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EVALUATING LEBESGUE CONSTANTS BY CHEBYSHEV POLYNOMIAL MESHES ON CUBE, SIMPLEX, AND BALL*

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Abstract. We show that product Chebyshev polynomial meshes can be used, in a fully discrete way, to evaluate with rigorous error bounds the Lebesgue constant, i.e., the maximum of the Lebesgue function, for a class of polynomial projectors on cube, simplex, and ball, including interpolation, hyperinterpolation, and weighted least-squares approximation. Several examples are presented and possible generalizations outlined. A numerical software package implementing the method is freely available online.

Key words. multivariate polynomial meshes, cube, simplex, ball, polynomial projectors, interpolation, least-squares, hyperinterpolation, polynomial optimization, Lebesgue constant

AMS subject classifications. 65D05, 65D10, 65K05

1. Introduction. Starting from the seminal paper by Calvi and Levenberg [16], the notion of *polynomial (admissible) mesh* has been emerging in the last years as a fundamental theoretical and computational tool in multivariate polynomial approximation. Let $K \subset \mathbb{R}^d$ (or $K \subset \mathbb{C}^d$) be a polynomial determining compact set, i.e., polynomials vanishing there vanish everywhere in $\mathbb{R}^d$. For example, any compact set with nonempty interior is polynomial determining since the zero set of any nonzero polynomial has null Lebesgue measure (the presence of interior points being however not necessary, e.g., the classical ternary Cantor set in $[0,1]$ is polynomial determining). Denote by $\mathbb{P}_n = \mathbb{P}_n^d$ the space of d-variate polynomials of total degree not exceeding n and by

$$N = N_n = \dim(\mathbb{P}_n) = \binom{n+d}{n}$$

its dimension. We recall that an *admissible polynomial mesh* of K is a sequence of finite subsets $A_n \subset K$ such that

$$\|p\|_K \leq c\|p\|_{A_n} , \quad \forall p \in \mathbb{P}_n ,$$

with $\mathrm{card}(A_n) = O(n^\alpha)$, $\alpha \geq d$, and c a constant *independent* of n. Here and below, $\|\cdot\|_Y$ denotes the sup-norm on a continuous or discrete compact set Y. Observe that $\mathrm{card}(A_n) \geq N$ necessarily holds. Indeed, each A_n is $\mathbb{P}_n$-determining, since polynomials vanishing on A_n vanish everywhere on K. Such a mesh is termed *optimal* when $\alpha = d$.

To give only a flavour of the topic, we recall that polynomial meshes are invariant by affine transformations, are stable under small perturbations, and can be assembled by finite union, finite product, and algebraic transformations, starting from known instances. These include cubes (boxes), simplices, and balls, but also more general linear and curved polytopes as well as convex bodies and more general compact domains satisfying Markov polynomial inequalities. On the other hand, when available, optimal polynomial meshes are preferable in applications due to their low cardinality, for example in the extraction of extremal points such as Fekete-like and Leja-like points. We do not even attempt here to give a comprehensive

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survey of the already considerable literature on polynomial meshes, referring the reader, e.g., to [2, 3, 8, 12, 18, 33, 42, 49] and the references therein.

In the present paper, by exploiting the connection with the polynomial optimization methods studied for example in [43, 53, 54], we develop theoretical estimates together with a numerical algorithm to approximate the value of the uniform norms of fitting operators (Lebesgue constants). Indeed, the computation of Lebesgue constants is a matter of optimization of Lebesgue functions, which ultimately corresponds to the computation of the maximum modulus of fitting polynomials. The method works in a fully discrete way on special polynomial meshes of product Chebyshev type on cubes, simplices, and balls, producing approximations of the Lebesgue constant from above and below, with rigorous error bounds. We also point out that, by the finite union property of polynomial meshes, the method can be readily applied, with the same error bounds, to complicated geometrical objects relevant in applications, namely single polytopes (via triangulation), union of polytopes, and union of balls.

The computation of Lebesgue constants is important in applications in order to investigate the quality of the sampling nodes, e.g., for interpolation. In the literature this step is typically made, for example within spectral and high-order methods for numerical PDEs, by empirical approaches, namely by the evaluation of Lebesgue functions on increasingly finer discretizations. In the present paper we provide, apparently for the first time, a fully discrete method with rigorous error bounds as well as the corresponding numerical codes.

The paper is organized as follows. In Section 2 we state and prove the main theoretical results, and we also outline an extended (but less accurate) approach via general polynomial meshes on multidimensional compact sets. In Section 3 we discuss some computational and implementation issues, and we present several examples concerning the evaluation of the size of Lebesgue constants for polynomial interpolation and least-squares approximation (including hyperinterpolation) on cube, simplex, and ball.

2. Lebesgue constants by Chebyshev meshes.

In the sequel, the following constant will play a key role

$$c_m = \frac{1}{\cos(\frac{\pi}{2m})}, \quad m > 1 \ .$$

Moreover, we shall denote by $\mathcal{C}_k$ the set of k Chebyshev zeros in $(-1,1)$, $\cos((2j-1)\frac{\pi}{2k})$, $1 \le j \le k$ (Chebyshev points), or the set of $k+1$ Chebyshev extrema in $[-1,1]$, $\cos(j\frac{\pi}{k})$, $0 \le j \le k$ (Chebyshev–Lobatto points).

2.1. The cube.

We first discuss the case of the d-cube. By invertible affine transformation, the result can be immediately extended to any d-box with invariance of the mesh constant c. The following result has been proved in [43], essentially following [6], using the notion of *Dubiner distance* on a compact set (which is tailored to polynomial spaces; cf., e.g., [10, 55] and the references therein).

LEMMA 2.1. *Let $\mathcal{C}_k$ be Chebyshev or Chebyshev–Lobatto points in $[-1,1]$. Then the sequence of product Chebyshev grids $A_n^m = (\mathcal{C}_{mn})^d$, $n = 1, 2, \ldots$, for a fixed $m > 1$, is an admissible polynomial mesh for the d-cube $[-1,1]^d$ with constant $c = c_m$.*

We can now state and prove a basic result on the approximation of Lebesgue constants by polynomial meshes on the d-cube.

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PROPOSITION 2.2. *Let $A_n^m = (\mathcal{C}_{mn})^d$ be a product Chebyshev admissible mesh of $K = [-1,1]^d$ as in Lemma 2.1, and let $L_n : C(K) \to \mathbb{P}_n$ be a linear projection operator such that*

$$L_n f(x) = \sum_{i=1}^{M} f(\xi_i)\, \varphi_i(x) \;, \tag{2.1}$$

where $\Xi = \{\xi_i\} \subset K$ and $\{\varphi_i\}$ is a set of possibly not independent generators of $\mathbb{P}_n$, i.e., $\mathbb{P}_n = \mathrm{span}\{\varphi_1, \ldots, \varphi_M\}$, with $M \geq N = \dim(\mathbb{P}_n)$. Moreover, let

$$\lambda_n(x) = \sum_{i=1}^{M} |\varphi_i(x)|$$

be the "Lebesgue function" of L_n and

$$\|L_n\| = \sup_{f \neq 0} \frac{\|L_n f\|_K}{\|f\|_K} = \|\lambda_n\|_K$$

its "Lebesgue constant". Then the following estimate holds for every $m > 1$:

$$\|\lambda_n\|_{A_n^m} \leq \|L_n\| \leq c_m \|\lambda_n\|_{A_n^m} \;. \tag{2.2}$$

Proof. First we prove that $\|L_n\| = \|\lambda_n\|_K$ for any projection operator of the form (2.1) on a general compact set K. Indeed, the inequality $\|L_n f\|_K \leq \|f\|_K \|\lambda_n\|_K$ is immediate, whereas the existence of a function $g^* \in C(K)$ such that $g^*(\xi_i) = \mathrm{sign}(\varphi_i(x^*))$, where $\|\lambda_n\|_K = \lambda_n(x^*)$, with $x^* \in K$ and $\|g^*\|_K = 1$, is guaranteed by a quite general topological result, namely the celebrated Tietze extension theorem of continuous functions from a closed subset of a normal topological space, preserving the range; cf., e.g., [27, Ch.7, Thm.5.1]. Indeed, defining a function g on the sampling set Ξ such that $g(\xi_i) = \mathrm{sign}(\varphi_i(x^*))$, with $1 \leq i \leq M$, then since g is trivially continuous on the closed discrete subset $\Xi \subset K$, there exists an extension $g^* \in C(K)$ taking values in $[-1,1]$ with $L_n g^*(x^*) = \lambda_n(x^*)$.

Now, applying Lemma 2.1 to the polynomial $L_n f$ we get

$$\|L_n f\|_K \leq c_m \|L_n f\|_{A_n^m} \;.$$

On the other hand, from the estimate $|L_n f(x)| \leq \|f\|_\Xi \lambda_n(x) \leq \|f\|_K \lambda_n(x)$ it follows that $\|L_n f\|_K \leq c_m \|f\|_K \|\lambda_n\|_{A_n^m}$, from which we immediately get

$$\|L_n\| = \|\lambda_n\|_K \leq c_m \|\lambda_n\|_{A_n^m} \;,$$

and thus (2.2) since $\|\lambda_n\|_K \geq \|\lambda_n\|_{A_n^m}$ by inclusion. □

REMARK 2.3. Notice that $c_m \to 1$ and thus, if the sampling set Ξ is independent of m, $\|\lambda_n\|_{A_n^m} \to \|L_n\|$ as $m \to \infty$. Moreover, from (2.2) we also get a relative error estimate, namely

$$0 \leq \frac{\|L_n\| - \|\lambda_n\|_{A_n^m}}{\|L_n\|} \leq c_m - 1 \sim \frac{\pi^2}{8m^2} \approx \frac{1.23}{m^2} \;, \tag{2.3}$$

i.e., $\|\lambda_n\|_{A_n^m}$ approximates the Lebesgue constant from below with an $\mathcal{O}(\frac{1}{m^2})$ relative error. On the other hand, (2.2) gives also the rigorous and *computable* absolute error estimate

$0 \le \|L_n\| - \|\lambda_n\|_{A_n^m} \le (c_m - 1)\|\lambda_n\|_{A_n^m}$. We finally notice that (2.2) is an *interval approximation* of the Lebesgue constant. Hence, we can use the midpoint approximation

$$\frac{|\,\|L_n\| - \Lambda_n^m\,|}{\|L_n\|} \le \frac{c_m - 1}{2}\ , \qquad \Lambda_n^m = \|\lambda_n\|_{A_n^m}\,\frac{1+c_m}{2}\ , \tag{2.4}$$

which improves the error estimates by a factor $\frac{1}{2}$.

REMARK 2.4. The structure of projection operators like (2.1) includes interpolation operators at unisolvent nodes $\Xi = \{\xi_1, \dots, \xi_N\} \subset K$, where, denoting by $V_n = [p_j(\xi_i)]_{ij}$, $1 \le i, j \le N$, the Vandermonde-like matrix in any fixed polynomial basis $\mathrm{span}\{p_1, \dots, p_N\} = \mathbb{P}_n$, we have that

$$\varphi_j(x) = \ell_j(x) = \frac{\det(V_n(\xi_1, \dots, \xi_{j-1}, x, \xi_{j+1}, \dots, \xi_N))}{\det(V_n(\xi_1, \dots, \xi_{j-1}, \xi_j, \xi_{j+1}, \dots, \xi_N))}$$

are the corresponding Lagrange cardinal polynomials. Also included are discrete weighted least-squares operators at $\mathbb{P}_n$-determining nodes $\Xi = \{\xi_1, \dots, \xi_M\} \subset K$ with positive weights $W = \{w_1, \dots, w_M\}$, $M > N$. Indeed, denoting by $\{\pi_k\}$, $1 \le k \le N$, the orthonormal polynomials with respect to the corresponding discrete scalar product

$$(f, g)_{\ell^2_W(\Xi)} = \sum_{j=1}^{M} w_j f(\xi_j) g(\xi_j),$$

we have that

$$L_n f(x) = \sum_{k=1}^{N} (f, \pi_k)_{\ell^2_W(\Xi)}\, \pi_k(x) = \sum_{j=1}^{M} f(\xi_j) w_j K_n(x, \xi_j)\ , \tag{2.5}$$

i.e.,

$$\varphi_j(x) = w_j K_n(x, \xi_j)\ , \tag{2.6}$$

where $K_n(x, y) = \sum_{k=1}^{N} \pi_k(x)\pi_k(y)$ is the reproducing kernel of the discrete scalar product. Notice that in this case (unless $M = N$ where the least-squares approximation coincides with interpolation) the φ_j are linearly dependent, thus forming a set of generators of $\mathbb{P}_n$.

We stress that the structure (2.5) also includes *hyperinterpolation* operators, a topic that has seen an increasing interest as a valid alternative to multivariate polynomial interpolation after the seminal paper [47] by Sloan in the mid '90s; cf., e.g., [1, 22, 30, 31, 50, 56, 58] and the references therein. Indeed, hyperinterpolation operators are substantially truncated Fourier-like expansions in series of orthogonal polynomials with respect to a continuous measure with density. The scalar products in L^2 are substituted by discrete scalar products, corresponding to a suitable positive quadrature formula being exact in $\mathbb{P}_{2n}$.

2.2. The simplex. We consider now the d-simplex

$$T_d = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : 0 \le x_d \le \cdots \le x_1 \le 1\}\ ,$$

along with the d-dimensional Duffy-like transformation $\mathcal{D} : [-1,1]^d \to T_d$, which can be defined as

$$x_i = \mathcal{D}_i(t) = \prod_{j=1}^{i} \left(\tfrac{t_j}{2} + \tfrac{1}{2}\right)\ , \qquad 1 \le i \le d\ ,\ \ t = (t_1, \dots, t_d) \in [-1,1]^d\ ; \tag{2.7}$$

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cf., e.g., [48]. We again notice that the results can be immediately extended to any simplex with invariance of the mesh constant c by an invertible affine transformation.

LEMMA 2.5. *Let $\mathcal{C}_k$ be the Chebyshev or Chebyshev–Lobatto points in $[-1,1]$ and $\mathcal{D}$ the Duffy-like transformation $[-1,1]^d \to T_d$ in (2.7), where T_d is the d-simplex. Then the sequence $A_n^m = \mathcal{D}((\mathcal{C}_{mn})^d)$, $n = 1, 2, \ldots$, for a fixed $m > 1$, is an admissible polynomial mesh for T_d with constant $c = (c_m)^d$.*

Proof. Let us denote by $\mathbb{P}_n^1$ the space of univariate real algebraic polynomials of degree not exceeding n. For every $p \in \mathbb{P}_n$ we have that $\|p\|_{T_d} = \|p \circ \mathcal{D}\|_{[-1,1]^d}$. Now, since $\mathcal{D}$ is a d-linear (surjective) mapping, it hold that $p \circ \mathcal{D} \in \bigotimes_{k=1}^d \mathbb{P}_n^1$, the space of tensorial polynomials of degree not exceeding n. Reasoning component by component and using iteratively the univariate version of Lemma 2.1, it is immediate to write the inequality

$$\|p \circ \mathcal{D}\|_{[-1,1]^d} \leq (c_m)^d \|p \circ \mathcal{D}\|_{(\mathcal{C}_{mn})^d} = (c_m)^d \|p\|_{A_n^m} . \qquad \square$$

PROPOSITION 2.6. *Let $A_n^m = \mathcal{D}((\mathcal{C}_{mn})^d)$ be a polynomial admissible mesh of the d-simplex $K = T_d$ as in Lemma 2.5 and $L_n : C(K) \to \mathbb{P}_n$ a linear projection operator with the structure defined in Proposition 2.2. Then the following estimate holds for every $m > 1$:*

$$\|\lambda_n\|_{A_n^m} \leq \|L_n\| \leq (c_m)^d \|\lambda_n\|_{A_n^m} .$$

Proof. We can proceed exactly as in the proof of Proposition 2.2 by simply substituting c_m with $(c_m)^d$. $\square$

REMARK 2.7. We observe that, since $(c_m)^d \to 1$ as $m \to \infty$, the same assertions of Remark 2.3 are valid with $(c_m)^d$ replacing c_m. The only difference is that the estimate (2.3) is asymptotically increased by a factor d, namely

$$\begin{aligned} 0 &\leq \frac{\|L_n\| - \|\lambda_n\|_{A_n^m}}{\|L_n\|} \leq (c_m)^d - 1 = (c_m^{d-1} + \cdots + c_m + 1)(c_m - 1) \\ &\leq \left(\left(\frac{m}{m-1}\right)^{d-1} + \cdots + \frac{m}{m-1} + 1 \right)(c_m - 1) \sim \frac{d\pi^2}{8m^2} \approx 1.23\,\frac{d}{m^2} \end{aligned}$$

for d fixed and $m \to \infty$, where we have used the elementary inequality

$$\cos(\theta) = \sin(\tfrac{\pi}{2} - \theta) \geq 1 - \frac{2\theta}{\pi}, \qquad 0 \leq \theta \leq \frac{\pi}{2}.$$

REMARK 2.8. It is worth observing that, by the finite union extension property of admissible polynomial meshes (cf., e.g., [16]), the results above are valid via “triangulation” (subdivision into non overlapping simplices) on any polytope, e.g., on any polygon in $d = 2$ or polyhedron in $d = 3$. We recall that for $d \geq 3$ in nonconvex instances, the triangulation could require extra vertices in view of the well-known Schönardt counterexample [45]. On the other hand, for the same reason it is also valid on any union of (even overlapping) polytopes, where again the mesh is the union of the single polytope meshes obtained via triangulation (in practice this avoids tracking and triangulating the union polytope, which can be very complicated).

2.3. The ball. As a third relevant case, we discuss the unit Euclidean ball, i.e., $B_d = \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$, by no loss of generality since it is affinely equivalent to any other ball by translation and scaling with invariance of the mesh constant c. Again, the results can be extended to a finite union of possibly overlapping balls, a geometrical object that is relevant in applications, e.g., in the field of molecular modelling [40].

Below we denote by $\mathbb{P}^1_n$ the space of univariate real algebraic polynomials of degree not exceeding n and by $\mathbb{T}^1_n([a,b])$ the space of univariate real trigonometric polynomials of degree not exceeding n, i.e., $\mathrm{span}\{1, \sin(j\theta), \cos(j\theta)\,,\ j = 1, \dots, n\}$, restricted to a subinterval $[a, b]$ of period $b - a \leq 2\pi$.

Moreover, we make use of *generalized spherical coordinates* in B_d, $d \geq 2$ (the case $B_1 = [-1, 1]$ is treated in Section 2.1), corresponding to the surjective transformation

$$\mathcal{G} : J = [0,1] \times [0,\pi]^{d-2} \times [0, 2\pi] \to B_d$$

defined by

$$x_j = r\cos(\theta_j)\prod_{k=1}^{j-1}\sin(\theta_k)\ ,\ 1 \leq j \leq d-1, \qquad x_d = r\sin(\theta_{d-1})\prod_{k=1}^{d-2}\sin(\theta_k),$$

where $r \in [0, 1]$, $\theta_k \in [0, \pi]$, $1 \leq k \leq d - 2$, $\theta_{d-1} \in [0, 2\pi]$; cf., e.g., [5]. These coordinates coincide with the usual polar coordinates for the disk B_2 and the spherical coordinates for the 3-ball B_3.

First, we state a basic lemma on a norming inequality for univariate trigonometric polynomials in the subperiodic case, whose proof can be found in [54].

LEMMA 2.9. *Let $\mathcal{C}_k$ be the Chebyshev or Chebyshev–Lobatto points in $[-1, 1]$, and let $\sigma_{a,b} : [-1, 1] \to [a, b]$, $b - a \leq 2\pi$, be the invertible map*

$$\sigma_{a,b}(u) = 2\arcsin(\alpha u) + \beta\ , \qquad u \in [-1,1]\ ,\ \alpha = \sin\left(\frac{b-a}{4}\right)\ ,\ \beta = \frac{b+a}{2}\ .$$

Then the following inequality holds:

$$\|\phi\|_{a,b} \leq c_m\|\phi\|_{\sigma_{a,b}(\mathcal{C}_{2mn})}\ , \quad \forall\phi \in \mathbb{T}_n([a,b])\ .$$

REMARK 2.10. We observe that the Chebyshev-like angles $\sigma_{a,b}(\mathcal{C}_{2mn})$ cluster at the interval endpoints for $b - a < 2\pi$ (subperiodic instances), whereas they are equally spaced for $b - a = 2\pi$ (periodic case).

LEMMA 2.11. *Let $\mathcal{C}_k$ be the Chebyshev or Chebyshev–Lobatto points in $[-1, 1]$, and consider the composed transformation*

$$\mathcal{S} = \mathcal{G} \circ \mathcal{U} : [-1,1]^d \to B_d\ , \tag{2.8}$$

$$\mathcal{U}(u_1, u_2, \dots, u_{d-1}, u_d) = \left(\frac{u_1}{2} + \frac{1}{2}, \sigma_{0,\pi}(u_2), \dots, \sigma_{0,\pi}(u_{d-1}), \sigma_{0,2\pi}(u_d)\right)\ ,$$

where B_d is the d-ball. Then the sequence $A^m_n = \mathcal{S}(\mathcal{C}_{mn} \times (\mathcal{C}_{2mn})^{d-1})$, $n = 1, 2, \dots$, for a fixed $m > 1$, is an admissible polynomial mesh for B_d with constant $c = (c_m)^d$.

Proof. For every $p \in \mathbb{P}_n$, it is easily seen by basic trigonometric identities that the composed function $p \circ \mathcal{G}$ belongs to an algebraic-trigonometric tensorial space on the box $J = [0, 1] \times [0, \pi]^{d-1} \times [0, 2\pi] = \mathcal{U}([-1, 1]^d)$, namely

$$p \circ \mathcal{G} \in \mathbb{P}^1_n \bigotimes \mathbb{T}_n([0,\pi]) \bigotimes \cdots \bigotimes \mathbb{T}_n([0,\pi]) \bigotimes \mathbb{T}_n([0,2\pi])\ .$$

Moreover, $\|p\|_{B_d} = \|p \circ \mathcal{G}\|_J$ by the surjectivity of $\mathcal{G}$. Then, reasoning component by component by using the univariate version of Lemma 2.1 and iteratively Lemma 2.9, it is immediate to write

$$\begin{aligned}\|p\|_{B_d} &= \|p \circ \mathcal{G}\|_J \leq (c_m)^d\|p \circ \mathcal{G}\|_{\mathcal{U}(\mathcal{C}_{mn}\times(\mathcal{C}_{2mn})^{d-1})}\\ &= (c_m)^d\|p \circ \mathcal{S}\|_{\mathcal{C}_{mn}\times(\mathcal{C}_{2mn})^{d-1}} = (c_m)^d\|p\|_{A^m_n}\ . \qquad \square\end{aligned}$$

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PROPOSITION 2.12. *Let $A_n^m = \mathcal{S}(\mathcal{C}_{mn} \times (\mathcal{C}_{2mn})^{d-1})$ be a polynomial admissible mesh of the d-ball $K = B_d$ as in Lemma 2.11, and let $L_n : C(K) \to \mathbb{P}_n$ be a linear projection operator with the structure defined in Proposition 2.2. Then the following estimate holds for every $m > 1$:*

$$\|\lambda_n\|_{A_n^m} \leq \|L_n\| \leq (c_m)^d \|\lambda_n\|_{A_n^m} \; . \tag{2.9}$$

Proof. Again (see Proposition 2.6), we can proceed exactly as in the proof of Proposition 2.2 simply by substituting c_m with $(c_m)^d$. ◻

We can finally observe that in view of (2.9), the considerations in Remark 2.7 apply also to the case of the ball.

2.4. General polynomial meshes. In the case of more general compact sets, we can still get a fully discrete, but less accurate, approximation of the Lebesgue constants by polynomial meshes based on the approach developed in [53] for polynomial optimization.

PROPOSITION 2.13. *Let $K \subset \mathbb{R}^d$ be a compact set, $\{A_n\}$ a polynomial admissible mesh of K, and $L_n : C(K) \to \mathbb{P}_n$ a linear projection operator with the structure defined in Proposition 2.2. Then the following estimates hold for every $m \geq 1$:*

$$\|\lambda_n\|_{A_{nm}} \leq \|L_n\| \leq c^{\frac{1}{m}} \|\lambda_n\|_{A_{nm}} \; . \tag{2.10}$$

Proof. First, observe that by definition of a polynomial mesh, for every $p \in \mathbb{P}_n$ and for every $m \geq 1$ we can write

$$\|p\|_K \leq c^{\frac{1}{m}} \|p\|_{A_{nm}} \; ,$$

since $p^m \in \mathbb{P}_{mn}$ and $\|p^m\|_K = \|p\|_K^m \leq C\|p^m\|_{A_{nm}} = C\|p\|_{A_{nm}}^m$. Then we can reason as in the proof of Proposition 2.2 to reach the conclusion with $c^{\frac{1}{m}}$ substituting c_m. ◻

REMARK 2.14. Notice that $c^{\frac{1}{m}} \to 1$, and thus again, if the sampling set Ξ is independent of m, $\|\lambda_n\|_{A_{nm}} \to \|L_n\|$ as $m \to \infty$. Moreover, from (2.10) we also get a relative error estimate, namely

$$\frac{\|L_n\| - \|\lambda_n\|_{A_{nm}}}{\|L_n\|} \leq e^{\frac{\log(c)}{m}} - 1 \leq e^{\frac{\log(c)}{m}} \, \frac{\log(c)}{m}$$

by the mean value theorem and the monotonicity of the exponential function. This is an $\mathcal{O}(\frac{1}{m})$ relative approximation of the Lebesgue constant by $\|\lambda_n\|_{A_{nm}}$. Again, (2.10) gives also the rigorous and *computable* absolute error estimate

$$0 \leq \|L_n\| - \|\lambda_n\|_{A_{nm}} \leq (c^{\frac{1}{m}} - 1)\|\lambda_n\|_{A_{nm}}.$$

Notice that also in this general case we may resort to the midpoint approximation $\Lambda_{A_{nm}} = \|\lambda_n\|_{A_{nm}} \frac{1+c^{\frac{1}{m}}}{2}$, whose relative error estimate is improved by a factor $\frac{1}{2}$, namely

$$\frac{|\,\|L_n\| - \Lambda_{A_{nm}}\,|}{\|L_n\|} \leq \frac{c^{\frac{1}{m}} - 1}{2}.$$

3. Numerical examples.

3.1. Computational issues. We make some observations on the main computational issues. Dealing with polynomial projectors, most computations can be seen as a matter of numerical linear algebra by standard algorithms applied to the relevant Vandermonde-like matrices in a polynomial basis $[p_1(x),\dots,p_N(x)]$ of $\mathbb{P}_n$. In order to control the conditioning of such matrices, which can become unacceptable already at moderate degrees with the standard monomial basis, we have chosen to use a total-degree product Chebyshev basis corresponding to the minimal enclosing Cartesian box, say $[a_1,b_1]\times\dots\times[a_d,b_d]\supset K$, namely $p_k(x)=\prod_{s=1}^{d}T_{m_s}(\alpha_s x_s+\beta_s)$, where $\alpha_s=\frac{b_s-a_s}{2}$, $\beta_s=\frac{b_s+a_s}{2}$, and the function $T_{m_s}(\cdot)=\cos(m_s\arccos(\cdot))$ is the standard Chebyshev polynomial of the second kind for degree m_s. Here, $k=1,\dots,N$ corresponds to some ordering (for example, a lexicographical ordering) of the d-tuples $(m_1,\dots,m_d)$, $0\le m_1+\dots+m_d\le n$. Alternatively, one could adopt an orthonormal polynomial basis with respect to some measure, when explicitly known, such as the Logan-Shepp or the Zernike basis for the disk [57] or the Dubiner basis for the 2-simplex [26].

In view of (2.5)–(2.6), the Lebesgue constant on the mesh of either an interpolation or weighted least-squares projection operator can be computed via the discrete reproducing kernel, i.e., via the discrete orthogonal polynomial basis $\{\pi_k\}$. Indeed, consider the Vandermonde-like matrix $V_n(\Xi)=[p_k(\xi_j)]_{jk}$, where $\Xi=\{\xi_j\}$ are either the interpolation or the weighted least-squares sampling points. By a QR factorization of the corresponding weighted matrix, with Q (rectangular) orthogonal and R square upper-triangular, namely

$$\operatorname{diag}(\sqrt{W})\,V_n(\Xi)=QR\,,$$

a discrete orthonormal basis in $\ell^2_W(\Xi)$ is simply

$$[\pi_1(x),\dots,\pi_N(x)]=[p_1(x),\dots,p_N(x)]R^{-1}\,.$$

Then, using a matrix formulation and denoting by $A_n^m=\{a_i\}$ the nodes of the polynomial mesh and by $V_n(A_n^m)=[p_k(a_i)]_{ik}$ the corresponding Vandermonde-like matrix, in view of (2.5)–(2.6) and observing that we have to perform a matrix column scaling by the least-squares weights, we can write

$$\begin{aligned}
[\varphi_j(a_i)]_{ij}&=[w_jK_n(a_i,\xi_j)]_{ij}=\left[w_j\sum_k\pi_k(a_i)\pi_k(\xi_j)\right]_{ij}\\
&=[\pi_k(a_i)]_{ik}\,[\pi_k(\xi_j)]^t_{jk}\,\operatorname{diag}(W)=V_n(A_n^m)R^{-1}(V_n(\Xi)R^{-1})^t\operatorname{diag}(W)\\
&=V_n(A_n^m)R^{-1}(\operatorname{diag}(\sqrt{W})V_n(\Xi)R^{-1})^t\operatorname{diag}(\sqrt{W})\\
&=V_n(A_n^m)R^{-1}Q^t\operatorname{diag}(\sqrt{W})\,,
\end{aligned}$$

from which we get

$$\|\lambda_n\|_{A_n^m}=\max_i\sum_j|\varphi_j(a_i)|=\|V_n(A_n^m)R^{-1}Q^t\operatorname{diag}(\sqrt{W})\|_\infty\,. \tag{3.1}$$

We recall again that the present formulation includes interpolation, where we simply have $\operatorname{card}(\{\xi_j\})=N=\dim(\mathbb{P}_n^d)$ and unit weights.

Some computational observations are in order. Despite the use of some orthogonal polynomial basis instead of the standard monomial basis, by increasing n, the matrix $V_n(\Xi)$ can become more and more ill-conditioned and such ill-conditioning is inherited by the

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triangular factor R. To give an idea of the conditioning with different polynomial bases, we report in Figure 3.1 the condition number of Vandermonde-like matrices on a given interpolation set of a 2-simplex (Waldron points) and a disk (approximate Lebesgue points); cf., [11, 37]. In this respect, in our implementation, a direct inversion of R by the Matlab command `inv` in (3.1) is not adopted already at moderate degrees, whereas the command $/R$ is better suited to manage ill-conditioning, at least up to condition numbers with order around the reciprocal of machine precision.

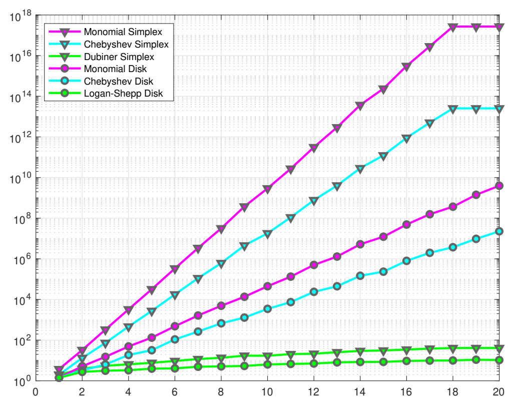

FIG. 3.1. *Conditioning of Vandermonde-like matrices with different polynomial bases on interpolation sets of a 2-simplex and a disk, for degree $n = 1, 2, \ldots, 20$ (the value stalling for the monomial basis at the highest degrees is due to the Matlab* `cond` *function).*

To have an idea of the approximation quality of the Lebesgue constant, we report here a table of the relative error estimates using the midpoint approximation; cf., (2.4), with c_m^d replacing c_m for the simplex and ball. Observe that, in order to get an error below 10%, that is, in order to recover the Lebesgue constant with one correct figure and thus computing accurately its order of magnitude (which is the relevant parameter in applications), one can take $m = 3$ for the d-cube, $m = 4$ in $d = 2$ for the simplex and disk, and $m = 5$ in $d = 3$ for the simplex and ball. In all cases the error is around 8%.

TABLE 3.1
Relative error estimates in the approximation of the Lebesgue constants on Chebyshev polynomial meshes A_n^m: d-cube (first row), 2-simplex and disk (second row), 3-simplex and ball (third row).

m	2	3	4	5	6	7	8	9	10
$c_m - 1$	11%	7.7%	4.1%	2.6%	1.8%	1.3%	0.98%	0.77 %	0.62%
$c_m^2 - 1$	50%	17%	8.6%	5.3%	3.6%	2.6%	2.0%	1.6%	1.3%
$c_m^3 - 1$	90%	27%	13%	8.1%	5.5%	4.0%	3.0%	2.4%	1.9%

We observe that the present fully discrete approach for Lebesgue constants evaluation, being based on product Chebyshev meshes of cardinality $\mathcal{O}((mn)^d)$, suffers from the curse of dimensionality and hence is essentially a low-dimensional tool. Considering, for example, the polynomial meshes corresponding to (transformed) grids of Chebyshev points, which are in the interior of the domains, the cardinalities are exactly $(mn)^d$ for the d-cube and the d-simplex and $2^{d-1}(mn)^d$ for the d-ball. To have an idea of the sizes, we display in Figure 3.2

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the values of the cardinality corresponding to the choices of m suggested above for a range of polynomial degrees in dimension $d = 2$ and $d = 3$.

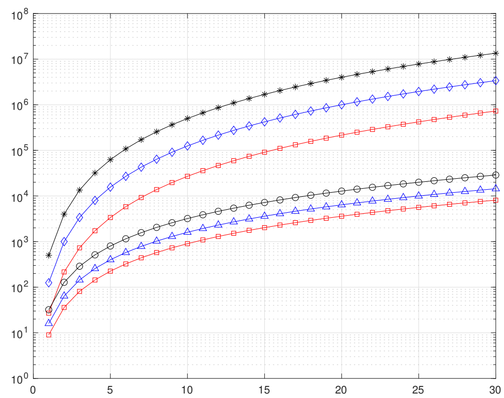

FIG. 3.2. *Cardinalities (log scale) of some Chebyshev polynomial meshes in dimension $d = 2, 3$ for degree $n = 1, 2, \ldots, 30$ (the Lebesgue constant is approximated at less than 10%, cf. Table 3.1): $(mn)^d$ with $m = 3$ (d-cube, $\Box$), $m = 4$ (2-simplex, $\triangle$), and $m = 5$ (3-simplex, $\Diamond$); $2^{d-1}(mn)^d$ with $m = 4$ (disk, o) and $m = 5$ (ball, $*$).*

Below we show a number of numerical tests in dimension 1, 2, and 3. The corresponding numerical codes and demos, implemented in Matlab, are freely available at [32].

3.2. Univariate interpolation points. In the univariate case, the Lebesgue constants for the interpolation on intervals have been extensively studied, with a number of theoretical results and estimates; cf., e.g., [14, 36] and the references therein. In order to test our method, we compare here the computed Lebesgue constants of Chebyshev points, Gauss–Legendre–Lobatto points (which are known to be Fekete points, i.e., points that maximize the absolute value of the Vandermonde determinant), and Gauss–Legendre points. The Lebesgue constant of the first two is known to be $\mathcal{O}(\log(n))$, whereas the third is $\mathcal{O}(\sqrt{n})$. Moreover, we also compute the Lebesgue constant of equally spaced points, which is known to grow exponentially; cf. [38]. The results are collected in Figure 3.3.

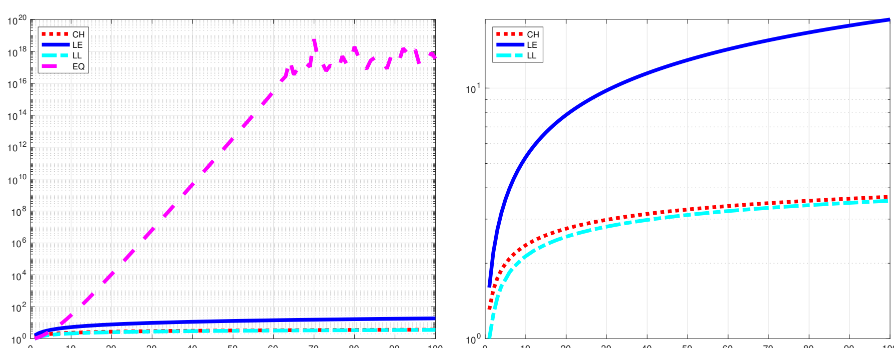

FIG. 3.3. *Left: Lebesgue constants of some point sets on the interval $[-1, 1]$ for degrees $n = 1, \ldots, 100$: Chebyshev (red dots), Legendre (blue line), Legendre–Lobatto (cyan dashes), and equispaced points (magenta dashes). Right: the three lowest curves in detail. In these experiments, $m = 3$ with a relative error $\approx 7.7\%$.*

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3.3. Comparing interpolation points on the square. We compare here the computed Lebesgue constants of some well-known families of interpolation points on the square. The results are collected in Figure 3.4.

The Padua points on the square, discovered in 2005 [15], are the union of two suitable Chebyshev subgrids. They are the first and, until now, the only explicitly known optimal point set for total-degree multivariate polynomial interpolation. For such points it has indeed been proved that the Lebesgue constant is $\mathcal{O}(\log^2(n))$; cf. [7].

The Morrow–Patterson points support one of the few known minimal positive cubature formulas, namely a formula with N nodes that has degree of exactness $2n$ for the product Chebyshev measure of the second kind; cf. [39]. Hence, the hyperinterpolation polynomial of degree not greater than n at these points, in view of minimality, turns out to be exactly the interpolation polynomial by [47, Lemma 3]. For such points it is proved that the Lebesgue constant is $\mathcal{O}(n^3)$, and it is conjectured that the actual growth is $\mathcal{O}(n^2)$; cf. [21].

For the purpose of comparison we also compute the Lebesgue constant of N Halton points. In view of a recent result in [19], N uniformly distributed (random) points are almost surely unisolvent, meaning that the probability of $\det(V_n(\Xi)) = 0$ is null. However, we expect that the Lebesgue constant has exponential growth, as observed numerically. A theoretical explanation is that a subexponential growth would imply weak-* convergence of the uniform discrete probability measure supported at the interpolation points to the potential theoretic equilibrium measure of the compact set (that in this case is the product Chebyshev measure); cf. [4]. On the contrary, with uniform random and Halton points, there is weak-* convergence to the Lebesgue measure (a fact that is at the base of Monte Carlo and Quasi-Monte Carlo integration). The exponential growth is clearly visible in Figure 3.4 (left) and in all the figures with uniform or Halton points as a trend up to some oscillations (in Figure 3.3 (left) where high degrees are considered, the behavior becomes numerically erratic when the Lebesgue constant goes beyond a size around 10^{17}).

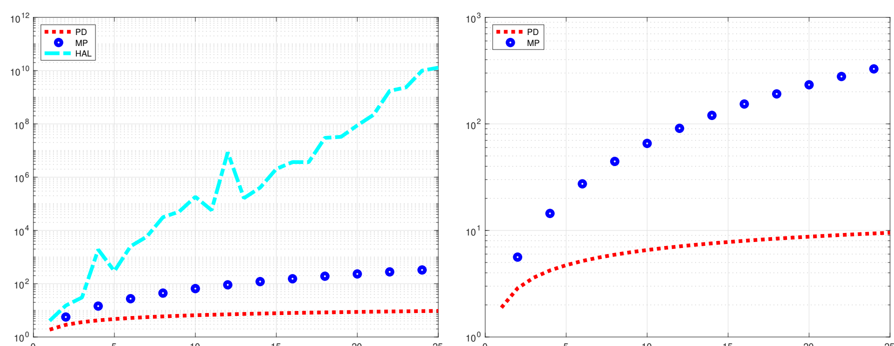

FIG. 3.4. *Left: Lebesgue constants of some total-degree interpolation point sets on the square* $[-1,1]^2$ *for degrees* $n = 1, \ldots, 25$*: Padua points (red dots), Morrow–Patterson (blue dots), Halton points (cyan dashes). Right: the two lowest curves in detail. In these experiments,* $m = 3$ *with a relative error* $\approx 7.7\%$.

3.4. Waldron points on the simplex. Good total-degree interpolation points on the simplex are relevant in the numerical solution of PDEs by spectral and high-order methods and have been extensively investigated for this reason—in most cases numerically; cf., e.g., [59] and the references therein.

Quite recently, a promising theoretical approach has been proposed constructing the so-called Waldron points that are obtained by looking for an appropriate spacing with respect to a

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distance related to the equilibrium measure of the domain (such as the Baran distance; cf. [11]). In Figure 3.5 we compare the Lebesgue constant of the Waldron points for the 2-simplex with that of the so-called Simplex Points (SIMP), a triangular grid corresponding to equally spaced points in the Euclidean distance on the equilateral triangle, and two families of points corresponding to a greedy minimization of the Lebesgue constant, namely the Approximate Lebesgue Points (ALP) and the Symmetric Approximate Lebesgue Points (SALP). The SALP are useful in the framework of spectral element methods for PDEs; cf. [13, 44].

We see that, as expected, with the Simplex Points there is an exponential growth. On the other hand, the Lebesgue constant of ALP and SALP are slowly increasing, whereas that of the Waldron points increase slowly up to about degree $n = 10$ and then turns to a manifestly exponential growth (though slower than with the Simplex Points).

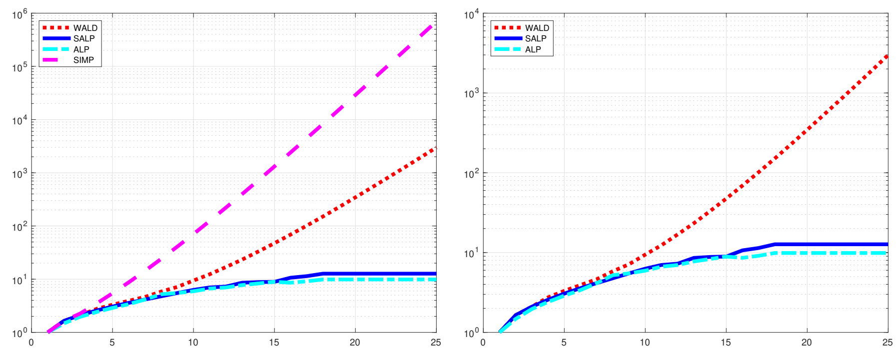

FIG. 3.5. *Left: Lebesgue constants of some point sets on the unit simplex with vertices* $(0,0)$, $(1,0)$, $(0,1)$, *for degrees* $n = 1, \ldots, 25$*: ALP (cyan dashes), SALP (blue line), Waldron points (red dots), simplex points (magenta dashes). Right: the three lowest curves in detail. In these experiments,* $m = 4$ *with a relative error* $\approx 8.6\%$.

3.5. Approximate Fekete and Leja points. The cases where good interpolation sets are known analytically are very few. Already in dimension $d = 2$, to our knowledge there is no known explicit family for the disk, and in $d = 3$ the same can be said for cube and ball.

On the other hand, interpolation points with slowly increasing Lebesgue constant can be determined numerically. Examples include the Approximate Fekete Points (AFP) and the Discrete Leja Points (DLP), both corresponding to a greedy maximization of the Vandermonde determinant modulus, typically extracting such points from polynomial meshes by numerical linear algebra algorithms; cf., e.g., [8, 9]. Moreover, for example in [13, 37], Approximate Lebesgue Points (ALP) have been computed once and for all on the square, simplex, and disk for restricted-degree ranges, working heuristically just with polynomial meshes and suitable greedy algorithms. Other point sets with low Lebesgue constant can be computed by the optimization algorithm proposed in [52].

For the purpose of illustration, in Figures 3.6–3.8, we plot the computed Lebesgue constants for interpolation at AFP, DLP, and Halton points on the disk, cube, and ball. The AFP and DLP have been computed on the same polynomial meshes A_n^m used for the Lebesgue constant evaluation. On the disk we also consider the Carnicer–Godes interpolation points [17] and the Approximate Lebesgue Points (ALP) computed in [37].

3.6. Standard and weighted least-squares operators. As observed in Remark 2.4, discrete least-squares operators, one of the very basic tools of computational mathematics in both the standard (equally weighted) and the weighted case, fall into the class of projectors where we can evaluate the Lebesgue constant, i.e., their uniform norm, by polynomial meshes.

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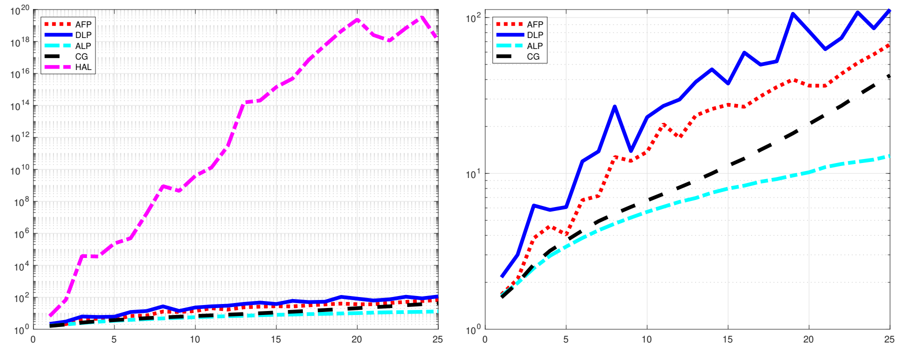

FIG. 3.6. *Left: Lebesgue constants of some point sets on the unit disk B_2, for degrees $n = 1, \ldots, 25$: AFP (red dots), DLP (blue dots), ALP (cyan dashes), Carnicer–Godes (black dashes), Halton points (magenta dashes). Right: the four lowest curves in detail. In these experiments, $m = 4$ with a relative error $\approx 8.6\%$.*

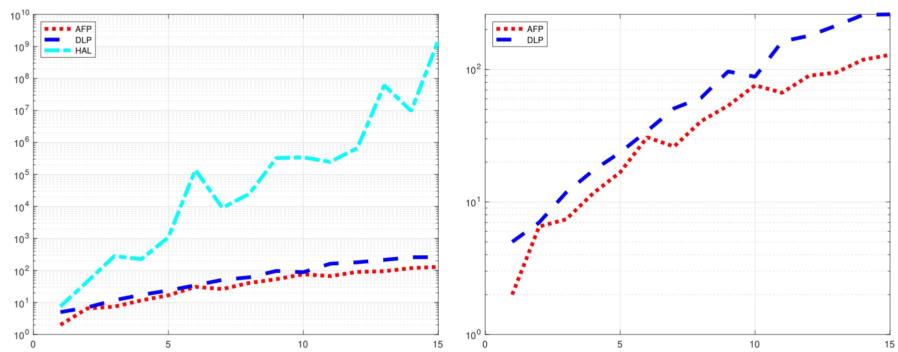

FIG. 3.7. *Left: Lebesgue constants of some point sets on the cube $[-1, 1]^3$, for degrees $n = 1, \ldots, 15$: AFP (red dots), DLP (blue dashes), Halton points (cyan dashes). Right: the two lowest curves in detail. In these experiments, $m = 3$ with a relative error $\approx 7.7\%$.*

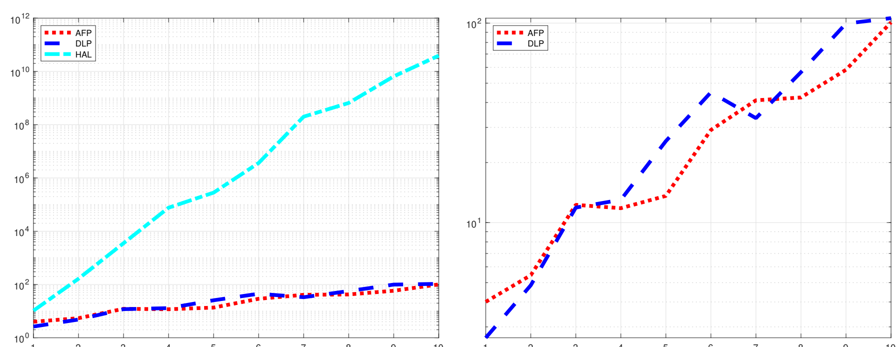

FIG. 3.8. *Left: Lebesgue constants of some point sets on the unit ball B_3, for degrees $n = 1, \ldots, 10$: AFP (red dots), DLP (blue dashes), Halton points (cyan dashes). Right: the two lowest curves in detail. In these experiments, $m = 5$ with a relative error $\approx 8.1\%$.*

It is worth recalling that a connection of the (standard) least-squares approach with polynomial meshes was already pointed out in [16], where it was shown that if the sampling set is a polynomial mesh on K, say $\Xi = A_n$ with constant c, then $\|L_n\| \leq c\sqrt{\mathrm{card}(A_n)}$.

This is however only a rough bound (as observed there), whereas estimating the actual size is important in applications.

Concerning weighted least-squares operators, we may also recall that they include, for example, hyperinterpolation operators (see Remark 2.4) as well as instances coming from the recent topic of "compression" of discrete measures. Roughly summarizing, given a discrete measure with large support, such a compression corresponds to extracting a subset of re-weighted points from the support such that the corresponding discrete measure keeps the same polynomial moments up to a given degree. This topic has been receiving an increasing attention in the literature over the last decade in both the probabilistic and the deterministic setting; cf., e.g., [12, 28, 29, 35, 41, 49, 51] and the references therein. In particular, for discrete least-squares approximations of degree n on a sampling set $\Xi = \{\xi_j\}$, moment matching has to be imposed up to degree $2n$, thus preserving orthogonal polynomials and reproducing kernels. This can be obtained by seeking a sparse nonnegative solution to the underdetermined moment-matching system $V_{2n}^t(\Xi)w = V_{2n}^t(\Xi)u$, where $V_{2n}(\Xi) = [p_k(\xi_j)]_{jk}$ is the corresponding Vandermonde-like matrix in a polynomial basis of $\mathbb{P}_{2n}$ and $u = (1, \dots, 1)^t$. Such a solution with no more than $N_{2n} = \dim(\mathbb{P}_{2n})$ nonzero components exists by the well-known Carathéodory theorem on conical combinations applied to the columns of the matrix and can be computed by solving the NonNegative Least-Squares (NNLS) problem

$$\min_{u \geq 0} \|V_{2n}^t(\Xi)w - V_{2n}^t(\Xi)u\|_2$$

via the Lawson–Hanson NNLS-solver [34] and its accelerated variants, such as that based on the recently developed "deviation maximization" criterion instead of column pivoting in the underlying QR factorizations; cf. [20, 23, 24, 25, 46]. The nonzero components of w then determine a compressed support $\Xi_n \subset \Xi$ with $N_n \leq \mathrm{card}(\Xi_n) \leq N_{2n}$, where the weighted least-squares polynomial can be computed; cf. [41].

For the purpose of illustration, in Figures 3.9–3.10 we compare on the square and disk the Lebesgue constants of hyperinterpolation and of polynomial least-squares approximation on Halton points and on Chebyshev polynomial meshes for the degrees $n = 1, \dots, 20$ together with their compressed versions. To give an idea of the compression ratios, those on the disk are reported in Table 3.2. We recall that in the case of hyperinterpolation it is theoretically known that the Lebesgue constant is $\mathcal{O}(n^2)$ for the square and $\mathcal{O}(n)$ for the disc with the Lebesgue measure and $\mathcal{O}(\log^2(n))$ for the square with the product Chebyshev measure; cf., e.g., [21, 56, 58].

It is numerically manifest that Lebesgue constants of full and compressed least-squares operators have substantially the same size with a remarkable reduction of the sampling cardinality for the latter. Moreover, the Lebesgue constant of the least-squares approximation on Chebyshev meshes and of hyperinterpolation with the product Chebyshev measure turn out to be very close; see Figure 3.9. This is not really surprising since the uniform discrete measure supported at univariate Chebyshev points is an algebraic quadrature formula for the (normalized) Chebyshev measure, and such a behavior extends to a product-like framework. Hence, standard discrete least-squares approximation on a Chebyshev mesh of the square is equivalent to hyperinterpolation with respect to the product Chebyshev measure, whose Lebesgue constant is expected to be $\mathcal{O}(\log^2(n))$ in view of [58]. The same can be said for the compressed least-squares approximation since this also corresponds to an algebraic quadrature with the same moments up to degree $2n$. Notice that the Lebesgue functions of standard and compressed least-squares approximation are not coincident, but both correspond to hyperinterpolation with respect to the product Chebyshev measure.

A similar argument applies in interpreting Figure 3.10 since standard discrete least-squares approximation on a polar Chebyshev mesh of the disk is equivalent to hyperinterpolation with

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respect to its equilibrium measure

$$\frac{dx_1\,dx_2}{\pi\sqrt{1-x_1^2-x_2^2}} = \frac{r\,dr\,d\theta}{\pi\sqrt{1-r^2}},$$

whose Lebesgue constant is expected to be $\mathcal{O}(\sqrt{n})$ in view of [56].

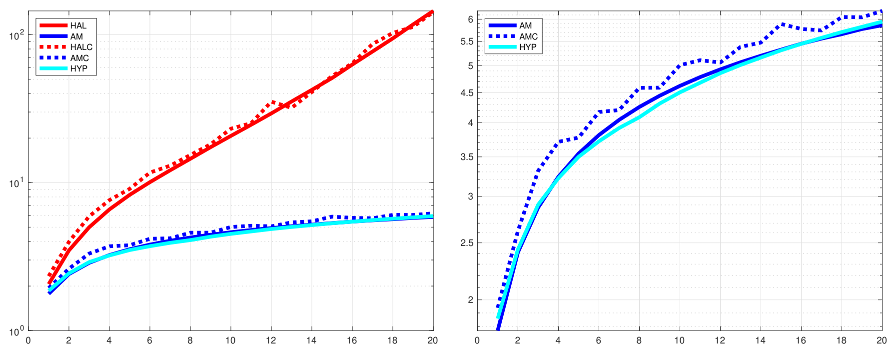

FIG. 3.9. *Left: Lebesgue constants of the least-squares operator on the square* $[-1,1]^2$*, for degrees* $n = 1,\ldots,20$*, on:* 10^4 *Halton points (HAL, red dots) and compressed version (HALC, red dashes); Chebyshev mesh* $A_{20}^5 = \mathcal{C}_{100} \times \mathcal{C}_{100}$ *with* 10^4 *points (blue line, AM20) and compressed version (AM20C, blue dashes); hyperinterpolation with product Gauss–Chebyshev quadrature (HYP, cyan line). Right: the three lowest curves in detail. In these experiments,* $m = 3$ *with a relative error* $\approx 7.7\%$.

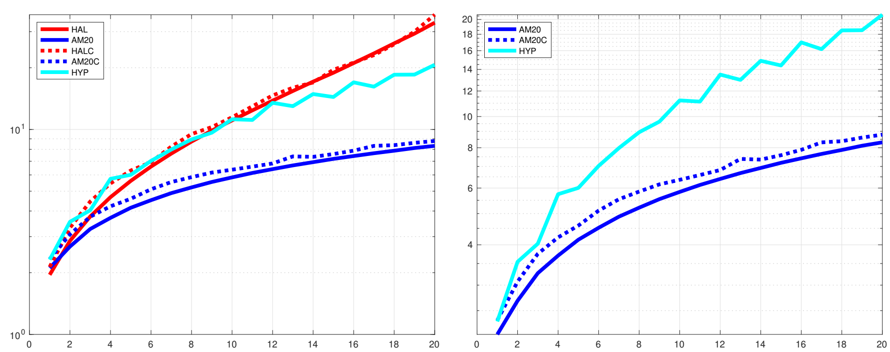

FIG. 3.10. *Left: Lebesgue constants of the least-squares operators on the unit disk* B_2*, for degrees* $n = 1,\ldots,20$*, on: 12800 Halton points (HAL, red dots) and compressed version (HALC, red dashes); Chebyshev mesh* $A_{20}^4 = \mathcal{S}(\mathcal{C}_{80} \times \mathcal{C}_{160})$ *with 12800 points (cf. (2.8)) (AM20, blue line) and compressed version (AM20C, blue dashes); hyperinterpolation via low cardinality rules for the Lebesgue measure (HYP, cyan line). Right: the three lowest curves in detail. In these experiments,* $m = 4$ *with a relative error* $\approx 8.6\%$.

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TABLE 3.2
Cardinalities and sampling compression ratios for compressed polynomial LS operators of degree n on 12800 points of the disk.

n	2	4	6	8	10	12	14	16	18	20
card $= N_{2n}$	15	45	91	153	231	325	435	561	703	861
cmp ratio	853	284	141	84	55	40	29	23	18	15

ANA&A (A. Sommariva and M. Vianello) and the UMI Group TAA "Approximation Theory and Applications" (A. Sommariva). The work was partially supported by the National Science Centre, Poland, grant No. 2017/25/B/ST1/00906 (L. Białas-Cież).

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