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Contents

Abstract.
Of all the possible projection methods for solving large-scale Lyapunov matrix equations, Galerkin approaches remain much more popular than minimal residual ones. This is mainly due to the different nature of the projected problems stemming from these two families of methods. While a Galerkin approach leads to the solution of a low-dimensional matrix equation per iteration, a matrix least-squares problem needs to be solved per iteration in a minimal residual setting. The significant computational cost of these least-squares problems has steered researchers towards Galerkin methods in spite of the appealing properties of minimal residual schemes. In this paper we introduce a framework that allows for modifying the Galerkin approach by low-rank, additive corrections to the projected matrix equation problem with the two-fold goal of attaining monotonic convergence rates similar to those of minimal residual schemes while maintaining essentially the same computational cost of the original Galerkin method. We analyze the well-posedness of our framework and determine possible scenarios where we expect the residual norm attained by two low-rank-modified variants to behave similarly to the one computed by a minimal residual technique. A panel of diverse numerical examples shows the behavior and potential of our new approach.

Key Words.
Lyapunov equation, matrix equation, block Krylov subspace, model order reduction

AMS Subject Classifications.
65F45, 65F50, 65F10, 65N22, 65J10

Abstract.
Low-rank tensor formats allow for efficient handling of high-dimensional objects. In many applications, it is crucial to preserve the non-negativity in the approximation, for instance, by constraining all cores to be non-negative. Common alternating strategies reduce the high-dimensional problem to a sequence of low-dimensional subproblems but often suffer from slow convergence and persistence in local minima. In order to counteract this, we propose a new quasi-orthogonalization strategy as an intermediate step between the alternating minimization steps that preserves non-negativity. It allows one to improve the expressivity in each individual factor by modifying the current factorization within the equivalence class representing the same tensor.

Key Words.
non-negative factorization, orthogonalization, M-matrices, low-rank tensors, alternating least-squares, high-dimensional problems

AMS Subject Classifications.
15-06, 65F06, 65D40

58 Relaxation of the rank-1 tensor approximation using different norms.
Hassan Bozorgmanesh.

Abstract.
The best rank-1 approximation of a real mth-order tensor is equal to solving m 2-norm optimization problems that each corresponds to a factor of the best rank-1 approximation. In this paper, these problems are relaxed by using the Frobenius and L_1-norms instead of the 2-norm. It is shown that the solution for the Frobenius relaxation of optimization problems is the leading eigenvector of a positive semi-definite matrix which is closely related to higher-order singular value decomposition and the solution of the L_1-relaxation can be obtained efficiently by summing over all modes of the associated tensor but one. The numerical examples show that these relaxations can be used to initialize the alternating least-squares (ALS) method and they are reasonably close to the solutions obtained by the ALS method.

Key Words.
rank-1 approximation, relaxation, tensors, maximum Z-eigenvalue

AMS Subject Classifications.
15A18, 15A69

72 CP decomposition and low-rank approximation of antisymmetric tensors.
Erna Begović Kovač and Lana Periša.

Abstract.
For antisymmetric tensors, the paper examines a low-rank approximation that is represented via only three vectors. We describe a suitable low-rank format and propose an alternating least-squares structure-preserving algorithm for finding such an approximation. Moreover, we show that this approximation problem is equivalent to the problem of finding the best multilinear low-rank antisymmetric approximation and, consequently, equivalent to the problem of finding the best unstructured rank-1 approximation. The case of partial antisymmetry is also discussed. The algorithms are implemented in the Julia programming language and their numerical performance is discussed.

Key Words.
CP decomposition, antisymmetric tensors, low-rank approximation, structure-preserving algorithm, Julia

AMS Subject Classifications.
15A69

95 Using LDL^{T} factorizations in Newton's method for solving general large-scale algebraic Riccati equations.
Jens Saak and Steffen W. R. Werner.

Abstract.
Continuous-time algebraic Riccati equations can be found in many disciplines in

different forms. In the case of small-scale dense coefficient matrices, stabilizing solutions can be computed to all possible formulations of the Riccati equation. This is not the case when it comes to large-scale sparse coefficient matrices. In this paper, we provide a reformulation of the Newton–Kleinman iteration scheme for continuous-time algebraic Riccati equations using indefinite symmetric low-rank factorizations. This allows the application of the method to the case of general large-scale sparse coefficient matrices. We provide convergence results for several prominent realizations of the equation and show in numerical examples the effectiveness of the approach.

Key Words.
Riccati equation, Newton's method, large-scale sparse matrices, low-rank factorization, indefinite terms

AMS Subject Classifications.
15A24, 49M15, 65F45, 65H10, 93A15

Abstract.
We consider iteration for approximately solving large-scale algebraic Sylvester equations. Inside every iteration step of this iterative process, a pair of linear systems of equations has to be solved. We investigate the situation when those inner linear systems are solved inexactly by an iterative method such as, for example, preconditioned Krylov subspace methods. The main contribution of this work are thresholds for the required accuracies regarding the inner linear systems, which dictate when the employed inner Krylov subspace methods can be safely terminated. The goal is to save computational effort by solving the inner linear system as inaccurately as possible without endangering the functionality of the low-rank Sylvester–ADI method. Ideally, the inexact ADI method mimics the convergence behavior of the more expensive exact ADI method, where the linear systems are solved directly. Alongside the theoretical results, strategies for an actual practical implementation of the stopping criteria are also developed. Numerical experiments confirm the effectiveness of the proposed strategies.

Key Words.
Sylvester equation, alternating direction implicit, low-rank approximation, inner–outer methods

AMS Subject Classifications.
15A06, 15A24, 65F45, 65F55

Abstract.
A class of (block) rational Krylov-subspace-based projection methods for solving the large-scale continuous-time algebraic Riccati equation (CARE) $0 = \mathcal{R}(X) := A^H X + XA + C^H C - XBB^H X$ with a large, sparse A, and B and C of full low rank is proposed. The CARE is projected onto a block rational Krylov subspace $\mathcal{K}_j$ spanned by blocks of the form $(A^H - s_k I)^{-1} C^H$ for some shifts s_k, $k =$

$1, \ldots, j$. The considered projections do not need to be orthogonal and are built from the matrices appearing in the block rational Arnoldi decomposition associated to $\mathcal{K}_j$. The resulting projected Riccati equation is solved for the small square Hermitian Y_j. Then the Hermitian low-rank approximation $X_j = Z_j Y_j Z_j^H$ to X is set up where the columns of Z_j span $\mathcal{K}_j$. The residual norm $\|R(X_j)\|_F$ can be computed efficiently via the norm of a readily available $2p \times 2p$ matrix. We suggest reducing the rank of the approximate solution X_j even further by truncating small eigenvalues from X_j. This truncated approximate solution can be interpreted as the solution of the Riccati residual projected to a subspace of $\mathcal{K}_j$. This gives us a way to efficiently evaluate the norm of the resulting residual. Numerical examples are presented.

Key Words.
algebraic Riccati equation, large-scale matrix equation, (block) rational Krylov subspace, projection method

AMS Subject Classifications.
15A24, 65F15

163 A note on TT-GMRES for the solution of parametric linear systems.
Olivier Coulaud, Luc Giraud, and Martina Iannacito.

Abstract.
We study the solution of linear systems with tensor product structure using the Generalized Minimal RESidual (GMRES) algorithm. To manage the computational complexity of high-dimensional problems, our approach relies on low-rank tensor representation, focusing specifically on the Tensor Train format. We implement and experimentally study the TT-GMRES algorithm. Our analysis bridges the heuristic methods proposed for TT-GMRES by Dolgov [Russian J. Numer. Anal. Math. Modelling, 28 (2013), pp. 149–172] and the theoretical framework of inexact GMRES by Simoncini and Szyld [SIAM J. Sci. Comput. 25 (2003), pp. 454–477]. This approach is particularly relevant in a scenario where a $(d+1)$-dimensional problem arises from concatenating a sequence of d-dimensional problems, as in the case of a parametric linear operator or parametric right-hand-side formulation. Thus, we provide backward error bounds that link the accuracy of the computed $(d+1)$-dimensional solution to the numerical quality of the extracted d-dimensional solutions. This facilitates the prescription of a convergence threshold ensuring that the d-dimensional solutions extracted from the $(d+1)$-dimensional result have the desired accuracy once the solver converges. We illustrate these results with academic examples across varying dimensions and sizes. Our experiments indicate that TT-GMRES retains the theoretical rounding-error properties observed in matrix-based GMRES.

Key Words.
GMRES, inexact GMRES, backward stability, Tensor Train format

AMS Subject Classifications.
65F10, 15A69, 65G50

188 Operator-dependent prolongation and restriction for the parameter-dependent multigrid method using low-rank tensor formats.
Lars Grasedyck and Tim A. Werthmann.

Abstract.
Iterative solution methods, such as the parameter-dependent multigrid method, solve

linear systems arising from partial differential equations that depend on parameters. When parameters introduce non-smooth dependencies, such as jumping coefficients, the convergence of the parameter-dependent multigrid method declines or even results in divergence. The goal is to enhance robustness of this multigrid method to enable effective solutions of problems involving jumping coefficients.
An operator-dependent prolongation and restriction, inspired by block Gaussian elimination, is derived that fulfills the approximation property under exact arithmetic, thereby enhancing the method's robustness. Using an approximation of this prolongation and restriction directly in a low-rank tensor format is a trade-off between computational cost and guaranteed convergence. Numerical experiments provide empirical support for the effectiveness of the method, even when using a lower accuracy to compute the approximation within the low-rank format. The proposed operator-dependent prolongation and restriction improves the convergence of the parameter-dependent multigrid method in the presence of jumping coefficients.

Key Words.
multigrid, transfer operators, partial differential equations, iterative solvers, low-rank tensor format, jumping coefficients

AMS Subject Classifications.
65N55, 15A69