J_\gamma(\boldsymbol{E}_{\tau,h}, 2\widehat{\boldsymbol{E}}_{\tau,h})]_\nu + \langle M_0 [\![\boldsymbol{E}_{\tau,h}]\!]_{n-1}, 2\widehat{\boldsymbol{E}}^+_{\tau,h}(t_{n-1})\rangle_H \\
&\qquad\qquad - Q_n[\boldsymbol{M}_1 \boldsymbol{E}_{\tau,h}, 2\widehat{\boldsymbol{E}}_{\tau,h}]_\nu - Q_n[J_\gamma(\boldsymbol{E}_{\tau,h}, 2\widehat{\boldsymbol{E}}_{\tau,h})]_\nu \\
&\qquad\qquad + \langle \boldsymbol{M}_0 \boldsymbol{E}^-_{\tau,h}(t_{n-1}), 2\widehat{\boldsymbol{E}}^+_{\tau,h}(t_{n-1})\rangle_{\boldsymbol{H}}\Big) .
\end{aligned} \tag{B.5}
$$

Using the error equation (A.2) with test function $\boldsymbol{V}_{\tau,h} = 2\widehat{\boldsymbol{E}}_{\tau,h}$, we deduce from (B.5) that

$$
\begin{aligned}
&\sup_{t\in I_m} \langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}(t), \boldsymbol{E}_{\tau,h}(t)\rangle_{\boldsymbol{H}} \\
&\qquad \le C\Big(- Q_n[\partial_t \boldsymbol{M}_0 + \boldsymbol{M}_1 + \boldsymbol{A}_h)\boldsymbol{Z}, 2\widehat{\boldsymbol{E}}_{\tau,h}]_\nu - Q_n[J_\partial(\boldsymbol{Z}, 2\widehat{\boldsymbol{E}}_{\tau,h}) \\
&\qquad\qquad + J_\gamma(\boldsymbol{Z}, 2\widehat{\boldsymbol{E}}_{\tau,h})]_\nu - \langle \boldsymbol{M}_0 [\![\boldsymbol{Z}]\!]_{n-1}, 2\widehat{\boldsymbol{E}}^+_{\tau,h}(t_{n-1})\rangle_{\boldsymbol{H}} \\
&\qquad\qquad - Q_n[\boldsymbol{M}_1 \boldsymbol{E}_{\tau,h}, 2\widehat{\boldsymbol{E}}_{\tau,h}]_\nu - Q_n[J_\gamma(\boldsymbol{E}_{\tau,h}, 2\widehat{\boldsymbol{E}}_{\tau,h})]_\nu \\
&\qquad\qquad + \langle \boldsymbol{M}_0 \boldsymbol{E}^-_{\tau,h}(t_{n-1}), 2\widehat{\boldsymbol{E}}^+_{\tau,h}(t_{n-1})\rangle_{\boldsymbol{H}}\Big) .
\end{aligned} \tag{B.6}
$$

Next, we bound the right-hand side in (B.6). For this, we use the boundedness of $\boldsymbol{M}_1$ and that $\langle \boldsymbol{M}_0 \boldsymbol{Z}^-(t_{n-1}), \widehat{\boldsymbol{E}}^+_{\tau,h}(t_{n-1})\rangle_{\boldsymbol{H}} = 0$ by the definition of $\boldsymbol{Z}$ in (4.3) and I_τ in (3.7). Further, by the non-negativity and selfadjointness of $\boldsymbol{M}_0$, there holds for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}$ that

$$
\langle \boldsymbol{M}_0 \boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{H}} = \langle \boldsymbol{M}_0^{1/2}\boldsymbol{u}, \boldsymbol{M}_0^{1/2}\boldsymbol{v}\rangle_{\boldsymbol{H}} \le \langle \boldsymbol{M}_0^{1/2}\boldsymbol{u}, \boldsymbol{M}_0^{1/2}\boldsymbol{u}\rangle_{\boldsymbol{H}} \langle \boldsymbol{M}_0^{1/2}\boldsymbol{v}, \boldsymbol{M}_0^{1/2}\boldsymbol{v}\rangle_{\boldsymbol{H}} .
$$

Similarly to (A.9), we then get from (B.6) that

$$
\begin{aligned}
&\sup_{t\in I_n} \langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}(t), \boldsymbol{E}_{\tau,h}(t)\rangle_H \\
&\qquad \le C\Big(\|\partial_t \boldsymbol{M}_0(\boldsymbol{U} - \widehat{I}^{k+1}_\tau \Pi_h \boldsymbol{U})\|^2_{\tau,\nu,n} + \|\boldsymbol{M}_1 \boldsymbol{Z}\|^2_{\tau,\nu,n} + \|\boldsymbol{A}_h \boldsymbol{Z}\|^2_{\tau,\nu,n} \\
&\qquad\qquad + |J^{\boldsymbol{n}}_\partial(\boldsymbol{Z},\boldsymbol{Z})|_{\tau,\nu,n} + |J_\partial(\boldsymbol{Z},\boldsymbol{Z})|_{\tau,\nu,n} + \|\boldsymbol{M}_0 \boldsymbol{Z}^+(t_{n-1})\|^2_{\boldsymbol{H}}\Big) \\
&\qquad + \alpha_1 \langle \boldsymbol{M}_0 \boldsymbol{E}^-_{\tau,h}(t_{n-1}), \boldsymbol{E}^-_{\tau,h}(t_{n-1})\rangle_{\boldsymbol{H}} + \alpha_2 \|\boldsymbol{M}_1\|^2 \|\boldsymbol{E}_{\tau,h}\|^2_{\tau,\nu,n} \\
&\qquad + \alpha_3 |J_\gamma(\boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h})|_{\tau,\nu,n} + \beta_1 \langle 2\boldsymbol{M}_0 \widehat{\boldsymbol{E}}^+_{\tau,h}(t_{n-1}), 2\widehat{\boldsymbol{E}}^+_{\tau,h}(t_{n-1})\rangle_{\boldsymbol{H}} \\
&\qquad + \beta_2 Q_n[2\widehat{\boldsymbol{E}}_{\tau,h}, 2\widehat{\boldsymbol{E}}_{\tau,h}]_\nu + \beta_3 Q_n[J_\gamma(2\widehat{\boldsymbol{E}}_{\tau,h}, 2\widehat{\boldsymbol{E}}_{\tau,h})]_\nu ,
\end{aligned} \tag{B.7}
$$

with some constants $\alpha_i, \beta_i > 0$, for $i = 1, 2, 3$. In (B.7), the term

$$
\begin{aligned}
G_n(\boldsymbol{E}_{\tau,h}) := {}& \alpha_1 \langle \boldsymbol{M}_0 \boldsymbol{E}^-_{\tau,h}(t_{n-1}), \boldsymbol{E}^-_{\tau,h}(t_{n-1})\rangle_{\boldsymbol{H}} + \alpha_2 \|\boldsymbol{M}_1\|^2 \|\boldsymbol{E}_{\tau,h}\|^2_{\tau,\nu,n} \\
& + \alpha_3 |J_\gamma(\boldsymbol{E}_{\tau,h}, \boldsymbol{E}_{\tau,h})|_{\tau,\nu,n}
\end{aligned}
$$

is bounded by means of (4.8), which still holds if $\boldsymbol{E}^-_{\tau,h}(t_N)$ is replaced by $\boldsymbol{E}^-_{\tau,h}(t_n)$, for $n = 1, \dots, N-1$. Further, noting that (cf. [51, Corollary 1.5])

$$\frac{\tau_n}{t_{n,\mu} - t_{n-1}} \leq \frac{\tau_n}{t_{n,0} - t_{n-1}} \leq \frac{1}{\delta}\,, \qquad \text{for } \; n \in \{1, \dots, N\}\,,$$

for some $\delta > 0$ depending on ν and T only, we have that

$$Q_n[\widehat{\boldsymbol{E}}_{\tau,h}, \widehat{\boldsymbol{E}}_{\tau,h}]_\nu \leq \frac{1}{\delta^2} \|\boldsymbol{E}_{\tau,h}\|^2_{\tau,\nu,n}\,, \tag{B.8a}$$

$$Q_n[J_\gamma(\widehat{\boldsymbol{E}}_{\tau,h}, \widehat{\boldsymbol{E}}_{\tau,h})]_\nu \leq \frac{1}{\delta^2} |J_\gamma(\boldsymbol{E}_{\tau,h}, E_{\tau,h})|^2_{\tau,\nu,n}\,, \tag{B.8b}$$

$$\langle \boldsymbol{M}_0 \widehat{\boldsymbol{E}}^+_{\tau,h}(t_{n-1}), \widehat{\boldsymbol{E}}^+_{\tau,h}(t_{n-1})\rangle_{\boldsymbol{H}} \leq \frac{1}{\delta^2} \sup_{t \in I_n} \langle \boldsymbol{M}_0 \boldsymbol{E}_{\tau,h}(t), \boldsymbol{E}_{\tau,h}(t)\rangle_H\,. \tag{B.8c}$$

Finally, combining (B.7) for a sufficiently small choice of β_1 to β_3 with (B.8) and using (4.8) proves the assertion (4.10). □

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