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A STABLE BIE METHOD FOR THE LAPLACE EQUATION WITH NEUMANN BOUNDARY CONDITIONS IN DOMAINS WITH PIECEWISE SMOOTH BOUNDARIES*

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Abstract. This paper deals with a new boundary integral equation method for the numerical solution of the exterior Neumann problem for the Laplace equation in planar domains with corners. Using the single layer representation of the potential, the differential problem is reformulated in terms of a boundary integral equation (BIE) whose solution has singularities at the corners. A "modified" Nyström-type method based on a Gauss–Jacobi–Lobatto quadrature formula is proposed for its approximation. Convergence and stability results are proved in proper weighted spaces of continuous functions. Moreover, the use of a smoothing transformation allows one to increase the regularity of the solution and, consequently, the order of convergence of the method. The efficiency of the proposed method is illustrated by some numerical tests.

Key words. boundary integral equations, Neumann problem, domains with corners, Nyström method

AMS subject classifications. 65R20, 45E99, 45F15

1. Introduction. It is well known that in classical potential theory, the Laplace equation with Neumann boundary conditions can be reduced to an integral equation of the second kind defined on the boundary of the domain. In particular, in this paper we consider the exterior Neumann problem in an open bounded simply connected planar domain $\Omega \subset \mathbb{R}^2$ with a piecewise smooth boundary Γ,

$$\begin{aligned} \Delta u(x) &= 0, && x \in \mathbb{R}^2 \setminus \overline{\Omega}, \\ \frac{\partial u(x)}{\partial n(x)} &= f, && x \in \Gamma, \\ |u(x)| &= o(1), && \text{as } |x| \to \infty, \end{aligned} \tag{1.1}$$

where $n(x)$ denotes the outward-pointing unit normal vector to Γ at x. Let Γ be at least twice continuously differentiable with the exception of some points $P_1, \dots, P_n$ where the interior angle lies in $(0, \pi) \cup (\pi, 2\pi)$ (i.e., no cusps are allowed). Assuming that the Neumann data f are sufficiently smooth and satisfy the condition

$$\int_\Gamma f(y) dS(y) = 0 \tag{1.2}$$

($dS(y)$ is the element of arc length), the solution of (1.1) exists and is unique (see, for instance, [6, 23, 24]). By representing u in the form of a single layer potential, that is,

$$u(x) = -\frac{1}{2\pi} \int_\Gamma \psi(y) \log |x - y| dS(y), \qquad x \in \mathbb{R}^2 \setminus \overline{\Omega} \tag{1.3}$$

($|x-y|$ is the Euclidean distance between x and y), the single layer density function ψ provides the solution of (1.1) when inserted into (1.3), and it can be determined by solving the boundary

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integral equation

$$\frac{1}{2}\psi(x) + \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n(x)} \log|x-y|\psi(y)dS(y) = -f(x), \qquad x \in \Gamma, \tag{1.4}$$

coupled with the additional equation (see, for instance, [6, p. 152], [23, p. 351], [24, p. 73])

$$\int_\Gamma \psi(y)dS(y) = 0. \tag{1.5}$$

Let us define the operators $\mathcal{K}$ and $\mathcal{K}^*$ by

$$(\mathcal{K}\psi)(x) = \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n(y)} \log|x-y|\psi(y)dS(y), \qquad x \in \Gamma \setminus \{P_1, \ldots, P_n\},$$

and

$$(\mathcal{K}^*\psi)(x) = \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n(x)} \log|x-y|\psi(y)dS(y), \qquad x \in \Gamma \setminus \{P_1, \ldots, P_n\},$$

respectively, such that equation (1.4) can be written as ($\mathcal{I}$ the identity operator)

$$\left(\frac{1}{2}\mathcal{I} + \mathcal{K}^*\right)\psi = -f.$$

It is well known that when the boundary Γ is piecewise smooth as in this case study, both the kernel of the integral operator and the solution ψ of the BIE (1.4) are singular at the corners. Consequently, the operator $\frac{1}{2}\mathcal{I} + \mathcal{K}^*$ is not invertible as an operator on $C(\Gamma)$. Nevertheless (see [34]), the linear operators $\mathcal{K}, \mathcal{K}^* : L^p(\Gamma) \longrightarrow L^p(\Gamma)$, $1 < p < \infty$, are bounded, where, as usual, $L^p(\Gamma)$ denotes the real Banach space of all p-integrable functions on Γ equipped with the norm

$$\|f\|_{L^p(\Gamma)} = \left(\int_\Gamma |f(x)|^p \, dS(x)\right)^{\frac{1}{p}}.$$

By changing the order of integration one can see that, with respect to the dual product

$$\langle f, g\rangle = \int_\Gamma f(x)g(x)\, dS(x), \qquad f \in L^p(\Gamma),\ g \in L^{p'}(\Gamma), \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

the operators $\mathcal{K}$ and $\mathcal{K}^*$ are adjoint, i.e.,

$$\langle \mathcal{K}f, g\rangle = \langle f, \mathcal{K}^* g\rangle \qquad \text{for all } f \in L^p(\Gamma),\, g \in L^{p'}(\Gamma).$$

Moreover, we have the following particular property:

LEMMA 1.1 ([34], Theorem 4.9). *For $1 < p \leq 2$, the operator*

$$\frac{1}{2}\mathcal{I} + \mathcal{K}^* : L^p(\Gamma) \longrightarrow L^p(\Gamma)$$

is invertible.

The following identity holds (cf. [24, (6.23)] for the case of smooth Γ and [1, (8.4.123)] for the case studied here; note that in [1] the normal vector is directed into the interior of the domain):

$$1 - 2\mathcal{K}1 = 0. \tag{1.6}$$

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Hence, we get for the solution ψ of (1.4)

$$\langle 1, \psi\rangle = \frac{1}{2}\langle 1 + 2\mathcal{K}1, \psi\rangle = \left\langle 1, \frac{1}{2}\psi + \mathcal{K}^*\psi\right\rangle = -\langle 1, f\rangle = 0$$

if f satisfies (1.2). In other words, for $1 < p \leq 2$,

$$L_0^p(\Gamma) = \left\{\psi \in L^p(\Gamma) \; : \; \int_\Gamma \psi \, dS = 0\right\}$$

is an invariant subspace of $L^p(\Gamma)$ with respect to the operator $\frac{1}{2}\mathcal{I} + \mathcal{K}^*$, and the following corollary of Lemma 1.1 is proved.

COROLLARY 1.2. *For $1 < p \leq 2$, the operator*

$$\frac{1}{2}\mathcal{I} + \mathcal{K}^* : L_0^p(\Gamma) \longrightarrow L_0^p(\Gamma)$$

is also invertible.

A substantial body of literature exists dealing with numerical methods for the fast and accurate solution of boundary integral equations when the boundary of the domain is smooth. In the case of piecewise smooth boundaries, the presence of fixed strong singularities of the kernel of the operator $\mathcal{K}^*$ at the corner points as well as the singular behavior of the solution near the corners makes the proof of convergence and stability a rather delicate issue for many traditional numerical methods. Many papers are devoted to propose efficient methods for the numerical solution of elliptic problems in domains with corners (see, for instance, [7, 12, 18, 25, 32, 33] and, more recently, [2, 3, 4, 5, 6, 13, 14, 16, 17, 26] and the references therein), especially based on the use of piecewise polynomial approximations on graded meshes but also of global ones. In order to address the more challenging problems arising in this context, some modifications of the classical methods in a neighborhood of the corners are employed in order to achieve stability. Moreover, suitable regularization techniques are often introduced that improve the smoothness of the solution and, consequently, increase the convergence rate of the approximation. A short panoramic on some recent literature can be found in [13, 14, 26].

By introducing a decomposition of the boundary into the union of smooth arcs and a parametrization for each of them, the boundary integral equation (1.4) is rewritten as an equivalent system of integral equations on the interval $(0, 1)$. Then, the aim of a numerical procedure is to approximate the solution of the resulting system.

In particular, in [26] a proper smoothing transformation for improving the behavior of the unknown functions was introduced, and a Nyström discretization of the solution in L^2 was obtained by using the classical Gauss–Legendre quadrature formula, suitably modified near the corner points in order to assure convergence of the rule. Moreover, the well-conditioning of the linear systems arising from the discretization of the integral equations was achieved by means of an appropriate preconditioning technique. However, the theoretical proof of stability has remained an open challenge.

Without the introduction of a smoothing transformation, due to the singularities of the solution of equation (1.4), the analysis is only possible in weighted spaces of continuous functions. In this paper, following an idea in [27], we are going to adopt such a type of approach. Furthermore, we will show that, if we combine it with suitable regularization techniques, we can achieve higher rates of convergence.

More precisely, after having introduced a smoothing transformation, we move to solve an equivalent system that admits a unique solution in a weighted space of continuous functions.

Then, we approximate its solution by applying a Nyström-type method using a Gauss–Jacobi or Gauss–Jacobi–Lobatto quadrature formula for the discretization of the involved integral operators.

The performed "modified" version of the classical Nyström method allows us to achieve theoretical stability and convergence results for the proposed numerical procedure. Moreover, the linear systems that have to be solved for the method are well conditioned without resorting to any preconditioning technique.

This paper is organized as follows. Section 2 provides preliminary definitions, notation, and some useful results. In Section 3 we reformulate the BIE (1.4) as a system of integral equations on the interval $(0,1)$. In Section 4 the mapping properties of the involved integral operators are studied in proper weighted function spaces. Section 5 is devoted to the proposed numerical method, whose stability and convergence are established in Section 6. Section 7 contains the proofs of the theoretical results, and, finally, in Sections 8 and 9 we present and discuss several numerical tests.

2. Preliminaries.

2.1. Function spaces. Let $w(t)$ be a Jacobi weight function on the interval $(0,1)$. For $1 \le p < \infty$ and $w \in L^p = L^p(0,1)$, let $L^p_w \equiv L^p_w(0,1)$ denote the set of all real valued measurable functions F such that

$$\|F\|_{w,p} = \|Fw\|_p = \left(\int_0^1 |F(t)w(t)|^p dt\right)^{\frac{1}{p}} < +\infty.$$

When $p = \infty$ and w is a continuous function on $[0,1]$, we set $L^\infty_w = C_w$, with $C_w = C_w[0,1]$ the Banach space of all continuous functions $F : (0,1) \to \mathbb{R}$ for which Fw can be extended to a continuous function on $[0,1]$, endowed with the weighted norm

$$\|F\|_{w,\infty} = \sup_{t\in(0,1)} |F(t)|w(t).$$

For general $1 \le p \le \infty$, we consider the following Sobolev-type subspace of L^p_w of order $r \in \mathbb{N}$, $r \ge 1$:

$$W^p_r(w) = \left\{F \in L^p_w :\ F^{(r-1)} \in AC(0,1),\ \|F^{(r)}\varphi^r\|_{w,p} < +\infty\right\},$$

where $\varphi(t) = \sqrt{t(1-t)}$ and $AC(0,1)$ denotes the collection of all functions which are absolutely continuous on every closed subset of $(0,1)$, equipped with the norm

$$\|F\|_{W^p_r(w)} = \|F\|_{w,p} + \|F^{(r)}\varphi^r\|_{w,p}.$$

Moreover, we consider the product space

$$\boldsymbol{C}_w = \left\{\boldsymbol{F} = (F_1,\dots,F_n)^T\ :\ F_i \in C_w\right\}$$

equipped with the norm

$$\|\boldsymbol{F}\|_{w,\infty} = \max_{i=1,\dots,n} \|F_i\|_{w,\infty}$$

and its subspace $\boldsymbol{C}_{w,0}$ defined as

$$\boldsymbol{C}_{w,0} = \left\{\boldsymbol{F} = (F_1,\dots,F_n)^T \in \boldsymbol{C}_w\ :\ \sum_{i=1}^n \int_0^1 F_i(t)dt = 0\right\}.$$

Finally, for $w(t) \equiv 1$, we denote the space $\boldsymbol{C}_w$ simply by $\boldsymbol{C}$ and the norm $\|\boldsymbol{F}\|_{w,\infty}$ by

$$\|\boldsymbol{F}\|_\infty = \max_{i=1,\dots,n} \|F_i\|_\infty, \qquad \boldsymbol{F} = (F_1,\dots,F_n)^T \in \boldsymbol{C}.$$

2.2. Best approximation by polynomials. Denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m. For a function $F \in L^p_w$ with $w(t) = (1-t)^\gamma t^\delta$ a Jacobi weight on $(0,1)$ and $1 \le p \le \infty$, the error of the best approximation of F in L^p_w by polynomials of degree at most m is defined as

$$E_m(F)_{w,p} = \inf_{P \in \mathbb{P}_m} \|F - P\|_{w,p}.$$

We point out that, for $p = \infty$, the validity of the Weierstrass theorem (i.e., the set $\mathbb{P}$ of all polynomials is dense in the space under consideration) is guaranteed only in the subspace C^0_w of C_w consisting of all functions $F \in C_w$ such that

$$\lim_{t\to 0^+} (Fw)(t) = 0 \quad \text{if } \delta > 0 \qquad \text{and} \qquad \lim_{t \to 1^-} (Fw)(t) = 0 \quad \text{if } \gamma > 0.$$

It is well known that for functions F belonging to $W^p_1(w)$, the Favard inequality

$$E_m(F)_{w,p} \le \frac{\mathcal{C}}{m} E_{m-1}(F')_{\varphi w,p} \tag{2.1}$$

is fulfilled for a positive constant $\mathcal{C}$ independent of m and F (see, for example, [30, (2.5.22), p. 172]). By iteration of inequality (2.1), it follows that, for $F \in W^p_r(w)$, $r \ge 1$, the estimate

$$E_m(F)_{w,p} \le \frac{\mathcal{C}}{m^r} E_{m-r}(F^{(r)})_{\varphi^r w,p}, \qquad \mathcal{C} \ne \mathcal{C}(m,F), \tag{2.2}$$

holds true.

Here and in the sequel, $\mathcal{C}$ denotes a positive constant which may assume different values in different formulas. We write $\mathcal{C} = \mathcal{C}(a,b,\ldots)$ to state that $\mathcal{C}$ is dependent on the parameters $a, b, \ldots$ and $\mathcal{C} \ne \mathcal{C}(a,b,\ldots)$ to state that $\mathcal{C}$ is independent of them.

2.3. The quadrature formulas. For the numerical solution of the boundary integral equation we will use two quadrature formulas on the interval $[0,1]$, the Gaussian rule with respect the to Jacobi weight $w(s) = w^{\gamma,\delta}(s) = (1-s)^\gamma s^\delta$, $\gamma, \delta > -1$,

$$\int_0^1 F(s)w(s)ds = \sum_{j=1}^m w^G_{m,j} F(s^G_{m,j}) + e^G_m(F)$$

and the respective Gauss–Lobatto rule

$$\int_0^1 F(s)w(s)ds = \sum_{j=1}^m w^L_{m,j} F(s^L_{m,j}) + e^L_m(F),$$

where $s^G_{m,j}$, $j = 1,\ldots,m$, are the zeros of the polynomial $p_m\left(w^{\gamma,\delta}\right)$ of degree m and orthonormal on $(0,1)$ with respect to the weight $w^{\gamma,\delta}$ and where $s^L_{m,1} = 0$, $s^L_{m,m} = 1$, and $s^L_{m,2},\ldots,s^L_{m,m-1}$ are the zeros of the polynomial $p_{m-2}\left(w^{\gamma+1,\delta+1}\right)$ of degree $m-2$ and orthonormal on $(0,1)$ with respect to the weight $w^{\gamma+1,\delta+1}(s) = (1-s)^{\gamma+1}s^{\delta+1}$. The weights $w^G_{m,j}$ and $w^L_{m,j}$, $j = 1,\ldots,m$, are given by

$$w^J_{m,j} = \int_0^1 l^J_{m,j}(s)w(s)ds, \qquad J \in \{G, L\},$$

with $l^J_{m,j}(s)$ being the j-th fundamental Lagrange polynomial based on the points $s^J_{m,j}$, $j = 1,\ldots,m$. The algebraic degree of exactness of these m-point quadrature rules is $2m-1$

and $2m-3$, respectively. The following estimate is fulfilled by the remainder term $e_m^J(F)$ for any function $F \in W_r^1(w)$, $r \geq 1$:

$$
\begin{aligned}
|e_m^G(F)| &\leq \frac{\mathcal{C}}{m^r} E_{2m-1-r}\left(F^{(r)}\right)_{\varphi^r w,1}, \qquad \text{as well as} \\
|e_m^L(F)| &\leq \frac{\mathcal{C}}{m^r} E_{2m-3-r}\left(F^{(r)}\right)_{\varphi^r w,1},
\end{aligned} \tag{2.3}
$$

where $\varphi(s) = \sqrt{(1-s)s}$ and $\mathcal{C} \neq \mathcal{C}(m,F)$ (see [28]). For more details, see Section 5.

3. The BIE system. The integral equation (1.4) can be immediately converted into a system of boundary integral equations if we consider the following decomposition of the piecewise smooth curve: $\Gamma = \Gamma_1 \cup \{P_1\} \cup \ldots \Gamma_n \cup \{P_n\}$, where, for $i = 1, \ldots, n$, Γ_i is the i-th open arc connecting the two consecutive corner points P_{i-1} and P_i of the boundary (hereafter it is $P_0 \equiv P_n$). Moreover, let $\beta_i \in (0,\pi) \cup (\pi, 2\pi)$ be the interior angle that Γ forms at the i-th corner point P_i. In particular, in our analysis we assume that each arc Γ_i is straight in some neighborhood of the corner, although it should be possible to derive similar results without this restrictive assumption (see [12, 33] and the references therein).

Denoting by ψ_i and f_i the restrictions of the function ψ and f to the arc Γ_i of the boundary, we can reformulate (1.4) as the system

$$
\frac{1}{2}\psi_i(x) + \frac{1}{2\pi}\sum_{k=1}^n \int_{\Gamma_k} \frac{\partial}{\partial n(x)} \log|x-y| \psi_k(y) dS(y) = -f_i(x), \qquad x \in \Gamma_i,\ i = 1, \ldots, n. \tag{3.1}
$$

Then, condition (1.5) is written as

$$
\sum_{i=1}^n \int_{\Gamma_i} \psi_i(y) dS(y) = 0. \tag{3.2}
$$

Since the conditions

$$
\boldsymbol{f} := (f_1, \ldots, f_n)^T \in \boldsymbol{L}^p := L^p(\Gamma_1) \times \ldots \times L^p(\Gamma_n)
$$

and

$$
\boldsymbol{f} := (f_1, \ldots, f_n)^T \in \boldsymbol{L}_0^p := \left\{ \boldsymbol{g} = (g_1, \ldots, g_n)^T \in \boldsymbol{L}^p : \sum_{i=1}^n \int_{\Gamma_i} g_i(y)\, dS(y) = 0 \right\}
$$

are equivalent to $f \in L^p(\Gamma)$ and $f \in L_0^p(\Gamma)$, respectively, we conclude from Lemma 1.1 and Corollary 1.2 the following corollary. Note, that the norm in $\boldsymbol{L}^p$ can be defined by

$$
\|\boldsymbol{f}\|_{\boldsymbol{L}^p} = \left(\sum_{i=1}^n \|f_i\|_{L^p(\Gamma_i)}^p \right)^{\frac{1}{p}}.
$$

COROLLARY 3.1. *Let $1 < p \leq 2$. For every given data $\boldsymbol{f} = (f_1, \ldots, f_n)^T \in \boldsymbol{L}^p$, there is a unique solution $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n)^T \in \boldsymbol{L}^p$ of the system (3.1), which belongs to $\boldsymbol{L}_0^p$ if $\boldsymbol{f} \in \boldsymbol{L}_0^p$.*

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Let us rewrite the system (3.1) in matrix form. For this, define

$$\mathcal{J}_0 = \begin{bmatrix} \mathcal{I}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{I}_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{I}_n \end{bmatrix}$$

with $\mathcal{I}_i : L^2(\Gamma_i) \longrightarrow L^2(\Gamma_i)$ the identity operator and

$$\mathcal{K}_0 = \begin{bmatrix} \mathcal{K}_{11}^0 & \mathcal{K}_{12}^0 & \cdots & \mathcal{K}_{1n}^0 \\ \mathcal{K}_{21}^0 & \mathcal{K}_{22}^0 & \cdots & \mathcal{K}_{2n}^0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{K}_{n1}^0 & \mathcal{K}_{n2}^0 & \cdots & \mathcal{K}_{nn}^0 \end{bmatrix},$$

where

$$\left(\mathcal{K}_{ij}^0 \psi\right)(x) = \frac{1}{2\pi} \int_{\Gamma_j} \frac{\partial}{\partial n(x)} \log|x-y| \psi(y) dS(y), \qquad x \in \Gamma_i.$$

Now, setting $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n)^T$ and $\boldsymbol{f} = (f_1, \ldots, f_n)^T$, the system (3.1) can be written as

$$\left(\frac{1}{2}\mathcal{J}_0 + \mathcal{K}_0\right)\boldsymbol{\psi} = -\boldsymbol{f}, \tag{3.3}$$

where the operator $\frac{1}{2}\mathcal{J}_0 + \mathcal{K}_0 : \boldsymbol{L}^p \longrightarrow \boldsymbol{L}^p$ is linear, bounded, and invertible with the invariant subspace $\boldsymbol{L}_0^p$, $1 < p \le 2$.

For each closed arc $\overline{\Gamma}_i$, $i = 1, \ldots, n$, we consider a parametric representation

$$\boldsymbol{\gamma}_i(t) = (\xi_i(t), \eta_i(t)) \in \overline{\Gamma}_i, \qquad t \in [0,1], \tag{3.4}$$

traversing Γ_i such that the domain Ω is on the left of Γ_i. We assume that $\xi_i, \eta_i \in C^2[0,1]$, $|\boldsymbol{\gamma}_i'(t)| \ne 0$, for all $t \in [0,1]$, $\boldsymbol{\gamma}_i(0) = P_{i-1}$, $\boldsymbol{\gamma}_i(1) = P_i$, and

$$\begin{aligned} \boldsymbol{\gamma}_i(t) - \boldsymbol{\gamma}_i(0) &= c_{i-1} t, && t \in [0, \varepsilon], \\ \boldsymbol{\gamma}_i(t) - \boldsymbol{\gamma}_i(1) &= c_i e^{i\beta_i}(1-t), && t \in [1-\varepsilon, 1], \end{aligned} \tag{3.5}$$

for a sufficiently small ε, $0 < \varepsilon < 1/2$, certain complex constants c_{i-1} and c_i (points in $\mathbb{R}^2$ are identified with complex numbers in the usual way), and the imaginary unit i with $\mathrm{i}^2 = -1$.

Using the parametric representations (3.4) we can transform the BIE system (3.1) into an equivalent system of integral equations defined on the interval $(0,1)$. More precisely, substituting $x = \boldsymbol{\gamma}_i(t)$ and $y = \boldsymbol{\gamma}_j(s)$ in (3.1) and multiplying both sides of the i-th equation in (3.1) by $|\boldsymbol{\gamma}_i'(t)|$, we get

$$\frac{1}{2}\psi_i(\boldsymbol{\gamma}_i(t))|\boldsymbol{\gamma}_i'(t)| + \sum_{j=1}^{n} \frac{1}{2\pi} \int_0^1 k_{ij}(t,s)\psi_j(\boldsymbol{\gamma}_j(s))|\boldsymbol{\gamma}_j'(s)| ds = -f_i(\boldsymbol{\gamma}_i(t))|\boldsymbol{\gamma}_i'(t)|, \qquad i = 1, \ldots, n, \tag{3.6}$$

with the kernels $k_{ij}(t,s)$ given by

$$k_{ij}(t,s) = \begin{cases} \dfrac{\eta_i'(t)[\xi_i(t) - \xi_j(s)] - \xi_i'(t)[\eta_i(t) - \eta_j(s)]}{[\xi_i(t) - \xi_j(s)]^2 + [\eta_i(t) - \eta_j(s)]^2}, & i \ne j \text{ or } t \ne s, \\ \dfrac{1}{2} \dfrac{\eta_i'(t)\xi_i''(t) - \xi_i'(t)\eta_i''(t)}{[\xi_i'(t)]^2 + [\eta_i'(t)]^2}, & i = j \text{ and } t = s. \end{cases} \tag{3.7}$$

Note that, when $|i-j|=1$ or $|i-j|=n-1$, in a neighborhood of $(0,1)$ or $(1,0)$, the kernel $k_{ij}(t,s)$ behaves like a Mellin convolution kernel; otherwise it is a continuous function on the square $[0,1]\times[0,1]$. Using the transformations (3.4), equation (3.2) is converted into

$$\sum_{j=1}^{n}\int_0^1 \psi_j(\gamma_j(s))|\gamma_j'(s)|ds = 0.$$

If f is sufficiently smooth, then the behavior of the solution ψ of equation (1.4) near the corner points P_i of the curve Γ is well known (see, for instance, [6, 8] and the references therein). In particular, from [6, Remark 2.1 and p. 154] we get the following proposition:

PROPOSITION 3.2. *Let $f_i \in C^2(\overline{\Gamma}_i)$, $i=1,\dots,n$. Then, for the solutions ψ_i of the system (3.1), we have*

$$\begin{aligned} &\psi_i(x) = c_{i0}|x-P_{i-1}|^{s_{i-1}} + o(1) \qquad && \text{if } x\in\Gamma_i \text{ and } x\longrightarrow P_{i-1},\\ &\psi_i(x) = c_{i1}|x-P_i|^{s_i} + o(1) && \text{if } x\in\Gamma_i \text{ and } x\longrightarrow P_i, \end{aligned} \tag{3.8}$$

with certain constants c_{i0} and c_{i1} as well as

$$s_i = \min\left\{\frac{\pi}{\beta_i}, \frac{\pi}{2\pi-\beta_i}\right\} - 1, \qquad P_0 := P_n.$$

Since $-\frac{1}{2} < s_i < 0$ for every $i\in\{1,\dots,n\}$, the solution is unbounded at the corners of the boundary. For this reason, many papers presented in the literature propose to improve the behavior of the unknown functions by introducing suitable smoothing transformations (see, for instance, [12, 26, 32, 33]). As a consequence, the convergence rate of the approximating solutions is usually also increased. We consider a smoothing function $\Phi_q : [0,1]\longrightarrow[0,1]$ such that

$$\Phi_q(t) = \begin{cases} t^q, & t\in[0,\varepsilon],\\ 1-(1-t)^q, & t\in[1-\varepsilon,1], \end{cases} \qquad \text{and} \qquad \Phi_q'(t)>0,\ t\in(0,1), \tag{3.9}$$

for some small $\varepsilon>0$ and some integer parameter $q>0$ (see, for instance, [10, 11, 12, 25, 29, 31]). For the case $q=1$, we fix $\Phi_1(t)=t$, $t\in[0,1]$. Introducing the changes of variables $t=\Phi_q(t)$ and $s=\Phi_q(s)$ in (3.6) and (3.2) and multiplying the equations of the obtained system by $\Phi_q'(t)$, we obtain the new system

$$\frac{1}{2}\widetilde{\psi}_i(t) + \sum_{j=1}^{n}\frac{1}{2\pi}\int_0^1 K_{ij}(t,s)\widetilde{\psi}_j(s)ds = -\widetilde{f}_i(t), \qquad i=1,\dots,n, \tag{3.10}$$

$$\sum_{i=1}^{n}\int_0^1 \widetilde{\psi}_i(t)dt = 0, \tag{3.11}$$

where we have set

$$\begin{aligned} \widetilde{\psi}_i(t) &= \psi_i(\gamma_i(\Phi_q(t))\,|\gamma_i'(\Phi_q(t))|\,\Phi_q'(t),\\ K_{ij}(t,s) &= k_{ij}(\Phi_q(t),\Phi_q(s))\Phi_q'(t),\\ \widetilde{f}_i(t) &= f_i(\gamma_i(\Phi_q(t)))\,|\gamma_i'(\Phi_q(t))|\,\Phi_q'(t). \end{aligned} \tag{3.12}$$

Due to the use of the smoothing transformation (3.9), the behavior of the solutions $\widetilde{\psi}_i$, $i=1,\dots,n$, of the system (3.10) near the endpoints of the interval $[0,1]$ is given by

$$\widetilde{\psi}_i(t) = \begin{cases} c_{0i}t^{q(1+s_{i-1})-1} + t^{q-1}\widetilde{\psi}_{0i}(t), & t\in(0,\varepsilon],\\ c_{1i}(1-t)^{q(1+s_i)-1} + (1-t)^{q-1}\widetilde{\psi}_{1i}(1-t), & t\in[1-\varepsilon,1), \end{cases} \tag{3.13}$$

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with $\lim_{t\to 0^+} \widetilde{\psi}_{ri}(t) = 0$ and s_i given in (3.8). Note that, since $s_i > -\frac{1}{2}$, for $q \geq 2$ one has that the solutions $\widetilde{\psi}_i(t)$, $i = 1,\ldots,n$, are continuous on $[0,1]$ (if the given Neumann data are sufficiently smooth). Furthermore, the larger the value of q, the smoother the solution will be.

In order to give a more compact matrix representation of the system of integral equations (3.10), we introduce the integral operators

$$(\mathcal{K}_{ij}F)(t) = \frac{1}{2\pi}\int_0^1 K_{ij}(t,s)F(s)ds,$$

with the kernels $K_{ij}(t,s)$ given in (3.12) as well as the matrices of operators of dimension n

$$\mathcal{J} = \begin{bmatrix} \mathcal{I} & 0 & \cdots & 0 \\ 0 & \mathcal{I} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{I} \end{bmatrix},$$

with the identity operator $\mathcal{I} : L^p(0,1) \longrightarrow L^p(0,1)$ and

$$\mathcal{K} = \begin{bmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & \cdots & \mathcal{K}_{1n} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \cdots & \mathcal{K}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{K}_{n1} & \mathcal{K}_{n2} & \cdots & \mathcal{K}_{nn} \end{bmatrix}.$$

Then, the system (3.10) can be written as

$$\left(\frac{1}{2}\mathcal{J} + \mathcal{K}\right)\widetilde{\boldsymbol{\psi}} = -\widetilde{\boldsymbol{f}}, \tag{3.14}$$

where $\widetilde{\boldsymbol{\psi}} = (\widetilde{\psi}_1,\ldots,\widetilde{\psi}_n)^T$ and $\widetilde{\boldsymbol{f}} = (\widetilde{f}_1,\ldots,\widetilde{f}_n)^T$.

Once the unknowns $\widetilde{\psi}_i(t)$, $i = 1,\ldots,n$, of (3.10) have been determined, one is able to compute the solution $u(x)$ of the original problem (1.1). In fact, using the single layer potential representation (1.3) and taking into account the decomposition of the boundary $\Gamma = \Gamma_1 \cup \{P_1\} \cup \ldots \cup \Gamma_n \cup \{P_n\}$ and the parametric representations of its arcs given in (3.4), one has for all $x = (x_1,x_2) \in \mathbb{R}^2 \setminus \overline{\Omega}$

$$u(x_1,x_2) = -\frac{1}{2\pi}\sum_{i=1}^{n}\int_0^1 \widetilde{\psi}_i(t)\log|(x_1,x_2) - (\xi_i(\Phi_q(t)),\eta_i(\Phi_q(t)))|dt. \tag{3.15}$$

Hence, our aim becomes to solve the system (3.10) combined with condition (3.2). In order to establish its solvability in suitable function spaces, we are going to study the properties of the related operators in the next section. In the present section, we give first results on the solvability of the system (3.10), respectively (3.14).

Since $\mathcal{C}^{-1} \leq |\gamma_i'(t)| \leq \mathcal{C}$ for some positive constant $\mathcal{C}$, we have in case of $q = 1$

$$\|\psi_i\|^p_{L^p(\Gamma_i)} = \int_{\Gamma_i} |\psi(x)|^p \, dS(x) = \int_0^1 |\psi_i(\gamma_i(t))|^p |\gamma_i'(t)| \, dt$$
$$\sim \int_0^1 |\psi_i(\gamma_i(t))|^p |\gamma_i'(t)|^p \, dt = \|\widetilde{\psi}_i\|^p_{L^p(0,1)}.$$

Hence, the map $\mathcal{T} : \boldsymbol{L}^p \longrightarrow \boldsymbol{L}^p(0,1) := L^p(0,1) \times \ldots \times L^p(0,1)$ defined by

$$(\mathcal{T}\boldsymbol{f})(t) = (f_1(\gamma_1(t))|\gamma_1'(t)|,\ldots,f_n(\gamma_n(t))|\gamma_n'(t)|)^T$$

is an isomorphism with $\mathcal{T}^{-1}\boldsymbol{g} = \left(\mathcal{T}_1^{-1}g_1, \ldots, \mathcal{T}_n^{-1}g_n\right)^T$ and

$$\left(\mathcal{T}_i^{-1}g_i\right)(x) = \frac{g_i(\gamma_i^{-1}(x))}{\left|\gamma_i'(\gamma_i^{-1}(x))\right|}, \qquad x \in \Gamma_i,\ i \in \{1, \ldots, n\}.$$

In other words, $\boldsymbol{f} = (f_1, \ldots, f_n)^T$ belongs to $\boldsymbol{L}^p$ if and only if $\widetilde{\boldsymbol{f}} = \mathcal{T}\boldsymbol{f} \in \boldsymbol{L}^p(0,1)$, and the system (3.14) can be written as (cf. (3.3))

$$\mathcal{T}\left(\frac{1}{2}\mathcal{J}_0 + \mathcal{K}_0\right)\mathcal{T}^{-1}\widetilde{\boldsymbol{\psi}} = -\mathcal{T}\boldsymbol{f}.$$

Thus, a direct consequence of Corollary 3.1 is the following:

COROLLARY 3.3. *Let $1 < p \le 2$ and $q = 1$ (i.e., no smoothing transformation is applied). Then, for every $\widetilde{\boldsymbol{f}} \in \boldsymbol{L}^p(0,1)$ the system (3.10), respectively (3.14), has a unique solution $\widetilde{\boldsymbol{\psi}} \in \boldsymbol{L}^p(0,1)$ that belongs to*

$$\boldsymbol{L}_0^p(0,1) := \left\{\widetilde{\boldsymbol{g}} = (\widetilde{g}_1, \ldots, \widetilde{g}_n)^T \in \boldsymbol{L}^p(0,1) : \sum_{j=1}^n \int_0^1 \widetilde{g}_j(t)\,dt = 0\right\}$$

if $\widetilde{\boldsymbol{f}} \in \boldsymbol{L}_0^p(0,1)$.

In case of $q > 1$, we additionally have to take into account the isomorphic map $\mathcal{S} : \boldsymbol{L}^p(0,1) \longrightarrow \boldsymbol{L}^p_{v^\rho}(0,1)$ defined by

$$\mathcal{S}\widetilde{\boldsymbol{f}} = \left(\mathcal{S}\widetilde{f}_1, \ldots, \mathcal{S}\widetilde{f}_n\right)^T \qquad \text{and} \qquad \left(\mathcal{S}\widetilde{f}\right)(t) = \widetilde{f}(\Phi_q(t))\Phi_q'(t), \quad t \in (0,1),$$

where $\rho = \frac{1-q}{p'}$ and $v^\rho(t) = (1-t)^\rho t^\rho$ as well as $\boldsymbol{L}^p_{v^\rho}(0,1) = L^p_{v^\rho}(0,1) \times \ldots \times L^p_{v^\rho}(0,1)$ (see Section 2.1). The continuity of this map can be seen as follows. For example, consider

$$\begin{aligned}
\|f\|^p_{L^p(0,\varepsilon)} &= \int_0^\varepsilon |f(t)|^p\,dt = q\int_0^{\varepsilon^{\frac{1}{q}}} |f(s^q)|^p s^{q-1}\,ds \\
&= q^{1-p}\int_0^{\varepsilon^{\frac{1}{q}}} |qf(s^q)s^{q-1}|^p \left[s^{\frac{(q-1)(1-p)}{p}}\right]^p ds \\
&= q^{1-p}\int_0^{\varepsilon^{\frac{1}{q}}} |(\mathcal{S}f)(s)|^p \left[s^{\frac{1-q}{p'}}\right]^p ds \sim \|\mathcal{S}f\|^p_{L^p_{v^\rho}\left(0,\varepsilon^{\frac{1}{q}}\right)}.
\end{aligned}$$

For the interval $(1-\varepsilon, 1)$ we can proceed analogously. In $(\varepsilon, 1-\varepsilon)$, respectively $\left(\varepsilon^{\frac{1}{q}}, (1-\varepsilon)^{\frac{1}{q}}\right)$, the weight function does not play an essential role. Hence, finally for all $q \ge 1$ we have the following result:

COROLLARY 3.4. *Let $1 < p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. For every $\widetilde{\boldsymbol{f}} \in \boldsymbol{L}^p_{v^\rho}(0,1)$, with $\rho = \frac{1-q}{p'}$, the system (3.10), respectively (3.14), has a unique solution $\widetilde{\boldsymbol{\psi}} \in \boldsymbol{L}^p_{v^\rho}(0,1)$, which belongs to*

$$\boldsymbol{L}^p_{v^\rho,0}(0,1) := \left\{\widetilde{\boldsymbol{g}} = (\widetilde{g}_1, \ldots, \widetilde{g}_n)^T \in \boldsymbol{L}^p_{v^\rho}(0,1) : \sum_{j=1}^n \int_0^1 \widetilde{g}_j(t)\,dt = 0\right\}$$

if $\widetilde{\boldsymbol{f}} \in \boldsymbol{L}^p_{v^\rho,0}(0,1)$.

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4. The integral operators. In order to perform a stability and convergence analysis of the numerical method in suitable weighted spaces of continuous functions, we consider a Jacobi weight $v^\sigma(t) = (1-t)^\sigma t^\sigma$ such that $0 \le \sigma < 1$ and $q(1+s_i) - 1 + \sigma \ge 0$, for any $i = 1, \dots, n$. Let us note that the second condition means that, when $q = 1$ (that is, no smoothing transformation is implemented), it should be $-s_i \le \sigma$, $i = 1, \dots, n$. We study equation (3.14) in the space $\boldsymbol{C}_{v^\sigma}$ (cf. Section 2.1).

Let us recall known useful properties of the operators associated with the system of integral equations (3.10) or equation (3.14). We introduce two sufficiently smooth cut-off functions χ_0 and χ_1 defined on the interval $[0,1]$ such that

$$\begin{array}{lll} \chi_0(t) = 1, & t \in [0, \varepsilon/2], & \mathrm{supp}(\chi_0) \subset [0, \varepsilon], \\ \chi_1(t) = 1, & t \in [1-\varepsilon/2, 1], & \mathrm{supp}(\chi_1) \subset [1-\varepsilon, 1], \end{array} \tag{4.1}$$

for some $0 < \varepsilon < 1/2$ and $0 \le \chi_0(t), \chi_1(t) \le 1$, for $t \in [0,1]$. From now on we shall assume

$$\chi_0, \chi_1 \in C^k[0,1] \tag{4.2}$$

for some large enough $k \in \mathbb{N}$. Moreover, for a fixed angle $\beta \in (0, 2\pi) \setminus \{\pi\}$, we consider the following kernels of Mellin-convolution type

$$K_0^\beta(t,s) = \frac{q t^{q-1}(1-s)^q \sin\beta}{t^{2q} - 2t^q(1-s)^q \cos\beta + (1-s)^{2q}},$$
$$K_1^\beta(t,s) = \frac{q(1-t)^{q-1} s^q \sin\beta}{(1-t)^{2q} - 2(1-t)^q s^q \cos\beta + s^{2q}},$$

which can be represented in the form

$$K_0^\beta(t,s) = \frac{1}{1-s} k^\beta\left(\frac{t}{1-s}\right) \qquad \text{and} \qquad K_1^\beta(t,s) = \frac{1}{s} k^\beta\left(\frac{1-t}{s}\right), \tag{4.3}$$

with the function k^β defined by

$$k^\beta(\tau) = \frac{q\tau^{q-1}\sin\beta}{\tau^{2q} - 2\tau^q\cos\beta + 1} \tag{4.4}$$

and the corresponding integral operators

$$(\mathcal{K}_r^\beta F)(t) = \frac{1}{2\pi}\int_0^1 K_r^\beta(t,s)F(s)ds, \qquad r = 0, 1. \tag{4.5}$$

For $t \in [1-\varepsilon, 1]$ and $s \in [0, \varepsilon]$, by using (3.7) and (3.5), we get

$$k_{i,i+1}(t,s) = \frac{s\,\sin\beta_i}{(1-t)^2 - 2(1-t)s\,\cos\beta_i + s^2}.$$

Analogously,

$$k_{i,i-1}(t,s) = \frac{(1-s)\sin\beta_{i-1}}{(1-t)^2 - 2t(1-s)\,\cos\beta_{i-1} + s^2}, \qquad t \in [0,\varepsilon],\ s \in [1-\varepsilon, 1].$$

Hence, taking into account (3.12) and (3.9), for $t \in [1-\varepsilon, 1]$, $s \in [0, \varepsilon]$ and $t \in [0, \varepsilon]$, $s \in [1-\varepsilon, 1]$, we have

$$K_{i,i+1}(t,s) = K_1^{\beta_i}(t,s) \qquad \text{and} \qquad K_{i,i-1}(t,s) = K_0^{\beta_{i-1}}(t,s), \tag{4.6}$$

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respectively. Since, by our assumptions and definition (4.1) of the functions χ_0 and χ_1, the functions $K_{i,i+1}(t,s) - \chi_1(t)K_{i,i+1}(t,s)\chi_0(s)$ and $K_{i,i-1}(t,s) - \chi_0(t)K_{i,i-1}(t,s)\chi_1(s)$ are continuous on the whole square $[0,1]^2$, each integral operator $\mathcal{K}_{ij}$ with i and j such that $|i-j| = 1$ or $|i-j| = n-1$, if $n \geq 2$, or the single operator $\mathcal{K}_{11}$, if $n = 1$, can be decomposed into the sum of an operator of Mellin type (4.5) with fixed singularities at the point $(0,1)$ or $(1,0)$ and a compact operator as established in the following lemma (see [27, Theorem 2]; cf. analogous considerations in [12, 33]):

LEMMA 4.1. *If $n \geq 2$, then for each couple of indices (i,j) with $i,j \in \{1,\dots,n\}$ and $|i-j| = 1$ or $|i-j| = n-1$, the following equalities hold:*

1. $\mathcal{K}_{ij} = \chi_0 \mathcal{K}_0^{\beta_{i-1}} \chi_1 + \mathcal{E}_{ij}, \quad j = i-1$ *or* $j = 1,\ i = n,$
2. $\mathcal{K}_{ij} = \chi_1 \mathcal{K}_1^{\beta_i} \chi_0 + \mathcal{E}_{ij}, \qquad j = i+1$ *or* $j = n,\ i = 1,$

where χ_r, $r = 0,1$, are the cut-off functions given in (4.1) *and $\mathcal{E}_{ij}$ is a compact operator on $C_{v^\sigma}[0,1]$ for every $\sigma \in [0,1)$. If $n = 1$, then the integral operator $\mathcal{K}_{11}$ satisfies*

3. $\mathcal{K}_{11} = \chi_0 \mathcal{K}_0^{\beta_1} \chi_1 + \chi_1 \mathcal{K}_1^{\beta_1} \chi_0 + \mathcal{E}_{11},$

with χ_r, $r = 0,1$, defined by (4.1) *and $\mathcal{E}_{11}$ a compact operator on $C_{v^\sigma}[0,1]$ for every $\sigma \in [0,1)$.*

The operators $\mathcal{K}_{10}$, $\mathcal{E}_{10}$ and $\mathcal{K}_{n,n+1}$, $\mathcal{E}_{n,n+1}$ have to be understood as $\mathcal{K}_{1n}$, $\mathcal{E}_{1n}$ and $\mathcal{K}_{n1}$, $\mathcal{E}_{n1}$, respectively. Let us note that the Mellin operators $\mathcal{K}_0^\beta$ and $\mathcal{K}_1^\beta$ in (4.5) are not bounded on the space $C[0,1]$. Anyway, since the function $k^\beta(\tau)$ in (4.4) satisfies the condition

$$\int_0^\infty \tau^{-1+\sigma} |k^\beta(\tau)| d\tau < \infty$$

if $0 < \sigma < 1$, it follows that for $F \in C_{v^\sigma}$ with $v^\sigma(t) = (1-t)^\sigma t^\sigma$, the weighted functions $v^\sigma \mathcal{K}_r^\beta F$, $r = 0,1$, can be extended by continuity at the singular points as follows (cf. the proof of Theorem 4.2):

$$(v^\sigma \mathcal{K}_0^\beta F)(t) = \begin{cases} \dfrac{(v^\sigma F)(1)}{2\pi} \displaystyle\int_0^\infty \tau^{-1+\sigma} k^\beta(\tau) d\tau, & t = 0, \\ \dfrac{v^\sigma(t)}{2\pi} \displaystyle\int_0^1 K_0^\beta(t,s) F(s) ds, & t \in (0,1], \end{cases} \tag{4.7}$$

and

$$(v^\sigma \mathcal{K}_1^\beta F)(t) = \begin{cases} \dfrac{v^\sigma(t)}{2\pi} \displaystyle\int_0^1 K_1^\beta(t,s) F(s) ds, & t \in [0,1), \\ \dfrac{(v^\sigma F)(0)}{2\pi} \displaystyle\int_0^\infty \tau^{-1+\sigma} k^\beta(\tau) d\tau, & t = 1. \end{cases} \tag{4.8}$$

With the help of the operator matrices

$$\mathcal{M} = \begin{bmatrix} 0 & \mathcal{M}_{12} & 0 & \cdots & \cdots & \mathcal{M}_{1n} \\ \mathcal{M}_{21} & 0 & \mathcal{M}_{23} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \mathcal{M}_{n-1,n-2} & 0 & \mathcal{M}_{n-1,n} \\ \mathcal{M}_{n1} & 0 & \cdots & 0 & \mathcal{M}_{n,n-1} & 0 \end{bmatrix}, \tag{4.9}$$

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where, for $i, j = 1, \dots, n$,

$$\mathcal{M}_{ij} = \begin{cases} \chi_0 \mathcal{K}_0^{\beta_{i-1}} \chi_1, & j = i-1, \\ \chi_1 \mathcal{K}_1^{\beta_i} \chi_0, & j = i+1, \end{cases}$$

$\mathcal{M}_{10} = \mathcal{M}_{1n}$, $\mathcal{M}_{n,n+1} = \mathcal{M}_{n1}$, and

$$\widetilde{\mathcal{K}} = \begin{bmatrix} \mathcal{K}_{11} & \mathcal{E}_{12} & \mathcal{K}_{13} & \cdots & \cdots & \mathcal{E}_{1n} \\ \mathcal{E}_{21} & \mathcal{K}_{22} & \mathcal{E}_{23} & \cdots & \cdots & \mathcal{K}_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathcal{K}_{n-1,1} & \mathcal{K}_{n-1,2} & \vdots & \mathcal{E}_{n-1,n-2} & \mathcal{K}_{n-1,n-1} & \mathcal{E}_{n-1,n} \\ \mathcal{E}_{n1} & \mathcal{K}_{n2} & \cdots & \mathcal{K}_{n-1,n-2} & \mathcal{E}_{n,n-1} & \mathcal{K}_{nn} \end{bmatrix}, \tag{4.10}$$

we write equation (3.14), by taking into account Lemma 4.1, in the form

$$\left(\frac{1}{2}\mathcal{J} + \mathcal{M} + \widetilde{\mathcal{K}}\right)\widetilde{\boldsymbol{\psi}} = -\widetilde{\boldsymbol{f}}. \tag{4.11}$$

In the special case $n = 1$, (4.11) holds true with

$$\mathcal{M} = \mathcal{M}_{11} = \chi_0 \mathcal{K}_0^{\beta_1} \chi_1 + \chi_1 \mathcal{K}_1^{\beta_1} \chi_0, \qquad \widetilde{\mathcal{K}} = \mathcal{E}_{11}. \tag{4.12}$$

As announced in Section 3, the method we are going to propose is based on the numerical solution of the system of integral equations (3.10) (or, equivalently, (3.14) or (4.11)) combined with the additional condition (3.11) followed by the numerical evaluation of the potential given in (3.15).

THEOREM 4.2. *For $0 < \sigma < 1$, the linear operator $\mathcal{M} : \boldsymbol{C}_{v^\sigma} \longrightarrow \boldsymbol{C}_{v^\sigma}$ is bounded with*

$$\|\mathcal{M}\|_{\boldsymbol{C}_{v^\sigma} \to \boldsymbol{C}_{v^\sigma}} \le \frac{(1-\varepsilon)^{-\sigma}}{2\pi} \max_{i=1,\dots,n} \int_0^\infty \tau^{-1+\sigma} |k^{\beta_i}(\tau)| d\tau, \tag{4.13}$$

with ε as in (4.1).

LEMMA 4.3. *Let $\beta \in (0, 2\pi) \setminus \{\pi\}$ and $0 < \sigma < 1$. Then the function $k^\beta(\tau)$ defined in (4.4) satisfies the inequality*

$$\int_0^\infty \tau^{-1+\sigma} |k^\beta(\tau)| d\tau < \pi \tag{4.14}$$

if and only if $1 - q \min\left\{\frac{\pi}{\beta}, \frac{\pi}{2\pi - \beta}\right\} < \sigma$.

Now, we are able to prove the following theorem establishing the solvability of the system (3.10), respectively (3.14), in the weighted space $\boldsymbol{C}_{v^\sigma}$. For this, we set (cf. (3.8) and (3.13))

$$\mu_i = q(1 + s_i) - 1 = q \min\left\{\frac{\pi}{\beta_i}, \frac{\pi}{2\pi - \beta_i}\right\} - 1, \qquad i = 1, \dots, n. \tag{4.15}$$

THEOREM 4.4. *Let $-\min_{i=1,\dots,n} \mu_i < \sigma < 1$. Then, for each given array of functions $\widetilde{\boldsymbol{f}} \in \boldsymbol{C}_{v^\sigma}$, the system (3.10), respectively (3.14), has a unique solution $\widetilde{\boldsymbol{\psi}} \in \boldsymbol{C}_{v^\sigma}$, which belongs to $\boldsymbol{C}_{v^\sigma,0}$ if $\widetilde{\boldsymbol{f}} \in \boldsymbol{C}_{v^\sigma,0}$.*

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By $\boldsymbol{C}_w^0$ we refer to the subspace of $\boldsymbol{C}_w$ defined by

$$\boldsymbol{C}_w^0 = \left\{ \boldsymbol{f} = (f_1, \ldots, f_n)^T : f_i \in C_w^0,\, i = 1, \ldots, n \right\}$$

(cf. Section 2.2). Moreover, let $\boldsymbol{C}_{w,0}^0 = \boldsymbol{C}_w^0 \cap \boldsymbol{C}_{w,0}$. As a consequence of the continuity of the functions given by the formulas (4.7) and (4.8) (cf. the proof of Theorem 4.2), we have that the space $\boldsymbol{C}_{v^\sigma}^0$ is an invariant subspace of $\boldsymbol{C}_{v^\sigma}$ with respect to the operator $\frac{1}{2}\mathcal{J} + \mathcal{K}$ in (3.14).

REMARK 4.5. Since $-\frac{1}{2} < s_i < 0$ (see (3.8)), the condition $\sigma > -\mu_i = 1 - q(1 + s_i)$ is fulfilled if $\sigma \geq 1 - \frac{q}{2}$. Consequently, the assertion of Theorem 4.4 is in any case true if $1 - \frac{q}{2} \leq \sigma < 1$ (without knowing the numbers s_i). If the solution of the boundary value problem satisfies (3.13), then the solution in $\boldsymbol{C}_{v^\sigma}$ of (3.14) is in $\boldsymbol{C}_{v^\sigma}^0$. Thus, it makes sense to consider (3.14) together with the Nyström method in $\boldsymbol{C}_{v^\sigma}^0$ if the Neumann data f satisfy the assumptions of Proposition 3.2.

Obviously, the operator $\mathcal{V}^\sigma : \boldsymbol{C}_{v^\sigma} \longrightarrow \boldsymbol{C}$,

$$\boldsymbol{f} = (f_1, \ldots, f_n)^T \mapsto \begin{bmatrix} v^\sigma \mathcal{I} & 0 & \cdots & 0 \\ 0 & v^\sigma \mathcal{I} & \ldots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & v^\sigma \mathcal{I} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

is an isometrical isomorphism. In case of the application of the Gauss–Lobatto rule for the approximate solution of the BIE system, for the description and investigation as well as for the implementation of the method, it is more convenient to consider the following equivalent problem in $\boldsymbol{C}$ instead of equation (3.14), respectively (4.11), in the space $\boldsymbol{C}_{v^\sigma}$,

$$\mathcal{V}^\sigma \left(\frac{1}{2}\mathcal{J} + \mathcal{K} \right) \mathcal{V}^{-\sigma} \widehat{\boldsymbol{\psi}} = -\widehat{\boldsymbol{f}}, \tag{4.16}$$

with $\mathcal{V}^{-\sigma} = (\mathcal{V}^\sigma)^{-1}$ and the new unknown $\widehat{\boldsymbol{\psi}} = \mathcal{V}^\sigma \widetilde{\boldsymbol{\psi}}$ as well as the new right-hand side $\widehat{\boldsymbol{f}} = \mathcal{V}^\sigma \widetilde{\boldsymbol{f}}$.

COROLLARY 4.6. *If*

$$-\min_{i=1,\ldots,n} \mu_i < \sigma < 1, \tag{4.17}$$

then the system (4.16) *has a unique solution* $\widehat{\boldsymbol{\psi}} \in \boldsymbol{C}$ *for each given array of functions* $\widehat{\boldsymbol{f}} \in \boldsymbol{C}$.

REMARK 4.7. It should also be possible to consider equation (3.10), respectively (3.14), in the space $\boldsymbol{C}_\Sigma^0$ associated with the parameter set $\Sigma = \{\sigma_1, \ldots, \sigma_n = \sigma_0\}$ and defined by

$$\boldsymbol{C}_\Sigma^0 = \left\{ (\widetilde{f}_1, \ldots, \widetilde{f}_n)^T : f_i \in C_{v_i}^0,\, i \in \{1, \ldots, n\} \right\},$$

where $v_i(t) = (1-t)^{\sigma_i} t^{\sigma_{i-1}}$ with $\sigma_i + \mu_i > 0$, for $i \in \{1, \ldots, n\}$. Here in this paper, for reasons of simplicity we restrict ourselves to the case $\sigma_1 = \ldots \sigma_n = \sigma$.

5. The method. In this section we are going to describe the proposed numerical method for solving the system (3.10), respectively (4.11). The method is a Nyström-type method based on the application of a suitable Gauss–Jacobi or a Gauss–Jacobi–Lobatto quadrature formula. More precisely, we consider the quadrature rules

$$\int_0^1 F(s) v^{-\sigma}(s) ds = \sum_{h=1}^m w_{m,h} F(s_{m,h}) + e_m(F), \tag{5.1}$$

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where $s_{m,h} = s^J_{m,h}$ and $w_{m,h} = w^J_{m,h}$, $h = 1, \dots, m$, $J \in \{G, L\}$, denote the nodes and the weights of the Gauss and the Gauss–Lobatto rule with respect to the Jacobi weight $v^{-\sigma}(s) = (1-s)^{-\sigma}s^{-\sigma}$ and where $e_m(F) = e^J_m(F)$ is the remainder term (cf. Section 2.3). An error estimate for the absolute quadrature error $|e_m(F)|$ is given in (2.3).

As already mentioned at the end of the previous section, it is advantageously for the method to study the equivalent equation (4.16). From now on we will assume that the hypotheses of Theorem 4.4, respectively Corollary 4.6, are fulfilled.

We represent $\mathcal{K}$ in (4.16) as the sum $\mathcal{M} + \widetilde{\mathcal{K}}$ (cf. (4.11)). We apply the quadrature formula (5.1) in order to approximate the integral operators

$$(v^\sigma \mathcal{K}_{ij} v^{-\sigma} f)(t) = \frac{v^\sigma(t)}{2\pi} \int_0^1 K_{ij}(t,s) v^{-\sigma}(s) f(s) ds, \qquad i,j = 1, \dots, n,$$

with the kernels $K_{ij}(t,s)$ defined by (3.12) and

$$(v^\sigma \mathcal{E}_{ij} v^{-\sigma} f)(t) = \frac{v^\sigma(t)}{2\pi} \int_0^1 E_{ij}(t,s) v^{-\sigma}(s) f(s) ds,$$

where the kernels $E_{ij}(t,s)$ are given by (see Lemma 4.1)

$$E_{ij}(t,s) = \begin{cases} K_{ij}(t,s) - \chi_0(t) K_0^{\beta_{i-1}}(t,s)\chi_1(s), & j = i-1, \\ K_{ij}(t,s) - \chi_1(t) K_1^{\beta_i}(t,s)\chi_0(s), & j = i+1. \end{cases} \tag{5.2}$$

We obtain the following quadrature operators, defined on the space C:

$$(\mathcal{K}^\sigma_{ij,m} f)(t) = \frac{v^\sigma(t)}{2\pi} \sum_{h=1}^m w_{m,h} K_{ij}(t, s_{m,h}) f(s_{m,h}), \quad t \in [0,1],$$

$$(\mathcal{E}^\sigma_{ij,m} f)(t) = \frac{v^\sigma(t)}{2\pi} \sum_{h=1}^m w_{m,h} E_{ij}(t, s_{m,h}) f(s_{m,h}), \quad t \in [0,1].$$

Concerning the approximation of the non-compact weighted operators $v^\sigma \mathcal{M}_{ij} v^{-\sigma}$ involved in the matrix $\mathcal{V}^\sigma \mathcal{M} \mathcal{V}^{-\sigma}$, we first consider the quadrature operators

$$(\mathcal{K}^{\beta,\sigma}_{r,m} f)(t) = \frac{v^\sigma(t)}{2\pi} \sum_{h=1}^m w_{m,h} K^\beta_r(t, s_{m,h}) f(s_{m,h}), \qquad r = 0, 1, \tag{5.3}$$

obtained by applying the quadrature rule (5.1) to the integrals defining the operators $v^\sigma \mathcal{K}^\beta_r v^{-\sigma}$, $r = 0, 1$ (see (4.5)). Then, fixing a positive constant c and a small $\epsilon > 0$ together with a breaking point

$$t_m = \frac{c}{m^{2-2\epsilon}}, \tag{5.4}$$

we introduce the "modified" operators

$$\left(\widetilde{\mathcal{K}}^{\beta,\sigma}_{0,m} f\right)(t) = \begin{cases} \frac{1}{t_m}\left[t\left(\mathcal{K}^{\beta,\sigma}_{0,m} f\right)(t_m) + (t_m - t)\left(v^\sigma \mathcal{K}^\beta_0 v^{-\sigma} f\right)(0)\right], & t \in [0, t_m), \\ \left(\mathcal{K}^{\beta,\sigma}_{0,m} f\right)(t), & t \in [t_m, 1], \end{cases} \tag{5.5}$$

and

$$(5.6)\quad \left(\widetilde{\mathcal{K}}^{\beta,\sigma}_{1,m} f\right)(t) = \begin{cases} \left(\mathcal{K}^{\beta,\sigma}_{1,m} f\right)(t), & t \in [0, 1-t_m], \\ \dfrac{1}{t_m}\Big[(1-t)\left(\mathcal{K}^{\beta,\sigma}_{1,m} f\right)(1-t_m) + (t-1+t_m)\left(v^\sigma \mathcal{K}^\beta_1 v^{-\sigma} f\right)(1)\Big], & \\ & t \in (1-t_m, 1], \end{cases}$$

approximating $v^\sigma \mathcal{K}^\beta_0 v^{-\sigma}$ and $v^\sigma \mathcal{K}^\beta_1 v^{-\sigma}$, respectively. Note that, due to (4.7) and (4.8) as well as formula (7.1), we have in (5.5) and (5.6)

$$\left(v^\sigma \mathcal{K}^\beta_0 v^{-\sigma} f\right)(0) = \frac{f(1)\,\sin((1-\rho)(\pi-\beta))}{4\pi \sin(\rho\pi)}$$

and

$$\left(v^\sigma \mathcal{K}^\beta_1 v^{-\sigma} f\right)(1) = \frac{f(0)\,\sin((1-\rho)(\pi-\beta))}{4\pi \sin(\rho\pi)},$$

with $\rho = 1 - \frac{1-\sigma}{q}$. Taking this into account we can make the following remark:

REMARK 5.1. If we consider the system (3.10) in $\boldsymbol{C}^0_{v^\sigma}$ or, which is equivalent, (4.16) in

$$\boldsymbol{C}^0 = \left\{(f_1, \dots, f_n)^T \in \boldsymbol{C} : f_i(0) = f_i(1) = 0,\ i = 1, \dots, n\right\},$$

then the definition of $\widetilde{\mathcal{K}}^{\beta,\sigma}_{r,m}$, $r = 0, 1$, in (5.5) and (5.6) becomes simpler, namely

$$\left(\widetilde{\mathcal{K}}^{\beta,\sigma}_{0,m} f\right)(t) = \begin{cases} \frac{t}{t_m}\left(\mathcal{K}^{\beta,\sigma}_{0,m} f\right)(t_m), & t \in [0, t_m), \\ \left(\mathcal{K}^{\beta,\sigma}_{0,m} f\right)(t), & t \in [t_m, 1], \end{cases}$$

and

$$\left(\widetilde{\mathcal{K}}^{\beta,\sigma}_{1,m} f\right)(t) = \begin{cases} \left(\mathcal{K}^{\beta,\sigma}_{1,m} f\right)(t), & t \in [0, 1-t_m], \\ \dfrac{1}{t_m}\Big[(1-t)\left(\mathcal{K}^{\beta\sigma}_{1,m} f\right)(1-t_m)\Big], & t \in (1-t_m, 1]. \end{cases}$$

Finally, by using definitions (5.5) and (5.6) as well as the cut-off functions χ_0 and χ_1 from (4.1), we set, for $n \geq 2$ and $i, j = 1, \dots, n$ such that $|i-j| = 1$,

$$(5.7)\quad \mathcal{M}^\sigma_{ij,m} = \begin{cases} \chi_0 \widetilde{\mathcal{K}}^{\beta_{i-1},\sigma}_{0,m} \chi_1, & j = i-1, \\ \chi_1 \widetilde{\mathcal{K}}^{\beta_i,\sigma}_{1,m} \chi_0, & j = i+1, \end{cases}$$

$\mathcal{M}^\sigma_{10,m} \equiv \mathcal{M}^\sigma_{1n,m}$, $\mathcal{M}^\sigma_{n,n+1,m} \equiv \mathcal{M}^\sigma_{n1,m}$, and, for $n = 1$,

$$(5.8)\quad \mathcal{M}^\sigma_{11,m} = \chi_0 \widetilde{\mathcal{K}}^{\beta_1,\sigma}_{0,m} \chi_1 + \chi_1 \widetilde{\mathcal{K}}^{\beta_1,\sigma}_{1,m} \chi_0.$$

Now, for $n > 1$ we introduce the matrices of approximating operators

$$(5.9)\quad \mathcal{M}^\sigma_m = \begin{bmatrix} 0 & \mathcal{M}^\sigma_{12,m} & 0 & \cdots & \cdots & \mathcal{M}^\sigma_{1n,m} \\ \mathcal{M}^\sigma_{21,m} & 0 & \mathcal{M}^\sigma_{23,m} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \mathcal{M}^\sigma_{n-1,n-2,m} & 0 & \mathcal{M}^\sigma_{n-1,n,m} \\ \mathcal{M}^\sigma_{n1,m} & \cdots & \cdots & 0 & \mathcal{M}^\sigma_{n,n-1,m} & 0 \end{bmatrix}$$

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and
(5.10)

$$\mathcal{K}_m^\sigma = \begin{bmatrix} \mathcal{K}^\sigma_{11,m} & \mathcal{E}^\sigma_{12,m} & \mathcal{K}^\sigma_{13,m} & \cdots & \cdots & \mathcal{E}^\sigma_{1n,m} \\ \mathcal{E}^\sigma_{21,m} & \mathcal{K}^\sigma_{22,m} & \mathcal{E}^\sigma_{23,m} & \cdots & \cdots & \mathcal{K}^\sigma_{2n,m} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathcal{K}^\sigma_{n-1,1,m} & \mathcal{K}^\sigma_{n-1,2,m} & \vdots & \mathcal{E}^\sigma_{n-1,n-2,m} & \mathcal{K}^\sigma_{n-1,n-1,m} & \mathcal{E}^\sigma_{n-1,n,m} \\ \mathcal{E}^\sigma_{n1,m} & \mathcal{K}^\sigma_{n2,m} & \cdots & \mathcal{K}^\sigma_{n-1,n-2,m} & \mathcal{E}^\sigma_{n,n-1,m} & \mathcal{K}^\sigma_{nn,m} \end{bmatrix},$$

while, in case of $n = 1$, we set

$$\mathcal{M}_m^\sigma = \mathcal{M}^\sigma_{11,m}, \qquad \mathcal{K}_m^\sigma = \mathcal{E}^\sigma_{11,m}. \tag{5.11}$$

The "modified" Nyström method consists in approximating the unknown solution $\widehat{\boldsymbol{\psi}} = \left(\widehat{\psi}_1, \ldots, \widehat{\psi}_n\right)^T$ of the system (4.16) by the solution $\widehat{\boldsymbol{\psi}}_m = (\widehat{\psi}_{m,1}, \ldots, \widehat{\psi}_{m,n})^T$ of the approximating system

$$\left(\frac{1}{2}\mathcal{J} + \mathcal{M}_m^\sigma + \mathcal{K}_m^\sigma\right)\widehat{\boldsymbol{\psi}}_m = -\widehat{\boldsymbol{f}}. \tag{5.12}$$

Collocating each equation of (5.12) at the nodes of the quadrature formula $s_{m,j}$, $j = 1, \ldots, m$, we obtain that the function values $\widehat{\boldsymbol{\psi}}_m(s_{m,i}) = \left(\widehat{\psi}_{m,1}(s_{m,i}), \ldots, \widehat{\psi}_{m,n}(s_{m,i})\right)^T$, $i = 1, \ldots, m$, have to satisfy the system

$$\left((\frac{1}{2}\mathcal{J} + \mathcal{M}_m^\sigma + \mathcal{K}_m^\sigma)\widehat{\boldsymbol{\psi}}_m \right)(s_{m,j}) = -\widehat{\boldsymbol{f}}(s_{m,j}), \qquad j = 1, \ldots, m. \tag{5.13}$$

Once (5.13) is solved, one can compute the Nyström interpolant

$$\widehat{\boldsymbol{\psi}}_m(t) = -2\left[\widehat{\boldsymbol{f}}(t) + \left(\mathcal{M}_m^\sigma \widehat{\boldsymbol{\psi}}_m\right)(t) + \left(\mathcal{K}_m^\sigma \widehat{\boldsymbol{\psi}}_m\right)(t)\right], \qquad t \in [0, 1], \tag{5.14}$$

which is then the solution of (5.12).

In the final step of our numerical procedure we are going to approximate the solution $u(x)$ of the boundary value problem (1.1) represented as a single layer potential (1.3). For all $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \overline{\Omega}$, taking into account (3.15), we propose to approximate $u(x_1, x_2)$ by means of the function
(5.15)

$$u_m(x_1, x_2) = -\frac{1}{2\pi}\sum_{i=1}^{n}\sum_{j=1}^{m} w_{m,j}\widehat{\psi}_{m,i}(s_{m,j}) \log|(x_1, x_2) - (\xi_i(\Phi_q(s_{m,j})), \eta_i(\Phi_q(s_{m,j}))|,$$

where $w_{m,j}$ and $s_{m,j}$ are the weights and knots of the quadrature formula (5.1). We observe that the quantities $(\widehat{\psi}_{m,i})(s_{m,j})$ in (5.15) are obtained from the solution

$$\begin{aligned}&\left(\widehat{\boldsymbol{\psi}}_m(s_{m,1}), \ldots, \widehat{\boldsymbol{\psi}}_m(s_{m,m})\right)^T \\ &\quad = \left(\widehat{\psi}_{m,1}(s_{m,1}), \ldots, \widehat{\psi}_{m,n}(s_{m,1}), \ldots, \widehat{\psi}_{m,1}(s_{m,m}), \ldots, \widehat{\psi}_{m,n}(s_{m,m})\right)^T\end{aligned}$$

of the linear system (5.13).

6. Stability and convergence. The proof of stability and convergence of the proposed method will be carried out by using the properties of the operator sequences $\{\mathcal{M}_m^\sigma\}_m$ and $\{\mathcal{K}_m^\sigma\}_m$ established in the next theorems. From now on we will assume that $t_m \le \varepsilon$, with ε as in (4.1) and t_m as in (5.4). Moreover, we know that if condition (4.17) is satisfied, then $\varepsilon \in \left(0, \frac{1}{2}\right)$ can be chosen small enough such that (4.13) is true. Hence, we can formulate the following condition which should be satisfied in what follows:

CONDITION A. Condition (4.17) is fulfilled, and $\varepsilon \in \left(0, \frac{1}{2}\right)$ is chosen small enough, such that (4.13) is satisfied.

THEOREM 6.1. *Let $\mathcal{M}$ and $\mathcal{M}_m^\sigma$ be defined as in (4.9) (or (4.12)) and (5.9) (or (5.11)), respectively. If Condition A is satisfied, then the linear operators $\mathcal{M}_m^\sigma : \boldsymbol{C} \longrightarrow \boldsymbol{C}$ are uniformly bounded with*

$$\limsup_{m} \|\mathcal{M}_m^\sigma\|_{\boldsymbol{C}\to\boldsymbol{C}} < \frac{1}{2} \tag{6.1}$$

and strongly convergent to $\mathcal{V}^\sigma \mathcal{M} \mathcal{V}^{-\sigma} : \boldsymbol{C} \longrightarrow \boldsymbol{C}$, i.e.,

$$\lim_{m\to\infty} \left\|(\mathcal{M}_m^\sigma - \mathcal{V}^\sigma \mathcal{M} \mathcal{V}^{-\sigma})\boldsymbol{f}\right\|_\infty = 0 \qquad \forall\, \boldsymbol{f} \in \boldsymbol{C}. \tag{6.2}$$

THEOREM 6.2. *Let $\widetilde{\mathcal{K}}$ and $\mathcal{K}_m^\sigma$ be defined as in (4.10) (or (4.12)) and (5.10) (or (5.11)), respectively. Then the linear operators $\mathcal{K}_m^\sigma : \boldsymbol{C} \longrightarrow \boldsymbol{C}$ form a collectively compact set $\{\mathcal{K}_m^\sigma : m = 1, 2, \ldots\}$ strongly convergent to $\mathcal{V}^\sigma \widetilde{\mathcal{K}} \mathcal{V}^{-\sigma} : \boldsymbol{C} \longrightarrow \boldsymbol{C}$, i.e.,*

$$\lim_{m\to\infty} \left\|(\mathcal{K}_m^\sigma - \mathcal{V}^\sigma \widetilde{\mathcal{K}} \mathcal{V}^{-\sigma})\boldsymbol{f}\right\|_\infty = 0 \qquad \forall\, \boldsymbol{f} \in \boldsymbol{C}. \tag{6.3}$$

From Theorems 6.1 and 6.2 the following main results can be deduced.

THEOREM 6.3. *Let Condition A be fulfilled and $\widehat{\boldsymbol{f}} \in \boldsymbol{C}$. Then, for all sufficiently large m, the operators $\frac{1}{2}\mathcal{J} + \mathcal{M}_m^\sigma + \mathcal{K}_m^\sigma : \boldsymbol{C} \longrightarrow \boldsymbol{C}$ are invertible, and their inverses are uniformly bounded. Moreover, for the solutions $\widehat{\boldsymbol{\psi}} \in \boldsymbol{C}$ of the system (4.16) and $\widehat{\boldsymbol{\psi}}_m$ of the system (5.12), one has*

$$\lim_{m\to\infty} \left\|\widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}}_m\right\|_\infty = 0. \tag{6.4}$$

REMARK 6.4. Theorems 6.1, 6.2, and 6.3 remain true if we replace the space $\boldsymbol{C}$ by $\boldsymbol{C}^0$ since for all involved operators the space $\boldsymbol{C}^0$ is an invariant subspace of $\boldsymbol{C}$.

REMARK 6.5. Let us observe that, if the function $\widetilde{\boldsymbol{f}}$ at the right-hand side of equation (3.14) belongs to $\boldsymbol{C}_{v^\sigma,0}$, then its solution $\widetilde{\boldsymbol{\psi}} = \mathcal{V}^{-\sigma}\widehat{\boldsymbol{\psi}}$ belongs to $\boldsymbol{C}_{v^\sigma,0}$. For the sequence of approximating solutions to (3.14),

$$\widetilde{\boldsymbol{\psi}}_m = \left(\widetilde{\psi}_{m,1}, \ldots, \widetilde{\psi}_{m,n}\right)^T := \mathcal{V}^{-\sigma}\widehat{\boldsymbol{\psi}} = \mathcal{V}^{-\sigma}\left(\widehat{\psi}_{m,1}, \ldots, \widehat{\psi}_{m,n}\right)^T,$$

one has that

$$\begin{aligned}\left|\sum_{i=1}^n \int_0^1 \widetilde{\psi}_i(t)dt - \sum_{i=1}^n \int_0^1 \widetilde{\psi}_{m,i}(t)dt\right| &\le \sum_{i=1}^n \int_0^1 \left|\widehat{\psi}_i(t) - \widehat{\psi}_{m,i}(t)\right| v^{-\sigma}(t)dt \\ &\le \left\|\widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}}_m\right\|_\infty n \int_0^1 v^{-\sigma}(t)dt,\end{aligned}$$

from which, by virtue of (6.4), we can deduce that

$$\lim_{m\to\infty} \sum_{i=1}^n \int_0^1 \widetilde{\psi}_{m,i}(t)dt = 0$$

holds true.

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THEOREM 6.6. *Let us assume that the hypotheses of Theorem 6.3 are satisfied. We further suppose that the closed arcs $\overline{\Gamma}_i$, $i = 1, \dots, n$, are curves of class $C^{\ell+2}$ for some integer $\ell \geq 2$ and that for some $r \in \mathbb{N}$ the conditions $\ell \geq r$ and $\sigma + \mu \geq \frac{r}{2}$ with $\mu = \min\{\mu_1, \dots, \mu_n\}$ and μ_i, $i = 1, \dots, n$, given in (4.15) hold. If the functions χ_0, χ_1 defined in (4.1) belong to $C^k[0,1]$ for some $k \geq \ell$ and if (cf. (3.1)) $f_i \in C^\ell(\Gamma_i)$, $i = 1, \dots, n$, then the error estimate*

$$\left\|\widehat{\psi} - \widehat{\psi}_m\right\|_\infty \leq \frac{\mathcal{C}}{m^\nu} \tag{6.5}$$

is fulfilled with $\nu = \min\{r, k\epsilon, 2(1-\epsilon)(\sigma+\mu)\}$ and a positive constant $\mathcal{C}$ independent of m.

The following theorem provides an error estimate for the approximating potential given by (5.15):

THEOREM 6.7. *Let the assumptions of Theorem 6.6 be fulfilled, and let u be the solution of the Neumann problem (1.1). Then, for any $x \in \mathbb{R}^2 \setminus \overline{\Omega}$, the approximate solution $u_m(x)$ in (5.15) satisfies the pointwise error estimate*

$$|u(x) - u_m(x)| \leq \frac{\mathcal{C}\,\rho_1(R)}{m^\nu} + \frac{\mathcal{C}\,\rho_2(R)}{m^r}, \tag{6.6}$$

with $\mathcal{C} \neq \mathcal{C}(m)$, $\nu = \min\{r, k\epsilon, 2(1-\epsilon)(\sigma+\mu)\}$, $R = \operatorname{dist}(x,\Gamma) = \inf\{|x-y| : y \in \Gamma\}$, and

$$\rho_1(R) = \begin{cases} \mathcal{O}(\log R), & R \longrightarrow \infty, \\ \mathcal{O}(-\log R), & R \longrightarrow 0, \end{cases} \qquad \rho_2(R) = \begin{cases} \mathcal{O}(\log R), & R \longrightarrow \infty, \\ \mathcal{O}\left(R^{-r}\right), & R \longrightarrow 0. \end{cases}$$

7. Proofs. This section provides proofs of the previous theorems and lemmas.

Proof of Theorem 4.2. First, we prove that, for any fixed angle $\beta \in (0, 2\pi) \setminus \{\pi\}$ and $0 < \sigma < 1$, the operator $v^\sigma \chi_0 \mathcal{K}_0^\beta \chi_1$ is a map from C_{v^σ} to C. Similar arguments apply to $v^\sigma \chi_1 \mathcal{K}_1^\beta \chi_0$. For any $F \in C_{v^\sigma}$, the continuity of $(v^\sigma \mathcal{K}_0^\beta F)(t)$ in $(0,1]$ trivially follows from the continuity of the kernel $K_0^\beta(t,s)$ for $t + s > 0$. The continuity of $(v^\sigma \mathcal{K}_0^\beta F)(t)$ at $t = 0$ is a consequence of the definition (4.7) and the following considerations. We have

$$2\pi\left(v^\sigma \mathcal{K}_0^\beta F\right)(t) = (1-t)^\sigma t^\sigma \left(\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^1\right) k^\beta\left(\frac{t}{1-s}\right)\frac{F(s)\,ds}{1-s} =: I_1(t) + I_2(t)$$

and

$$\begin{aligned} I_2(t) &= (1-t)^\sigma t^\sigma \int_{2t}^\infty \tau^{-1} k^\beta(\tau) F\left(1 - \frac{t}{\tau}\right) d\tau \\ &= (1-t)^\sigma \int_0^\infty \tau^{\sigma-1} k^\beta(\tau)\left(1 - \frac{t}{\tau}\right)^{-\sigma} (v^\sigma F)\left(1 - \frac{t}{\tau}\right)\chi_{[2t,\infty)}(\tau)\,d\tau, \end{aligned}$$

where $\chi_{[2t,\infty)}$ denotes the characteristic function of the interval $[2t, \infty)$. Since

$$\left|\tau^{\sigma-1}k^\beta(\tau)\left(1-\frac{t}{\tau}\right)^{-\sigma}(v^\sigma F)\left(1-\frac{t}{\tau}\right)\chi_{[2t,\infty)}(\tau)\right| \leq 2^\sigma \tau^{\sigma-1}k^\beta(\tau)\left\|v^\sigma F\right\|_\infty,$$

we can apply Lebesgue's dominated convergence theorem and obtain

$$\lim_{t\to 0^+} I_2(t) = (v^\sigma F)(1)\int_0^\infty \tau^{\sigma-1}k^\beta(\tau)\,d\tau = 2\pi\left(v^\sigma \mathcal{K}_0^\beta F\right)(0).$$

Moreover, since $\left|k^\beta(t)\right| = \dfrac{qt^{q-1}|\sin\beta|}{(t^q-\cos\beta)^2+\sin^2\beta} \le \dfrac{q\,t^{q-1}}{|\sin\beta|}$, $t>0$, we can estimate

$$\begin{aligned}|I_1(t)| &\le (1-t)^\sigma \int_0^{\frac{1}{2}} \left(\frac{t}{1-s}\right)^\sigma k^\beta\left(\frac{t}{1-s}\right) \frac{(1-s)^\sigma s^\sigma |F(s)|\,ds}{(1-s)s^\sigma} \\ &\le \mathcal{C}(1-t)^\sigma t^{\sigma+q-1} \int_0^{\frac{1}{2}} \frac{ds}{s^\sigma(1-s)^{\sigma+q}} \|v^\sigma F\|_\infty \longrightarrow 0 \quad \text{if } t\longrightarrow 0^+.\end{aligned}$$

Taking smooth cut-off functions χ_0 and χ_1 on the interval $[0,1]$, it follows that, if $F\in C_{v^\sigma}$, then the function $v^\sigma\chi_0\mathcal{K}_0^\beta\chi_1 F$ is continuous on $[0,1]$. Consequently the matrix operator $\mathcal{M}$ is a map from the space $\boldsymbol{C}_{v^\sigma}$ into $\boldsymbol{C}_{v^\sigma}$. In order to prove that it is bounded and satisfies (4.13), let us estimate the following norm for a general array of functions $\boldsymbol{F}=(F_1,\ldots,F_n)\in\boldsymbol{C}_{v^\sigma}$:

$$\|\mathcal{M}\boldsymbol{F}\|_{v^\sigma,\infty} = \max_{i=1,\ldots,n}\left\|\chi_0\mathcal{K}_0^{\beta_{i-1}}\chi_1F_{i-1}+\chi_1\mathcal{K}_1^{\beta_i}\chi_0F_{i+1}\right\|_{v^\sigma,\infty},$$

with the notation $\beta_0=\beta_n$, $F_0=F_n$, $F_{n+1}=F_1$.

Let $0<\varepsilon<1/2$ be the small quantity involved in the definition (4.1) of the functions χ_0 and χ_1. Using the substitution $1-s=\frac{t}{\tau}$, we have

$$\begin{aligned}&2\pi\sup_{t\in(0,\varepsilon]}\left|(v^\sigma\chi_0\mathcal{K}_0^{\beta_{i-1}}\chi_1F_{i-1}+v^\sigma\chi_1\mathcal{K}_1^{\beta_i}\chi_0F_{i+1})(t)\right| \\ &= 2\pi\sup_{t\in(0,\varepsilon]}\left|v^\sigma(t)\chi_0(t)(\mathcal{K}_0^{\beta_{i-1}}\chi_1F_{i-1})(t)\right| \\ &\le \sup_{t\in(0,\varepsilon]} t^\sigma\int_{1-\varepsilon}^1 \frac{1}{1-s}\left|k^{\beta_{i-1}}\left(\frac{t}{1-s}\right)\right|\left|(v^\sigma F_{i-1})(s)\right|(1-s)^{-\sigma}s^{-\sigma}ds \\ &\le (1-\varepsilon)^{-\sigma}\sup_{t\in(0,\varepsilon]}\int_0^1\frac{1}{1-s}\left|k^{\beta_{i-1}}\left(\frac{t}{1-s}\right)\right|\left|(v^\sigma F_{i-1})(s)\right|\left(\frac{t}{1-s}\right)^\sigma ds \\ &\le \|F_{i-1}\|_{v^\sigma,\infty}(1-\varepsilon)^{-\sigma}\sup_{t\in(0,\varepsilon]}\int_t^\infty \tau^{-1+\sigma}\left|k^{\beta_{i-1}}(\tau)\right|d\tau \\ &\le \|F_{i-1}\|_{v^\sigma,\infty}(1-\varepsilon)^{-\sigma}\int_0^\infty \tau^{-1+\sigma}\left|k^{\beta_{i-1}}(\tau)\right|d\tau.\end{aligned}$$

Proceeding in an analogous way, using the change of variable $s=\frac{1-t}{\tau}$, one has

$$\begin{aligned}&2\pi\sup_{t\in[1-\varepsilon,1)}\left|(v^\sigma\chi_0\mathcal{K}_0^{\beta_{i-1}}\chi_1F_{i-1}+v^\sigma\chi_1\mathcal{K}_1^{\beta_i}\chi_0F_{i+1})(t)\right| \\ &= 2\pi\sup_{t\in[1-\varepsilon,1)}\left|v^\sigma(t)\chi_1(t)(\mathcal{K}_1^{\beta_i}\chi_0F_{i+1})(t)\right| \\ &\le \sup_{t\in[1-\varepsilon,1)}(1-t)^\sigma\int_0^\varepsilon\frac{1}{s}\left|k^{\beta_i}\left(\frac{1-t}{s}\right)\right|\left|(v^\sigma F_{i+1})(s)\right|(1-s)^{-\sigma}s^{-\sigma}ds \\ &\le (1-\varepsilon)^{-\sigma}\sup_{t\in[1-\varepsilon,1)}\int_0^1\frac{1}{s}\left|k^{\beta_i}\left(\frac{1-t}{s}\right)\right|\left|(v^\sigma F_{i+1})(s)\right|\left(\frac{1-t}{s}\right)^\sigma ds \\ &\le \|F_{i+1}\|_{v^\sigma,\infty}(1-\varepsilon)^{-\sigma}\int_0^\infty\tau^{-1+\sigma}\left|k^{\beta_i}(\tau)\right|d\tau.\end{aligned}$$

Finally, observing that

$$(v^\sigma\chi_0\mathcal{K}_0^{\beta_{i-1}}\chi_1F_{i-1}+v^\sigma\chi_1\mathcal{K}_1^{\beta_i}\chi_0F_{i+1})(t)=0 \qquad \forall t\in(\varepsilon,1-\varepsilon),$$

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we deduce that

$$\|\mathcal{M}\boldsymbol{F}\|_{v^\sigma,\infty} \le \|\boldsymbol{F}\|_{v^\sigma,\infty} \frac{(1-\varepsilon)^{-\sigma}}{2\pi} \max_{i=1,\dots,n} \int_0^\infty \tau^{-1+\sigma} \left|k^{\beta_i}(\tau)\right| d\tau,$$

from which the assertion follows. □

Proof of Lemma 4.3. Using the change of variable $x = \tau^q$ and setting, for brevity, $\rho = 1 - \frac{1-\sigma}{q}$, by virtue of [15, formula 3.252, 12.]), we can write

$$\begin{aligned}
\int_0^\infty \tau^{-1+\sigma}|k^\beta(\tau)|d\tau &= |\sin\beta| \int_0^\infty \frac{x^{-1+\rho}}{x^2+2x\cos(\pi-\beta)+1}dx \\
&= |\sin(\beta)|(-\pi)\csc(\pi-\beta)\csc(\rho\pi)\sin((\rho-1)(\pi-\beta)) \\
&= \pi\frac{|\sin\beta|}{\sin\beta}\frac{\sin((1-\rho)(\pi-\beta))}{\sin(\rho\pi)}.
\end{aligned} \tag{7.1}$$

Thus, inequality (4.14) is equivalent to

$$\left|\frac{\sin((1-\rho)(\pi-\beta))}{\sin(\rho\pi)}\right| < 1. \tag{7.2}$$

Due to our assumptions we have $0 < \rho < 1$ and $\beta \in (0, 2\pi) \setminus \{\pi\}$. In case of $0 < \beta < \pi$ and $0 < \rho \le \frac{1}{2}$, relation (7.2) is equivalent to

$$(1-\rho)(\pi-\beta) < \rho\pi \qquad \text{or} \qquad (1-\rho)(\pi-\beta) > \pi - \rho\pi,$$

which is the same as $1 - \frac{\pi}{2\pi-\beta} < \rho \le \frac{1}{2}$. In case of $0 < \beta < \pi$ and $\frac{1}{2} \le \rho < 1$, condition (7.2) is equivalent to

$$(1-\rho)(\pi-\beta) > \rho\pi \qquad \text{or} \qquad (1-\rho)(\pi-\beta) < \pi - \rho\pi,$$

which means nothing else than $\frac{1}{2} \le \rho < 1$. Analogously, in case of $\pi < \beta < 2\pi$, we obtain that inequality (7.2) is equivalent to $1 - \frac{\pi}{\beta} < \rho < 1$. Consequently, for $\beta \in (0, 2\pi) \setminus \{\pi\}$ and $0 < \rho < 1$, condition (7.2) is equivalent to

$$\left(0 < \beta < \pi \;\text{ and }\; 1 - \frac{\pi}{2\pi-\beta} < \rho < 1\right) \quad \text{or} \quad \left(\pi < \beta < 2\pi \;\text{ and }\; 1 - \frac{\pi}{\beta} < \rho < 1\right),$$

which can also be written as the inequality $1 - \min\left\{\frac{\pi}{\beta}, \frac{\pi}{2\pi-\beta}\right\} < \rho < 1$ or, equivalently, as $1 - q\,\min\left\{\frac{\pi}{\beta}, \frac{\pi}{2\pi-\beta}\right\} < \sigma < 1$. □

Proof of Theorem 4.4. In view of Lemma 4.3 we can choose $\varepsilon \in \left(0, \frac{1}{2}\right)$ such that

$$(1-\varepsilon)^{-\sigma} \max_{i=1,\dots,n} \int_0^\infty \tau^{-1+\sigma}|k^{\beta_i}(\tau)|d\tau < \pi$$

is satisfied. Taking further into account (4.13), we have

$$\|\mathcal{M}\|_{\boldsymbol{C}_{v^\sigma}\to\boldsymbol{C}_{v^\sigma}} < \frac{1}{2}.$$

Applying the Neumann series theorem, we can deduce the invertibility of the operator $\frac{1}{2}\mathcal{J}+\mathcal{M}$ as a map from $\boldsymbol{C}_{v^\sigma}$ into $\boldsymbol{C}_{v^\sigma}$. Since the operator $\widetilde{\mathcal{K}} : \boldsymbol{C}_{v^\sigma} \to \boldsymbol{C}_{v^\sigma}$ is compact, being a matrix of compact operators (cf. Lemma 4.1), the operator $\frac{1}{2}\mathcal{J}+\mathcal{K} = \frac{1}{2}\mathcal{J}+\mathcal{M}+\widetilde{\mathcal{K}} : \boldsymbol{C}_{v^\sigma} \longrightarrow \boldsymbol{C}_{v^\sigma}$

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is a Fredholm operator of index 0. Thus, to prove its invertibility we have to show that the equation

$$\left(\frac{1}{2}\mathcal{J}+\mathcal{K}\right)\widetilde{\boldsymbol{\psi}}=0 \tag{7.3}$$

has only the trivial solution $\widetilde{\boldsymbol{\psi}}=0$ in $\boldsymbol{C}_{v^\sigma}$. Fix a number $p\in(1,2]$ such that $p<\frac{q}{q-(1-\sigma)}$. Such a number exists because of $\frac{q}{q-(1-\sigma)}>1$. Then, a function $\psi\in C_{v^\sigma}$ belongs also to $L^p_{v^\rho}(0,1)$ with $\rho=\frac{1-q}{p'}$. This is due to the estimate

$$\|\psi\|^p_{L^p_{v^\rho}}=\int_0^1\left[v^{\rho-\sigma}(t)\right]^p\left|(v^\sigma\psi)(t)\right|^p\,dt\le\int_0^1\left[v^{\rho-\sigma}(t)\right]^p\,dt\,\|\psi\|^p_{v^\sigma,\infty}=\mathcal{C}\,\|\psi\|^p_{v^\sigma,\infty},$$

which is true if $(\rho-\sigma)p>-1$. However, the last inequality is equivalent to $p<\frac{q}{q-(1-\sigma)}$. Consequently, a solution $\widetilde{\boldsymbol{\psi}}\in\boldsymbol{C}_{v^\sigma}$ of (7.3) also belongs to $\boldsymbol{L}^p_{v^\rho}(0,1)$ and is, due to Corollary 3.4, identically 0. □

LEMMA 7.1. *Let $K_0^\beta(t,s)$ and $K_1^\beta(t,s)$ be the kernels defined in (4.3), and assume that the functions χ_0 and χ_1 in (4.1) satisfy (4.2) for some $k\in\mathbb{N}$. Then, for any nonnegative integer $r\le k$, the following inequalities hold true:*

$$\left\|\frac{\partial^r}{\partial s^r}\left(K_0^\beta(t,\cdot)\chi_1\right)\varphi^r v^{-\sigma}\right\|_1\le\frac{\mathcal{C}}{t^{\frac{r}{2}+\sigma}},\qquad t\in(0,1], \tag{7.4}$$

$$\left\|\frac{\partial^r}{\partial s^r}\left(K_1^\beta(t,\cdot)\chi_0\right)\varphi^r v^{-\sigma}\right\|_1\le\frac{\mathcal{C}}{(1-t)^{\frac{r}{2}+\sigma}},\qquad t\in[0,1), \tag{7.5}$$

with $\varphi(s)=\sqrt{s(1-s)}$ and $\mathcal{C}$ a positive constant independent of t.

Proof. We limit ourselves to proving (7.4). The proof of (7.5) is analogous. We start by observing that

$$K_0^\beta(t,s)=\frac{qt^{q-1}}{2\mathrm{i}}\left[\frac{1}{t^q-e^{\mathrm{i}\beta}(1-s)^q}-\frac{1}{t^q-e^{-\mathrm{i}\beta}(1-s)^q}\right],$$

with $\mathrm{i}^2=-1$, and, consequently, for any integer $j\ge1$ it holds

$$\frac{\partial^j}{\partial s^j}K_0^\beta(t,s)=\frac{qt^{q-1}}{2\mathrm{i}}\sum_{h=1}^{j}c_h(q)h!(1-s)^{hq-j}\left[\frac{e^{\mathrm{i}\beta h}}{\left(t^q-e^{\mathrm{i}\beta}(1-s)^q\right)^{h+1}}-\frac{e^{-\mathrm{i}\beta h}}{\left(t^q-e^{-\mathrm{i}\beta}(1-s)^q\right)^{h+1}}\right]$$

with suitable constants $c_h(q)$, $h=1,\dots,j$, depending on the parameter q. Moreover, since

$$\begin{aligned}\frac{1}{2\mathrm{i}}&\left[\frac{e^{\mathrm{i}\beta h}}{\left(t^q-e^{\mathrm{i}\beta}(1-s)^q\right)^{h+1}}-\frac{e^{-\mathrm{i}\beta h}}{\left(t^q-e^{-\mathrm{i}\beta}(1-s)^q\right)^{h+1}}\right]\\&=-\frac{\displaystyle\sum_{i=0}^{h+1}\binom{h+1}{i}t^{qi}(-1)^{h+1-i}(1-s)^{q(h+1-i)}\sin\left((1-i)\beta\right)}{\left[t^{2q}-2t^q(1-s)^q\cos\beta+(1-s)^{2q}\right]^{h+1}},\end{aligned}$$

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we have

$$\begin{aligned}
&\left|\frac{\partial^j}{\partial s^j}K_0^\beta(t,s)\right| \\
&\le qt^{q-1}\sum_{h=1}^{j}|c_h(q)|\,h!(1-s)^{hq-j}\,\frac{\sum_{i=0}^{h+1}\binom{h+1}{i}t^{qi}(1-s)^{q(h+1-i)}}{\left[t^{2q}-2t^q(1-s)^q\cos\beta+(1-s)^{2q}\right]^{h+1}} \\
&= qt^{q-1}\sum_{h=1}^{j}|c_h(q)|\,h!(1-s)^{hq-j}\,\frac{\left[t^q+(1-s)^q\right]^{h+1}}{\left[t^{2q}-2t^q(1-s)^q\cos\beta+(1-s)^{2q}\right]^{h+1}}.
\end{aligned} \tag{7.6}$$

Now, noting that

$$\left\|\frac{\partial^r}{\partial s^r}\left(K_0^\beta(t,\cdot)\chi_1\right)\varphi^r v^{-\sigma}\right\|_1 \le \sum_{j=0}^{r}\binom{r}{j}\left\|\frac{\partial^j}{\partial s^j}K_0^\beta(t,\cdot)\chi_1^{(r-j)}\varphi^r v^{-\sigma}\right\|_1,$$

we are going to estimate $\left\|\frac{\partial^j}{\partial s^j}K_0^\beta(t,\cdot)\chi_1^{(r-j)}\varphi^r v^{-\sigma}\right\|_1$. Using (7.6) and the change of variable $1-s=\tau t$, one has

$$\begin{aligned}
&\left\|\frac{\partial^j}{\partial s^j}K_0^\beta(t,\cdot)\chi_1^{(r-j)}\varphi^r v^{-\sigma}\right\|_1 = \int_{1-\varepsilon}^1\left|\frac{\partial^j}{\partial s^j}K_0^\beta(t,s)\right|\left|\chi_1^{(r-j)}(s)\right|\varphi^r(s)v^{-\sigma}(s)ds \\
&\quad\le \mathcal{C}(1-\varepsilon)^{-\sigma}\int_0^1\left|\frac{\partial^j}{\partial s^j}K_0^\beta(t,s)\right|(1-s)^{\frac{r}{2}-\sigma}ds \\
&\quad\le \mathcal{C}(1-\varepsilon)^{-\sigma}qt^{q-1}\sum_{h=1}^{j}|c_h(q)|\,h!\int_0^1\frac{\left[t^q+(1-s)^q\right]^{h+1}(1-s)^{hq-j+\frac{r}{2}-\sigma}}{\left[t^{2q}-2t^q(1-s)^q\cos\beta+(1-s)^{2q}\right]^{h+1}}ds \\
&\quad= \mathcal{C}(1-\varepsilon)^{-\sigma}\frac{q}{t^{j-\frac{r}{2}+\sigma}}\sum_{h=1}^{j}|c_h(q)|\,h!\int_0^{\frac{1}{t}}\frac{(1+\tau^q)^{h+1}\tau^{hq-j+\frac{r}{2}-\sigma}}{(\tau^{2q}-2\tau^q\cos\beta+1)^{h+1}}d\tau \\
&\quad\le \mathcal{C}(1-\varepsilon)^{-\sigma}\frac{r!}{t^{\frac{r}{2}+\sigma}},
\end{aligned}$$

where we used the estimate

$$\begin{aligned}
\int_0^{\frac{1}{t}}\frac{(1+\tau^q)^{h+1}\tau^{hq-j+\frac{r}{2}-\sigma}}{(\tau^{2q}-2\tau^q\cos\beta+1)^{h+1}}d\tau &\le \mathcal{C}\left(1+\int_1^{\frac{1}{t}}\tau^{\frac{r}{2}-\sigma-j-q}d\tau\right) \\
&\le \mathcal{C}\begin{cases}\left(\frac{1}{t}\right)^{\frac{r}{2}-\sigma-j-q+1}, & \frac{r}{2}-\sigma-j-q+1>0,\\ \log\frac{1}{t}, & \frac{r}{2}-\sigma-j-q+1=0,\\ \mathcal{C}, & \frac{r}{2}-\sigma-j-q+1<0.\end{cases}
\end{aligned} \tag{7.7}$$

Hence, we can deduce

$$\left\|\frac{\partial^r}{\partial s^r}\left(K_0^\beta(t,\cdot)\chi_1\right)\varphi^r v^{-\sigma}\right\|_1 \le \frac{\mathcal{C}}{t^{\frac{r}{2}+\sigma}},$$

with $\mathcal{C}$ a positive constant dependent on q, r, ε, and σ, but independent of t. □

LEMMA 7.2. *For some integer $\ell \geq 2$ and $i = 1, \ldots, n$, let $\xi_i, \eta_i : [0,1] \longrightarrow \mathbb{R}$ in (3.4) belong to $C^{\ell+2}[0,1]$. Moreover, assume that, for the right-hand sides of the system (3.1), we have $f_i \in C^{\ell}(\Gamma_i)$, $i = 1, \ldots, n$, and that $\chi_0, \chi_1 \in C^{\ell}[0,1]$ as well as $\sigma + \mu > 0$ with $\mu = \min\{\mu_1, \ldots, \mu_n\}$. Then, for $r = 0, 1, \ldots, \ell$, the unique solution $\widehat{\boldsymbol{\psi}} = \left(\widehat{\psi}_1, \ldots, \widehat{\psi}_n\right)^T \in \boldsymbol{C}$ of (4.16) is r-times continuously differentiable on $(0,1)$ and satisfies the estimates*

$$\left|\widehat{\psi}_i^{(r)}(t)\right| \leq \mathcal{C}\, t^{\sigma+\mu_{i-1}-r}(1-t)^{\sigma+\mu_i-r} \leq \mathcal{C}\, v^{\sigma+\mu-r}(t), \qquad 0 < t < 1. \tag{7.8}$$

Proof. If $\widehat{\boldsymbol{\psi}} = \left(\widehat{\psi}_1, \ldots, \widehat{\psi}_n\right)^T \in \boldsymbol{C}$ is the solution of equation (4.16), then, for $i = 1, \ldots, n$,

$$\begin{aligned}\widehat{\psi}_i &= -2\widehat{f}_i - 2\sum_{j=1}^{n} v^{\sigma}\mathcal{K}_{ij}v^{-\sigma}\widehat{\psi}_j \\ &= -2\widehat{f}_i - 2v^{\sigma}\mathcal{M}_{i,i-1}v^{-\sigma}\widehat{\psi}_{i-1} - 2v^{\sigma}\mathcal{M}_{i,i+1}v^{-\sigma}\widehat{\psi}_{i+1} \\ &\qquad - 2v^{\sigma}\mathcal{E}_{i,i-1}v^{-\sigma}\widehat{\psi}_{i-1} - 2v^{\sigma}\mathcal{E}_{i,i+1}v^{-\sigma}\widehat{\psi}_{i+1} - 2\sum_{j=1, j\neq i\pm 1}^{n} v^{\sigma}\mathcal{K}_{ij}v^{-\sigma}\widehat{\psi}_j.\end{aligned} \tag{7.9}$$

Since $\widehat{f}_i(t) = v^{\sigma}(t)g_i\left(\Phi_q(t)\right)\phi_q'(t)$ with $g_i(t) = f_i(\gamma_i(t))|\gamma_i'(t)|$ (cf. (3.12)), for $r = 0, \ldots, \ell$ and $0 < t < 1$, the estimate

$$\left|\widehat{f}_i^{(r)}(t)\right| \leq \mathcal{C}\, t^{\sigma+q-1-r}(1-t)^{\sigma+q-1-r} \tag{7.10}$$

is true. Analogously, for the kernel function $K_{ij}(t,s)$ in (3.12) we have, for $r = 0, \ldots, \ell$,

$$\left|\frac{\partial^r}{\partial t^r}\left[v^{\sigma}(t)K_{ij}(t,s)\right]\right| \leq \mathcal{C}\, t^{\sigma+q-1-r}(1-t)^{\sigma+q-1-r}, \qquad \begin{array}{l} i = 1, \ldots, n,\ j \neq i \pm 1, \\ 0 < t, s < 1. \end{array}$$

The same is true for the kernel functions

$$K_{i,i+1}(t,s) - \chi_1(t)K_{i,i+1}(t,s)\chi_0(s) \quad \text{and} \quad K_{i,i-1}(t,s) - \chi_0(t)K_{i,i-1}(t,s)\chi_1(s)$$

of the operators $\mathcal{E}_{i,i\pm 1}$. This implies

$$\left|\left(v^{\sigma}\mathcal{K}_{ij}v^{-\sigma}\widehat{\psi}_j\right)^{(r)}(t)\right| \leq \mathcal{C}\, t^{\sigma+q-1-r}(1-t)^{\sigma+q-1-r}, \qquad \begin{array}{l} i = 1, \ldots, n,\ j \neq i \pm 1, \\ 0 < t < 1, \end{array} \tag{7.11}$$

and

$$\left|\left(v^{\sigma}\mathcal{E}_{i,i\pm 1}v^{-\sigma}\widehat{\psi}_j\right)^{(r)}(t)\right| \leq \mathcal{C}\, t^{\sigma+q-1-r}(1-t)^{\sigma+q-1-r}, \qquad \begin{array}{l} i = 1, \ldots, n, \\ 0 < t < 1. \end{array} \tag{7.12}$$

Recall that $\mathcal{M}_{i,i-1} = \chi_0\mathcal{K}_0^{\beta_{i-1}}\chi_1$ and $\mathcal{M}_{i,i+1} = \chi_1\mathcal{K}_1^{\beta_i}\chi_0$ (cf. (4.5) and (4.6)).

Analogously to the proof of Lemma 7.1, for $j \geq 0$ we have

$$\begin{aligned}&\frac{\partial^j}{\partial t^j}K_0^{\beta}(t,s) \\ &= \frac{1}{2\mathrm{i}}\sum_{h=0}^{j} d_h(q)t^{(h+1)q-j-1}\left[\frac{1}{\left(t^q - e^{\mathrm{i}\beta}(1-s)^q\right)^{h+1}} - \frac{1}{\left(t^q - e^{-\mathrm{i}\beta}(1-s)^q\right)^{h+1}}\right]\end{aligned}$$

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$$= -\sum_{h=0}^{j}\left\{ d_h(q)t^{(h+1)q-j-1} \quad \times \right.$$
$$\left. \frac{\sum_{i=0}^{h+1}\begin{pmatrix} h+1 \\ i \end{pmatrix} t^{qi}(-1)^{h+1-i}(1-s)^{q(h+1-i)}\sin((h+1-i)\beta)}{[t^{2q}-2t^q(1-s)^q\cos\beta+(1-s)^{2q}]^{h+1}} \right\},$$

with certain constants $d_h(q)$ depending on q. Consequently,

(7.13)

$$\left|\frac{\partial^j}{\partial t^j}K_0^\beta(t,s)\right| \le \sum_{h=0}^{j}|d_h(q)|t^{(h+1)q-j-1}\frac{\sum_{i=0}^{h+1}\begin{pmatrix} h+1 \\ i \end{pmatrix} t^{qi}(1-s)^{q(h+1-i)}}{[t^{2q}-2t^q(1-s)^q\cos\beta+(1-s)^{2q}]^{h+1}}$$
$$= \sum_{h=0}^{j}|d_h(q)|t^{(h+1)q-j-1}\frac{[t^q+(1-s)^q]^{h+1}}{[t^{2q}-2t^q(1-s)^q\cos\beta+(1-s)^{2q}]^{h+1}}.$$

Now,

$$\left(v^\sigma\chi_0\mathcal{K}_0^{\beta_{i-1}}\chi_1 v^{-\sigma}\widehat{\psi}_{i-1}\right)^{(r)}(t)$$
$$= \frac{1}{2\pi}\sum_{j=0}^{r}\begin{pmatrix} r \\ j \end{pmatrix}(v^\sigma\chi_0)^{(r-j)}(t)\int_{1-\varepsilon}^{1}\frac{\partial^j}{\partial t^j}K_0^\beta(t,s)v^{-\sigma}(s)\widehat{\psi}_{i-1}(s)ds,$$

where, in virtue of (7.13) and (3.13) (cf. also (7.7)),

$$\left|\int_{1-\varepsilon}^{1}\frac{\partial^j}{\partial t^j}K_0^\beta(t,s)v^{-\sigma}(s)\widehat{\psi}_{i-1}(s)ds\right|$$
$$\le \mathcal{C}\sum_{h=0}^{j}t^{(h+1)q-j-1}\int_0^1\frac{[t^q+(1-s)^q]^{h+1}(1-s)^{\mu_{i-1}}ds}{[t^{2q}-2t^q(1-s)^q\cos\beta+(1-s)^{2q}]^{h+1}}$$
$$= \mathcal{C}\,t^{\mu_{i-1}-j}\sum_{h=0}^{j}\int_0^{t^{-1}}\frac{(1+\tau^q)^{h+1}\tau^{\mu_{i-1}}d\tau}{(1-2\tau^q\cos\beta+\tau^{2q})^{h+1}}$$
$$\le \mathcal{C}\,t^{\mu_{i-1}-j}\sum_{h=0}^{j}\left(1+\int_1^{t^{-1}}\tau^{\mu_{i-1}-q(h+1)}d\tau\right)$$
$$\le \mathcal{C}\,t^{\mu_{i-1}-j}\sum_{h=0}^{j}\left(1+t^{q(h+1)-\mu_{i-1}-1}\right) \le \mathcal{C}\,t^{\mu_{i-1}-j}.$$

Hence

$$\left|\left(v^\sigma\chi_0\mathcal{K}_0^{\beta_{i-1}}\chi_1 v^{-\sigma}\widehat{\psi}_{i-1}\right)^{(r)}(t)\right| \le \mathcal{C}\,t^{\sigma+\mu_{i-1}-r}. \tag{7.14}$$

In the same manner we get

$$\left|\left(v^\sigma\chi_1\mathcal{K}_0^{\beta_i}\chi_0 v^{-\sigma}\widehat{\psi}_{i+1}\right)^{(r)}(t)\right| \le \mathcal{C}\,(1-t)^{\sigma+\mu_i-r}. \tag{7.15}$$

Since $q-1<\mu_i$, $i=1,\ldots,n$, from (7.9), (7.10), (7.11), (7.12), (7.14), and (7.15) we conclude the assertion (7.8). □

Proof of Theorem 6.1. Let us consider the case $n\ge 2$. Then the proof immediately follows in the simpler case $n=1$. The linearity of the operators $\mathcal{M}_{ij,m}^\sigma$, the entries of the matrix $\mathcal{M}_m^\sigma$, is a trivial consequence of their definition (see (5.5)–(5.8)) and of the linearity of the operators $\mathcal{K}_r^\beta$ and $\mathcal{K}_{r,m}^\beta$ (see (4.5) and (5.3)). In order to prove that $\mathcal{M}_m^\sigma \boldsymbol{f}$ belongs to the space $\boldsymbol{C}$ for any array of functions $\boldsymbol{f}\in\boldsymbol{C}$, we take into account that the cut-off functions χ_0 and χ_1 defined by (4.1) are assumed to be smooth, such that we only have to show that, for any angle β and any function $f\in C$, the functions $\widetilde{\mathcal{K}}_{0,m}^{\beta,\sigma} f$ and $\widetilde{\mathcal{K}}_{1,m}^{\beta,\sigma} f$ are continuous at the point t_m and at the point $1-t_m$, respectively. By definition (5.5) and (5.6) we have

$$\lim_{t\to t_m^-}\left(\widetilde{\mathcal{K}}_{0,m}^{\beta,\sigma} f\right)(t)=\left(\mathcal{K}_{0,m}^{\beta,\sigma} f\right)(t_m)=\lim_{t\to t_m^+}\left(\widetilde{\mathcal{K}}_{0,m}^{\beta,\sigma} f\right)(t)$$

and

$$\lim_{t\to(1-t_m)^-}\left(\widetilde{\mathcal{K}}_{1,m}^{\beta,\sigma} f\right)(t)=\left(v^\sigma\mathcal{K}_{1,m}^{\beta,\sigma} f\right)(1-t_m)=\lim_{t\to(1-t_m)^+}\left(\widetilde{\mathcal{K}}_{1,m}^{\beta,\sigma} f\right)(t).$$

Now let us prove (6.1). Since, for $\boldsymbol{f}=(f_1,\ldots,f_n)^T\in\boldsymbol{C}$ one has

$$\|\mathcal{M}_m^\sigma \boldsymbol{f}\|_\infty=\max_{i=1,\ldots,n}\left\|\mathcal{M}_{i,i-1,m}^\sigma f_{i-1}+\mathcal{M}_{i,i+1,m}^\sigma f_{i+1}\right\|_\infty, \tag{7.16}$$

with the following meaning of the notation: $\mathcal{M}_{1,0,m}^\sigma\equiv\mathcal{M}_{1,n,m}^\sigma$, $\mathcal{M}_{n,n+1,m}^\sigma\equiv\mathcal{M}_{n,1,m}^\sigma$ and $f_0\equiv f_n$, $f_{n+1}\equiv f_1$.

We proceed by estimating, for a fixed $i\in\{1,\ldots,n\}$, the norm

$$\begin{aligned}&\|\mathcal{M}_{i,i-1,m}^\sigma f_{i-1}+\mathcal{M}_{i,i+1,m}^\sigma f_{i+1}\|_\infty\\&=\max\left\{\sup_{t\in[0,\varepsilon]}\left|\left(\mathcal{M}_{i,i-1,m}^\sigma f_{i-1}+\mathcal{M}_{i,i+1,m}^\sigma f_{i+1}\right)(t)\right|,\right.\\&\qquad\sup_{t\in[\varepsilon,1-\varepsilon]}\left|\left(\mathcal{M}_{i,i-1,m}^\sigma f_{i-1}+\mathcal{M}_{i,i+1,m}^\sigma f_{i+1}\right)(t)\right|,\\&\qquad\left.\sup_{t\in[1-\varepsilon,1]}\left|\left(\mathcal{M}_{i,i-1,m}^\sigma f_{i-1}+\mathcal{M}_{i,i+1,m}^\sigma f_{i+1}\right)(t)\right|\right\}.\end{aligned}$$

Taking into account (4.1) and (5.7), we can write

$$\begin{aligned}&\left\|\mathcal{M}_{i,i-1,m}^\sigma f_{i-1}+\mathcal{M}_{i,i+1,m}^\sigma f_{i+1}\right\|_\infty\\&\le\max\left\{\sup_{t\in[0,\varepsilon]}\left|\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma}\chi_1 f_{i-1}\right)(t)\right|,\sup_{t\in[1-\varepsilon,1]}\left|\left(\widetilde{\mathcal{K}}_{1,m}^{\beta_i,\sigma}\chi_0 f_{i+1}\right)(t)\right|\right\}.\end{aligned}\tag{7.17}$$

Now, being $t_m\le\varepsilon$, we have

$$\begin{aligned}&\sup_{t\in[0,\varepsilon]}\left|\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma}\chi_1 f_{i-1}\right)(t)\right|\\&=\max\left\{\sup_{t\in[0,t_m]}\left|\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma}\chi_1 f_{i-1}\right)(t)\right|,\sup_{t\in[t_m,\varepsilon]}\left|\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma}\chi_1 f_{i-1}\right)(t)\right|\right\}.\end{aligned}\tag{7.18}$$

For the first term one has

$$\begin{aligned}&\sup_{t\in[0,t_m]}\left|\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma}\chi_1 f_{i-1}\right)(t)\right|\\&=\max\left\{\left|\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}\chi_1 v^{-\sigma}f_{i-1}\right)(0)\right|,\left|\left(v^\sigma\mathcal{K}_{0,m}^{\beta_{i-1}}\chi_1 v^{-\sigma}f_{i-1}\right)(t_m)\right|\right\},\end{aligned}$$

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and, consequently, by virtue of (7.18) it follows that

$$\sup_{t\in[0,\varepsilon]}\left|\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma}\chi_1 f_{i-1}\right)(t)\right|$$
$$\leq \max\left\{\left|\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}\chi_1 v^{-\sigma}f_{i-1}\right)(0)\right|,\ \sup_{t\in[t_m,\varepsilon]}\left|\left(v^\sigma\mathcal{K}_{0,m}^{\beta_{i-1}}\chi_1 v^{-\sigma}f_{i-1}\right)(t)\right|\right\}.$$

Let us estimate both terms in the brackets. In the proof of Theorem 4.2 we have shown

$$\left|\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}\chi_1 v^{-\sigma}f_{i-1}\right)(0)\right| \leq \frac{\|f_{i-1}\|_\infty}{2\pi}\int_0^\infty \tau^{-1+\sigma}k^{\beta_{i-1}}(\tau)\,d\tau.$$

Moreover, for $t\in[t_m,1]$,

$$\left|\left(v^\sigma\mathcal{K}_{0,m}^{\beta_{i-1}}\chi_1 v^{-\sigma}f_{i-1}\right)(t)\right|$$
$$\leq \frac{v^\sigma(t)}{2\pi}\sum_{j=1}^m w_{m,j}\left|K_0^{\beta_{i-1}}(t,s_{m,j})\right|\left|f_{i-1})(s_{m,j})\right|\chi_1(s_{m,j})$$
$$\leq \|f_{i-1}\|_\infty\frac{v^\sigma(t)}{2\pi}\left[\int_0^1\left|K_0^{\beta_{i-1}}(t,s)\right|\chi_1(s)v^{-\sigma}(s)\,ds+\left|e_m\left(\left|K_0^{\beta_{i-1}}(t,\cdot)\right|\chi_1\right)\right|\right],$$

where (see the proof of Theorem 4.2)

$$v^\sigma(t)\int_0^1\left|K_0^{\beta_{i-1}}(t,s)\right|\chi_1(s)v^{-\sigma}(s)\,ds \leq (1-\varepsilon)^{-\sigma}\int_0^\infty \tau^{-1+\sigma}k^{\beta_{i-1}}(\tau)\,d\tau,$$

and, by virtue of the error estimate (2.3), (5.4), and also (7.4),

$$v^\sigma(t)\left|e_m\left(\left|K_0^{\beta_{i-1}}(t,\cdot)\right|\chi_1\right)\right| \leq \frac{\mathcal{C}}{m}\frac{1}{t^{\frac{1}{2}}} \leq \frac{\mathcal{C}}{m}\frac{1}{t_m^{\frac{1}{2}}} = \frac{\mathcal{C}}{m^\epsilon},\quad t\in[t_m,\varepsilon],\ \ \mathcal{C}\neq\mathcal{C}(m).$$

Summarizing, we have proved that

$$\begin{aligned}\sup_{t\in[0,\varepsilon]}&\left|\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma}\chi_1 f_{i-1}\right)(t)\right|\\ &\leq \frac{\|f_{i-1}\|_\infty}{2\pi}\left[(1-\varepsilon)^{-\sigma}\int_0^\infty \tau^{-1+\sigma}k^{\beta_{i-1}}(\tau)\,d\tau+\frac{\mathcal{C}}{m^\epsilon}\right].\end{aligned}\tag{7.19}$$

Proceeding in an analogous way it can also be proved that

$$\begin{aligned}\sup_{t\in[1-\varepsilon,1]}&\left|\left(\mathcal{K}_{1,m}^{\beta_i,\sigma}\chi_0 F_{i+1}\right)(t)\right|\\ &\leq \frac{\|f_{i+1}\|_\infty}{2\pi}\left[(1-\varepsilon)^{-\sigma}\int_0^\infty \tau^{-1+\sigma}\left|k^{\beta_i}(\tau)\right|d\tau+\frac{\mathcal{C}}{m^\epsilon}\right].\end{aligned}\tag{7.20}$$

Combining (7.19) and (7.20) with (7.16) and (7.17), we get

$$\|\mathcal{M}_m^\sigma\boldsymbol{f}\|_\infty \leq \frac{\|\boldsymbol{f}\|_\infty}{2\pi}\left[(1-\varepsilon)^{-\sigma}\max_{i=1,\dots,n}\int_0^\infty \tau^{-1+\sigma}|k^{\beta_i}(\tau)|d\tau+\frac{\mathcal{C}}{m^\epsilon}\right].$$

Finally, taking into account Lemma 4.3, we can immediately deduce the estimate (6.1).

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In order to prove (6.2), we will apply the Banach-Steinhaus theorem to the sequence of the operators $\mathcal{M}_m^\sigma : \boldsymbol{C} \to \boldsymbol{C}$, $m \in \mathbb{N}$, which is uniformly bounded as just has been shown. We consider the product space of polynomials

$$\boldsymbol{P} = \left\{ \boldsymbol{p} = (p_1, \ldots, p_n)^T : p_i \in \mathbb{P} \right\},$$

where $\mathbb{P}$ denotes the set of all algebraic polynomials defined on the interval $[0,1]$. Since $\boldsymbol{P}$ is a dense subspace of $\boldsymbol{C}$, it is sufficient to prove that

$$\lim_{m\to\infty} \left\| (\mathcal{M}_m^\sigma - \mathcal{V}^\sigma \mathcal{M} \mathcal{V}^{-\sigma}) \boldsymbol{p} \right\|_\infty = 0 \qquad \forall\, \boldsymbol{p} \in \boldsymbol{P}. \tag{7.21}$$

For $\boldsymbol{p} = (p_1, \ldots, p_n)^T \in \boldsymbol{P}$ we observe that

$$\begin{aligned}
&\left\| (\mathcal{M}_m^\sigma - \mathcal{V}^\sigma \mathcal{M} \mathcal{V}^{-\sigma}) \boldsymbol{p} \right\|_\infty \\
&= \max_{i=1,\ldots,n} \left\| \chi_0 \left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma} - v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma} \right) \chi_1 p_{i-1} + \chi_1 \left(\widetilde{\mathcal{K}}_{1,m}^{\beta_i,\sigma} - v^\sigma \mathcal{K}_1^{\beta_i} v^{-\sigma} \right) \chi_0 p_{i+1} \right\|_\infty ,
\end{aligned}$$

and, for any fixed $1 \le i \le n$,

$$\begin{aligned}
&\left\| \chi_0 \left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma} - v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma} \right) \chi_1 p_{i-1} + \chi_1 \left(\widetilde{\mathcal{K}}_{1,m}^{\beta_i,\sigma} - v^\sigma \mathcal{K}_1^{\beta_i} v^{-\sigma} \right) \chi_0 p_{i+1} \right\|_\infty \\
&\quad \le \max \left\{ \sup_{t\in[0,\varepsilon]} \left| \left(\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma} - v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma} \right) \chi_1 p_{i-1} \right)(t) \right| , \right. \\
&\qquad\qquad \left. \sup_{t\in[1-\varepsilon,1]} \left| \left(\left(\widetilde{\mathcal{K}}_{1,m}^{\beta_i,\sigma} - v^\sigma \mathcal{K}_1^{\beta_i} v^{-\sigma} \right) \chi_0 p_{i+1} \right)(t) \right| \right\}.
\end{aligned}$$

Now, being $t_m \le \varepsilon$, we are going to separately estimate

$$\begin{aligned}
&\sup_{t\in[0,t_m]} \left| \left(\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma} - v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma} \right) \chi_1 p_{i-1} \right)(t) \right| , \\
&\sup_{t\in[t_m,\varepsilon]} \left| \left(\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma} - v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma} \right) \chi_1 p_{i-1} \right)(t) \right|
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\in[1-\varepsilon,1-t_m]} \left| \left(\left(\widetilde{\mathcal{K}}_{1,m}^{\beta_i,\sigma} - v^\sigma \mathcal{K}_1^{\beta_i} v^{-\sigma} \right) \chi_0 p_{i+1} \right)(t) \right| , \\
&\sup_{t\in[1-t_m,1]} \left| \left(\left(\widetilde{\mathcal{K}}_{1,m}^{\beta_i,\sigma} - v^\sigma \mathcal{K}_1^{\beta_i} v^{-\sigma} \right) \chi_0 p_{i+1} \right)(t) \right| .
\end{aligned}$$

For $t \in [0, t_m]$, we have

$$\begin{aligned}
&\left| \left(\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma} - v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma} \right) \chi_1 p_{i-1} \right)(t) \right| \\
&\quad \le \left| \frac{t}{t_m} \left[\left(v^\sigma \mathcal{K}_{0,m}^{\beta_{i-1}} v^{-\sigma} \chi_1 p_{i-1} \right)(t_m) - \left(v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma} \chi_1 p_{i-1} \right)(0) \right] \right| \\
&\qquad + \left| \left(v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma} \chi_1 p_{i-1} \right)(0) - \left(v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma} \chi_1 p_{i-1} \right)(t) \right| \\
&\quad \le \frac{t}{t_m} \left| \left(v^\sigma \mathcal{K}_{0,m}^{\beta_{i-1}} v^{-\sigma} \chi_1 p_{i-1} \right)(t_m) - \left(v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma} \chi_1 p_{i-1} \right)(t_m) \right|
\end{aligned}$$

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$$+\frac{t}{t_m}\left|\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1p_{i-1}\right)(t_m)-\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1p_{i-1}\right)(0)\right|$$
$$+\left|\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1p_{i-1}\right)(0)-\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1p_{i-1}\right)(t)\right|,$$

and, analogously, for $t\in[1-t_m,1]$,

$$\begin{aligned}
&\left|\left(\left(\widetilde{\mathcal{K}}_{1,m}^{\beta_i,\sigma}-v^\sigma\mathcal{K}_0^{\beta_i}v^{-\sigma}\right)\chi_0p_{i+1}\right)(t)\right|\\
&\quad\le\left|\frac{1-t}{t_m}\left[\left(v^\sigma\mathcal{K}_{1,m}^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(1-t_m)-\left(v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_1p_{i+1}\right)(1)\right]\right|\\
&\qquad\qquad+\left|\left(v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(1)-\left(v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(t)\right|\\
&\quad\le\frac{1-t}{t_m}\left|\left(v^\sigma\mathcal{K}_{1,m}^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(1-t_m)-\left(v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(1-t_m)\right|\\
&\qquad\qquad+\frac{1-t}{t_m}\left|\left(v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(1-t_m)-\left(v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(1)\right|\\
&\qquad\qquad+\left|\left(v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(1)-\left(v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(t)\right|.
\end{aligned}$$

Since $v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1p_{i-1}$ and $v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_0p_{i+1}$ are continuous functions (see the proof of Theorem 4.2), we get

$$\begin{aligned}
&\lim_{m\to\infty}\sup_{t\in[0,t_m]}\frac{t}{t_m}\left|\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1p_{i-1}\right)(t_m)-\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1p_{i-1}\right)(0)\right|=0,\\
&\lim_{m\to\infty}\sup_{t\in[0,t_m]}\left|\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1p_{i-1}\right)(0)-\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1p_{i-1}\right)(t)\right|=0,\\
&\lim_{m\to\infty}\sup_{t\in[1-t_m,1]}\frac{1-t}{t_m}\left|\left(v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(1-t_m)-\left(v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(1)\right|=0,\\
&\lim_{m\to\infty}\sup_{t\in[1-t_m,1]}\left|\left(v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(1)-\left(v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\chi_0p_{i+1}\right)(t)\right|=0.
\end{aligned}$$

Moreover, proceeding as in [27, Proof of Lemma 1], taking into account the property (7.4) of the kernel $K_0^{\beta_{i-1}}(t,s)$ and the quadrature error estimate (2.3), for any $t\ge t_m$, one has

$$\begin{aligned}
&\left|\left(v^\sigma\mathcal{K}_{0,m}^{\beta}v^{-\sigma}\chi_1p\right)(t)-\left(v^\sigma\mathcal{K}_0^{\beta}v^{-\sigma}\chi_1p_{i-1}\right)(t)\right|\\
&=\frac{v^\sigma(t)}{2\pi}\left|e_m\left(K_0^\beta(t,\cdot)\chi_1p\right)\right|\\
&\le\frac{\mathcal{C}\,v^\sigma(t)}{m^k}\int_0^1\left|\frac{\partial^k}{\partial t^k}\left[K_0^\beta(t,s)\chi_1(s)p(s)\right]\right|\varphi^r(s)v^{-\sigma}(s)ds\\
&\le\frac{\mathcal{C}\,v^\sigma(t)}{m^k}\sum_{j=0}^k\binom{k}{j}\int_{1-\varepsilon}^1\left|\frac{\partial^j}{\partial s^j}\left[K_0^\beta(t,s)\chi_1(s)\right]\right|\varphi^j(s)\left|p^{(k-j)}(s)\right|\varphi^{k-j}(s)v^{-\sigma}(s)ds\\
&\le\frac{\mathcal{C}\,v^\sigma(t)}{m^k}\sum_{j=0}^k\binom{k}{j}\left\|\frac{\partial^j}{\partial s^j}\left(K_0^\beta(t,\cdot)\chi_1\right)\varphi^jv^{-\sigma}\right\|_1\\
&\le\frac{\mathcal{C}}{m^k}\sum_{j=0}^k\binom{k}{j}t^{-\frac{j}{2}}\le\frac{\mathcal{C}}{m^k}t^{-\frac{k}{2}},
\end{aligned}$$

where $\mathcal{C}$ does not depend on m and t and it has been assumed that $\chi_1 \in C^k[0,1]$. Hence, in particular, we can deduce that

$$\sup_{t_m \le t \le 1} \left| \left(v^\sigma \mathcal{K}_{0,m}^{\beta_{i-1}} \chi_1 p_{i-1} \right)(t) - \left(v^\sigma \mathcal{K}_0^{\beta_{i-1}} \chi_1 F_{i-1} \right)(t) \right| \le \frac{\mathcal{C}}{m^{k\epsilon}} \longrightarrow 0 \qquad \text{if } m \longrightarrow \infty.$$

In a similar way, for $t \le 1 - t_m$, assuming that $\chi_0 \in C^k([0,1])$ and also using property (7.5) of the kernel $K_1^{\beta_i}(t,s)$, one has

$$\sup_{0 \le t \le 1-t_m} \left| \left(v^\sigma \mathcal{K}_{1,m}^{\beta_i} \chi_0 p_{i+1} \right)(t) - \left(v^\sigma \mathcal{K}_1^{\beta_i} \chi_0 p_{i+1} \right)(t) \right| \le \frac{\mathcal{C}}{m^{k\epsilon}} \longrightarrow 0 \qquad \text{if } m \longrightarrow \infty.$$

Combining all the previous results, the convergence (7.21) follows. □

Proof of Theorem 6.2. We first observe that the operators $\mathcal{K}_m^\sigma$ map $\boldsymbol{C}$ into $\boldsymbol{C}$, and the set $\{\mathcal{K}_m^\sigma : m \in \mathbb{N}\}_m$ is collectively compact, if the sets of operators

$$\left\{\mathcal{K}_{ij,m}^\sigma : m \in \mathbb{N}\right\}_m \qquad \text{for any pair of indices } (i,j) \text{ such that } |i-j| \ne 1 \qquad \text{and}$$
$$\left\{\mathcal{E}_{ij,m}^\sigma : m \in \mathbb{N}\right\}_m \qquad \text{for any pair of indices } (i,j) \text{ such that } |i-j| = 1$$

are collectively compact. Here the operators $\mathcal{E}_{10,m}^\sigma$ and $\mathcal{E}_{n,n+1,m}^\sigma$ have to be understood as $\mathcal{E}_{1n,m}^\sigma$ and $\mathcal{E}_{n1,m}^\sigma$, respectively. Moreover, if for any $\boldsymbol{f} = (f_1, \dots, f_n)^T \in \boldsymbol{C}$ one has

$$\lim_{m\to\infty} \left\| \left(\mathcal{K}_{ij,m}^\sigma - v^\sigma \mathcal{K}_{ij} v^{-\sigma} \right) p_j \right\|_\infty = 0 \quad \text{and} \quad \lim_{m\to\infty} \left\| \left(\mathcal{E}_{ij,m}^\sigma - v^\sigma \mathcal{E}_{ij} v^{-\sigma} \right) p_j \right\|_\infty = 0,$$

$i,j = 1, \dots, n$, then (6.3) is fulfilled. Since the kernels $K_{ij}(t,s)$ ($|i-j| \ne 1$) and $E_{ij}(t,s)$ ($|i-j| = 1$) are continuous on the square $[0,1] \times [0,1]$, one can take into account the estimate (2.3) of the quadrature error and proceed as in the proof of [27, Lemma 2] to obtain the assertion (cf. also [21, Section 3] or [22, Section 6.2]). □

Proof of Theorem 6.3. First, we observe that from (6.1) and (6.2) we can deduce that the linear operators $\frac{1}{2}\mathcal{J} + \mathcal{M}_m^\sigma : \boldsymbol{C} \longrightarrow \boldsymbol{C}$ are bounded and strongly convergent to $\frac{1}{2}\mathcal{J} + \mathcal{V}^\sigma \mathcal{M} \mathcal{V}^{-\sigma} : \boldsymbol{C} \longrightarrow \boldsymbol{C}$. Moreover, using the geometric series theorem, we deduce that for sufficiently large m, say $m \ge m_0$, the inverse operators $\left(\frac{1}{2}\mathcal{J} + \mathcal{M}_m^\sigma\right)^{-1} : \boldsymbol{C} \longrightarrow \boldsymbol{C}$ exist and are uniformly bounded with

$$\left\| \left(\frac{1}{2}\mathcal{J} + \mathcal{M}_m^\sigma \right)^{-1} \right\|_{\boldsymbol{C}\to\boldsymbol{C}} \le \frac{1}{\frac{1}{2} - \sup_{k \ge m_0} \|\mathcal{M}_k^\sigma\|_{\boldsymbol{C}\to\boldsymbol{C}}}.$$

Hence, since the sequence of operators $\{\mathcal{K}_m^\sigma\}_m$ is collectively compact and pointwise convergent to the operator $\mathcal{V}^\sigma \widetilde{\mathcal{K}} \mathcal{V}^{-\sigma}$ in $\boldsymbol{C}$ (see Theorem 6.2), it follows that (see, for instance, [25, Theorem 10.8 and Problem 10.3]) for all sufficiently large m, the operators $\left(\frac{1}{2}\mathcal{J} + \mathcal{M}_m^\sigma + \mathcal{K}_m^\sigma\right)^{-1} : \boldsymbol{C} \to \boldsymbol{C}$ exist and are uniformly bounded. Consequently, if $\widehat{\boldsymbol{\psi}}$ and $\widehat{\boldsymbol{\psi}}_m$ are the solutions of (4.16) and (5.12), respectively, then

$$\widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}}_m = \left(\frac{1}{2}\mathcal{J} + \mathcal{M}_m^\sigma + \mathcal{K}_m^\sigma \right)^{-1} \left[(\mathcal{M}_m^\sigma - \mathcal{V}^\sigma \mathcal{M} \mathcal{V}^{-\sigma}) \widehat{\boldsymbol{\psi}} + (\mathcal{K}_m^\sigma - \mathcal{V}^\sigma \widetilde{\mathcal{K}} \mathcal{V}^{-\sigma}) \widehat{\boldsymbol{\psi}} \right], \tag{7.22}$$

and from (6.2) and (6.3) we immediately deduce (6.4). □

Proof of Theorem 6.6. In order to prove the estimate (6.5), first we observe that from (7.22), taking also into account that the operators $\left(\frac{1}{2}\mathcal{J} + \mathcal{M}_m^\sigma + \mathcal{K}_m^\sigma\right)^{-1} : \boldsymbol{C} \to \boldsymbol{C}$ are uniformly

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bounded, one can deduce

$$\|\widehat{\boldsymbol{\psi}}-\widehat{\boldsymbol{\psi}}_m\|_\infty \le \mathcal{C}\Big[\big\|(\mathcal{M}_m^\sigma-\mathcal{V}^\sigma\mathcal{M}\mathcal{V}^{-\sigma})\widehat{\boldsymbol{\psi}}\big\|_\infty + \big\|(\mathcal{K}_m^\sigma-\mathcal{V}^\sigma\widetilde{\mathcal{K}}\mathcal{V}^{-\sigma})\widehat{\boldsymbol{\psi}}\big\|_\infty\Big], \qquad \mathcal{C}\neq\mathcal{C}(m). \tag{7.23}$$

The estimate of $\big\|(\mathcal{M}_m^\sigma-\mathcal{V}^\sigma\mathcal{M}\mathcal{V}^{-\sigma})\widehat{\boldsymbol{\psi}}\big\|_\infty$ can be reduced to estimating (see the proof of Theorem 6.1) the quantities

$$\begin{aligned}&\sup_{t\in[0,\varepsilon]}\left|\left(\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma}-v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\right)\chi_1\widehat{\psi}_{i-1}\right)(t)\right|,\\ &\sup_{t\in[1-\varepsilon,1]}\left|\left(\left(\widetilde{\mathcal{K}}_{1,m}^{\beta_i,\sigma}-v^\sigma\mathcal{K}_1^{\beta_i}v^{-\sigma}\right)\chi_0\widehat{\psi}_{i+1}\right)(t)\right|,\end{aligned} \tag{7.24}$$

for every fixed $1\le i\le n$. Here and in the sequel we set $\widehat{\psi}_0\equiv\widehat{\psi}_n$ and $\widehat{\psi}_{n+1}\equiv\widehat{\psi}_1$. We are going to estimate the first quantity in (7.24). For the second quantity one can proceed analogously.

Following the same steps as in the proof of Theorem 6.1, for all m large enough such that $t_m\le\varepsilon$, we are going to estimate

$$\begin{aligned}&\sup_{t\in[0,t_m]}\left|\left(\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma}-v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\right)\chi_1\widehat{\psi}_{i-1}\right)(t)\right|,\\ &\sup_{t\in[t_m,\varepsilon]}\left|\left(\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma}-v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\right)\chi_1\widehat{\psi}_{i-1}\right)(t)\right|\end{aligned} \tag{7.25}$$

separately. For $t\in[0,t_m]$, we have

$$\left|\left(\left(\widetilde{\mathcal{K}}_{0,m}^{\beta_{i-1},\sigma}-v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\right)\chi_1\widehat{\psi}_{i-1}\right)(t)\right|\le A_1(t)+A_2(t)+A_3(t),$$

where

$$\begin{aligned}A_1(t)&=\frac{t}{t_m}\left|\left(\mathcal{K}_{0,m}^{\beta_{i-1},\sigma}\chi_1\widehat{\psi}_{i-1}\right)(t_m)-\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1\widehat{\psi}_{i-1}\right)(t_m)\right|,\\ A_2(t)&=\frac{t}{t_m}\left|\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1\widehat{\psi}_{i-1}\right)(t_m)-\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1\widehat{\psi}_{i-1}\right)(0)\right|,\\ A_3(t)&=\left|\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1\widehat{\psi}_{i-1}\right)(0)-\left(v^\sigma\mathcal{K}_0^{\beta_{i-1}}v^{-\sigma}\chi_1\widehat{\psi}_{i-1}\right)(t)\right|.\end{aligned}$$

For the term $A_2(t)$, we get

$$\begin{aligned}A_2(t)&\le\frac{t}{2\pi t_m}\int_{1-\varepsilon}^1\frac{1}{1-s}k^{\beta_{i-1}}\left(\frac{t_m}{1-s}\right)\left(\frac{t_m}{1-s}\right)^\sigma\left|\widehat{\psi}_{i-1}(s)\right|(1-t_m)^\sigma s^{-\sigma}ds\\ &\overset{(7.8)}{\le}\mathcal{C}\int_{1-\varepsilon}^1\frac{1}{1-s}k^{\beta_{i-1}}\left(\frac{t_m}{1-s}\right)\left(\frac{t_m}{1-s}\right)^\sigma(1-s)^{\sigma+\mu}(1-t_m)^\sigma s^{-\sigma}ds\\ &\overset{\tau=\frac{t_m}{1-s}}{=}\mathcal{C}(1-t_m)^\sigma(1-\varepsilon)^{-\sigma}t_m^{\sigma+\mu}\int_{\varepsilon^{-1}t_m}^\infty k^{\beta_{i-1}}(\tau)\tau^{-\mu-1}d\tau\\ &\le\mathcal{C}\,t_m^{\sigma+\mu}\int_0^\infty k^{\beta_{i-1}}(\tau)\tau^{-\mu-1}d\tau\le\frac{\mathcal{C}}{m^{2(1-\epsilon)(\sigma+\mu)}},\end{aligned}$$

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with $\mu = \min\{\mu_1, \ldots, \mu_n\}$. Here and in what follows, $\mathcal{C}$ denotes a positive constant independent of m. That the last integral in this estimate is finite follows from the following observation: Note that the integral in (7.1) is finite for $0 < \rho < 2$, which is equivalent to $1-q < \sigma < 1+q$ or, for $-\mu$ instead of σ, $-q-1 < \mu < q-1$. This is true, since $\mu_i = q(1+s_i) - 1$ and $-\frac{1}{2} < s_i < 0$.

For the estimate of $A_3(t)$ we can proceed as above (replacing t_m with t), and, since $t \le t_m$, we obtain

$$A_3(t) \le \frac{\mathcal{C}}{m^{2(1-\epsilon)(\sigma+\mu)}}.$$

In order to estimate both $A_1(t)$ and the second term in (7.25), we assume that $t \ge t_m$, and, using (2.3), we get

$$\begin{aligned}
&\left| \left(\left(\mathcal{K}_{0,m}^{\beta_{i-1},\sigma} - v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma} \right) \chi_1 \widehat{\psi}_{i-1} \right)(t) \right| \\
&= v^\sigma(t) \left| e_m^J \left(K_0^{\beta_{i-1}}(t,\cdot) \chi_1 \widehat{\psi}_{i-1} \right) \right| \\
&\le \frac{\mathcal{C}\, v^\sigma(t)}{m} E_{2m-4} \left(\left(K_0^{\beta_{i-1}}(t,\cdot) \chi_1 \widehat{\psi}_{i-1} \right)' \right)_{\varphi v^{-\sigma},1} \\
&\le \frac{\mathcal{C}\, v^\sigma(t)}{m} \left[E_{2m-4} \left(\left(K_0^{\beta_{i-1}}(t,\cdot) \chi_1 \right)' \widehat{\psi}_{i-1} \right)_{\varphi v^{-\sigma},1} \right. \\
&\qquad\qquad \left. + E_{2m-4} \left(K_0^{\beta_{i-1}}(t,\cdot) \chi_1 \widehat{\psi}'_{i-1} \right)_{\varphi v^{-\sigma},1} \right] \\
&=: C(t) + D(t).
\end{aligned}$$

Now, we have that

$$\begin{aligned}
C(t) \le \frac{\mathcal{C}\, v^\sigma(t)}{m} &\left[2E_{m-2} \left(\frac{\partial}{\partial s} \left(K_0^{\beta_{i-1}}(t,\cdot) \chi_1 \right) \right)_{\varphi v^{-\sigma},1} \left\| \widehat{\psi}_{i-1} \right\|_\infty \right. \\
&\qquad \left. + \left\| \frac{\partial}{\partial s} \left(K_0^{\beta_{i-1}}(t,\cdot) \chi_1 \right) \varphi v^{-\sigma} \right\|_1 E_{m-2}(\widehat{\psi}_{i-1})_\infty \right] \\
=: C_1(t) + C_2(t).&
\end{aligned}$$

Taking into account the Favard inequality (2.2) and the estimate (7.4), we can estimate under our assumptions

$$\begin{aligned}
C_1(t) &\le \frac{\mathcal{C}\, v^\sigma(t)}{m^k} \left\| \frac{\partial^k}{\partial s^k} \left(K_0^{\beta_{i-1}}(t,\cdot) \chi_1 \right) \varphi^k v^{-\sigma} \right\|_1 \\
&\le \frac{\mathcal{C}\, v^\sigma(t)}{m^k} \frac{1}{t^{\frac{k}{2}+\sigma}} \le \frac{\mathcal{C}}{m^k} \frac{1}{t^{\frac{k}{2}}} \le \frac{\mathcal{C}}{m^{k\epsilon}}
\end{aligned} \tag{7.26}$$

and, due to (7.8) and $\frac{r}{2} \le \sigma + \mu$,

$$C_2(t) \le \frac{\mathcal{C}\, v^\sigma(t)}{m^{1+r}} \frac{1}{t^{\frac{1}{2}+\sigma}} \left\| \left(\widehat{\psi}_{i-1} \right)^{(r)} \varphi^r \right\|_\infty \le \frac{\mathcal{C}}{m^{1+r}} \frac{1}{t^{\frac{1}{2}}} \le \frac{\mathcal{C}}{m^{r+\epsilon}}.$$

For $D(t)$ we get

$$D(t) \le \frac{\mathcal{C}}{m} v^\sigma(t) \left[2E_{m-2} \left(K_0^{\beta_{i-1}}(t,\cdot) \chi_1 \right)_{v^{-\sigma},1} \left\| \left(\widehat{\psi}_{i-1} \right)' \varphi \right\|_\infty \right.$$

$$+ \left\| K_0^{\beta_{i-1}}(t,\cdot)\chi_1 v^{-\sigma} \right\|_1 E_{m-2}\left(\widehat{\psi}'_{i-1}\right)_{\varphi,\infty} \Bigg]$$
$$=: D_1(t) + D_2(t),$$

and, by virtue of (2.2), (7.4), and (7.8), we can estimate the quantities $D_1(t)$ and $D_2(t)$ as follows:

$$\begin{aligned} D_1(t) &\le \frac{\mathcal{C}\, v^\sigma(t)}{m^{k+1}} \left\| \frac{\partial^k}{\partial s^k}\left(K_0^{\beta_{i-1}}(t,\cdot)\chi_1\right)\varphi^k v^{-\sigma}\right\|_1 \\ &\le \frac{\mathcal{C}\, v^\sigma(t)}{m^{k+1}} \frac{1}{t^{\frac{k}{2}+\sigma}} \le \frac{\mathcal{C}}{m^{k+1}}\frac{1}{t^{\frac{k}{2}}} \le \frac{\mathcal{C}}{m^{1+k\epsilon}}, \\ D_2(t) &\le \frac{\mathcal{C}\, v^\sigma(t)}{m^r}\frac{1}{t^\sigma}\left\|\widehat{\psi}_{i-1}^{(r)}\varphi^r\right\|_\infty = \frac{\mathcal{C}}{m^r}. \end{aligned} \tag{7.27}$$

By combining the proved estimates for $A_2(t)$ and $A_3(t)$ with the estimates (7.26)–(7.27), we obtain that

$$\sup_{t\in[0,t_m]}\left|\left(\left(\mathcal{K}_{0,m}^{\beta_{i-1},\sigma} - v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma}\right)\chi_1\widehat{\psi}_{i-1}\right)(t)\right| \le \frac{\mathcal{C}}{m^{\min\{r,k\epsilon,2(1-\epsilon)(\sigma+\mu)\}}}.$$

Moreover, from (7.26)–(7.27) it also follows that

$$\sup_{t\in[t_m,\varepsilon]}\left|\left(\left(\mathcal{K}_{0,m}^{\beta_{i-1},\sigma} - v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma}\right)\chi_1\widehat{\psi}_{i-1}\right)(t)\right| \le \frac{\mathcal{C}}{m^{\min\{r,k\epsilon\}}},$$

and, consequently, we can conclude that

$$\sup_{t\in[0,\varepsilon]}\left|\left(\left(\mathcal{K}_{0,m}^{\beta_{i-1},\sigma} - v^\sigma \mathcal{K}_0^{\beta_{i-1}} v^{-\sigma}\right)\chi_1\widehat{\psi}_{i-1}\right)(t)\right| \le \frac{\mathcal{C}}{m^{\min\{r,k\epsilon,2(1-\epsilon)(\sigma+\mu)\}}}. \tag{7.28}$$

In a similar way, it can be proved that

$$\sup_{t\in[1-\varepsilon,1]}\left|\left(\left(\mathcal{K}_{1,m}^{\beta_i,\sigma} - v^\sigma \mathcal{K}_1^{\beta_i} v^{-\sigma}\right)\chi_0\widehat{\psi}_{i+1}\right)(t)\right| \le \frac{\mathcal{C}}{m^{\min\{r,k\epsilon,2(1-\epsilon)(\sigma+\mu)\}}}.$$

Now, let us observe that the rate of convergence of the term $\left\|(\mathcal{K}_m^\sigma - \mathcal{V}^\sigma\mathcal{K}\mathcal{V}^{-\sigma})\widehat{\boldsymbol{\psi}}\right\|_\infty$ in (7.23) depends on the smoothness of the curves $\overline{\Gamma}_i$ and on the behavior of the solutions on the interval $(0,1)$ (see (7.8)). More precisely, first we recall that the curves $\overline{\Gamma}_i$ are of class $C^{\ell+2}$ for some $\ell\in\mathbb{N}$ ($\ell\ge 2$). By examining the formulas (3.12) for the kernels $K_{ij}(t,s)$, we deduce that $K_{ij}\in C^\ell([0,1]\times[0,1])$ with $|i-j|\ne 1$ and $|i-j|\ne n-1$. Moreover, for the kernels $E_{ij}(t,s)$ with $|i-j|=1$ and $|i-j|=n-1$ (see Lemma 4.1 and (5.2) for their definition) one has that $E_{ij}\in C^\ell([0,1]\times[0,1])$, too. Consequently, by (2.3) we have

$$\begin{aligned} &\left|(\mathcal{K}_{ij,m}^\sigma - v^\sigma\mathcal{K}_{ij}v^{-\sigma})\widehat{\psi}_j(t)\right| = v^\sigma(t)\left|e_m^J\left((K_{ij}(t,\cdot)\widehat{\psi}_j\right)\right| \\ &\le \frac{\mathcal{C}\, v^\sigma(t)}{m}E_{2m-4}\left(\left(K_{ij}(t,\cdot)\widehat{\psi}_j\right)'\right)_{\varphi v^{-\sigma},1} \\ &\le \frac{\mathcal{C}\, v^\sigma(t)}{m}\left[E_{2m-4}\left((K_{ij}(t,\cdot))'\,\widehat{\psi}_j\right)_{\varphi v^{-\sigma},1} + E_{2m-4}\left(K_{ij}(t,\cdot)\widehat{\psi}'_j\right)_{\varphi v^{-\sigma},1}\right] \\ &=: G(t) + H(t). \end{aligned}$$

We estimate $G(t)$ and $H(t)$ by

$$G(t) \le \frac{\mathcal{C}\, v^\sigma(t)}{m}\left[2\left\|(K_{ij}(t,\cdot))'\right\|_\infty E_{m-2}\big(\widehat{\psi}_j\big)_{\varphi v^{-\sigma},1} + E_{m-2}\left((K_{ij}(t,\cdot))'\right)_\infty \left\|\widehat{\psi}_j\right\|_{\varphi v^{-\sigma},1}\right] =: G_1(t) + G_2(t)$$

and

$$H(t) \le \frac{\mathcal{C}\, v^\sigma(t)}{m}\left[2\left\|K_{ij}(t,\cdot)\right\|_\infty E_{m-2}\big(\widehat{\psi}'_j\big)_{\varphi v^{-\sigma},1} + E_{m-2}\left(K_{ij}(t,\cdot)\right)_\infty \left\|\widehat{\psi}'_j\right\|_{\varphi v^{-\sigma},1}\right] =: H_1(t) + H_2(t).$$

Now, using (2.2), (7.8), and taking into account that under our hypotheses, $\mu_i - \frac{r}{2} > -1$, we get, for $r_0 = \min\{r+1, \ell\}$,

$$\begin{aligned}
G_1(t) &\le \frac{\mathcal{C}\, v^\sigma(t)}{m^{r_0+1}} \left\|(K_{ij}(t,\cdot))'\right\|_\infty E_{m-r-3}\left(\widehat{\psi}_j^{(r_0)}\right)_{\varphi^{r_0+1}v^{-\sigma},1} \\
&\le \frac{\mathcal{C}}{m^{r_0+1}} \left\|\widehat{\psi}_j^{(r_0)}\right\|_{\varphi^{r_0+1}v^{-\sigma},1} \le \frac{\mathcal{C}}{m^{r_0+1}}, \\
G_2(t) &\le \frac{\mathcal{C}\, v^\sigma(t)}{m^\ell} E_{m-\ell-1}\left((K_{ij}(t,\cdot))^{(\ell)}\right)_{\varphi^{\ell-1},\infty} \left\|\widehat{\psi}_j\right\|_{\varphi v^{-\sigma},1} \\
&\le \frac{\mathcal{C}\, v^\sigma(t)}{m^\ell} \left\|(K_{ij}(t,\cdot))^{(\ell)}\right\|_{\varphi^{\ell-1},\infty} \le \frac{\mathcal{C}}{m^\ell}, \\
H_1(t) &\le \frac{\mathcal{C}\, v^\sigma(t)}{m^r} \left\|K_{ij}(t,\cdot)\right\|_\infty \left\|\widehat{\psi}_j^{(r)}\right\|_{\varphi^r v^{-\sigma},1} \le \frac{\mathcal{C}}{m^r}, \\
H_2(t) &\le \frac{\mathcal{C}\, v^\sigma(t)}{m^{\ell+1}} \left\|(K_{ij}(t,\cdot))^{(\ell)}\right\|_{\varphi^\ell,\infty} \left\|\widehat{\psi}'_j\right\|_{\varphi v^{-\sigma},1} \le \frac{\mathcal{C}}{m^{\ell+1}},
\end{aligned}$$

from which one can deduce that, for $|i-j| \ne 1$ and $|i-j| \ne n-1$,

$$\left|\left((\mathcal{K}^\sigma_{ij,m} - v^\sigma \mathcal{K}_{ij} v^{-\sigma})\widehat{\psi}_j\right)(t)\right| \le \frac{\mathcal{C}}{m^r}, \qquad \mathcal{C} \ne \mathcal{C}(m,t).$$

In a similar way, for $|i-j| = 1$ and $|i-j| = n-1$, one can prove that

$$\left|\left((\mathcal{E}^\sigma_{ij,m} - v^\sigma \mathcal{E}_{ij} v^{-\sigma})\widehat{\psi}_j\right)(t)\right| \le \frac{\mathcal{C}}{m^r}, \qquad \mathcal{C} \ne \mathcal{C}(m,t). \tag{7.29}$$

The assertion (6.5) follows immediately from (7.23) taking into account (7.28)–(7.29). ☐

Proof of Theorem 6.7. For the sake of brevity, for any $i = 1, \ldots, n$, $t \in [0,1]$, and $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \overline{\Omega}$, we define the function $L_i = L_i^x : [0,1] \longrightarrow \mathbb{R}$ by

$$L_i(t) = \log|(x_1, x_2) - (\xi_i(\Phi_q(t)), \eta_i(\Phi_q(t))|.$$

With this notation we can write (see (5.15))

$$
\begin{aligned}
|u(x)-u_m(x)| &\le \sum_{i=1}^{n}\left|\int_0^1 \widehat{\psi}_i(t)v^{-\sigma}(t)L_i(t)dt-\sum_{j=1}^{m} w_{m,j}(\widehat{\psi}_{m,i})(s_{m,j})L_i(s_{m,j})\right|\\
&\le \sum_{i=1}^{n}\left|\int_0^1 \widehat{\psi}_i(t)v^{-\sigma}(t)L_i(t)dt-\sum_{j=1}^{m} w_{m,j}\widehat{\psi}_i(s_{m,j})L_i(s_{m,j})\right|\\
&\quad+\sum_{i=1}^{n}\left|\sum_{j=1}^{m} w_{m,j}(\widehat{\psi}_i-\widehat{\psi}_{m,i})(s_{m,j})L_i(s_{m,j})\right|\\
&=:\sum_{i=1}^{n}(A_i+B_i).
\end{aligned}
$$

Now, by the error estimate (2.3), for any $i\in\{1,\dots,n\}$, we get

$$
\begin{aligned}
A_i &= \left|\int_0^1 \widehat{\psi}_i(t)v^{-\sigma}(t)L_i(t)dt-\sum_{j=1}^{m} w_{m,j}(\widehat{\psi}_i)(s_{m,j})L_i(s_{m,j})\right|\\
&\le \frac{\mathcal{C}}{m}E_{2m-4}\left(\left(\widehat{\psi}_iL_i\right)'\right)_{\varphi v^{-\sigma},1}\\
&\le \frac{\mathcal{C}}{m}\left[E_{2m-4}\big(\widehat{\psi}_i'L_i\big)_{\varphi v^{-\sigma},1}+E_{2m-4}\big(\widehat{\psi}_iL_i'\big)_{\varphi v^{-\sigma},1}\right]\\
&=: A_{i,1}+A_{i,2}.
\end{aligned}
$$

Note that

$$
\|L_i\|_\infty=\log\max\left\{R_i,\frac{1}{R_i}\right\}\qquad\text{and}\qquad \|L_i^{(r)}\|_\infty\le\frac{\mathcal{C}}{R_i^r},\qquad r=1,\dots,\ell+2,
$$

where $R_i=\operatorname{dist}(x,\Gamma_i)=\inf\{|x-y|:y\in\Gamma_i\}$. By similar arguments as in the proof of Theorem 6.6, the following estimates for the quantities $A_{i,1}$ and $A_{i,2}$ can be obtained ($\ell_0=r,\dots,\ell$):

$$
\begin{aligned}
A_{i,1} &\le \frac{\mathcal{C}}{m}\left[2\|L_i\|_\infty E_{m-2}\big(\widehat{\psi}_i'\big)_{\varphi v^{-\sigma},1}+E_{m-2}(L_i)_\infty\|\widehat{\psi}_i'\|_{\varphi v^{-\sigma},1}\right]\\
&\le \frac{\mathcal{C}}{m^r}\|L_i\|_\infty\|\widehat{\psi}_i^{(r)}\|_{\varphi^r v^{-\sigma},1}+\frac{\mathcal{C}}{m^{\ell_0+1}}\|L_i^{(\ell_0)}\|_{\varphi^{\ell_0},\infty}\|\widehat{\psi}_i'\|_{\varphi v^{-\sigma},1}\\
&\overset{1+\mu>\frac{r}{2}}{\le} \frac{\mathcal{C}}{m^r}\log\max\left\{R_i,\frac{1}{R_i}\right\}+\frac{\mathcal{C}}{m^{\ell_0+1}R_i^{\ell_0}}\\
A_{i,2} &\le \frac{\mathcal{C}}{m}\left[2\|L_i'\|_\infty E_{m-2}\big(\widehat{\psi}_i\big)_{\varphi v^{-\sigma},1}+E_{m-2}(L_i')_\infty\|\widehat{\psi}_i\|_{\varphi v^{-\sigma},1}\right]\\
&\overset{r_0:=\min\{r+1,\ell\}}{\le} \frac{\mathcal{C}}{m^{r_0+1}}\|L_i'\|_\infty E_{m-r-3}\big(\widehat{\psi}_i^{(r_0)}\big)_{\varphi^{r_0+1}v^{-\sigma},1}\\
&\qquad+\frac{\mathcal{C}}{m^{\ell_0}}E_{m-\ell_0-1}\big(L_i^{(\ell_0)}\big)_{\varphi^{\ell_0-1},\infty}\|\widehat{\psi}_i\|_{\varphi v^{-\sigma},1}\\
&\le \frac{\mathcal{C}}{m^{r_0}R_i}+\frac{\mathcal{C}}{m^{\ell_0}R_i^{\ell_0}},
\end{aligned}
$$

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from which it follows that (it suffices to choose $\ell_0 = r$ since m^{-r} is the best rate we can get)

$$A_i \leq \frac{\mathcal{C}}{m^r} \max\left\{\log\max\left\{R_i, \frac{1}{R_i}\right\}, \frac{1}{R_i}, \frac{1}{R_i^r}\right\}, \qquad \mathcal{C} \neq \mathcal{C}(m). \tag{7.30}$$

Concerning the quantity B_i, for any fixed $i = 1, \ldots, n$, we can write

$$\begin{aligned}
B_i &= \left|\sum_{j=1}^{m} w_{m,j}\big(\widehat{\psi}_i - \widehat{\psi}_{i,m}\big)(s_{m,j}) L_i(s_{m,j})\right| \\
&\leq \sum_{j=1}^{m} w_{m,j}\left|\big(\widehat{\psi}_i - \widehat{\psi}_{i,m}\big)(s_{m,j})\right| |L_i(s_{m,j})| \\
&\leq \|L_i\|_\infty \left\|\widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}}_m\right\|_\infty \int_0^1 v^{-\sigma}(t)dt \\
&\leq \mathcal{C}\, \log\max\left\{R_i, \frac{1}{R_i}\right\} \left\|\widehat{\boldsymbol{\psi}} - \widehat{\boldsymbol{\psi}}_m\right\|_\infty, \qquad \mathcal{C}_i \neq \mathcal{C}_i(m).
\end{aligned}$$

Finally, combining the last estimate with (7.30) and the error estimate (6.5), we can deduce the assertion (6.6). □

8. What about condition (3.11)? Finally, let us discuss the question how to handle condition (3.11), which is equivalent to condition (1.5), in our numerical procedure. Beside equation (1.4), this condition is essential for the single layer potential (1.3) being a solution of the exterior Neumann problem (1.1). But, in general, the solution $\widehat{\boldsymbol{\psi}}_m$ of (5.12), i.e., the Nyström interpolant (5.14), does not automatically satisfy

$$\sum_{i=1}^{n} \int_0^1 \widehat{\psi}_{m,i}(t) v^{-\sigma}(t)dt = 0$$

or, which is the same, does not automatically lie in $\boldsymbol{C}_{v^\sigma,0}$. By Remark 6.5 we only know that $\widehat{\boldsymbol{\psi}}_m(t)$ fulfills this condition asymptotically.

Using the parametric representations (3.4) of the boundary pieces Γ_i, we see that relation (1.6) can be written in the form

$$\frac{1}{2} = \frac{1}{2\pi}\sum_{j=1}^{n}\int_0^1 \frac{[\xi_j(s) - \xi_i(t)]\,\eta_j'(s) - [\eta_j(s) - \eta_i(t)]\,\xi_j'(s)}{[\xi_j(s) - \xi_i(t)]^2 + [\eta_j(s) - \eta_i(t)]^2}\, ds = \frac{1}{2\pi}\sum_{j=1}^{n}\int_0^1 k_{ji}(s,t)\, ds,$$

$0 < t < 1$, $i = 1, \ldots, n$, where the kernel functions $k_{ij}(t, s)$ are defined in (3.7). If we further apply the smoothing transformation (3.9), then we get the relation

$$\frac{1}{\pi}\sum_{j=1}^{n}\int_0^1 K_{ji}(s,t)\, ds = 1, \qquad 0 < t < 1, \tag{8.1}$$

with $K_{ij}(t, s)$ defined in (3.12).

For a moment, assume that we apply the Nyström method directly to the system (4.16) without the modifications (5.5) and (5.6) and that for this we use the Gaussian rule in (5.1), i.e., $0 < s_{m,1} < \ldots < s_{m,m} < 1$. In other words, let $\widehat{\boldsymbol{\psi}}_m = \big(\widehat{\psi}_{m,1}, \ldots, \widehat{\psi}_{m,n}\big)$ be a solution of the system

(8.2)

$$\frac{1}{2}\widehat{\psi}_{m,i}(t) + \frac{v^\sigma(t)}{2\pi}\sum_{j=1}^{n}\sum_{k=1}^{m} w_{m,k} K_{ij}(t, s_{m,k})\widehat{\psi}_{m,j}(s_{m,k}) = -\widehat{f}_i(t), \qquad \begin{array}{l} 0 \leq t \leq 1, \\ i = 1, \ldots, n, \end{array}$$

or, equivalently,

$$\frac{1}{2}\widetilde{\psi}_{m,i}(t) + \frac{1}{2\pi}\sum_{j=1}^{n}\sum_{k=1}^{m} w_{m,k}K_{ij}(t,s_{m,k})\widehat{\psi}_{m,j}(s_{m,k}) = -\widetilde{f}_i(t), \qquad \begin{array}{l} 0<t<1, \\ i=1,\dots,n. \end{array}$$

Let us now investigate when the respective Nyström interpolant $\widetilde{\boldsymbol{\psi}}_m = \left(\widetilde{\psi}_{m,1},\dots,\widetilde{\psi}_{m,n}\right)^T$, with

$$\widetilde{\psi}_{m,i}(t) = -\frac{1}{\pi}\sum_{j=1}^{n}\sum_{k=1}^{m} w_{m,k}K_{ij}(t,s_{m,k})\widehat{\psi}_{m,j}(s_{m,k}) - 2\widetilde{f}_i(t), \quad i=1,\dots,n,$$

belongs to $\boldsymbol{C}_{v^\sigma,0}$. For this, compute

$$\begin{aligned}
&\sum_{i=1}^{n}\int_0^1 \widetilde{\psi}_{m,i}(t)\,dt \\
&= -\frac{1}{\pi}\sum_{j=1}^{n}\sum_{k=1}^{m} w_{m,k}\sum_{i=1}^{n}\int_0^1 K_{ij}(t,s_{m,k})\,dt\,\widehat{\psi}_{m,j}(s_{m,k}) - 2\sum_{i=1}^{n}\int_0^1 \widetilde{f}_i(t)\,dt.
\end{aligned}$$

If we assume that $\widetilde{\boldsymbol{f}} \in \boldsymbol{C}_{v^\sigma,0}$ and if we use relation (8.1), then we conclude

$$\sum_{i=1}^{n}\int_0^1 \widetilde{\psi}_{m,i}(t)\,dt = -\sum_{j=1}^{n}\sum_{k=1}^{m}\widehat{\psi}_{m,j}(s_{m,k}).$$

Thus, under the condition $\widetilde{\boldsymbol{f}} \in \boldsymbol{C}_{v^\sigma,0}$ we have $\widetilde{\boldsymbol{\psi}}_m \in \boldsymbol{C}_{v^\sigma,0}$ if and only if

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_{m,j}\widehat{\psi}_{m,i}(s_{m,j}) = 0. \tag{8.3}$$

is satisfied.

Hence, although we use the Nyström method with the modifications (5.5) and (5.6), condition (8.3) should be a good choice to approximate the original condition

$$\sum_{i=1}^{n}\int_0^1 \widetilde{\psi}_{m,i}(t)dt = 0.$$

However, adding the discrete condition (8.3) to the equations of the linear system (5.13) we obtain an overdetermined system for $nm+1$ equations in the nm unknowns $\widehat{\psi}_{m,i}(s_{m,j})$, $i=1,\dots,n$, $j=1,\dots,m$. Hence, in order to get a square system, an equation has to be dropped. Numerical results show that a good choice consists in omitting the first or the last equation in (5.13) and that the numerical results concerning both the approximation error and the condition number of the system matrix are better than for the system (5.13) itself (see the results for Example 9.1 in the following section). Although we cannot prove stability and convergence for such a modified system of equations, for the further examples we only present numerical results employing this adaption.

The reason for introducing the modifications (5.5) as well as (5.6) and not studying the classical Nyström method (8.2) is that we cannot prove the stability of that method; for the modified method its proof is based on the inequality (6.1). Alternative methods (without such modifications) could be collocation-quadrature methods with weighted polynomials as ansatz functions studied in weighted L^2-spaces and with the help of C^*-algebra techniques (cf. [19, 20]).

9. Numerical results. In this section we present some numerical tests in which the method described in Section 5 is applied for approximating the solution of the exterior Neumann problem (1.1) in some planar domains with corners. The Neumann data f are given after a test harmonic function u is chosen as the exact solution. Once the linear system (5.13), (8.3) is solved, we approximate the single layer potential u in (1.3) by means of the function u_m defined in (5.15), and we compute the absolute errors

$$e_m(x_1,x_2) = |u(x_1,x_2) - u_m(x_1,x_2)| \tag{9.1}$$

at some points $(x_1,x_2) \in \mathbb{R}^2 \setminus \overline{\Omega}$. Moreover, the condition numbers in the infinity norm, $\mathrm{cond}(A_m)$, of the coefficient matrix A_m of the linear system (5.13), (8.3) are computed and reported in the following tables. For some tests, we also give the errors in the computation of the single layer density function $\widetilde{\boldsymbol{\psi}}$. Since the exact solution $\widehat{\boldsymbol{\psi}} = (\widehat{\psi}_1,\ldots,\widehat{\psi}_n)^T$ of the system (4.16) is unknown, we use as substitute for it the approximation $\widehat{\boldsymbol{\psi}}_M = (\widehat{\psi}_{M,1},\ldots,\widehat{\psi}_{M,n})^T$ with $M = 512$ or $M = 1024$. Then, we report the errors

$$err_m = \max_{i=1,\ldots,n} err_{i,m}, \qquad err_{i,m} = \max_{j=1,\ldots,100} \left|\widehat{\psi}_{M,i}(t_j) - \widehat{\psi}_{m,i}(t_j)\right|, \tag{9.2}$$

where $t_1,\ldots,t_{100}$ are equispaced points in the interval $(0,1)$, and we present the related estimated orders of convergence

$$eoc = \frac{\log\left(err_m/err_{2m}\right)}{\log 2}. \tag{9.3}$$

In the practical implementation, following [32, 33], we have taken the smoothing function Φ_q as

$$\Phi_q(t) = \frac{\int_0^t x^{q-1}(1-x)^{q-1}dx}{\int_0^1 x^{q-1}(1-x)^{q-1}dx}, \qquad 0 \le t \le 1, \quad q > 1.$$

The values of the involved parameters have been chosen in such a way to minimize the errors as well as to maximize the order of convergence. More precisely, following some criteria proposed in [27] (see also [9]) and taking into account the asymptotic behavior of the solution described by (3.13), one can choose $c = 10^{(\sigma+\mu)-1}$ and $\epsilon = \frac{2(\sigma+\mu)}{k+2(\sigma+\mu)}$.

EXAMPLE 9.1. We consider a family of heart-shaped domains bounded by the curves

$$\boldsymbol{\gamma}(t) = \begin{bmatrix} \cos\left(1+\frac{\beta}{\pi}\right)\pi t & -\sin\left(1+\frac{\beta}{\pi}\right)\pi t \\ \sin\left(1+\frac{\beta}{\pi}\right)\pi t & \cos\left(1+\frac{\beta}{\pi}\right)\pi t \end{bmatrix} \begin{bmatrix} \tan\frac{\beta}{2} \\ 1 \end{bmatrix} - \begin{bmatrix} \tan\frac{\beta}{2} \\ \cos\pi t \end{bmatrix}, \quad t \in [0,1],$$

where $\beta \in (\pi, 2\pi)$ is the interior angle of the single outward-pointing corner $P = (0,0)$ (see Figure 9.1). For this test we choose as exact solution the harmonic function

$$u(x_1,x_2) = \log|(x_1,x_2) - (0.5,0)| - \log|(x_1,x_2) - (0.2,0)|$$

and as interior angle at the corner point $\beta = \frac{5}{3}\pi$. In Tables 9.1–9.5 we report the numerical results obtained by applying the Gaussian quadrature formula and in Tables 9.8–9.12 those by applying the Gauss–Lobatto rule. We can observe that the results are very similar. In particular, in Tables 9.4–9.5 and 9.11–9.12, we compare the errors $e_m(x_1,x_2)$ (obtained by choosing $q = 2$) with the errors $\bar{e}_m(x_1,x_2)$ presented in [14] and the errors $\tilde{e}_m(x_1,x_2)$ given in [26] (m is the dimension of the solved linear system). Taking also into account that in [14] some

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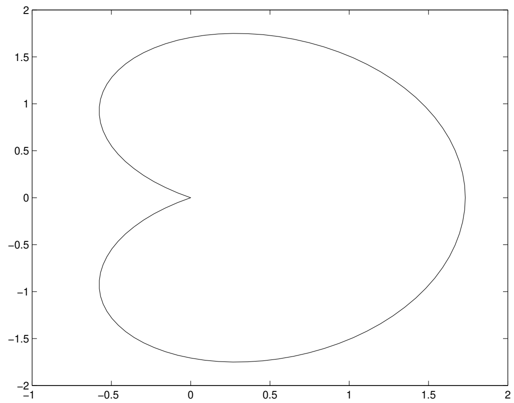

FIG. 9.1. Γ *in Example 9.1 with* $\beta = \frac{5}{3}\pi$.

TABLE 9.1
Example 9.1: Errors err_m *in (9.2), eoc in (9.3), and condition numbers of* A_m *using the Gaussian rule.*

	$q=1$			$q=2$			$q=3$		
m	err_m	eoc	cond(A_m)	err_m	eoc	cond(A_m)	err_m	eoc	cond(A_m)
8	8.45e-03		8.64e+02	3.94e-01		1.23e+03	1.10e-01		1.05e+03
		2.80			2.27			-1.94	
16	1.21e-03		1.25e+03	8.15e-02		2.52e+03	4.27e-01		2.66e+03
		3.75			4.39			4.97	
32	8.94e-05		1.38e+03	3.87e-03		2.89e+03	1.35e-02		3.38e+03
		1.11			4.73			5.52	
64	4.12e-05		1.42e+03	1.44e-04		2.97e+03	2.95e-04		3.60e+03
		2.24			4.79			6.93	
128	8.71e-06		1.43e+03	5.23e-06		2.99e+03	2.40e-06		3.66e+03
		2.61			4.85			7.14	
256	1.42e-06		1.43e+03	1.80e-07		3.00e+03	1.69e-08		3.67e+03

TABLE 9.2
Example 9.1: Errors $e_m(x_1, x_2)$ *in (9.1) obtained by using the Gaussian rule.*

	$q=1$		$q=2$		$q=3$	
m	$e_m(-0.1,0)$	$e_m(3,3)$	$e_m(-0.1,0)$	$e_m(3,3)$	$e_m(-0.1,0)$	$e_m(3,3)$
8	1.43e-01	1.24e-02	4.67e-01	3.53e-02	7.40e-01	4.87e-02
16	2.12e-02	1.56e-04	3.62e-02	2.51e-03	1.19e-01	8.17e-03
32	1.33e-03	2.92e-08	7.10e-04	9.37e-06	2.17e-03	8.44e-05
64	2.14e-05	2.29e-14	3.07e-06	8.37e-11	4.77e-06	9.22e-09
128	1.03e-08	9.29e-16	1.74e-10	1.20e-15	4.49e-10	1.99e-15
256	2.24e-14	4.09e-16	1.34e-14	8.25e-16	8.88e-15	2.08e-17

additional computational cost has to be paid in order to compute the right-hand side of the linear system, we can observe the better behaviour of the newly proposed procedure here.

The chosen value for the parameter σ is $\sigma = 0.99$. Thus, as $\sigma + \mu = 0.59$ for $q = 1$, $\sigma + \mu = 1.19$ for $q = 2$, and $\sigma + \mu = 1.79$ for $q = 3$, the expected theoretical order of convergence for the approximation errors for the single layer density is $\nu = 1$ for $q = 1$, $\nu = 2$ for $q = 2$, and $\nu = 3$ for $q = 3$.

For comparison with Tables 9.1 and 9.2, in Tables 9.6 and 9.7 we present the results obtained for the system (5.13), i.e., without taking into account the additional condition (3.11).

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TABLE 9.3
Example 9.1: Errors $e_m(x_1, x_2)$ in (9.1) obtained by using the Gaussian rule.

	$q=1$		$q=2$		$q=3$	
m	$e_m(-40,-50)$	$e_m(100,-100)$	$e_m(-40,-50)$	$e_m(100,-100)$	$e_m(-40,-50)$	$e_m(100,-100)$
8	9.45e-04	4.81e-04	2.10e-03	1.07e-03	2.87e-03	1.47e-03
16	1.62e-05	8.27e-06	2.38e-04	1.21e-04	6.50e-04	3.31e-04
32	2.98e-09	1.51e-09	8.98e-07	4.57e-07	8.46e-06	4.30e-06
64	2.55e-15	1.30e-15	9.28e-12	4.72e-12	9.08e-10	4.62e-10
128	4.02e-16	1.33e-15	3.43e-16	1.98e-15	2.18e-15	3.69e-15
256	1.24e-16	7.91e-16	1.52e-15	2.35e-15	8.89e-16	4.50e-16

TABLE 9.4
Example 9.1: Errors $\bar{e}_m(x_1, x_2)$ in [14], $\tilde{e}_m(x_1, x_2)$ in [26], and $e_m(x_1, x_2)$ in (9.1) obtained by using the Gaussian rule.

(x_1, x_2)	$(-0.1, 0)$			$(3, 3)$		
m	$\bar{e}_m$	$\tilde{e}_m$	e_m	$\bar{e}_m$	$\tilde{e}_m$	e_m
51	6.62e-03	1.53e-05	2.52e-05	2.35e-05	7.04e-07	1.00e-08
99	6.95e-03	1.18e-09	1.41e-08	1.89e-04	1.97e-12	1.07e-15
195	6.78e-04	1.66e-15	2.22e-16	1.81e-05	3.46e-16	1.11e-16

TABLE 9.5
Example 9.1: Errors $\bar{e}_m(x_1, x_2)$ in [14], $\tilde{e}_m(x_1, x_2)$ in [26], and $e_m(x_1, x_2)$ in (9.1) obtained by using the Gaussian rule.

(x_1, x_2)	$(-40, -50)$			$(100, -100)$		
m	$\bar{e}_m$	$\tilde{e}_m$	e_m	$\bar{e}_m$	$\tilde{e}_m$	e_m
51	4.52e-05	7.53e-08	9.92e-10	5.9e-05	3.83e-08	5.05e-10
99	1.12e-05	1.95e-13	1.15e-15	5.3e-06	9.87e-14	1.87e-15
195	1.05e-06	2.85e-16	2.62e-15	5.3e-07	8.51e-16	2.09e-15

TABLE 9.6
Example 9.1: Errors err_m in (9.2), eoc in (9.3), and condition numbers of A_m for the system (5.13).

	$q=1$			$q=2$			$q=3$		
m	err_m	*eoc*	cond(A_m)	err_m	*eoc*	cond(A_m)	err_m	*eoc*	cond(A_m)
8	7.41e-02		5.45e+03	2.09e-01		6.67e+03	1.24e-01		9.17e+03
		0.88			1.23			-1.60	
16	4.01e-02		6.92e+03	8.86e-02		1.43e+04	3.78e-01		1.87e+04
		1.43			1.87			5.19	
32	1.48e-02		7.41e+03	2.41e-02		1.95e+04	1.03e-02		2.50e+04
		1.37			2.31			4.92	
64	5.70e-03		7.64e+03	4.86e-03		2.14e+04	3.39e-04		2.74e+04
		1.41			2.43			3.55	
128	2.13e-03		7.82e+03	9.01e-04		2.21e+04	2.89e-05		2.85e+04
		1.73			2.65			3.70	
256	6.41e-04		8.00e+03	1.43e-04		2.26e+04	2.22e-06		2.93e+04

EXAMPLE 9.2. We consider the teardrop domain Ω bounded by the curve Γ parameterized by

$$\boldsymbol{\gamma}(t) = \left(2\sin \pi t, -\tan\frac{\beta}{2}\sin 2\pi t\right), \qquad t \in [0,1],$$

where $\beta \in (0, \pi)$ denotes the interior angle of the single outward-pointing corner $P = (0,0)$ (see Figure 9.2). The numerical results shown in Tables 9.13–9.17 are obtained for $\beta = \frac{2}{3}\pi$

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TABLE 9.7
Example 9.1: Errors $e_m(x_1, x_2)$ in (9.1) for the system (5.13).

	$q = 1$		$q = 2$		$q = 3$	
m	$e_m(-0.1, 0)$	$e_m(3, 3)$	$e_m(-0.1, 0)$	$e_m(3, 3)$	$e_m(-0.1, 0)$	$e_m(3, 3)$
8	6.42e-01	8.32e-02	6.83e-01	1.39e-01	4.47e-01	7.90e-03
16	3.29e-01	4.34e-02	6.92e-01	1.00e-01	1.29e-01	2.75e-02
32	1.13e-01	1.57e-02	1.79e-01	2.53e-02	2.45e-02	3.67e-03
64	4.52e-02	6.35e-03	3.60e-02	5.07e-03	2.51e-03	3.53e-04
128	1.92e-02	2.70e-03	6.87e-03	9.66e-04	2.14e-04	3.02e-05
256	8.30e-03	1.16e-03	1.30e-03	1.82e-04	1.78e-05	2.50e-06

TABLE 9.8
Example 9.1: Errors err_m, eoc, and condition numbers of A_m obtained by using the Gauss–Lobatto rule.

	$q = 1$			$q = 2$			$q = 3$		
m	err_m	eoc	cond(A_m)	err_m	eoc	cond(A_m)	err_m	eoc	cond(A_m)
8	3.74e-02		8.73e+02	2.82e-01		8.59e+02	3.30e-01		8.34e+02
		2.13			3.88			-0.91	
16	8.51e-03		1.32e+03	1.90e-02		2.03e+03	6.22e-01		2.88e+03
		2.16			4.79			5.14	
32	1.89e-03		1.42e+03	6.86e-04		5.03e+03	1.76e-02		4.08e+03
		2.34			3.17			5.67	
64	3.72e-04		1.43e+03	7.57e-05		2.55e+03	3.45e-04		3.60e+03
		2.44			4.86			7.13	
128	6.82e-05		1.44e+03	2.60e-06		2.58e+03	2.46e-06		3.66e+03
		2.66			4.89			7.19	
256	1.07e-05		1.43e+03	8.77e-08		2.58e+03	1.68e-08		3.67e+03

TABLE 9.9
Example 9.1: Errors $e_m(x_1, x_2)$ obtained by using the Gauss–Lobatto rule.

	$q = 1$		$q = 2$		$q = 3$	
m	$e_m(-0.1, 0)$	$e_m(3, 3)$	$e_m(-0.1, 0)$	$e_m(3, 3)$	$e_m(-0.1, 0)$	$e_m(3, 3)$
8	4.23e-01	2.03e-02	6.83e-01	4.89e-02	9.85e-01	4.49e-02
16	3.44e-02	2.54e-04	8.45e-02	3.62e-03	1.76e-01	1.08e-02
32	2.46e-03	4.82e-08	4.51e-04	1.33e-05	2.98e-03	1.09e-04
64	3.20e-05	2.68e-14	3.03e-06	1.19e-10	3.80e-06	1.21e-08
128	1.29e-08	5.48e-16	7.89e-11	1.05e-15	1.28e-10	1.40e-15
256	5.55e-15	8.60e-16	2.13e-14	1.42e-15	2.06e-14	1.17e-15

TABLE 9.10
Example 9.1: Errors $e_m(x_1, x_2)$ obtained by using the Gauss–Lobatto rule.

	$q = 1$		$q = 2$		$q = 3$	
m	$e_m(-40, -50)$	$e_m(100, -100)$	$e_m(-40, -50)$	$e_m(100, -100)$	$e_m(-40, -50)$	$e_m(100, -100)$
8	1.38e-03	7.04e-04	2.91e-03	1.49e-03	2.65e-03	1.36e-03
16	2.70e-05	1.37e-05	3.28e-04	1.67e-04	8.21e-04	4.19e-04
32	5.07e-09	2.58e-09	1.27e-06	6.49e-07	1.11e-05	5.68e-06
64	1.72e-15	3.88e-16	1.32e-11	6.75e-12	1.20e-09	6.14e-10
128	1.24e-15	2.77e-17	2.51e-16	9.26e-16	4.65e-16	1.63e-15
256	1.94e-16	8.74e-16	3.91e-16	1.55e-15	7.81e-16	9.75e-17

TABLE 9.11
Example 9.1: Errors $\bar{e}_m(x_1,x_2)$, $\tilde{e}_m(x_1,x_2)$, and $e_m(x_1,x_2)$ obtained by using the Gauss–Lobatto rule.

(x_1,x_2)	$(-0.1,0)$			$(3,3)$		
m	$\bar{e}_m$	$\tilde{e}_m$	e_m	$\bar{e}_m$	$\tilde{e}_m$	e_m
51	6.62e-03	1.53e-05	2.30e-05	2.35e-05	7.04e-07	1.42e-08
99	6.95e-03	1.18e-09	7.05e-09	1.89e-04	1.97e-12	2.48e-15
195	6.78e-04	1.66e-15	1.27e-14	1.81e-05	3.46e-16	8.25e-16

TABLE 9.12
Example 9.1: Errors $\bar{e}_m(x_1,x_2)$, $\tilde{e}_m(x_1,x_2)$, and $e_m(x_1,x_2)$ obtained by using the Gauss–Lobatto rule.

(x_1,x_2)	$(-40,-50)$			$(100,-100)$		
m	$\bar{e}_m$	$\tilde{e}_m$	e_m	$\bar{e}_m$	$\tilde{e}_m$	e_m
51	4.52e-05	7.53e-08	1.41e-09	5.9e-05	3.83e-08	1.41e-09
99	1.12e-05	1.95e-13	7.31e-16	5.3e-06	9.87e-14	7.31e-16
195	1.05e-06	2.85e-16	8.47e-16	5.3e-07	8.51e-16	8.47e-16

and with the boundary data corresponding to the exact solution

$$u(x_1,x_2) = \arctan\left(\frac{x_2-0.2}{x_1-0.8}\right) - \arctan\left(\frac{x_2}{x_1-0.8}\right).$$

The method has been implemented taking $\sigma = 0.96$ and the parameters c and ϵ according to the criteria previously described. The expected theoretical order of convergence for the approximation errors for the single layer density is $\nu = 1$ for $q = 1$ (being $2(1-\epsilon)(\sigma+\mu) \approx 1.4$), $\nu = 2$ for $q = 2$ (being $2(1-\epsilon)(\sigma+\mu) \approx 2.84$), and $\nu = 4$ for $q = 3$ (being $2(1-\epsilon)(\sigma+\mu) \approx 4.23$). We observe that, in cases $q = 2$ and $q = 3$, there is no improvement of err_m when m changes from 128 to 256. That's why a reliable computation of eoc in that cases is not possible.

TABLE 9.13
Example 9.2: Errors err_m in (9.2), eoc in (9.3), and condition numbers of A_m obtained by using the Gauss–Lobatto rule.

	$q=1$			$q=2$			$q=3$		
m	err_m	eoc	cond(A_m)	err_m	eoc	cond(A_m)	err_m	eoc	cond(A_m)
8	1.86e-02		6.23e+01	4.55e-02		3.60e+01	7.32e-02		2.59e+01
		3.60			2.20			1.88	
16	1.53e-03		9.60e+01	9.88e-03		1.45e+02	1.98e-02		1.65e+02
		9.71			6.20			4.59	
32	1.82e-06		1.06e+02	1.33e-04		1.88e+02	8.22e-04		2.56e+02
		9.12			13.73			10.50	
64	3.27e-09		1.08e+02	9.80e-09		1.99e+02	5.67e-07		2.82e+02
		5.84			11.37			19.35	
128	5.69e-11		1.09e+02	3.69e-12		2.02e+02	8.49e-13		2.89e+02
		5.85							
256	9.82e-13		1.09e+02	3.69e-12		2.03e+02	8.01e-13		2.91e+02

By comparing the values of $\mathrm{cond}(A_m)$ reported in Table 9.13 with those given in [26, Table 10], we can observe that the method proposed in the present paper gives rise to linear systems which are better conditioned with condition numbers not increasing with the dimension m. In Tables 9.16–9.17 one can see a comparison of the numerical results obtained for $q = 2$ with the analogous ones obtained in [14] and [26], which shows the better behavior of the proposed method.

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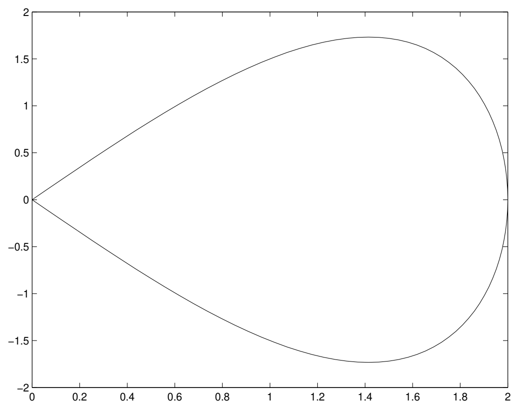

FIG. 9.2. Γ *in Example 9.2 with* $\beta = \frac{2}{3}\pi$.

TABLE 9.14
Example 9.2: Errors $e_m(x_1, x_2)$ *in (9.1) obtained by using the Gauss–Lobatto rule.*

	$q=1$		$q=2$		$q=3$	
m	$e_m(-0.1,0)$	$e_m(3,3)$	$e_m(-0.1,0)$	$e_m(3,3)$	$e_m(-0.1,0)$	$e_m(3,3)$
8	2.95e-01	2.02e-02	9.13e-02	2.74e-02	6.01-01	4.11e-02
16	7.64e-03	3.05e-04	4.12e-02	5.09e-03	7.19-02	9.08e-03
32	1.39e-08	4.92e-07	6.38e-04	3.20e-05	3.81-03	2.68e-04
64	2.95e-08	8.24e-11	1.54e-08	8.43e-09	1.58-06	1.48e-07
128	4.37e-10	1.17e-12	1.88e-15	5.34e-16	4.92-13	1.34e-13
256	6.60e-12	1.61e-14	2.66e-15	6.10e-16	4.46-15	6.66e-16

TABLE 9.15
Example 9.2: Errors $e_m(x_1, x_2)$ *in (9.1) obtained by using the Gauss–Lobatto rule.*

	$q=1$		$q=2$		$q=3$	
m	$e_m(-40,-50)$	$e_m(100,-100)$	$e_m(-40,-50)$	$e_m(100,-100)$	$e_m(-40,-50)$	$e_m(100,-100)$
8	1.47e-03	6.12e-04	1.68e-03	8.55e-04	2.58e-03	1.02e-03
16	4.00e-05	4.23e-05	4.14e-04	2.37e-04	6.94e-04	4.68e-04
32	6.32e-09	5.29e-08	4.63e-06	3.89e-06	2.88e-05	2.01e-05
64	4.79e-12	2.41e-12	1.84e-10	2.35e-10	4.19e-09	1.30e-08
128	6.86e-14	3.48e-14	4.36e-16	4.72e-16	8.02e-15	2.27e-16
256	3.62e-15	2.22e-15	7.55e-16	7.58e-16	1.24e-16	1.01e-17

We also solved the exterior Neumann problem (1.1) for degenerate teardrop domains Ω with the interior angle β at the corner point close to 0 and π and the boundary data f given by the normal derivative of the potential

$$u(x_1, x_2) = \arctan\left(\frac{x_2 - 0.001}{x_1 - 1.0}\right) - \arctan\left(\frac{x_2 + 0.001}{x_1 - 1.2}\right).$$

In Table 9.18 we report the largest absolute error E_m obtained (with $q = 2$) while approximating the solution u by u_m at a set of 600 points sampled randomly in the box with corners $(3, 4)$ and $(6, 0)$. We can observe that, if the interior angle at the corner becomes very close to 0 and π, then a high accuracy in the approximation of the potential in the exterior domain can yet be obtained, but a larger computational cost is required.

TABLE 9.16
Example 9.2: Errors $\bar{e}_m(x_1,x_2)$ in [14], $\tilde{e}_m(x_1,x_2)$ in [26], and $e_m(x_1,x_2)$ in (9.1) obtained by using the Gauss–Lobatto rule.

(x_1,x_2)	$(-0.1,0)$			$(3,3)$		
m	$\bar{e}_m$	$\tilde{e}_m$	e_m	$\bar{e}_m$	$\tilde{e}_m$	e_m
51	4.15e-03	9.80e-05	1.88e-06	6.39e-04	6.01e-07	1.59e-07
99	4.80e-05	4.34e-10	1.38e-12	8.81e-06	3.57e-10	4.15e-13
195	3.70e-06	1.63e-15	8.32e-16	2.54e-07	2.08e-17	8.46e-16

TABLE 9.17
Example 9.2: Errors $\bar{e}_m(x_1,x_2)$ in [14], $\tilde{e}_m(x_1,x_2)$ in [26], and $e_m(x_1,x_2)$ in (9.1) obtained by using the Gauss–Lobatto rule.

(x_1,x_2)	$(-40,-50)$			$(100,-100)$		
m	$\bar{e}_m$	$\tilde{e}_m$	e_m	$\bar{e}_m$	$\tilde{e}_m$	e_m
51	1.47e-03	3.64e-07	5.42e-09	1.80e-03	4.04e-07	1.48e-08
99	4.34e-06	1.31e-11	2.03e-14	5.57e-06	3.73e-12	4.92e-15
195	1.98e-08	7.50e-17	1.31e-15	8.05e-09	1.81e-16	1.66e-15

TABLE 9.18
Example 9.2: Errors E_m for nearly degenerate teardrop domains obtained by using the Gauss–Lobatto rule.

β	m	E_m
$\pi/10$	512	8.95e-15
$\pi/20$	1024	2.44e-14
$\pi/50$	2048	4.83e-14
$\pi/100$	4096	1.62e-13
$\pi-\pi/10$	1024	4.24e-15
$\pi-\pi/20$	2048	1.06e-14
$\pi-\pi/50$	4096	5.72e-14
$\pi-\pi/100$	8192	5.95e-14

EXAMPLE 9.3. In this example we consider the family of boomerang-shaped domains (see Figure 9.3) with the boundary Γ parameterized by

$$\boldsymbol{\gamma}(t)=\left(2\sin 3\pi t,-\tan\frac{\beta}{2}\sin 2\pi t\right),\qquad t\in[0,1],$$

with $\beta\in(\pi,2\pi)$ the interior angle at the single corner point $P=(0,0)$. We test the numerical method described in Section 5 in the case where the interior angle is $\beta=\frac{3}{2}\pi$ and the boundary data f taken to be the normal derivative of the harmonic function

$$u(x_1,x_2)=\log|(x_1,x_2)-(1.0,0.0)|-\log|(x_1,x_2)-(1.1,0.1)|.$$

In order to compare the numerical results obtained by applying our method with those presented in [3, Table 3] and in [26, Table 13], we compute the maximum absolute error, denoted by $E_m(3)$, at 300 points on the circle of radius 3 centered at the origin, and we report them in Table 9.19 along with the condition numbers $\mathrm{cond}(A_m)$ and the weighted errors err_m in the approximation of the single layer density function.

We can observe that the method proposed here (in particular in the case $q=1$) seems to provide more accurate approximations of the potential by solving linear systems of smaller dimension. We also highlight that the sequence of the condition numbers of the coefficient matrices is uniformly bounded, similarly to what happens in [3] (see Table 3) but differently to what appears in [26] (see Table 13), where they increase when the dimension of the linear system grows.

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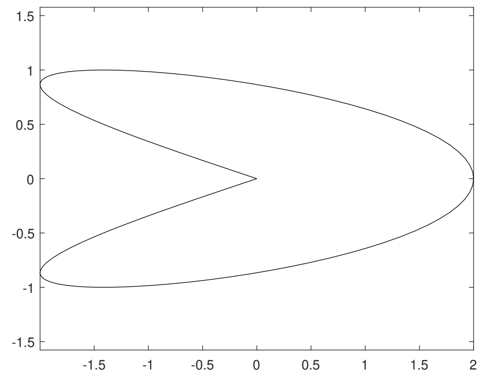

FIG. 9.3. Γ *in Example 9.3 with* $\beta = \frac{3}{2}\pi$.

TABLE 9.19
Example 9.3: Errors err_m in (9.2), $E_m(3)$, and condition numbers of A_m obtained by using the Gauss–Lobatto rule.

	$q=1$			$q=2$		
m	err_m	$E_m(3)$	cond(A_m)	err_m	$E_m(3)$	cond(A_m)
16	2.33e-01	5.80e-02	2.8849e+03	1.50e+00	1.54e-01	3.9353e+03
32	7.64e-02	1.45e-02	2.4428e+03	1.01e-01	7.33e-02	5.5076e+03
64	6.97e-04	4.34e-04	2.9172e+03	1.91e-03	9.61e-03	6.8494e+03
128	2.55e-07	1.40e-07	2.9146e+03	1.02e-06	7.19e-06	6.8321e+03
256	2.07e-12	1.03e-14	2.9477e+03	1.42e-12	4.05e-10	6.8162e+03

TABLE 9.20
Example 9.3: Errors $E_m(3)$ for nearly degenerate boomerang domains obtained by using the Gauss–Lobatto rule.

β	m	E_m
$\pi+\pi/10$	512	1.51e-15
$\pi+\pi/20$	2048	4.39e-15
$\pi+\pi/50$	2048	2.25e-14
$\pi+\pi/100$	4096	2.82e-14
$2\pi-\pi/10$	2048	2.28e-14
$2\pi-\pi/20$	4096	1.64e-14
$2\pi-\pi/50$	8192	1.05e-13
$2\pi-\pi/100$	16384	1.48e-13

We also consider the cases of degenerate boomerang domains Ω with the interior angle β at the corner point close to π and 2π. For each case the boundary data f were taken to be the normal derivative of the function

$$u(x_1, x_2) = \log|(x_1, x_2) - (1.0, 0.0)| - \log|(x_1, x_2) - (1.1, 0)|.$$

In Table 9.20 we report the largest absolute error E_m obtained (with $q = 2$) while approximating the solution u by u_m at a set of 600 points sampled randomly in the box with corners $(3, 4)$ and $(6, 0)$.

EXAMPLE 9.4. In this example we consider the domain Ω in Figure 9.4. Its boundary is formed by four circular arcs of radius 3.64, centered at ± 1 and ± 3, respectively. We assume

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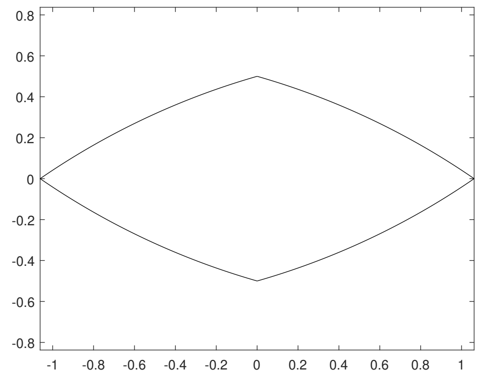

FIG. 9.4. Γ *in Example 9.4.*

TABLE 9.21
Example 9.4: Errors err_m, weighted norms $\|\tilde{\psi}_m\|_{v^\sigma,\infty}$, and condition numbers of A_m obtained by using the Gauss–Lobatto rule.

m	err_m		$\|\tilde{\psi}_m\|_{v^\sigma,\infty}$		$\text{cond}(A_m)$	
	$q=1$	$q=2$	$q=1$	$q=2$	$q=1$	$q=2$
8	8.81e-05	7.19e-04	2.873791227268631e-01	4.366457033169382e-01	1.6530e+04	3.3362e+04
16	6.94e-07	3.24e-06	2.874203356049211e-01	4.364611068575046e-01	1.6745e+04	3.3389e+04
32	6.36e-09	1.38e-08	2.874203390392681e-01	4.364608976297081e-01	1.6718e+04	3.3236e+04
64	6.34e-09	3.85e-09	2.874203390536005e-01	4.364608976393504e-01	1.6722e+04	3.3131e+04
128	6.34e-09	3.86e-09	2.874203391111713e-01	4.364608976393615e-01	1.6714e+04	3.3009e+04
256	6.34e-09	3.86e-09	2.874203390536796e-01	4.364608976407774e-01	1.6708e+04	3.2900e+04

TABLE 9.22
Example 9.4: Errors $E_m(\rho)$ obtained by using the Gauss–Lobatto rule.

m	$E_m(1.1)$		$E_m(2)$		$E_m(6)$		$E_m(20)$	
	$q=1$	$q=2$	$q=1$	$q=2$	$q=1$	$q=2$	$q=1$	$q=2$
16	4.98e-07	1.38e-06	1.81e-07	2.33e-07	5.56e-08	8.01e-08	1.65e-08	2.40e-08
32	1.38e-08	8.02e-10	4.82e-09	3.21e-11	1.45e-09	6.86e-12	4.33e-10	1.97e-12
64	3.92e-10	8.16e-14	1.20e-10	4.49e-15	3.66e-11	3.10e-15	1.08e-11	3.36e-15
128	1.03e-11	1.87e-14	3.10e-12	7.82e-15	9.43e-13	5.13e-15	2.82e-13	4.87e-15
256	2.37e-13	1.41e-14	6.94e-14	7.91e-15	2.38e-14	5.17e-15	1.24e-14	5.83e-15

as exact solution of the Neumann problem (1.1) the function

$$u(x_1, x_2) = \frac{x_1}{x_1^2 + x_2^2}.$$

The numerical results obtained by applying our BIE method are reported in Tables 9.21–9.22. Here $\|\widetilde{\psi}_m\|_{v^\sigma,\infty}$ denotes a discrete version of the weighted norm of the approximating solution $\widetilde{\psi}_m$. Both the values of the weighted errors err_m and the values of $\|\widetilde{\psi}_m\|_{v^\sigma,\infty}$ in Table 9.21 show the convergence of the sequence of the approximating solutions in the weighted space $\boldsymbol{C}_{v^\sigma}$. We can observe that a certain accuracy is already achieved by solving a linear system of dimension $m = 64$, although it does not seem to increase with larger m. In this case the condition numbers $\text{cond}(A_m)$ of the linear systems are not very small but we highlight that they turn out uniformly bounded with respect to m.

Finally, in Table 9.22 we report the maximum absolute error $E_m(\rho)$ in the computation of the potential u at 200 points on a circle of radius ρ centered at the origin, computed for different values of ρ. In all the cases we can note that using a smoothing transformation produces an acceleration of the convergence, both in points of the exterior domain which are closer to the boundary Γ ($\rho = 1.1$) and in increasingly distant points ($\rho = 2, 6, 20$).

10. Conclusions. A Nyström-type method based on global approximation is proposed for the numerical solution of the exterior Neumann problem for the Laplace equation in domains with corners. By using the single layer potential representation of the solution and a suitable decomposition of the piecewise smooth boundary, the differential problem is converted into a system of boundary integral equations whose solutions can be unbounded at the corner points. For this reason, we aim to approximate the unknowns in suitable weighted spaces of functions. The Nyström discretization of the system is obtained by applying a proper Gauss or a Gauss–Lobatto quadrature formula to the integral operators, suitably modified near the corner points in order to assure stability and convergence of the method. A smoothing transformation is also introduced in order to increase the rate of convergence.

Finally, an approximation of the single layer potential is computed by using the (weighted) samples at the quadrature knots of the approximating solutions of the BIEs system. Such values are obtained simply by solving a linear system. Several numerical examples confirm the applicability of the proposed method and illustrate its performance. The well-conditioning of the linear systems, which the method leads to, is also shown through numerical evidence.

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