

FULLY ALGEBRAIC DOMAIN DECOMPOSITION PRECONDITIONERS WITH ADAPTIVE SPECTRAL BOUNDS*

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Abstract. In this article a new family of preconditioners is introduced for symmetric positive definite linear systems. The new preconditioners, called the AWG preconditioners (for Algebraic-Woodbury-GenEO), are constructed algebraically. By this we mean that only the knowledge of the matrix \mathbf{A} for which the linear system is being solved is required. Thanks to the GenEO spectral coarse space technique, the condition number of the preconditioned operator is bounded theoretically from above. This upper bound can be made smaller by enriching the coarse space with more spectral modes.

The novelty is that, unlike in previous work on the GenEO coarse spaces, no knowledge of a partially non-assembled form of \mathbf{A} is required. Indeed, the spectral coarse space technique is not applied directly to \mathbf{A} but to a low-rank modification of \mathbf{A} of which a suitable non-assembled form is known by construction. The extra cost is a second (and to this day rather expensive) coarse solve in the preconditioner. One of the AWG preconditioners has already been presented in a short preprint by Spillane [Domain Decomposition Methods in Science and Engineering XXVI, Springer, Cham, 2022, pp. 745–752]. This article is the first full presentation of the larger family of AWG preconditioners. It includes proofs of the spectral bounds as well as numerical illustrations.

Key words. preconditioner, domain decomposition, coarse space, algebraic, linear system, Woodbury matrix identity

AMS subject classifications. 65F10, 65N30, 65N55

1. Introduction. Throughout this article we consider the problem of finding $\mathbf{x}_* \in \mathbb{R}^n$ that is the solution of the following linear system:

$$(1.1) \quad \mathbf{A}\mathbf{x}_* = \mathbf{b}, \quad \text{where } \mathbf{A} \in \mathbb{R}^{n \times n} \text{ is symmetric positive definite (spd),}$$

for a given right-hand side $\mathbf{b} \in \mathbb{R}^n$.

The applications we have in mind are ones for which \mathbf{A} is sparse and the number n of unknowns is very large. Hence, we study parallel solvers and, more specifically, preconditioners for the preconditioned conjugate gradient (PCG) method [44, Section 9.2]. Our objective is to propose a new preconditioner \mathbf{H} for solving (1.1) such that the condition number of $\mathbf{H}\mathbf{A}$ is bounded from above by a *small enough* constant chosen by the user. This guarantees that the PCG method converges in *sufficiently few* iterations [51, Lemma C.10]. Two-level domain decomposition preconditioners satisfying such a nice property already exist, specifically, the spectral coarse space methods described below. These methods, however, rely on the knowledge of some partially unassembled form of the matrix \mathbf{A} . The additional challenge tackled in this work is that the new preconditioner must rely only on the knowledge of the matrix \mathbf{A} . This is the meaning of the word *algebraic*.

Very generally speaking, domain decomposition methods partition the domain Ω in which the solution is sought into smaller spaces Ω^s , indexed by $s \in \llbracket 1, N \rrbracket := \{1, \dots, N\}$, and characterized by prolongation matrices $\mathbf{R}^{s\top}$ that satisfy $\sum_{s=1}^N \text{range}(\mathbf{R}^{s\top}) = \Omega$. One-level domain decomposition preconditioners then approximate \mathbf{A}^{-1} by a sum (interpolated by the $\mathbf{R}^{s\top}$) of inverses of well-chosen problems \mathbf{A}^s . Two-level domain decomposition methods have an additional space called the coarse space generated by the columns of a matrix $\mathbf{R}^{0\top}$. A coarse problem \mathbf{A}^0 is solved in the coarse space. As an example, an application of the

*Received June 18, 2021. Accepted November 26, 2023. Published online on April 5, 2024. Recommended by Johannes Kraus.

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two-level additive preconditioner to a vector $\mathbf{x} \in \mathbb{R}^n$ takes the form: $\sum_{s=0}^N \mathbf{y}^s$, where

$$\mathbf{y}^s = \mathbf{R}^{s\top} (\tilde{\mathbf{A}}^s)^{-1} \mathbf{R}^s \mathbf{x} \quad \text{and} \quad \mathbf{y}^0 = \mathbf{R}^{0\top} (\tilde{\mathbf{A}}^0)^{-1} \mathbf{R}^0 \mathbf{x}.$$

\uparrow prolongation (by 0) to Ω \uparrow local solve \uparrow restriction to Ω^s \uparrow coarse interpolation \uparrow coarse solve \uparrow interpolation back into Ω

The notation in the previous line likely seems natural to readers familiar with domain decomposition. The coarse contribution \mathbf{y}^0 is written separately to insist on the fact that it plays a different role to the other \mathbf{y}^s . Usually, the coarse space is computed for a given choice of the \mathbf{R}^s and $\tilde{\mathbf{A}}^s$.

The choice of the coarse space is a very crucial topic in domain decomposition. Over the last decade, the range of symmetric positive definite problems for which two-level domain decomposition preconditioners can be made scalable and robust by a good choice of the coarse space has become very large by the development of so-called spectral coarse spaces. The following list gives an overview of some of these contributions: [1, 18, 19, 20, 21, 22, 26, 37, 38, 48, 49] for Additive Schwarz, [55] for Additive Average Schwarz, [1, 50] for BDD and FETI, [10, 15, 29, 30, 31, 32, 41, 56] for BDDC and/or FETI-DP, [25] for Optimized Schwarz, and [35] in the context of boundary element methods.

In this article it is particularly referred to the GenEO coarse spaces [48, 49, 50], to which one of the authors contributed. The abstract theory of coarse spaces of the GenEO family in [46] is applied within the definition and analysis of the new Algebraic-Woodbury-GenEO (AWG) preconditioners.

The spectral coarse spaces mentioned above, and in particular the GenEO coarse spaces, are computed by partially solving one or two generalized eigenvalue problems in each subdomain, then selecting either the lowest- or highest-frequency eigenvectors, and prolongating them to the global domain. Until very recently, all of the spectral coarse spaces for which there are no assumptions on the shapes of subdomains and the distribution of material parameters require the knowledge of a set of symmetric positive semi-definite (spsd) matrices \mathbf{N}^s that satisfy

$$(1.2) \quad \exists C > 0 \text{ such that } \sum_{s=1}^N \langle \mathbf{x}, \mathbf{R}^{s\top} \mathbf{N}^s \mathbf{R}^s \mathbf{x} \rangle \leq C \langle \mathbf{x}, \mathbf{A} \mathbf{x} \rangle \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

The matrices \mathbf{N}^s enter directly into the coarse space construction *via* the choice of a matrix pencil for the generalized eigenvalue problems. In other words they play an essential role.

For matrices arising from discretized PDEs, the above assumption is natural. The matrices \mathbf{N}^s are not expensive to compute as long as it is known at assembly that they are required. Indeed, as an example, if \mathbf{A} arises from the finite element discretization (with basis functions $\{\phi_k\}_{k=1,\dots,n}$) of the Laplace equation over some domain $\omega \subset \mathbb{R}^2$ or 3 , then the coefficients in \mathbf{A} are $\int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j$. Assuming that the degrees of freedom selected by the restriction matrix \mathbf{R}^s are those in some $\omega^s \subset \omega$, \mathbf{N}^s can be taken to be the matrix whose coefficients are $\int_{\omega^s} \nabla \phi_i \cdot \nabla \phi_j$, for the basis functions ϕ_i and ϕ_j whose support intersects ω^s . This is how condition (1.2) is usually satisfied. These \mathbf{N}^s are sometimes called the local Neumann matrices. Partial assembly over subdomains is neither conceptually hard nor computationally expensive, and the purpose of this article is not to rule it out when it is possible. There are, however, many cases where only \mathbf{A} is known or available without changes to the code. Then, the unassembled information is simply lost, and the GenEO coarse spaces cannot be computed. This is quite a common scenario: the problem may have been assembled by another user or with another piece of software. In this case only black box algorithms can be used.

Direct solvers, like MUMPS [5, 6], belong to the category of black box solvers, and they are the most efficient ones up to a certain problem size. In the field of domain decomposition,

Li and Saad propose an algebraic preconditioner under the name DD-LR, for Domain Decomposition based Low-Rank [34]. The original matrix is rewritten in a particular form inspired by domain decomposition. The inverse of \mathbf{A} can then be expressed in terms of the components in that formulation thanks to the Woodbury matrix identity. Finally, a low-rank approximation of one of the terms is performed in order to get an approximation of \mathbf{A}^{-1} that can serve as a preconditioner. Our AWG preconditioners also exploit the Woodbury matrix identity, but the modification of \mathbf{A} that it is applied to is entirely different. Since this manuscript was first submitted, Daas et al. proposed an algebraic coarse space for sparse normal equations, i.e., of the form $\mathbf{M}^\top \mathbf{M}$, by defining a particular algebraic splitting in the sense of (1.2) [4]. The splitting is then adapted to general sparse matrices by essentially taking the square root of the local matrices in the splitting [3]. The construction is rather expensive, and the size of the coarse space can become significantly larger than that of the (classical, non algebraic) GenEO method. For a scalar elliptic problem, a different algebraic coarse space is presented in [27] for which the two-level overlapping Schwarz method is robust. Generalized eigenvalue problems are solved on the interfaces between subdomains.

Also closely related to domain decomposition are the multigrid algorithms, a very well-established set of solvers that approximate the solution by iterating over the original (fine) problem as well as coarser and coarser approximations of it. The original multigrid algorithm [11, 14] is often referred to as geometric multigrid as it requires a predetermined hierarchy of grids and discretizations. The objective of Algebraic Multigrid (AMG) is to make multigrid applicable in cases where less information about the problem is accessible or some assumptions are not satisfied; see [13, 43] for the original contributions or [54] for a unified presentation and theory of many multigrid algorithms. There have been many developments in algebraic multigrid that aim at robustness with respect to the problem parameters, geometry, heterogeneous coefficients, etc. In smoothed aggregation AMG [52], the kernel or near-kernel of the problem is preserved throughout the levels, and this is how the coarse basis vectors are chosen. Geometric information can be needed to describe the near-kernel vectors, e.g., the rigid-body modes if the problem arises from linear elasticity. Spectral AMGe [16], where the e stands for elements, bares strong connections with the spectral coarse spaces in domain decomposition and precedes them. In spectral AMGe, the coarse vectors are low-frequency eigenvectors of the agglomerate stiffness matrices. An assumption of the form (1.2) is required to define these agglomerate stiffness matrices. Auxiliary space multigrid [33] is based on exact two-by-two block factorizations of local stiffness matrices. The coarse-grid correction step, as used in classical multigrid methods, however is replaced by a correction that involves the application of an auxiliary space preconditioner, a technique that arises also in domain decomposition. Aggregation-based algebraic multigrid (AGMG) [39] is truly algebraic in the sense that only \mathbf{A} is given to the solver. AGMG then uses a strength function to define aggregates for which the local convergence rate on each aggregate is bounded by a given threshold. The analysis in [36] requires only an appropriate splitting of the system matrix \mathbf{A} that can be constructed in a systematic way when the matrix is diagonally dominant. Another fully algebraic multigrid approach is the Bootstrap AMG method [12]. Bootstrap AMG performs some steps of the algorithm, detects the algebraically smooth parts of the error, and self-improves.

Our ambition is to propose an algorithm as easy to apply as the Algebraic Multigrid algorithms. In this article, a new family of preconditioners, called AWG, is proposed and analyzed for the cases where \mathbf{A} is already assembled. There is no assumption other than \mathbf{A} is spd. AWG is a domain decomposition preconditioner with two coarse spaces. The methodology is the following:

- The problem matrix \mathbf{A} is split into symmetric, but possibly indefinite, matrices \mathbf{B}^s as $\mathbf{A} = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{B}^s \mathbf{R}^s$.
- The positive parts \mathbf{A}_+^s of the matrices \mathbf{B}^s are computed and assembled to form a global matrix $\mathbf{A}_+ = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{A}_+^s \mathbf{R}^s$. By construction, a splitting of \mathbf{A}_+ into spsd matrices, i.e., a suitable partially unassembled form of \mathbf{A}_+ , is known. In other words, with $\mathbf{N}^s = \mathbf{A}_+^s$ and $C = 1$, (1.2) is satisfied. Consequently, two-level preconditioners with GenEO coarse spaces can be computed for \mathbf{A}_+ by applying the abstract theory in [46].
- Finally, the Woodbury matrix identity relates the inverses of \mathbf{A} and of \mathbf{A}_+ and makes apparent that a good preconditioner for \mathbf{A} can be obtained by adding a second coarse space to a GenEO preconditioner for \mathbf{A}_+ .

A full theory for the condition number of the new preconditioned operators is given as well as numerical illustrations. The outline of the remainder of this article is as follows. In Section 2, some elements of the Abstract Schwarz theory [51] are recalled in their fully algebraic form. For readers less familiar with domain decomposition, the general form of a one-level and a two-level domain decomposition preconditioner is given. In Section 3, the new operator \mathbf{A}_+ is introduced, and four preconditioners with their GenEO coarse spaces are considered for \mathbf{A}_+ . For each one, the spectral bounds are given by applying a result from [46]. Then, in Section 4, \mathbf{A} is viewed as a modification of \mathbf{A}_+ , and the Woodbury matrix identity is applied. This makes apparent how to add a second coarse space to the preconditioners for \mathbf{A}_+ in order to get a preconditioner for \mathbf{A} that satisfies nice convergence bounds. Each of these new preconditioners is indexed by one or two parameters (or thresholds) that can be adjusted to decrease the condition number of the preconditioned operator by enriching the GenEO coarse space with more spectral modes. Some comments are also made about the implementation of the new preconditioners. Finally, Section 5 presents some numerical results with the objective of confirming the theoretical results and illustrating the practical behavior of the new AWG preconditioners.

2. Abstract Schwarz framework in the algebraic setting. An algebraic version of the abstract Schwarz framework is introduced in this section. This means that all the domain decomposition-type operators are written only in terms of vectors in \mathbb{R}^n .

2.1. Subdomains. Let $\Omega = \llbracket 1, n \rrbracket$ be the set of all indices in \mathbb{R}^n .

DEFINITION 2.1 (Partition of Ω). *A set $(\Omega^s)_{s=1, \dots, N}$ of $N \in \mathbb{N}$ subsets of $\Omega = \llbracket 1, n \rrbracket$ is called a partition of Ω if*

$$\Omega = \bigcup_{s=1}^N \Omega^s.$$

The partition is said to have at least minimal overlap if Assumption 1 is satisfied.

ASSUMPTION 1 (Minimal overlap). *For any pair of indices $(i, j) \in \llbracket 1, n \rrbracket^2$, denoting by A_{ij} the coefficient of \mathbf{A} at the i -th line and j -th column,*

$$A_{ij} \neq 0 \Rightarrow (\exists s \in \llbracket 1, N \rrbracket \text{ such that } \{i, j\} \subset \Omega^s).$$

The usual global-to-local restriction matrices are defined next.

DEFINITION 2.2. *For each $s = 1, \dots, N$, let n^s be the cardinality of Ω^s . Then, let $\mathbf{R}^s \in \mathbb{R}^{n^s \times n}$ be the restriction matrix defined by the following condition: \mathbf{R}^s is zero everywhere except for the block formed by the columns in Ω^s , which is the $n^s \times n^s$ identity matrix.*

By simply performing the multiplications it can be proved that

$$\mathbf{R}^{s\top} \mathbf{R}^s \text{ is diagonal and that } \mathbf{R}^s \mathbf{R}^{s\top} = \mathbf{I}, \text{ the identity matrix in } \mathbb{R}^{n^s}.$$

2.2. Partition of unity, coloring constant. In the construction and analysis of the preconditioners, two more elements from the abstract Schwarz theory are needed: the partition of unity matrices and the coloring constants.

ASSUMPTION 2 (Partition of unity matrices). *Let $\{\mathbf{D}^s \in \mathbb{R}^{n^s \times n^s}; s = 1, \dots, N\}$ be a family of matrices that satisfies*

$$\mathbf{I} = \sum_{s=1}^N \mathbf{R}^s \mathbf{D}^s \mathbf{R}^s, \quad \text{with } \mathbf{I} \text{ the } n \times n \text{ identity matrix.}$$

One way of fulfilling Assumption 2 is to choose the following partition of unity matrices:

DEFINITION 2.3 (Possible choice of partition of unity \mathbf{D}^s). *First, let $\mathbf{D} \in \mathbb{R}^{n \times n}$ be the non-singular diagonal matrix defined by*

$$\mathbf{D} := \left(\sum_{t=1}^N \mathbf{R}^t \mathbf{R}^{t\top} \right)^{-1}.$$

Then, for each $s \in \llbracket 1, N \rrbracket$, let $\mathbf{D}^s \in \mathbb{R}^{n^s \times n^s}$ be defined by

$$\mathbf{D}^s := \mathbf{R}^s \mathbf{D} \mathbf{R}^{s\top}.$$

The matrices in the above definition satisfy Assumption 2; see [46, Lemma 4]. Their coefficients are the inverses of the multiplicity of each degree of freedom. Next, the coloring constant is defined in agreement with [51, Section 2.5.1]. The dependency of the coloring constant on the matrix with respect to which orthogonality is taken is written explicitly.

DEFINITION 2.4 (Coloring constant). *Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Let $\mathcal{N}(\mathbf{M}) \in \mathbb{N}$ be such that there exists a set $\{\mathcal{C}_j; 1 \leq j \leq \mathcal{N}(\mathbf{M})\}$ of pairwise disjoint subsets of $\llbracket 1, N \rrbracket$ satisfying*

$$\begin{aligned} \llbracket 1, N \rrbracket &= \bigcup_{1 \leq j \leq \mathcal{N}(\mathbf{M})} \mathcal{C}_j \quad \text{and} \\ \forall j \in \llbracket 1, \mathcal{N}(\mathbf{M}) \rrbracket : \{s, t\} \subset \mathcal{C}_j &\implies (\mathbf{R}^s \mathbf{M} \mathbf{R}^{t\top} = \mathbf{0} \text{ or } s = t). \end{aligned}$$

2.3. Abstract Schwarz preconditioners. One-level abstract Schwarz preconditioners are of the form

$$(2.1) \quad \mathbf{H} := \sum_{s=1}^N \mathbf{R}^s \mathbf{A}^s \mathbf{R}^s,$$

where for each $s = 1, \dots, N$, it is assumed that $\tilde{\mathbf{A}}^s \in \mathbb{R}^{n^s \times n^s}$ is an spsd matrix and where $\tilde{\mathbf{A}}^{s\dagger}$ is the pseudo-inverse of $\tilde{\mathbf{A}}^s$.

Two-level domain decomposition preconditioners have two extra ingredients compared to the one-level method that they are based on: a coarse space and a coarse solver. Let's assume that the coarse space is denoted by V^0 and that the interpolation operator \mathbf{R}^0 satisfies Assumption 3.

ASSUMPTION 3. *A basis for the coarse space V^0 is stored in the rows of the matrix \mathbf{R}^0 :*

$$V^0 = \text{range}(\mathbf{R}^0), \quad \mathbf{R}^0 \in \mathbb{R}^{n^0 \times n}, \quad n^0 = \dim(V^0), \quad n^0 < n.$$

The most common choice for the coarse solver, and the one that we wish to introduce, is the so called *exact* solver, where the word *exact* is with respect to the problem being solved. If

the matrix in the linear system is an spd matrix $\tilde{\mathbf{A}}$, then the matrix that is inverted during the coarse solve is $\mathbf{R}^0 \tilde{\mathbf{A}} \mathbf{R}^{0\top}$.

Even with the same $\tilde{\mathbf{A}}^s$ and \mathbf{R}^0 , there are still at least two two-level preconditioners with exact coarse spaces: the two-level additive preconditioner, denoted by \mathbf{H}_{ad} , and the hybrid preconditioner, denoted by \mathbf{H}_{hyb} and also called the deflated preconditioner. They are defined as follows:

$$\mathbf{H}_{\text{ad}} := \mathbf{H} + \mathbf{R}^{0\top} (\mathbf{R}^0 \tilde{\mathbf{A}} \mathbf{R}^{0\top})^{-1} \mathbf{R}^0$$

and

$$\mathbf{H}_{\text{hyb}} := \mathbf{\Pi} \mathbf{H} \mathbf{\Pi}^\top + \mathbf{R}^{0\top} (\mathbf{R}^0 \tilde{\mathbf{A}} \mathbf{R}^{0\top})^{-1} \mathbf{R}^0,$$

where

$$\mathbf{\Pi} := \mathbf{I} - \mathbf{R}^{0\top} (\mathbf{R}^0 \tilde{\mathbf{A}} \mathbf{R}^{0\top})^{-1} \mathbf{R}^0 \tilde{\mathbf{A}}, \quad \mathbf{I} \text{ is the } n \times n \text{ identity matrix.}$$

The generic notation $\tilde{\mathbf{A}}$ has very deliberately been used in the previous equations instead of \mathbf{A} . Indeed, the two-level preconditioner in this article is a preconditioner for a new matrix denoted by \mathbf{A}_+ . The next section gives the definition of \mathbf{A}_+ and the choices of $\tilde{\mathbf{A}}^s$ and V^0 that make the characterization of an abstract two-level preconditioner for \mathbf{A}_+ complete.

3. A new matrix \mathbf{A}_+ and its two-level GenEO preconditioners. This section introduces new operators and notation. Its outline is as follows.

- In Section 3.1, Assumption (1.2) is relaxed by allowing the symmetric matrices in the splitting of \mathbf{A} to be indefinite. Precisely, Assumption 4 stipulates that symmetric matrices \mathbf{B}^s are known such that $\mathbf{A} = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{B}^s \mathbf{R}^s$. Definition 3.1 provides one possible (algebraic) choice of \mathbf{B}^s .
- In Section 3.2, each \mathbf{B}^s is first split into an spd part and a symmetric negative semi-definite part as $\mathbf{B}^s = \mathbf{A}_+^s - \mathbf{A}_-^s$. All the local spd parts are assembled to form the new matrix $\mathbf{A}_+ = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{A}_+^s \mathbf{R}^s$. It is easy to prove that \mathbf{A}_+ is symmetric positive definite.
- In Section 3.3, one-level preconditioners are defined for \mathbf{A}_+ . They are one-level abstract Schwarz preconditioners in the sense of [51].
- In Section 3.4, two-level preconditioners are defined for \mathbf{A}_+ . Bounds for their condition numbers are given by applying the abstract GenEO theory [46]. This is possible because the new matrix \mathbf{A}_+ satisfies (1.2) with $\mathbf{N}^s = \mathbf{A}_+^s$ and $C = 1$. This very useful characteristic of \mathbf{A}_+ is precisely what led to its definition.

3.1. A splitting of \mathbf{A} into symmetric matrices. The matrices \mathbf{B}^s in the assumption below are the starting point for the new preconditioners. To make the construction complete, an example of such matrices is given below. It is (of course) constructed algebraically and is the one used in our numerical computations. This choice is, however, far from unique.

ASSUMPTION 4. *Let's assume there is a family of symmetric matrices $\mathbf{B}^s \in \mathbb{R}^{n^s \times n^s}$ for $s \in \llbracket 1, N \rrbracket$ such that*

$$\mathbf{A} = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{B}^s \mathbf{R}^s.$$

Such a family of matrices can always be chosen under the minimal overlap Assumption 1. Indeed, one possible choice is given in the next definition.

DEFINITION 3.1 (Possible choice of matrices \mathbf{B}^s). Let $\mathbf{S}(\mathbf{A})$ be the $n \times n$ Boolean matrix with the same sparsity pattern as \mathbf{A} :

$$(S(\mathbf{A}))_{ij} := \begin{cases} 1 & \text{if } A_{ij} \neq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for any } i, j \in \llbracket 1, n \rrbracket.$$

Then let \mathbf{M}_μ be the matrix that counts the number of subdomains that each pair of indices in $\{\{i, j\}; A_{ij} \neq 0\}$ belongs to:¹

$$\mathbf{M}_\mu := \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{R}^s \mathbf{S}(\mathbf{A}) \mathbf{R}^{s\top} \mathbf{R}^s,$$

and let \mathbf{B} be the Hadamard division of \mathbf{A} by \mathbf{M}_μ

$$B_{ij} := \begin{cases} A_{ij}/M_{\mu ij} & \text{if } A_{ij} \neq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for any } i, j \in \llbracket 1, n \rrbracket.$$

Finally, set \mathbf{B}^s to be the block of \mathbf{B} corresponding to the degrees of freedom in Ω^s :

$$\mathbf{B}^s := \mathbf{R}^s \mathbf{B} \mathbf{R}^{s\top}.$$

Note that from the previous definition, only the notation \mathbf{B}^s will be reused further on in the article. We next verify that these matrices \mathbf{B}^s are indeed suitable.

THEOREM 3.2. Let \mathbf{A} be an order- n spd matrix, let $(\Omega^s)_{s=1, \dots, N}$ represent the partition into subdomains, and let $(\mathbf{R}^s)_{s=1, \dots, N}$ be the set of restriction matrices from Definition 2.2. Under Assumption 1, the matrices \mathbf{B}^s from Definition 3.1 satisfy Assumption 4.

Proof. First, we justify the fact that \mathbf{M}_μ counts the multiplicity of the pairs of degrees of freedom $\{i, j\}$ for which $A_{ij} \neq 0$:

$$(M_\mu)_{ij} = \left(\sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{R}^s \mathbf{S}(\mathbf{A}) \mathbf{R}^{s\top} \mathbf{R}^s \right)_{ij} = \sum_{\{s; \{i, j\} \subset \Omega^s\}} \left(\mathbf{R}^{s\top} \mathbf{R}^s \mathbf{S}(\mathbf{A}) \mathbf{R}^{s\top} \mathbf{R}^s \right)_{ij},$$

so, for any $i, j \in \llbracket 1, n \rrbracket$,

$$(M_\mu)_{ij} = \sum_{\{s; \{i, j\} \subset \Omega^s\}} (S(\mathbf{A}))_{ij} = \begin{cases} \#\{s; \{i, j\} \subset \Omega^s\} & \text{if } A_{ij} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

With a similar calculation, it can then be verified that the \mathbf{B}^s form a splitting of \mathbf{A} :

$$\left(\sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{B}^s \mathbf{R}^s \right)_{ij} = \left(\sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{R}^s \mathbf{B} \mathbf{R}^{s\top} \mathbf{R}^s \right)_{ij} = \sum_{\{s; \{i, j\} \subset \Omega^s\}} B_{ij},$$

so, for any $i, j \in \llbracket 1, n \rrbracket$,

$$\left(\sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{B}^s \mathbf{R}^s \right)_{ij} = \begin{cases} \sum_{\{s; \{i, j\} \subset \Omega^s\}} A_{ij}/M_{\mu ij} = A_{ij} & \text{if } A_{ij} \neq 0, \\ 0 = A_{ij} & \text{otherwise.} \end{cases}$$

It can be concluded that $\mathbf{A} = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{B}^s \mathbf{R}^s$. \square

¹This interpretation of \mathbf{M}_μ is justified in the proof of Theorem 3.2.

REMARK 3.3. Notice that, if the minimal overlap condition is not satisfied, then there exists a pair of indices $\{i, j\}$ for which $A_{ij} \neq 0$ but $\{s; \{i, j\} \subset \Omega^s\} = \emptyset$. This leads to $\left(\sum_{s=1}^N \mathbf{R}^s \mathbf{B}^s \mathbf{R}^s\right)_{ij} = 0$ and shows that it is impossible that Assumption 4 be satisfied without the minimal overlap condition no matter how the matrices \mathbf{B}^s are chosen.

3.2. Definition of \mathbf{A}_+ . The first step in defining the very important matrix \mathbf{A}_+ is to split \mathbf{B}^s into a positive part and a negative semi-definite part.

DEFINITION 3.4 (Splitting of \mathbf{B}^s). *Let \mathbf{B}^s , for $s \in \llbracket 1, N \rrbracket$, be a family of matrices that satisfy Assumption 4. For each s , let a diagonalization of \mathbf{B}^s be written as*

$$\mathbf{B}^s = \mathbf{V}^s \mathbf{\Lambda}^s \mathbf{V}^{s\top} \quad \text{with } \mathbf{V}^s \text{ orthogonal and } \mathbf{\Lambda}^s \text{ diagonal.}$$

Assume, without loss of generality, that the diagonal values of $\mathbf{\Lambda}^s$ (which are the eigenvalues of \mathbf{B}^s) are sorted in non-decreasing order. Let n_+^s be the number of positive eigenvalues and $n_-^s = n^s - n_+^s$ be the number of non-positive eigenvalues. Let $\mathbf{V}_-^s \in \mathbb{R}^{n^s \times n_-^s}$, $\mathbf{V}_+^s \in \mathbb{R}^{n^s \times n_+^s}$, $\mathbf{\Lambda}_-^s \in \mathbb{R}^{n_-^s \times n_-^s}$, $\mathbf{\Lambda}_+^s \in \mathbb{R}^{n_+^s \times n_+^s}$ be the blocks of \mathbf{V}^s and $\mathbf{\Lambda}^s$ that satisfy

$$\mathbf{\Lambda}^s = \begin{pmatrix} \mathbf{\Lambda}_-^s & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_+^s \end{pmatrix}, \quad \mathbf{V}^s = [\mathbf{V}_-^s | \mathbf{V}_+^s], \quad \mathbf{\Lambda}_+^s \text{ is spd}, \quad -\mathbf{\Lambda}_-^s \text{ is spsd.}$$

Finally, define the two following matrices in $\mathbb{R}^{n^s \times n^s}$:

$$\mathbf{A}_+^s := \mathbf{V}_+^s \mathbf{\Lambda}_+^s \mathbf{V}_+^{s\top} \quad \text{and} \quad \mathbf{A}_-^s := -\mathbf{V}_-^s \mathbf{\Lambda}_-^s \mathbf{V}_-^{s\top}.$$

It is clear that, for each $s \in \llbracket 1, N \rrbracket$, both matrices \mathbf{A}_+^s and \mathbf{A}_-^s are spsd matrices and that $\mathbf{B}^s = \mathbf{A}_+^s - \mathbf{A}_-^s$. Next, global matrices are computed by assembling the local components with the usual restriction and prolongation operators \mathbf{R}^s and $\mathbf{R}^{s\top}$.

DEFINITION 3.5 (New matrices \mathbf{A}_+ and \mathbf{A}_-). *Let the global matrices $\mathbf{A}_+ \in \mathbb{R}^{n \times n}$ and $\mathbf{A}_- \in \mathbb{R}^{n \times n}$ be defined by*

$$\mathbf{A}_+ := \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{A}_+^s \mathbf{R}^s \quad \text{and} \quad \mathbf{A}_- := \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{A}_-^s \mathbf{R}^s.$$

THEOREM 3.6. *The matrices \mathbf{A}_+ and \mathbf{A}_- from Definition 3.5 satisfy the following three properties*

- (i) $\mathbf{A} = \mathbf{A}_+ - \mathbf{A}_-$,
- (ii) the matrix \mathbf{A}_- is symmetric positive semi-definite,
- (iii) the matrix \mathbf{A}_+ is symmetric positive definite.

Proof. Following Definition 3.5, it holds that $\mathbf{A}_+ = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{V}_+^s \mathbf{\Lambda}_+^s \mathbf{V}_+^{s\top} \mathbf{R}^s$ and $\mathbf{A}_- = -\sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{V}_-^s \mathbf{\Lambda}_-^s \mathbf{V}_-^{s\top} \mathbf{R}^s$, where the matrices $\mathbf{\Lambda}_+^s$ ($s \in \llbracket 1, N \rrbracket$) are diagonal matrices with positive entries and the matrices $\mathbf{\Lambda}_-^s$ ($s \in \llbracket 1, N \rrbracket$) are diagonal matrices with non-positive entries. Consequently, \mathbf{A}_+ and \mathbf{A}_- are both spsd, and item (ii) is proved. Moreover, by Definition 3.4 and Assumption 4, item (i) holds:

$$\begin{aligned} \mathbf{A}_+ - \mathbf{A}_- &= \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{V}_+^s \mathbf{\Lambda}_+^s \mathbf{V}_+^{s\top} \mathbf{R}^s + \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{V}_-^s \mathbf{\Lambda}_-^s \mathbf{V}_-^{s\top} \mathbf{R}^s \\ &= \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{B}^s \mathbf{R}^s = \mathbf{A}. \end{aligned}$$

Finally, it has already been argued that \mathbf{A}_+ is spsd, so, to prove item (iii), it remains only to confirm that the kernel of \mathbf{A}_+ is restricted to the zero vector. Let $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{A}_+\mathbf{x} = \mathbf{0}$. Then,

$$0 = \langle \mathbf{x}, \mathbf{A}_+\mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{A}_-\mathbf{x} \rangle \geq \langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle,$$

and this last term equals 0 only if $\mathbf{x} = \mathbf{0}$, completing the proof. \square

REMARK 3.7. The zero eigenvalues of \mathbf{B}^s are in Λ_-^s . Another possibility would be to put them into Λ_+^s .

The previously defined matrix \mathbf{A}_+ is an spd matrix for which we have knowledge of spsd local matrices $\mathbf{N}^s = \mathbf{A}_+^s$ that satisfy (1.2) with $C = 1$. This means that it fits right into the abstract GenEO theory [46], and hence, a variety of two-level preconditioners with guaranteed convergence rates can be defined. First, the one-level preconditioners to which to apply GenEO are chosen, and then the GenEO coarse spaces are given. In terms of the two ingredients still missing in the abstract two-level preconditioners from Section 2.3, we delineate in the following: the local solvers $\tilde{\mathbf{A}}^s$ are defined in Section 3.3, and the coarse interpolation operators \mathbf{R}^0 are defined in Section 3.4.

3.3. One-level preconditioners for \mathbf{A}_+ . In order to define a one-level preconditioner in our framework, it only remains to choose the matrices $\tilde{\mathbf{A}}^s$, i.e., the local solvers in the abstract form (2.1). Three types of local solvers are introduced, as all three are natural choices for \mathbf{A}_+ : the exact local solver, $\tilde{\mathbf{A}}^s = \mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top}$, the matrix in the spsd splitting with adequate weights, $\tilde{\mathbf{A}}^s = \mathbf{D}^{s-1} \mathbf{A}_+^s \mathbf{D}^{s-1}$, and what would be the exact solver if we were solving a problem with \mathbf{A} , since this is, after all, our endgame, $\tilde{\mathbf{A}}^s = \mathbf{R}^s \mathbf{A} \mathbf{R}^{s\top}$.

DEFINITION 3.8. Let three one-level preconditioners be defined by:

$$\begin{aligned} \mathbf{H}_+^{\text{AS}} &:= \sum_{s=1}^N \mathbf{R}^{s\top} (\mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top})^{-1} \mathbf{R}^s, \\ \mathbf{H}^{\text{AS}} &:= \sum_{s=1}^N \mathbf{R}^{s\top} (\mathbf{R}^s \mathbf{A} \mathbf{R}^{s\top})^{-1} \mathbf{R}^s, \end{aligned}$$

and

$$\mathbf{H}^{\text{NN}} := \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{D}^s (\mathbf{A}_+^s)^\dagger \mathbf{D}^s \mathbf{R}^s = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{D}^s \mathbf{V}_+^s (\Lambda_+^s)^{-1} \mathbf{V}_+^{s\top} \mathbf{D}^s \mathbf{R}^s,$$

where \mathbf{D}^s are the partition of unity matrices from Definition 2.3.

It is recalled that, for $s \in \llbracket 1, N \rrbracket$, \mathbf{A}_+^s , \mathbf{V}_+^s , and Λ_+^s were introduced in Definition 3.4, \mathbf{D}^s in Definition 2.3, and \mathbf{A}_+ in Definition 3.5.

LEMMA 3.9. The one-level preconditioners \mathbf{H}_+^{AS} , \mathbf{H}^{AS} , and \mathbf{H}^{NN} from Definition 3.8 are spd.

Proof. The preconditioners \mathbf{H}_+^{AS} and \mathbf{H}^{AS} are usual Additive Schwarz preconditioners for spd matrices, so they are spd. For the third preconditioner, it is obvious that \mathbf{H}^{NN} is spsd. Moreover, let $\mathbf{x} \in \text{Ker}(\mathbf{H}^{\text{NN}})$. Then,

$$0 = \langle \mathbf{x}, \mathbf{H}^{\text{NN}} \mathbf{x} \rangle = \sum_{s=1}^N \langle \mathbf{D}^s \mathbf{R}^s \mathbf{x}, \mathbf{A}_+^s \dagger \mathbf{D}^s \mathbf{R}^s \mathbf{x} \rangle.$$

For this to hold, each term in the sum of non-negative terms must also be zero, so,

$$\text{for any } s \in \llbracket 1, N \rrbracket : \quad \mathbf{D}^s \mathbf{R}^s \mathbf{x} \in \text{Ker}(\mathbf{A}_+^s \dagger) = \text{Ker}(\mathbf{A}_+^s).$$

Let's prolongate $(\mathbf{A}_+^s)\mathbf{D}^s\mathbf{R}^s\mathbf{x} = \mathbf{0}$ to the global domain with $\mathbf{R}^{s\top}$, sum over s , and inject the definition of \mathbf{D}^s (Definition 2.3 in which \mathbf{D} is diagonal) to obtain

$$\mathbf{0} = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{A}_+^s \mathbf{D}^s \mathbf{R}^s \mathbf{x} = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{A}_+^s \mathbf{R}^s \mathbf{D} \mathbf{R}^{s\top} \mathbf{R}^s \mathbf{x} = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{A}_+^s \mathbf{R}^s \mathbf{D} \mathbf{x} = \mathbf{A}_+ \mathbf{D} \mathbf{x}.$$

Finally, the non singularity of \mathbf{A}_+ and of \mathbf{D} allow us to conclude that $\mathbf{x} = \mathbf{0}$, which ends the proof. \square

REMARK 3.10. For the proof of the non-singularity of \mathbf{H}^{NN} , the definition of the partition of unity matrices \mathbf{D}^s was used, Definition 2.3. A general proof does not go through for all partitions of unity, i.e., if the \mathbf{D}^s in Definition 3.8 are replaced by another family of matrices satisfying Assumption 2. However, it is not likely that the \mathbf{D}^s could and would be chosen in a way that makes \mathbf{H}^{NN} singular. In other words, this is a technical restriction, and other choices of partition of unity matrices should definitely be explored. All parts of the article that are not related to \mathbf{H}^{NN} are not concerned by this technical restriction.

3.4. Two-level preconditioners for \mathbf{A}_+ with GenEO. Next, the GenEO coarse spaces that correspond to solving a linear system for \mathbf{A}_+ with each of the one-level preconditioners are introduced. The corresponding spectral bounds for the preconditioned operators are given. The proofs consist in giving the adequate references to [46]. The information is organized with one theorem per choice of one-level preconditioner. First, some very useful notation is chosen to designate a normalized basis of the high- (or low-) frequency eigenvectors with respect to a certain matrix pencil and a certain threshold.

DEFINITION 3.11. *Let $m \in \mathbb{N}^*$, let $\mathbf{M}_A \in \mathbb{R}^{m \times m}$ be an spsd matrix, and let $\mathbf{M}_B \in \mathbb{R}^{m \times m}$ be an spd matrix. Let $(\lambda_k, \mathbf{y}_k)_{k=1, \dots, m}$ be the (ordered and \mathbf{M}_B -normalized) eigenpairs of the generalized eigenproblem associated with the matrix pencil $(\mathbf{M}_A, \mathbf{M}_B)$, i.e.,*

$$\begin{aligned} (\lambda_k, \mathbf{y}_k) &\in \mathbb{R} \times \mathbb{R}^m, & \langle \mathbf{y}_k, \mathbf{M}_B \mathbf{y}_k \rangle &= 1, \\ \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m, & \text{ and } & \mathbf{M}_A \mathbf{y}_k &= \lambda \mathbf{M}_B \mathbf{y}_k. \end{aligned}$$

For any scalar $\tau > 0$, set $m_L := \min \{k \in \llbracket 0, m-1 \rrbracket; \lambda_{k+1} \geq \tau\}$ if $\lambda_m \geq \tau$ and $m_L := m$ otherwise. Then define the two following matrices by concatenating eigenvectors

$$\mathbf{Y}_L(\tau, \mathbf{M}_A, \mathbf{M}_B) := [\mathbf{y}_1 | \dots | \mathbf{y}_{m_L}] \quad \text{and} \quad \mathbf{Y}_H(\tau, \mathbf{M}_A, \mathbf{M}_B) := [\mathbf{y}_{m_L+1} | \dots | \mathbf{y}_m].$$

The subscripts $*_L$ and $*_H$ refer to the words *low* and *high* depending on which end of the spectrum is selected. The definition is in agreement with the definition in [46].

REMARK 3.12. It is never necessary to fully solve the generalized eigenvalue problem $\mathbf{M}_A \mathbf{y}_k = \lambda \mathbf{M}_B \mathbf{y}_k$. Instead, only the smallest, or the largest, eigenvalues are required as well as the corresponding eigenvectors. This can be performed by an iterative method, many of which are implemented in SLEPc [28]. A spectral transformation is performed within these iterative eigensolvers to rewrite the generalized eigenvalue problem in a form that can be solved by a power iteration method [23, Section 7.3]. To this end,

- the computation of $\mathbf{Y}_L(\tau, \mathbf{M}_A, \mathbf{M}_B)$ requires to be able to solve linear systems with \mathbf{M}_A and to multiply vectors by \mathbf{M}_B ,
- the computation of $\mathbf{Y}_H(\tau, \mathbf{M}_A, \mathbf{M}_B)$ requires to be able to solve linear systems with \mathbf{M}_B and to multiply vectors by \mathbf{M}_A .

THEOREM 3.13 (Two-level preconditioners with GenEO for \mathbf{H}_+^{AS}). *For any $\tau_b > 1$, let $V_{\text{AS},+}^0(\tau_b)$ be defined by*

$$V_{\text{AS},+}^0(\tau_b) := \sum_{s=1}^N \text{range} \left(\mathbf{R}^{s\top} \mathbf{Y}_L(\tau_b^{-1}), \mathbf{D}^{s-1} \mathbf{A}_+^s \mathbf{D}^{s-1}, \mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top} \right),$$

and assume that a corresponding interpolation matrix $\mathbf{R}_{\text{AS},+}^0(\tau_b)$ is defined to satisfy Assumption 3. Then, the coarse projector as well as the hybrid and additive preconditioners are defined naturally as

$$\begin{aligned} \mathbf{\Pi}_+^{\text{AS}}(\tau_b) &:= \mathbf{I} - \mathbf{R}_{\text{AS},+}^0(\tau_b)^\top (\mathbf{R}_{\text{AS},+}^0(\tau_b) \mathbf{A}_+ \mathbf{R}_{\text{AS},+}^0(\tau_b)^\top)^{-1} \mathbf{R}_{\text{AS},+}^0(\tau_b) \mathbf{A}_+, \\ \mathbf{H}_{+, \text{hyb}}^{\text{AS}}(\tau_b) &:= \mathbf{\Pi}_+^{\text{AS}}(\tau_b) \mathbf{H}_+^{\text{AS}} \mathbf{\Pi}_+^{\text{AS}}(\tau_b)^\top \\ &\quad + \mathbf{R}_{\text{AS},+}^0(\tau_b)^\top (\mathbf{R}_{\text{AS},+}^0(\tau_b) \mathbf{A}_+ \mathbf{R}_{\text{AS},+}^0(\tau_b)^\top)^{-1} \mathbf{R}_{\text{AS},+}^0(\tau_b), \\ \mathbf{H}_{+, \text{ad}}^{\text{AS}}(\tau_b) &:= \mathbf{H}_+^{\text{AS}} + \mathbf{R}_{\text{AS},+}^0(\tau_b)^\top (\mathbf{R}_{\text{AS},+}^0(\tau_b) \mathbf{A}_+ \mathbf{R}_{\text{AS},+}^0(\tau_b)^\top)^{-1} \mathbf{R}_{\text{AS},+}^0(\tau_b). \end{aligned}$$

The eigenvalues of the preconditioned operators are bounded as follows:

$$\begin{aligned} 1/\tau_b &\leq \lambda(\mathbf{H}_{+, \text{hyb}}^{\text{AS}}(\tau_b) \mathbf{A}_+) \leq \mathcal{N}(\mathbf{A}_+) \\ 1/((1 + 2\mathcal{N}(\mathbf{A}_+))\tau_b) &\leq \lambda(\mathbf{H}_{+, \text{ad}}^{\text{AS}}(\tau_b) \mathbf{A}_+) \leq \mathcal{N}(\mathbf{A}_+) + 1, \end{aligned}$$

where $\mathcal{N}(\mathbf{A}_+)$ is the coloring constant with respect to the matrix \mathbf{A}_+ (see Definition 2.4).

Proof. This results from an application of [46, Corollary 3] (for the hybrid preconditioner) and [46, Corollary 4] (for the additive preconditioner) under [46, Assumption 7]. The parameters are $\mathcal{N}' = 1$, $\mathbf{M}^s = \mathbf{D}^{s-1} \mathbf{A}_+^s \mathbf{D}^{s-1}$, $\tilde{\mathbf{A}}^s = \mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top}$, and the alternate formulation for the coarse space given in [46, Definition 5]. \square

Note that if $0 < \tau_b \leq 1$, then there is also a spectral result that is slightly longer to state involving min and max.

THEOREM 3.14 (Two-level preconditioner with GenEO for \mathbf{H}^{NN}). *For any $0 < \tau_\sharp < 1$, let $V_{\text{NN}}^0(\tau_\sharp)$ be defined by*

$$V_{\text{NN}}^0(\tau_\sharp) := \sum_{s=1}^N \text{range} \left(\mathbf{R}^{s\top} \mathbf{Y}_L(\tau_\sharp), \mathbf{D}^{s-1} \mathbf{A}_+^s \mathbf{D}^{s-1}, \mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top} \right),$$

and assume that a corresponding interpolation matrix $\mathbf{R}_{\text{NN}}^0(\tau_\sharp)$ is defined to satisfy Assumption 3. Then, the coarse projector as well as the hybrid preconditioner are defined naturally as

$$\begin{aligned} \mathbf{\Pi}^{\text{NN}}(\tau_\sharp) &:= \mathbf{I} - \mathbf{R}_{\text{NN}}^0(\tau_\sharp)^\top (\mathbf{R}_{\text{NN}}^0(\tau_\sharp) \mathbf{A}_+ \mathbf{R}_{\text{NN}}^0(\tau_\sharp)^\top)^{-1} \mathbf{R}_{\text{NN}}^0(\tau_\sharp) \mathbf{A}_+, \\ \mathbf{H}_{\text{hyb}}^{\text{NN}}(\tau_\sharp) &:= \mathbf{\Pi}^{\text{NN}}(\tau_\sharp) \mathbf{H}^{\text{NN}} \mathbf{\Pi}^{\text{NN}}(\tau_\sharp)^\top + \mathbf{R}_{\text{NN}}^0(\tau_\sharp)^\top (\mathbf{R}_{\text{NN}}^0(\tau_\sharp) \mathbf{A}_+ \mathbf{R}_{\text{NN}}^0(\tau_\sharp)^\top)^{-1} \mathbf{R}_{\text{NN}}^0(\tau_\sharp). \end{aligned}$$

The eigenvalues of the preconditioned operator are bounded as follows:

$$1 \leq \lambda(\mathbf{H}_{\text{hyb}}^{\text{NN}}(\tau_\sharp) \mathbf{A}_+) \leq \mathcal{N}(\mathbf{A}_+)/\tau_\sharp,$$

where $\mathcal{N}(\mathbf{A}_+)$ is the coloring constant with respect to the matrix \mathbf{A}_+ ; see Definition 2.4.

Proof. This results from an application of [46, Corollary 2] with $\mathcal{N}' = 1$, $\tau_b \rightarrow 1$ (as in [46, Section 5.2.3]), $\tilde{\mathbf{A}}^s = \mathbf{D}^{s-1} \mathbf{A}_+^s \mathbf{D}^{s-1}$, and $\mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top}$ as the exact local solver, because we are solving for \mathbf{A}_+ . \square

Note that if $\tau_{\sharp} \geq 1$, then there is also a spectral result that is slightly longer to state involving min and max. There is no spectral result for the two-level additive preconditioner, so it is not considered in the theorem.

THEOREM 3.15 (Two-level preconditioner with GenEO for \mathbf{H}^{AS}). *For any $\tau_b > 1$ and $0 < \tau_{\sharp} < 1$, let $V_{\text{AS}}^0(\tau_{\sharp}, \tau_b)$ be defined by*

$$V_{\text{AS}}^0(\tau_{\sharp}, \tau_b) := \sum_{s=1}^N \text{range} \left(\mathbf{R}^{s\top} \mathbf{Y}_L(\tau_b^{-1}, \mathbf{D}^{s-1} \mathbf{A}_+^s \mathbf{D}^{s-1}, \mathbf{R}^s \mathbf{A} \mathbf{R}^{s\top}) \right) \\ + \sum_{s=1}^N \text{range} \left(\mathbf{R}^{s\top} \mathbf{Y}_L(\tau_{\sharp}, \mathbf{R}^s \mathbf{A} \mathbf{R}^{s\top}, \mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top}) \right),$$

and assume that a corresponding interpolation matrix $\mathbf{R}_{\text{AS}}^0(\tau_{\sharp}, \tau_b)$ is defined to satisfy Assumption 3. Then, the coarse projector as well as the hybrid preconditioner are defined naturally as

$$\mathbf{\Pi}^{\text{AS}}(\tau_{\sharp}, \tau_b) := \mathbf{I} - \mathbf{R}_{\text{AS}}^0(\tau_{\sharp}, \tau_b)^{\top} (\mathbf{R}_{\text{AS}}^0(\tau_{\sharp}, \tau_b) \mathbf{A}_+ \mathbf{R}_{\text{AS}}^0(\tau_{\sharp}, \tau_b)^{\top})^{-1} \mathbf{R}_{\text{AS}}^0(\tau_{\sharp}, \tau_b) \mathbf{A}_+, \\ \mathbf{H}_{\text{hyb}}^{\text{AS}}(\tau_{\sharp}, \tau_b) := \mathbf{\Pi}^{\text{AS}}(\tau_{\sharp}, \tau_b) \mathbf{H}^{\text{AS}} \mathbf{\Pi}^{\text{AS}}(\tau_{\sharp}, \tau_b)^{\top} \\ + \mathbf{R}_{\text{AS}}^0(\tau_{\sharp}, \tau_b)^{\top} (\mathbf{R}_{\text{AS}}^0(\tau_{\sharp}, \tau_b) \mathbf{A}_+ \mathbf{R}_{\text{AS}}^0(\tau_{\sharp}, \tau_b)^{\top})^{-1} \mathbf{R}_{\text{AS}}^0(\tau_{\sharp}, \tau_b).$$

The eigenvalues of the preconditioned operator are bounded as follows:

$$1/\tau_b \leq \lambda(\mathbf{H}_{\text{hyb}}^{\text{AS}}(\tau_{\sharp}, \tau_b) \mathbf{A}_+) \leq \mathcal{N}(\mathbf{A}_+)/\tau_{\sharp},$$

where $\mathcal{N}(\mathbf{A}_+)$ is the coloring constant with respect to the matrix \mathbf{A}_+ ; see Definition 2.4.

Proof. This results from [46, Corollary 2] with $\mathcal{N}' = 1$, $\mathbf{M}^s = \mathbf{D}^{s-1} \mathbf{A}_+^s \mathbf{D}^{s-1}$, and $\tilde{\mathbf{A}}^s = \mathbf{R}^s \mathbf{A} \mathbf{R}^{s\top}$, where $\mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top}$ is the exact local solver (because we are solving for \mathbf{A}_+), and the alternate formulation for the coarse space given in [46, Definition 5]. \square

Note that if $0 < \tau_b \leq 1$, then there is also a spectral result that is slightly longer to state involving min and max. The choice $\tau_{\sharp} \geq 1$ would lead to having all the vectors in the coarse space, so this is excluded. There is no spectral result for the two-level additive preconditioner, so it is not considered in the theorem.

REMARK 3.16. For all four considered choices of two-level preconditioner, it holds that $\sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{D}^s \mathbf{V}_-^s$ is included in the coarse space because $\text{Ker}(\mathbf{D}^{s-1} \mathbf{A}_+^s \mathbf{D}^{s-1}) = \mathbf{D}^s \mathbf{V}_-^s$.

4. Algebraic Woodbury-GenEO (AWG) preconditioners for \mathbf{A} . In the previous section, four two-level preconditioners for \mathbf{A}_+ (indexed by one or two thresholds) and their spectral bounds were introduced. They are denoted by $\mathbf{H}_{+, \text{hyb}}^{\text{AS}}(\tau_b)$, $\mathbf{H}_{+, \text{ad}}^{\text{AS}}(\tau_b)$, $\mathbf{H}_{\text{hyb}}^{\text{NN}}(\tau_{\sharp})$, and $\mathbf{H}_{\text{hyb}}^{\text{AS}}(\tau_{\sharp}, \tau_b)$. In this section, \mathbf{H}_2 denotes any one of these. All eigenvalues of the preconditioned operator $\mathbf{H}_2 \mathbf{A}_+$ are contained in the interval $[\lambda_{\min}(\mathbf{H}_2 \mathbf{A}_+), \lambda_{\max}(\mathbf{H}_2 \mathbf{A}_+)]$. The subscript $*_2$ was chosen to refer to two-level preconditioners. We may now set aside the choice of a preconditioner for \mathbf{A}_+ and come back to our original problem of finding a preconditioner for \mathbf{A} .

We define the new AWG preconditioners for \mathbf{A} by viewing \mathbf{A} as a low-rank modification of \mathbf{A}_+ . Three new preconditioners are introduced. The first, in Section 4.2, has an inexact coarse solve that comes directly from the Woodbury matrix identity. The other two, in Section 4.3, correspond to attaching a second coarse space to \mathbf{H}_2 in a hybrid or an additive fashion.

4.1. Woodbury matrix identity for $\mathbf{A} = \mathbf{A}_+ - \mathbf{A}_-$. The new preconditioner for \mathbf{A} arises from the realization that \mathbf{A} can be viewed as a low-rank modification of \mathbf{A}_+ and adding a term to \mathbf{H}_2 accordingly.

THEOREM 4.1. *The rank of \mathbf{A}_- denoted by n_- satisfies $n_- \leq \sum_{s=1}^N n^s - n$.*

Proof. By definition of \mathbf{A}_+ and \mathbf{A}_- , it holds that

$$\text{rank}(\mathbf{A}_+) + \text{rank}(\mathbf{A}_-) \leq \sum_{s=1}^N n^s,$$

and $\text{rank}(\mathbf{A}_+) = n$ as \mathbf{A}_+ is non-singular. \square

In words, the rank of \mathbf{A}_- is at most the difference between the number of degrees of freedom and the number of degrees of freedom multiplied by their multiplicity. If there is little overlap (while still satisfying the minimal overlap condition), then the rank of \mathbf{A}_- is small compared to the rank n of \mathbf{A} , that is, $n_- \ll n$. In practice it is possible and desirable that n_- be much smaller even than $\sum_{s=1}^N n^s - n$. Following this observation it is natural to write \mathbf{A} as a modification of \mathbf{A}_+ .

To this end, let us introduce the factors in the diagonalization of \mathbf{A}_- .

DEFINITION 4.2. *Let $\mathbf{\Lambda}_- \in \mathbb{R}^{n_- \times n_-}$ and $\mathbf{V}_- \in \mathbb{R}^{n \times n_-}$ be the diagonal matrix and the orthogonal matrix that are obtained by removing the null part of \mathbf{A}_- from its diagonalization in such a way that*

$$\mathbf{A}_- = \mathbf{V}_- \mathbf{\Lambda}_- \mathbf{V}_-^\top.$$

REMARK 4.3. The diagonalization of \mathbf{A}_- is not actually required in the numerical implementation; see Section 4.4.

It now holds that $\mathbf{A} = \mathbf{A}_+ - \mathbf{V}_- \mathbf{\Lambda}_- \mathbf{V}_-^\top$, with \mathbf{A} , \mathbf{A}_+ , $\mathbf{\Lambda}_-$ spd matrices and \mathbf{V}_- a full rank matrix. The Woodbury matrix identity [53] applied to computing the inverse of \mathbf{A} gives

$$(4.1) \quad \mathbf{A}^{-1} = \mathbf{A}_+^{-1} + \mathbf{A}_+^{-1} \mathbf{V}_- (\mathbf{\Lambda}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1} \mathbf{V}_-)^{-1} \mathbf{V}_-^\top \mathbf{A}_+^{-1}.$$

REMARK 4.4. The Woodbury matrix identity is also called the Sherman-Morrison-Woodbury formula, for instance in [23, Section 2.1.4]. The formula is correct since \mathbf{A}_+ and $\mathbf{\Lambda}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1} \mathbf{V}_-$ are non-singular. Indeed, let $\mathbf{y} \in \mathbb{R}^{n_-}$, and assume further that $(\mathbf{\Lambda}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1} \mathbf{V}_-) \mathbf{y} = \mathbf{0}$. Then,

$$\mathbf{V}_- \mathbf{y} = \mathbf{V}_- \mathbf{\Lambda}_- \mathbf{V}_-^\top \mathbf{A}_+^{-1} \mathbf{V}_- \mathbf{y} \iff \mathbf{V}_- \mathbf{y} = \mathbf{A}_- \mathbf{A}_+^{-1} \mathbf{V}_- \mathbf{y}.$$

With $\mathbf{z} = \mathbf{A}_+^{-1} \mathbf{V}_- \mathbf{y}$, it then holds that $\mathbf{A}_+ \mathbf{z} = \mathbf{A}_- \mathbf{z}$, which is equivalent to $\mathbf{A} \mathbf{z} = \mathbf{0}$ and in turn to $\mathbf{z} = \mathbf{0}$ and $\mathbf{y} = \mathbf{0}$.

4.2. AWG preconditioner for \mathbf{A} with inexact coarse space. The Woodbury matrix identity leads, rather straightforwardly, to a new preconditioner for the original matrix \mathbf{A} that is defined in the following theorem.

THEOREM 4.5 (AWG preconditioner for \mathbf{A} with inexact coarse space). *Given a preconditioner \mathbf{H}_2 for \mathbf{A}_+ such that the eigenvalues of $\mathbf{H}_2 \mathbf{A}_+$ are in $[\lambda_{\min}(\mathbf{H}_2 \mathbf{A}_+), \lambda_{\max}(\mathbf{H}_2 \mathbf{A}_+)]$. Let the inexact AWG preconditioner for \mathbf{A} be defined as*

$$\mathbf{H}_{3,\text{inex}} := \mathbf{H}_2 + \mathbf{A}_+^{-1} \mathbf{V}_- (\mathbf{\Lambda}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1} \mathbf{V}_-)^{-1} \mathbf{V}_-^\top \mathbf{A}_+^{-1}.$$

Then the eigenvalues of the new preconditioned operator satisfy

$$(4.2) \quad \min(1, \lambda_{\min}(\mathbf{H}_2 \mathbf{A}_+)) \leq \lambda(\mathbf{H}_{3,\text{inex}} \mathbf{A}) \leq \max(1, \lambda_{\max}(\mathbf{H}_2 \mathbf{A}_+)).$$

Proof. The estimate for the eigenvalues of $\mathbf{H}_2\mathbf{A}_+$ is equivalent to

$$\lambda_{\min}(\mathbf{H}_2\mathbf{A}_+)\langle \mathbf{x}, \mathbf{A}_+^{-1}\mathbf{x} \rangle \leq \langle \mathbf{x}, \mathbf{H}_2\mathbf{x} \rangle \leq \lambda_{\max}(\mathbf{H}_2\mathbf{A}_+)\langle \mathbf{x}, \mathbf{A}_+^{-1}\mathbf{x} \rangle, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Adding, $\langle \mathbf{x}, \mathbf{A}_+^{-1}\mathbf{V}_- (\mathbf{D}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-)^{-1} \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{x} \rangle$ to each term, it holds that

$$\min(1, \lambda_{\min}(\mathbf{H}_2\mathbf{A}_+))\langle \mathbf{x}, \mathbf{A}_+^{-1}\mathbf{x} \rangle \leq \langle \mathbf{x}, \mathbf{H}_{3,\text{inex}}\mathbf{x} \rangle \leq \max(1, \lambda_{\max}(\mathbf{H}_2\mathbf{A}_+))\langle \mathbf{x}, \mathbf{A}_+^{-1}\mathbf{x} \rangle,$$

where the Woodbury matrix identity (4.1) was applied. This is equivalent to (4.2). \square

We have just introduced new preconditioners called the AWG preconditioners with an inexact coarse space. The plural in the previous sentence comes from the fact that there are many possible choices for \mathbf{H}_2 (including the four from Section 3) and that for each one there are parameters that can be adjusted. These new preconditioners are purely algebraic, and they have guaranteed spectral bounds when applied to solving linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$. The condition number of $\mathbf{H}_{3,\text{inex}}\mathbf{A}$ can be reduced by enriching the coarse space in \mathbf{H}_2 . The name *inexact* comes from the fact that $\mathbf{H}_{3,\text{inex}}\mathbf{A}$ has the form of a domain decomposition preconditioner with two coarse spaces, one is in \mathbf{H}_2 , and the other is in the term $\mathbf{A}_+^{-1}\mathbf{V}_- (\mathbf{A}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-)^{-1} \mathbf{V}_-^\top \mathbf{A}_+^{-1}$, where the coarse solve $(\mathbf{A}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-)^{-1}$ is inexact. Next, two other AWG preconditioners are defined which have an exact coarse spaces in the sense of the Abstract Schwarz theory: their coarse operator is of the form $\mathbf{R}^0\mathbf{A}\mathbf{R}^{0^\top}$.

4.3. Additive and hybrid AWG preconditioners for \mathbf{A} . Solving a problem with the preconditioner introduced in the previous section requires it to be computationally feasible to multiply by $(\mathbf{A}_+^{-1}\mathbf{V}_-)$ and its transpose. Looking at the Woodbury identity, we realize (and prove it in the lemma below) that $\text{range}(\mathbf{A}_+^{-1}\mathbf{V}_-) = \text{range}(\mathbf{A}_-^{-1}\mathbf{V}_-)$. This opens up new possibilities: if it is possible to compute $\text{range}(\mathbf{A}_-^{-1}\mathbf{V}_-)$, it is also possible to project \mathbf{A} -orthogonally onto the space that is ℓ_2 -orthogonal to $\text{range}(\mathbf{V}_-)$, which is exactly $\text{Ker}(\mathbf{A}_-)$; see $\mathbf{\Pi}_3$ in Definition 4.7 below. On that space, $\mathbf{A}_{|\text{Ker}(\mathbf{A}_-)} = (\mathbf{A}_+)|_{\text{Ker}(\mathbf{A}_-)}$, and we can fall back onto a known and efficient preconditioner \mathbf{H}_2 for \mathbf{A}_+ .

LEMMA 4.6. *With \mathbf{A}_+ from Definition 3.5 and \mathbf{V}_- from Definition 4.2, the following property holds*

$$\text{range}(\mathbf{A}_+^{-1}\mathbf{V}_-) = \text{range}(\mathbf{A}_-^{-1}\mathbf{V}_-).$$

Proof. It follows from the Woodbury identity that

$$\text{range}(\mathbf{A}_-^{-1}\mathbf{V}_-) = \text{range}\left(\mathbf{A}_+^{-1}\mathbf{V}_- (\mathbf{I} + (\mathbf{A}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-)^{-1} \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-)\right),$$

where \mathbf{I} is the $n_- \times n_-$ identity matrix. Proving the result in the lemma comes down to proving that the range of $(\mathbf{I} + (\mathbf{A}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-)^{-1} \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-)$ is the whole of \mathbb{R}^{n_-} , or, equivalently, that its kernel is restricted to $\mathbf{0}$ (the zero vector in \mathbb{R}^{n_-}). This last step is achieved as follows. Let $\mathbf{y} \in \mathbb{R}^{n_-}$,

$$\begin{aligned} & (\mathbf{I} + (\mathbf{A}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-)^{-1} \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-) \mathbf{y} = \mathbf{0} \\ \iff & (\mathbf{A}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-) \mathbf{y} = -\mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_- \mathbf{y} \iff \mathbf{A}_-^{-1}\mathbf{y} = \mathbf{0} \iff \mathbf{y} = \mathbf{0}. \quad \square \end{aligned}$$

DEFINITION 4.7. Let $\mathbf{W} \in \mathbb{R}^{n \times n_-}$ be such that $\text{range}(\mathbf{W}) = \text{range}(\mathbf{A}_+^{-1}\mathbf{V}_-)$, and let

$$\mathbf{\Pi}_3 := \mathbf{I} - \mathbf{W}(\mathbf{W}^\top \mathbf{A}\mathbf{W})^{-1} \mathbf{W}^\top \mathbf{A}.$$

Assume that \mathbf{H}_2 is a given preconditioner for \mathbf{A}_+ such that the eigenvalues of $\mathbf{H}_2\mathbf{A}_+$ are in the interval $[\lambda_{\min}(\mathbf{H}_2\mathbf{A}_+), \lambda_{\max}(\mathbf{H}_2\mathbf{A}_+)]$. Let two new preconditioner for \mathbf{A} be defined as

$$\mathbf{H}_{3,\text{ad}} := \mathbf{H}_2 + \mathbf{W}(\mathbf{W}^\top \mathbf{A}\mathbf{W})^{-1} \mathbf{W}^\top \quad (\text{Additive AWG preconditioner}),$$

and

$$\mathbf{H}_{3,\text{hyb}} := \mathbf{\Pi}_3 \mathbf{H}_2 \mathbf{\Pi}_3^\top + \mathbf{W}(\mathbf{W}^\top \mathbf{A}\mathbf{W})^{-1} \mathbf{W}^\top \quad (\text{Hybrid AWG preconditioner}).$$

THEOREM 4.8. Let $\mathbf{\Pi}_3$, $\mathbf{H}_{3,\text{ad}}$, and $\mathbf{H}_{3,\text{hyb}}$ be as in Definition 4.7. The operator $\mathbf{\Pi}_3$ is an \mathbf{A} -orthogonal projection operator that satisfies

$$(4.3) \quad \text{Ker}(\mathbf{\Pi}_3) = \text{range}(\mathbf{A}^{-1}\mathbf{V}_-) \quad \text{and} \quad \text{range}(\mathbf{\Pi}_3) = \text{Ker}(\mathbf{A}_-).$$

Moreover, the new preconditioned operators satisfy the spectral bounds :

$$(4.4) \quad \lambda_{\min}(\mathbf{H}_2\mathbf{A}_+) \leq \lambda(\mathbf{H}_2\mathbf{A}\mathbf{\Pi}_3) \leq \lambda_{\max}(\mathbf{H}_2\mathbf{A}_+) \text{ if } \lambda(\mathbf{H}_2\mathbf{A}\mathbf{\Pi}_3) \neq 0,$$

$$(4.5) \quad \min(1, \lambda_{\min}(\mathbf{H}_2\mathbf{A}_+)) \leq \lambda(\mathbf{H}_{3,\text{hyb}}\mathbf{A}) \leq \max(1, \lambda_{\max}(\mathbf{H}_2\mathbf{A}_+)),$$

$$\min(1, \lambda_{\min}(\mathbf{H}_2\mathbf{A}_+)) \leq \lambda(\mathbf{H}_{3,\text{ad}}\mathbf{A}) \leq (\lambda_{\max}(\mathbf{H}_2\mathbf{A}_+) + 1),$$

where we recall that \mathbf{H}_2 can be chosen as one of the two-level preconditioners from Section 3.4 in such a way that $\lambda_{\min}(\mathbf{H}_2\mathbf{A}_+)$ and $\lambda_{\max}(\mathbf{H}_2\mathbf{A}_+)$ are known and controlled by the choice of the coarse space.

Note that the theorem includes with (4.4) a bound for the projected and preconditioned operator $(\mathbf{H}_2\mathbf{A}\mathbf{\Pi}_3)$.

Proof. We begin by proving (4.3). Let $\mathbf{x} \in \mathbb{R}^n$. \mathbf{x} is in the kernel of $\mathbf{\Pi}_3$ if

$$\begin{aligned} \mathbf{\Pi}_3\mathbf{x} = 0 &\iff \mathbf{x} = \mathbf{W}(\mathbf{W}^\top \mathbf{A}\mathbf{W})^{-1} \mathbf{W}^\top \mathbf{A}\mathbf{x} \\ &\iff \mathbf{x} \in \text{range}(\mathbf{W}) = \text{range}(\mathbf{A}_+^{-1}\mathbf{V}_-) = \text{range}(\mathbf{A}^{-1}\mathbf{V}_-). \end{aligned}$$

The last equality comes from Lemma 4.6. Moreover, $\mathbf{\Pi}_3$ is an \mathbf{A} -orthogonal projection, so

$$\text{range}(\mathbf{\Pi}_3) = (\text{Ker}(\mathbf{\Pi}_3))^{\perp^{\mathbf{A}}} = (\text{range}(\mathbf{A}^{-1}\mathbf{V}_-))^{\perp^{\mathbf{A}}} = (\text{range}(\mathbf{V}_-))^{\perp^{\ell_2}} = \text{Ker}(\mathbf{V}_-^\top),$$

and $\text{Ker}(\mathbf{V}_-^\top) = \text{Ker}(\mathbf{A}_-)$ by definition of \mathbf{V}_- in Definition 4.2. A direct consequence of this result frequently used in the remainder of the proof is the identity $\mathbf{A}_+\mathbf{\Pi}_3 = \mathbf{A}\mathbf{\Pi}_3$.

We now prove the spectral bounds starting with (4.4) for the projected and preconditioned operator $\mathbf{H}_2\mathbf{A}\mathbf{\Pi}_3$. Let $(\lambda, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^n$ be an eigenpair of the matrix $\mathbf{H}_2\mathbf{A}\mathbf{\Pi}_3 (= \mathbf{H}_2\mathbf{A}_+\mathbf{\Pi}_3)$ meaning that

$$\mathbf{y} \neq \mathbf{0} \quad \text{and} \quad \mathbf{H}_2\mathbf{A}_+\mathbf{\Pi}_3\mathbf{y} = \lambda\mathbf{y}.$$

Taking the inner product by $\mathbf{A}_+\mathbf{\Pi}_3\mathbf{y}$ gives

$$\langle \mathbf{A}_+\mathbf{\Pi}_3\mathbf{y}, \mathbf{H}_2\mathbf{A}_+\mathbf{\Pi}_3\mathbf{y} \rangle = \lambda \langle \mathbf{A}_+\mathbf{\Pi}_3\mathbf{y}, \mathbf{y} \rangle = \lambda \langle \mathbf{A}\mathbf{\Pi}_3\mathbf{y}, \mathbf{\Pi}_3\mathbf{y} \rangle = \lambda \langle \mathbf{A}_+\mathbf{\Pi}_3\mathbf{y}, \mathbf{\Pi}_3\mathbf{y} \rangle.$$

Moreover, since \mathbf{A}_+ and \mathbf{H}_2 are spd, the spectral bound for $\mathbf{H}_2\mathbf{A}_+$ is equivalent to

$$(4.6) \quad \begin{aligned} \lambda_{\min}(\mathbf{H}_2\mathbf{A}_+) \langle \mathbf{x}, \mathbf{A}_+\mathbf{x} \rangle &\leq \langle \mathbf{x}, \mathbf{A}_+\mathbf{H}_2\mathbf{A}_+\mathbf{x} \rangle \\ &\leq \lambda_{\max}(\mathbf{H}_2\mathbf{A}_+) \langle \mathbf{x}, \mathbf{A}_+\mathbf{x} \rangle, \quad \forall \mathbf{x} \in \mathbb{R}^n. \end{aligned}$$

In particular, for $\mathbf{x} = \mathbf{\Pi}_3 \mathbf{y}$, we get

$$\begin{aligned} \lambda_{\min}(\mathbf{H}_2 \mathbf{A}_+) \langle \mathbf{\Pi}_3 \mathbf{y}, \mathbf{A}_+ \mathbf{\Pi}_3 \mathbf{y} \rangle &\leq \underbrace{\langle \mathbf{\Pi}_3 \mathbf{y}, \mathbf{A}_+ \mathbf{H}_2 \mathbf{A}_+ \mathbf{\Pi}_3 \mathbf{y} \rangle}_{=\lambda \langle \mathbf{A}_+ \mathbf{\Pi}_3 \mathbf{y}, \mathbf{\Pi}_3 \mathbf{y} \rangle} \\ &\leq \lambda_{\max}(\mathbf{H}_2 \mathbf{A}_+) \langle \mathbf{\Pi}_3 \mathbf{y}, \mathbf{A}_+ \mathbf{\Pi}_3 \mathbf{y} \rangle. \end{aligned}$$

Finally, there are two possibilities, either $\langle \mathbf{A}_+ \mathbf{\Pi}_3 \mathbf{y}, \mathbf{\Pi}_3 \mathbf{y} \rangle = 0$, so $\mathbf{y} \in \text{Ker}(\mathbf{\Pi}_3)$ and $\lambda = 0$, or $\langle \mathbf{A}_+ \mathbf{\Pi}_3 \mathbf{y}, \mathbf{\Pi}_3 \mathbf{y} \rangle \neq 0$ and $\lambda \in [\lambda_{\min}(\mathbf{H}_2 \mathbf{A}_+), \lambda_{\max}(\mathbf{H}_2 \mathbf{A}_+)]$. In other words (4.4) holds.

Next, we prove the spectral bound with the hybrid preconditioner $\mathbf{H}_{3,\text{hyb}}$ from (4.5). Let $\mathbf{x} \in \mathbb{R}^n$. We add the term

$$\langle \mathbf{x}, \mathbf{A} \mathbf{W} (\mathbf{W}^\top \mathbf{A} \mathbf{W})^{-1} \mathbf{W}^\top \mathbf{A} \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{A} (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x} \rangle = \langle (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x}, \mathbf{A} (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x} \rangle$$

to estimate (4.6) evaluated at $\mathbf{\Pi}_3 \mathbf{x}$ and obtain

$$\begin{aligned} \lambda_{\min}(\mathbf{H}_2 \mathbf{A}_+) \langle \mathbf{\Pi}_3 \mathbf{x}, \mathbf{A} \mathbf{\Pi}_3 \mathbf{x} \rangle + \langle (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x}, \mathbf{A} (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x} \rangle \\ \leq \langle \mathbf{\Pi}_3 \mathbf{x}, \mathbf{A} \mathbf{H}_2 \mathbf{A} \mathbf{\Pi}_3 \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{A} \mathbf{W} (\mathbf{W}^\top \mathbf{A} \mathbf{W})^{-1} \mathbf{W}^\top \mathbf{A} \mathbf{x} \rangle \\ \leq \lambda_{\max}(\mathbf{H}_2 \mathbf{A}_+) \langle \mathbf{\Pi}_3 \mathbf{x}, \mathbf{A} \mathbf{\Pi}_3 \mathbf{x} \rangle + \langle (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x}, \mathbf{A} (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x} \rangle, \end{aligned}$$

where $\mathbf{A}_+ \mathbf{\Pi}_3 = \mathbf{A} \mathbf{\Pi}_3$ has also been applied. This then implies that

$$\min(1, \lambda_{\min}(\mathbf{H}_2 \mathbf{A}_+)) \langle \mathbf{x}, \mathbf{A} \mathbf{x} \rangle \leq \langle \mathbf{x}, \mathbf{A} \mathbf{H}_{3,\text{hyb}} \mathbf{A} \mathbf{x} \rangle \leq \max(1, \lambda_{\max}(\mathbf{H}_2 \mathbf{A}_+)) \langle \mathbf{x}, \mathbf{A} \mathbf{x} \rangle,$$

and the eigenvalue estimate in the theorem holds because \mathbf{A} and $\mathbf{H}_{3,\text{hyb}}$ are spd.

Finally, we prove the spectral bound with the additive preconditioner $\mathbf{H}_{3,\text{ad}}$ from (4.5). The matrices \mathbf{A}_+ and \mathbf{H}_2 are both spd, so the fact that all eigenvalues are not larger than $\lambda_{\max}(\mathbf{H}_2 \mathbf{A}_+)$ is equivalent to

$$\langle \mathbf{x}, \mathbf{H}_2 \mathbf{x} \rangle \leq \lambda_{\max}(\mathbf{H}_2 \mathbf{A}_+) \langle \mathbf{x}, \mathbf{A}_+^{-1} \mathbf{x} \rangle, \quad \text{for any } \mathbf{x} \in \mathbb{R}^n.$$

Moreover, $\langle \mathbf{x}, \mathbf{A}_+^{-1} \mathbf{x} \rangle \leq \langle \mathbf{x}, \mathbf{A}^{-1} \mathbf{x} \rangle$ for any $\mathbf{x} \in \mathbb{R}^n$, so

$$\langle \mathbf{x}, \mathbf{H}_2 \mathbf{x} \rangle \leq \lambda_{\max}(\mathbf{H}_2 \mathbf{A}_+) \langle \mathbf{x}, \mathbf{A}^{-1} \mathbf{x} \rangle, \quad \text{for any } \mathbf{x} \in \mathbb{R}^n.$$

It also holds that

$$\langle \mathbf{x}, \mathbf{A} \mathbf{W} (\mathbf{W}^\top \mathbf{A} \mathbf{W})^{-1} \mathbf{W}^\top \mathbf{A} \mathbf{x} \rangle = \langle (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x}, \mathbf{A} (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x} \rangle \leq \langle \mathbf{x}, \mathbf{A} \mathbf{x} \rangle,$$

for any $\mathbf{x} \in \mathbb{R}^n$, or, equivalently,

$$\langle \mathbf{x}, \mathbf{W} (\mathbf{W}^\top \mathbf{A} \mathbf{W})^{-1} \mathbf{W}^\top \mathbf{x} \rangle \leq \langle \mathbf{x}, \mathbf{A}^{-1} \mathbf{x} \rangle, \quad \text{for any } \mathbf{x} \in \mathbb{R}^n.$$

Adding the last two results together gives us

$$\langle \mathbf{x}, \mathbf{H}_{3,\text{ad}} \mathbf{x} \rangle \leq (\lambda_{\max}(\mathbf{H}_2 \mathbf{A}_+) + 1) \langle \mathbf{x}, \mathbf{A}^{-1} \mathbf{x} \rangle, \quad \text{for any } \mathbf{x} \in \mathbb{R}^n,$$

or in other words, all eigenvalues of $\mathbf{H}_{3,\text{ad}} \mathbf{A}$ are not larger than $(\lambda_{\max}(\mathbf{H}_2 \mathbf{A}_+) + 1)$. For the smallest eigenvalue of $\mathbf{H}_{3,\text{ad}} \mathbf{A}$, we can look at $\mathbf{H}_{3,\text{ad}}$ in the abstract Schwarz framework. Indeed,

$$\mathbf{H}_{3,\text{ad}} = \mathbf{I} \mathbf{H}_2 \mathbf{I} + \mathbf{W} (\mathbf{W}^\top \mathbf{A} \mathbf{W})^{-1} \mathbf{W}^\top,$$

is an abstract Schwarz solver for the two subspaces \mathbb{R}^n and \mathbb{R}^{n-} with prolongation operators \mathbf{I} (the identity matrix in \mathbb{R}^n) and \mathbf{W} and with local solvers \mathbf{H}_2 and $(\mathbf{W}^\top \mathbf{A} \mathbf{W})^{-1}$. We know that \mathbf{H}_2 is spd, thus the classical stable splitting result from [51] applies (see also [46]): all eigenvalues of $\mathbf{H}_{3,\text{ad}} \mathbf{A}$ are larger than C_0^{-2} if, for any $\mathbf{x} \in \mathbb{R}^n$, there exist $\mathbf{z}_+ \in \mathbb{R}^n$ and $\mathbf{z}_- \in \mathbb{R}^{n-}$ that satisfy

$$\mathbf{z}_+ + \mathbf{W} \mathbf{z}_- = \mathbf{x} \quad \text{and} \quad \langle \mathbf{z}_+, \mathbf{H}_2^{-1} \mathbf{z}_+ \rangle + \langle \mathbf{z}_-, \mathbf{W}^\top \mathbf{A} \mathbf{W} \mathbf{z}_- \rangle \leq C_0^2 \langle \mathbf{x}, \mathbf{A} \mathbf{x} \rangle.$$

The following splitting is proposed: $\mathbf{z}_+ = \mathbf{\Pi}_3 \mathbf{x}$ and $\mathbf{z}_- = (\mathbf{W}^\top \mathbf{A} \mathbf{W})^{-1} \mathbf{W}^\top \mathbf{A} \mathbf{x}$. We first verify that they do split \mathbf{x} :

$$\mathbf{z}_+ + \mathbf{W} \mathbf{z}_- = (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x} + \mathbf{W} (\mathbf{W}^\top \mathbf{A} \mathbf{W})^{-1} \mathbf{W}^\top \mathbf{A} \mathbf{x} = \mathbf{x}.$$

We then verify the stability of the splitting:

$$\begin{aligned} & \langle \mathbf{z}_+, \mathbf{H}_2^{-1} \mathbf{z}_+ \rangle + \langle \mathbf{z}_-, \mathbf{W}^\top \mathbf{A} \mathbf{W} \mathbf{z}_- \rangle \\ &= \langle \mathbf{\Pi}_3 \mathbf{x}, \mathbf{H}_2^{-1} \mathbf{\Pi}_3 \mathbf{x} \rangle + \langle (\mathbf{W}^\top \mathbf{A} \mathbf{W})^{-1} \mathbf{W}^\top \mathbf{A} \mathbf{x}, \mathbf{W}^\top \mathbf{A} \mathbf{W} (\mathbf{W}^\top \mathbf{A} \mathbf{W})^{-1} \mathbf{W}^\top \mathbf{A} \mathbf{x} \rangle \\ &\leq (\lambda_{\min}(\mathbf{H}_2 \mathbf{A}_+))^{-1} \langle \mathbf{\Pi}_3 \mathbf{x}, \mathbf{A}_+ \mathbf{\Pi}_3 \mathbf{x} \rangle + \langle (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x}, \mathbf{A} (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x} \rangle \\ &\leq (\lambda_{\min}(\mathbf{H}_2 \mathbf{A}_+))^{-1} \langle \mathbf{\Pi}_3 \mathbf{x}, \mathbf{A} \mathbf{\Pi}_3 \mathbf{x} \rangle + \langle (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x}, \mathbf{A} (\mathbf{I} - \mathbf{\Pi}_3) \mathbf{x} \rangle \\ &\leq \max(1, \lambda_{\min}(\mathbf{H}_2 \mathbf{A}_+)^{-1}) \langle \mathbf{x}, \mathbf{A} \mathbf{x} \rangle. \end{aligned}$$

Finally, it has been proved that all eigenvalues of $\mathbf{H}_{3,\text{ad}}$ are larger than or equal to

$$\max(1, \lambda_{\min}(\mathbf{H}_2 \mathbf{A}_+)^{-1}) = \min(1, \lambda_{\min}(\mathbf{H}_2 \mathbf{A}_+)),$$

and that ends the proof. \square

Again, the AWG preconditioners $\mathbf{H}_{3,\text{ad}}$ and $\mathbf{H}_{3,\text{hyb}}$ are families of preconditioners that are computed algebraically and lead to guaranteed spectral bounds when applied to solving $\mathbf{A} \mathbf{x} = \mathbf{b}$. The condition numbers can be made smaller by enriching the coarse space in \mathbf{H}_2 .

4.4. Remarks on the implementation of the AWG preconditioners. Below some important remarks are made about the implementation of $\mathbf{H}_{3,\text{ad}}$ and $\mathbf{H}_{3,\text{hyb}}$.

1. It is not necessary to diagonalize \mathbf{A}_- as suggested by the definition of \mathbf{V}_- in Definition 4.2. Indeed, Definition 4.7 requires only a basis \mathbf{W} of range $(\mathbf{A}_+^{-1} \mathbf{V}_-)$ to generate the second coarse space. A natural choice is made by recalling that \mathbf{V}_- is generated by the eigenvectors of \mathbf{B}^s that correspond to negative eigenvalues (once prolonged to Ω by $\mathbf{R}^{s\top}$). If the matrices \mathbf{B}^s are non-singular, then the range of $(\mathbf{A}_+^{-1} \mathbf{V}_-)$ is also generated by

$$\mathbf{W} = \mathbf{A}_+^{-1} [\mathbf{R}^{1\top} \mathbf{V}_-^1 \mid \dots \mid \mathbf{R}^{N\top} \mathbf{V}_-^N],$$

with the \mathbf{V}_-^s from Definition 3.4. If the \mathbf{B}^s are singular, then it is necessary to first remove the columns in \mathbf{V}_-^s that correspond to zero eigenvalues. It may also be necessary (although we have not observed it in practice) to remove some linear dependencies between the columns. This is rather standard and can be done either when computing \mathbf{W} or when factorizing the coarse problem $\mathbf{W}^\top \mathbf{A} \mathbf{W}$.

2. The computation of \mathbf{W} is one of the bottlenecks of the algorithm: many systems must be solved for the global matrix \mathbf{A}_+ . In our current implementation these linear systems are solved one after the other with PCG preconditioned by \mathbf{H}_2 . Since \mathbf{H}_2 is a *good* preconditioner for \mathbf{A}_+ , this takes *few* iterations. It must be explored whether computational efficiency could be improved with block CG methods [40] or adaptive multipreconditioning [45].

3. Following Remark 3.12, all four choices of preconditioners \mathbf{H}_2 for \mathbf{A}_+ considered in this article require that the action of $(\mathbf{A}_+^s)^\dagger$ be implemented in order to compute the corresponding GenEO coarse space. Instead of computing the full diagonalization of \mathbf{B}^s , it is sufficient to compute its negative eigenvalues and corresponding orthonormalized set of eigenvectors, i.e., $\mathbf{\Lambda}_-^s$ and \mathbf{V}_-^s from Definition 3.4, and to recall that

$$\mathbf{A}_+^s = (\mathbf{I} - \mathbf{V}_-^s \mathbf{V}_-^{s\top}) \mathbf{B}^s (\mathbf{I} - \mathbf{V}_-^s \mathbf{V}_-^{s\top}),$$

which also implies that

$$\mathbf{A}_+^{s\dagger} = (\mathbf{I} - \mathbf{V}_-^s \mathbf{V}_-^{s\top}) \mathbf{B}^{s\dagger} (\mathbf{I} - \mathbf{V}_-^s \mathbf{V}_-^{s\top}).$$

Since \mathbf{B}^s is symmetric, it can be factorized using MUMPS [5, 6].

4. Following Remark 3.12, all four choices of preconditioners \mathbf{H}_2 for \mathbf{A}_+ considered require that the action of $\mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top}$ be implemented in order to compute the corresponding GenEO coarse space. As $\mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top}$ is a dense matrix, it is never assembled. Instead, the action of $\mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top}$ is computed as

$$\begin{aligned} \mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top} &= \mathbf{R}^s \mathbf{A} \mathbf{R}^{s\top} + \mathbf{R}^s \mathbf{A}_- \mathbf{R}^{s\top} \\ &= \mathbf{R}^s \mathbf{A} \mathbf{R}^{s\top} - \sum_{t=1}^N \mathbf{R}^s \mathbf{R}^t \mathbf{V}_-^t \mathbf{\Lambda}_-^t \mathbf{V}_-^{t\top} \mathbf{R}^t \mathbf{R}^{s\top}, \end{aligned}$$

where again $\mathbf{\Lambda}_-^t$ and \mathbf{V}_-^t are the ones from Definition 3.4. In the sum, all terms for which $\mathbf{R}^t \mathbf{R}^{s\top}$ is zero are zero.

5. If $\mathbf{H}_+^{\text{AS}} := \sum_{s=1}^N \mathbf{R}^s \mathbf{R}^{s\top} (\mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top})^{-1} \mathbf{R}^s$ is chosen as a one-level preconditioner for \mathbf{A}_+ , then it is necessary to compute the action of $(\mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top})^{-1}$. This is done by applying the Woodbury matrix identity to the formula just above.
6. The coarse solves of the form $(\mathbf{WAW}^\top)^{-1}$ and $(\mathbf{R}^0 \mathbf{A}_+ \mathbf{R}^{0\top})^{-1}$ are performed by Cholesky factorization using MUMPS [5, 6]. As an alternative, in cases where the coarse spaces become of very large dimension, multi-level methods could be considered. There is no theoretical obstruction to applying the AWG method recursively to $(\mathbf{WAW}^\top)^{-1}$ and the classical GenEO method recursively to $(\mathbf{R}^0 \mathbf{A}_+ \mathbf{R}^{0\top})^{-1}$. We leave the detailed proofs and implementation for future work.

5. Numerical results. In this section, numerical results are presented for the new AWG preconditioners with the exact coarse spaces $\mathbf{H}_{3,\text{ad}}$ and $\mathbf{H}_{3,\text{hyb}}$. The theoretical convergence bounds are verified, and the behavior of the new preconditioners is illustrated for the first time. Some comparisons to non-algebraic domain decomposition preconditioners with more standard GenEO coarse spaces are performed. The considered linear systems results from discretizing a two-dimensional linear elasticity problem with Q_1 finite elements. All details are provided below.

REMARK 5.1. The purpose of our numerical results is to verify and illustrate that the method behaves as predicted by theory. Its performance is estimated by measuring the number of iterations needed to converge and the size of the coarse spaces. We acknowledge that a true test of robustness and performance would require to solve an extensive set of problems with measures of the CPU time.

REMARK 5.2. The AWG preconditioner with inexact coarse space has not been included into the numerical study, but some numerical results can be found in [47]. The behavior of $\mathbf{H}_{3,\text{inex}}$ is not expected to differ much from the behavior of $\mathbf{H}_{3,\text{ad}}$ and $\mathbf{H}_{3,\text{hyb}}$. In particular

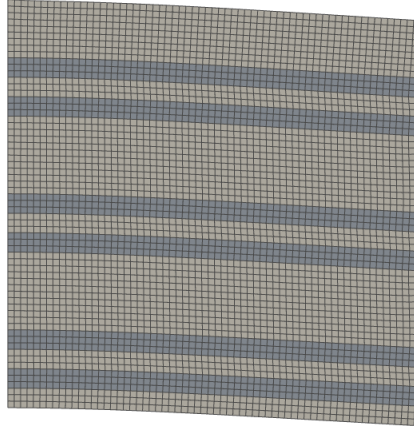


FIG. 5.1. The computational domain ω has been displaced by 10^5 multiplied by the solution \mathbf{x}_* . The colors show the distribution of coefficients as defined in (5.2). Dark: $E = E_1 = 10^{11}$ - Light: $E = E_2 = 10^7$.

they all share the same coarse spaces and have very similar convergence bounds (or exactly the same in the case of $\mathbf{H}_{3,\text{hyb}}$). In the future, when CPU time is considered, $\mathbf{H}_{3,\text{inex}}$ should be included in the comparison.

All the results presented were obtained with PETSc4py [17], a Python port to the PETSc libraries [7, 8, 9]. The eigensolves are performed by SLEPc [28], and the matrix factorizations (for the local and coarse problems) are performed by MUMPS [5, 6]. Our code is available on Github [24].

Let $\omega = [0, 3] \times [0, 3] \subset \mathbb{R}^2$ be the computational domain. Let $\partial\omega_D$ be the left-hand side boundary of ω , and let $\mathcal{V} = \{\mathbf{v} \in H^1(\omega)^2; \mathbf{v} = \mathbf{0} \text{ on } \partial\omega_D\}$. A solution $\mathbf{u} \in \mathcal{V}$ is sought such that

$$(5.1) \quad \int_{\omega} 2\mu\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx + \int_{\omega} L \operatorname{div}(\mathbf{u}) \operatorname{div}(\mathbf{v}) \, dx = \int_{\omega} \mathbf{g} \cdot \mathbf{v} \, dx, \quad \text{for all } \mathbf{v} \in \mathcal{V},$$

where, for $i, j = 1, 2$, $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$, δ_{ij} the Kronecker symbol, $\mathbf{g} = (0, -9.81)^\top$, and the Lamé coefficients are functions of Young's modulus E and Poisson's ratio ν : $\mu = \frac{E}{2(1+\nu)}$, $L = \frac{E\nu}{(1+\nu)(1-2\nu)}$. It is well known (see, e.g., [42]) that the solution of (5.1) in a heterogeneous medium is challenging due to ill-conditioning. Unless specified otherwise, the considered coefficient distribution is the following: for any $(x, y) \in \omega$,

$$(5.2) \quad \nu(x, y) = 0.3 \quad \text{and} \quad E(x, y) = \begin{cases} 10^{11} & \text{if } (\text{floor}(y) - y) \in [\frac{1}{7}, \frac{2}{7}] \cup [\frac{3}{7}, \frac{4}{7}], \\ 10^7 & \text{otherwise.} \end{cases}$$

The computational domain is discretized by a uniform mesh with element size $h = 1/21$, so there are

$$n = 8064 \text{ degrees of freedom (once removed the ones on } \partial\omega_D).$$

The boundary value problem is solved numerically with Q_1 finite elements. Let \mathcal{V}_h be the space of Q_1 finite elements that satisfy the Dirichlet boundary condition. Let $\{\phi_k\}_{k=1}^n$ be a basis of \mathcal{V}_h . The linear system to be solved is

$$\text{find } \mathbf{x}_* \in \mathbb{R}^n \text{ such that } \mathbf{A}\mathbf{x}_* = \mathbf{b},$$

with $A_{ij} = \int_{\omega} [2\mu\varepsilon(\phi_i) : \varepsilon(\phi_j) + L \operatorname{div}(\phi_i) \operatorname{div}(\phi_j)] dx$ and $\mathbf{b}_i = \int_{\omega} \mathbf{g} \cdot \phi_i dx$. The mesh, the solution, and the distribution of E are represented in Figure 5.1.

Unless specified otherwise, for each computation, the domain ω is split into

$$N = 9$$

unit square subdomains that overlap only at the interface, and this in turn gives the partition of the degrees of freedom into $\Omega^1, \dots, \Omega^9$. With Q_1 finite elements, restricting the overlap to the shared subdomain boundaries is enough to ensure the minimal overlap condition. There are 504 degrees of freedom shared by more than one subdomain. All linear systems are solved with PCG up to a relative residual tolerance of 10^{-10} . The preconditioner is specified for each test case.

The matrix $\mathbf{A}_+^{-1}\mathbf{V}_-$ is computed by solving n_- linear systems for \mathbf{A}_+^{-1} . This is also done with PCG preconditioned by \mathbf{H}_2 up to a relative residual tolerance of 10^{-10} , unless specified otherwise.

Except in the next paragraph, the AWG preconditioner under study is

$$\mathbf{H}_{3,\text{ad}} \text{ with } \mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{NN}}(\tau_{\sharp}) \text{ from Definition 4.7 and Theorem 3.14.}$$

Comparison of $\mathbf{H}_{3,\text{ad}}$, and $\mathbf{H}_{3,\text{hyb}}$ for all variants of \mathbf{H}_2 . The test case is solved with the eight AWG preconditioners for a fixed values of the threshold. Specifically, there are two ways of incorporating the second coarse space leading to $\mathbf{H}_{3,\text{ad}}$ (additive) and $\mathbf{H}_{3,\text{hyb}}$ (hybrid) as well as, for each one, four choices for \mathbf{H}_2 : $\mathbf{H}_{\text{hyb}}^{\text{NN}}(\tau_{\sharp})$, $\mathbf{H}_{\text{hyb}}^{\text{AS}}(\tau_{\sharp}, \tau_b)$, $\mathbf{H}_{+,\text{ad}}^{\text{AS}}(\tau_b)$, and $\mathbf{H}_{+,\text{hyb}}^{\text{AS}}(\tau_b)$. The thresholds for selecting eigenvalues in the GenEO coarse spaces are set to $\tau_b = 10$ and $\tau_{\sharp} = 0.1$. The results are shown in Table 5.1. As a matter of comparison, results with more classical (non-algebraic) domain decomposition preconditioners with GenEO coarse spaces presented in [46, Section 5] are also reported.

In all lines of the table corresponding to AWG, the quantity n_- (the size of the second coarse space) is the same, which is normal because the second coarse space depends only on \mathbf{A} . The size $\#V_0$ of the GenEO coarse space also appears to be the same for all choices of \mathbf{H}_2 . For $\mathbf{H}_{\text{hyb}}^{\text{NN}}(\tau_{\sharp})$, $\mathbf{H}_{\text{hyb}}^{\text{AS}}(\tau_{\sharp}, \tau_b)$, $\mathbf{H}_{+,\text{ad}}^{\text{AS}}(\tau_b)$, the same eigenvalue problem is being solved, so as long as $\tau_{\sharp} = \tau_b^{-1}$ this was expected. For the last choice $\mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{AS}}(\tau_{\sharp}, \tau_b)$, it is not entirely surprising that the size of the coarse space is not too different as there are connections between the GenEO eigenproblems, but a small difference in size would not have surprised us either.

For all AWG preconditioners, the extreme eigenvalues of the preconditioned operators behave as predicted. All AWG preconditioners reduce the condition number of the preconditioned operator to a very small value below 20 with the result that convergence to 10^{-10} occurs in at most 31 iterations. The GenEO coarse spaces constructed by AWG are of almost the same size as the classical (non algebraic) coarse spaces, which is very satisfying. Of course the AWG preconditioners bear the cost of the extra coarse space.

For $\mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{NN}}(\tau_{\sharp})$ the first 20 non-zero eigenvalues computed for the GenEO eigenproblem in each subdomain are plotted in Figure 5.2. It appears that choosing larger values of τ_{\sharp} than 10^{-1} could significantly increase the size of the coarse space without improving the condition number much as the eigenvalues are quite clustered. The eigenvectors that are selected for the coarse space are plotted in Figure 5.3. Since the unknowns are displacements, they have been represented by applying the deformation to the subdomain. The colors show the values of E . The influence of the hard (darker colored) layers can be seen, but it is not easy to make any conclusions about the eigenvectors.

In all that follows we focus on the following choice of preconditioner:

$$\mathbf{H}_{3,\text{ad}} \text{ with } \mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{NN}}(\tau_{\sharp}).$$

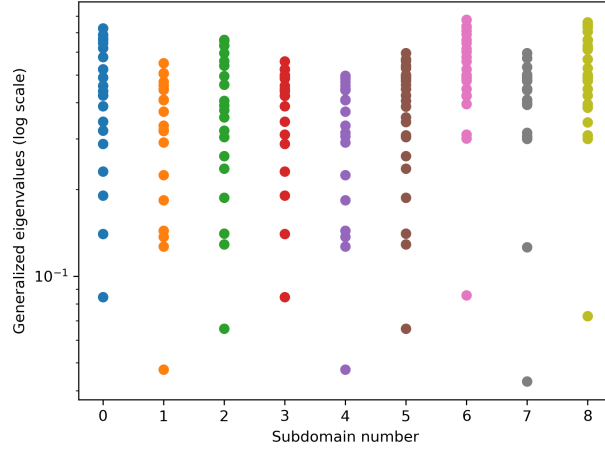


FIG. 5.2. For $\mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{NN}}(\tau_{\sharp})$: 20 smallest non-zero eigenvalues of the GenEO eigenproblem in each subdomain.

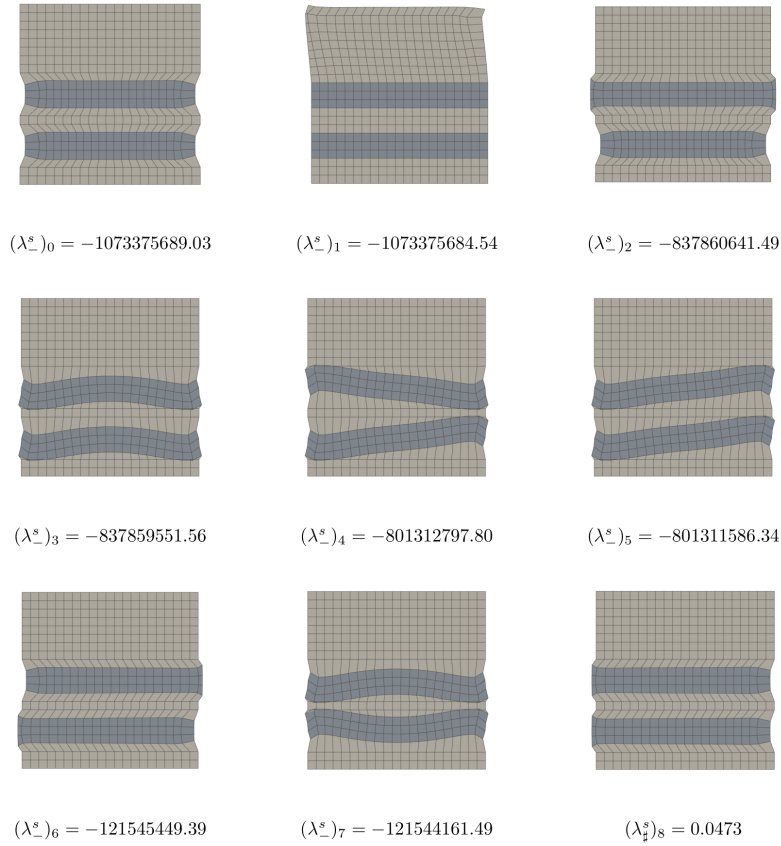


FIG. 5.3. For $\mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{NN}}(\tau_{\sharp})$: plot of the 9 vectors that are selected for the coarse space in the central subdomain ($s = 4$) represented as deformations. The colors correspond to the two values of E ; darker color is harder material. The first eight vectors correspond to a zero eigenvalue in the GenEO eigenproblem, i.e., to a negative eigenvalue λ_-^s of \mathbf{B}^s . The last vector corresponds to the first non-zero eigenvalue $(\lambda_{\sharp}^s)_8$ in the GenEO eigenproblem.

TABLE 5.1

Comparison between all AWG preconditioners for fixed τ_{\sharp} and τ_{\flat} . Data are also included for classical GenEO and the one-level method. κ : condition number of the preconditioned operator, It : number of iterations, λ_{\min} : smallest eigenvalue of the preconditioned operator, λ_{\max} : largest eigenvalue of the preconditioned operator, $\#V_0$: dimension of the GenEO coarse space, $n_- = \text{rank}(\mathbf{A}_-)$: dimension of the second coarse space.

	κ	It	λ_{\min}	λ_{\max}	$\#V_0$	n_-
New AWG preconditioners:						
$\mathbf{H}_{3,\text{ad}}$ with $\mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{NN}}(0.1)$	9.09	26	1.0	9.1	57	48
$\mathbf{H}_{3,\text{ad}}$ with $\mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{AS}}(0.1, 10)$	12.2	26	0.33	4.0	57	48
$\mathbf{H}_{3,\text{ad}}$ with $\mathbf{H}_2 = \mathbf{H}_{+, \text{hyb}}^{\text{AS}}(10)$	12.3	25	0.33	4.0	57	48
$\mathbf{H}_{3,\text{ad}}$ with $\mathbf{H}_2 = \mathbf{H}_{+, \text{ad}}^{\text{AS}}(10)$	16.8	31	0.24	4.0	57	48
$\mathbf{H}_{3,\text{hyb}}$ with $\mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{NN}}(0.1)$	9.09	27	1.0	9.1	57	48
$\mathbf{H}_{3,\text{hyb}}$ with $\mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{AS}}(0.1, 10)$	12.1	25	0.33	4.0	57	48
$\mathbf{H}_{3,\text{hyb}}$ with $\mathbf{H}_2 = \mathbf{H}_{+, \text{hyb}}^{\text{AS}}(10)$	12.2	25	0.33	4.0	57	48
$\mathbf{H}_{3,\text{hyb}}$ with $\mathbf{H}_2 = \mathbf{H}_{+, \text{ad}}^{\text{AS}}(10)$	16.7	29	0.24	4.0	57	48
Non-algebraic methods:						
Hybrid AS + GenEO ($\tau = 10$)	26.5	43	0.15	4.0	55	0
Additive AS + GenEO ($\tau = 10$)	50.0	58	0.080	4.0	55	0
BNN with GenEO ($\tau = 0.1$)	11.1	29	1.0	11.1	55	0
One-level AS	34772	> 150	0.000115	4.0	0	0

With this choice of \mathbf{H}_2 , the theory predicts that the smallest eigenvalue of the preconditioned operator is larger than 1, and this bound is observed to be sharp in Table 5.1 and more generally throughout our numerical experiments. For this reason, we no longer report on the extreme eigenvalues. Instead we only give values of the condition number κ .

Influence of the threshold τ_{\sharp} . For this test we study the influence of τ_{\sharp} . When τ_{\sharp} increases, more vectors are selected for the coarse space, and the condition number bound decreases. AWG is compared to classical Neumann Neumann GenEO, which is not algebraic. Figure 5.4 is a plot of the condition number of the preconditioned operator versus the size of the GenEO coarse space. Recall that GenEO has the disadvantage of not being algebraic, but the AWG method has the disadvantage of having a second coarse space of size 48. For AWG, the size of the coarse space cannot go below n_- because the kernels of \mathbf{A}_- are always selected. The study was performed by running the simulations for $\tau_{\sharp} \in [0, 0.001, 0.01, 0.05, 0.1, 0.2, 0.5]$. Two values of Poisson's ratio ν are considered $\nu = 0.3$ and $\nu = 0.4$. For $\nu = 0.3$, the AWG coarse space required to achieve a condition number of 10 is almost the same as that of the classical GenEO. When $\nu = 0.4$, more vectors are required for AWG.

Influence of Poisson's ratio ν . For this study, the value of ν varies between 0.2 and 0.49. Young's modulus is kept constant in the domain and is equal to $E = 10^{11}$. The threshold is $\tau_{\sharp} = 0.05$, so slightly smaller than previously. As shown in Table 5.2, increasing ν has quite a dramatic effect on n_- even without going near the incompressible limit. This is rather disappointing. The classical (non algebraic) GenEO does not suffer from this problem away from the incompressible limit $\nu \rightarrow 0.5$. The takeaway is that the matrices \mathbf{B}^s from Definition 2.3 have limitations, and that different choices should be explored.

REMARK 5.3. It is well-known that the Q_1 finite element method is not robust with respect to the incompressible limit $\nu \rightarrow 0.5$. Indeed, a phenomenon called locking appears, and the solution to the discretized system may not converge to the solution of the continuous linear elasticity system as the mesh size goes to 0. We refer the reader to [2] (and the references therein) for a presentation and analysis of the locking phenomenon and how to fix

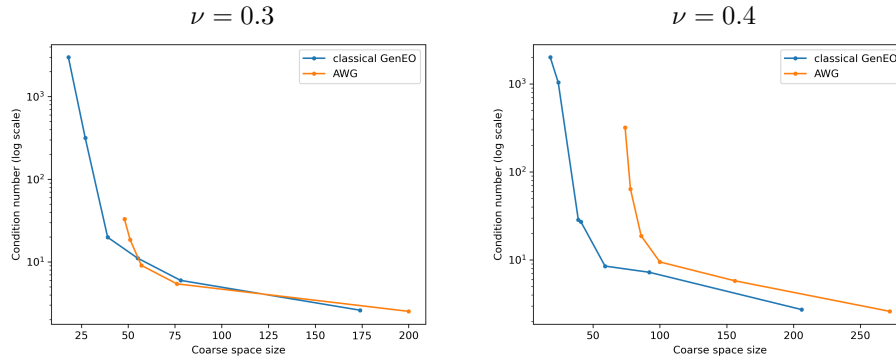


FIG. 5.4. Condition number with respect to the coarse space size for $\mathbf{H}_{3,\text{ad}}$ with $\mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{NN}}(\tau_{\sharp})$ and the comparable classical GenEO coarse space. $\tau_{\sharp} \in [0, 0.001, 0.01, 0.05, 0.1, 0.2, 0.5]$. Left: $\nu = 0.3$. Right: $\nu = 0.4$.

TABLE 5.2

The influence of Poisson's ratio ν is studied when E is constant and equal to 10^{11} . The threshold is $\tau_{\sharp} = 0.05$. ν : Poisson's ratio, κ : condition number of the preconditioned operator, It : number of iterations, $\#V_0$: dimension of the GenEO coarse space, $n_- = \text{rank}(\mathbf{A}_-)$: dimension of the second coarse space. Recall that classical GenEO is not algebraic.

AWG					Classical GenEO				
ν	κ	It	$\#V_0$	n_-	ν	κ	It	$\#V_0$	n_-
0.20	19.7	33	21	12	0.20	17.2	33	21	0
0.30	20.3	32	29	19	0.30	17.6	36	21	0
0.35	18.6	32	47	25	0.35	19.1	37	21	0
0.40	25.8	39	98	70	0.40	20.1	39	24	0
0.45	27.1	29	115	110	0.45	33.7	46	28	0
0.49	16.8	25	362	357	0.49	34.9	51	94	0

it. Nevertheless, the linear systems are well-posed, so it remains an interesting question to study how efficiently they can be solved.

Influence of E . The threshold is set back to $\tau_{\sharp} = 0.1$ and Poisson's ratio to $\nu = 0.3$ in all that follows. This time, the values E_1 and E_2 of Young's modulus in, respectively, the dark and light parts of ω in Figure 5.1 are varied. The results are illustrated in Table 5.3. We observe that all AWG condition numbers are between 8 and 12.2, so they are all very small, and fast convergence is guaranteed. The smallest coarse space size (both for the GenEO coarse space and the second coarse space) is for the case where E is constant throughout ω . The cases where $E_1 > E_2$ (hard layers in softer material) require smaller coarse spaces than the cases where $E_2 > E_1$, i.e., soft layers in harder material. Finally, the AWG coarse spaces are always larger than the (non algebraic) GenEO coarse spaces but not significantly in five cases out of seven.

Influence of the accuracy of \mathbf{W} . The second coarse space for the AWG preconditioners is computed by solving n_- linear systems: $\mathbf{A}_+ \setminus (\mathbf{R}^s \top \mathbf{v}_-^s)$ for the vectors \mathbf{v}_-^s that correspond to a negative eigenvalue of \mathbf{B}^s , $s \in \llbracket 1, N \rrbracket$. Until now, we have solved these with very high accuracy: the relative residual tolerance $rtol$ was set to 10^{-10} . In this experiment we vary $rtol$. Table 5.4 shows how increasing $rtol$ affects the condition number of \mathbf{A} preconditioned by AWG. Two cases have been studied with different Poisson's ratios: $\nu = 0.3$ and $\nu = 0.4$. Up to $rtol = 10^{-2}$ there is no change compared to $rtol = 10^{-10}$, and these intermediary

TABLE 5.3

The influence of E and of the jump between E_1 and E_2 is studied. (E_1, E_2) : values of Young's modulus in the layers of coefficients, κ : condition number of the preconditioned operator, It : number of iterations, $\#V_0$: dimension of the GenEO coarse space, $n_- = \text{rank}(\mathbf{A}_-)$: dimension of the second coarse space. Recall that classical GenEO is not algebraic.

AWG					Classical GenEO				
(E_1, E_2)	κ	It	$\#V_0$	n_-	(E_1, E_2)	κ	It	$\#V_0$	n_-
$(10^5, 10^{11})$	10.8	22	95	75	$(10^5, 10^{11})$	8.6	23	90	0
$(10^7, 10^{11})$	10.8	23	95	75	$(10^7, 10^{11})$	8.6	26	87	0
$(10^9, 10^{11})$	10.4	24	94	73	$(10^9, 10^{11})$	8.5	25	85	0
$(10^{11}, 10^{11})$	12.2	29	35	19	$(10^{11}, 10^{11})$	13.7	32	28	0
$(10^{11}, 10^9)$	8.0	26	59	48	$(10^{11}, 10^9)$	11.2	30	52	0
$(10^{11}, 10^7)$	9.0	26	57	48	$(10^{11}, 10^7)$	11.1	29	55	0
$(10^{11}, 10^5)$	8.4	29	57	48	$(10^{11}, 10^5)$	12.7	30	55	0

TABLE 5.4

Influence of the accuracy of $rtol$ up to which the linear systems with \mathbf{A}_+ preconditioned by \mathbf{H}_2 are solved during the setup of the second coarse basis \mathbf{W} . $rtol$: tolerance, κ : condition number of the preconditioned operator, It : number of iterations, $\#V_0$: dimension of the GenEO coarse space, $n_- = \text{rank}(\mathbf{A}_-)$: dimension of the second coarse space.

$\nu = 0.3$					$\nu = 0.4$				
$rtol$	κ	It	$\#V_0$	n_-	$rtol$	κ	It	$\#V_0$	n_-
10^{-10}	9.0	26	57	48	10^{-10}	9.4	29	100	74
10^{-2}	9.0	27	57	48	10^{-2}	9.4	30	100	74
0.05	11.1	31	57	48	0.05	12.0	33	100	74
0.1	12.2	32	57	48	0.1	17.4	36	100	74
0.5	400.8	40	57	48	0.5	1563.3	88	100	74
0.9	706.8	64	57	48	0.9	2142.1	100	100	74

results have not been reported in the table. In fact, up to $rtol = 0.5$, the condition number is hardly degraded. It must be kept in mind that this is a case with few subdomains, and what we observe to be a small change in κ could become more significant with more subdomains. Still, the conclusion is optimistic: the linear solves with \mathbf{A}_+ do not need to be overly precise.

Varying the number of harder layers. This time the number of layers of the harder coefficient varies. The case with six layers is the usual one from (5.2) represented in Figure 5.1. The case with nine layers is obtained by also setting $E = 10^{11}$ if $(\text{floor}(y) - y) \in [5/7, 6/7]$. The case with three layers is obtained by setting $E = 10^{11}$ only if $(\text{floor}(y) - y) \in [1/7, 2/7]$. Two additional cases with homogeneous hard and soft material are also considered. The results are shown in Table 5.5.

We first observe that distributions of E with more discontinuities require more coarse vectors. The AWG coarse space is always larger than the (non-algebraic) classical GenEO coarse space but not significantly. We also verify that homogeneous distributions of E have the same behavior for different values of E , and this is expected because \mathbf{A} and \mathbf{b} are linear in E .

Long domain with layers of coefficients. This test case (represented in Figure 5.5) is often studied in domain decomposition articles and presentations. Its drawback is that it does not have any crosspoints (degrees of freedom that belong to more than two subdomains) but all simulations up until now had crosspoints, and they do not appear to be an issue for AWG preconditioners. In this paragraph, weak scalability is studied. The parameters in the test case are:

TABLE 5.5

The number of layers of the harder coefficient varies. Two cases with homogeneous E are also considered. κ : condition number of the preconditioned operator, It : number of iterations, $\#V_0$: dimension of the GenEO coarse space, $n_- = \text{rank}(\mathbf{A}_-)$: dimension of the second coarse space. Recall that classical GenEO is not algebraic.

AWG					Classical GenEO				
E	κ	It	$\#V_0$	n_-	E	κ	It	$\#V_0$	n_-
$E = 10^{11}$	12.2	29	35	19	$E = 10^{11}$	13.7	32	28	0
9 layers	4.9	17	72	72	9 layers	4.8	20	69	0
6 layers	9.0	26	57	48	6 layers	11.1	29	55	0
3 layers	9.8	29	43	25	3 layer	9.9	31	35	0
$E = 10^7$	12.2	29	35	19	$E = 10^7$	13.7	32	28	0

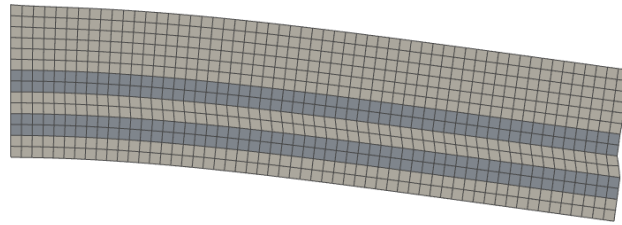


FIG. 5.5. *The computational domain ω has been displaced by 10^5 multiplied by the solution \mathbf{x}_* . The colors show the distribution of coefficients. Dark: $E = E_1 = 10^{11}$. Light: $E = E_2 = 10^7$.*

TABLE 5.6

The number N of subdomains increases, the problem size is proportional to the number of subdomains. Weak scalable behavior would be for the number of iterations to remain constant, and this is what is observed. N : number of subdomains, κ : condition number of the preconditioned operator, It : number of iterations, $\#V_0$: dimension of the GenEO coarse space, $n_- = \text{rank}(\mathbf{A}_-)$: dimension of the second coarse space. Recall that classical GenEO is not algebraic.

AWG					Classical GenEO				
N	κ	It	$\#V_0$	n_-	N	κ	It	$\#V_0$	n_-
2	12.6	15	8	8	2	9.5	15	7	0
4	9.8	16	26	20	4	11.9	19	19	0
8	9.0	15	62	44	8	12.6	23	43	0
15	8.8	15	125	86	15	12.8	27	85	0
29	8.7	17	251	170	29	12.8	28	169	0

- $N \in \{2; 4; 8; 15; 29\}$ (number of unit-square subdomains),
- $\omega = [N, 1]$ (computational domain parameterized by number of subdomains),
- $h = 1/14$ (mesh size),
- $\nu = 0.3$ (Poisson's ratio),
- $E = \begin{cases} E_1 = 10^{11} & \text{if } y \in [1/7; 2/7] \cup [3/7; 4/7], \\ E_2 = 10^7 & \text{otherwise,} \end{cases}$ (Young's modulus),
- $\mathbf{H}_{3,\text{ad}}$ with $\mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{NN}}(0.1)$ (preconditioner),
- $rtol = 10^{-10}$, the relative residual tolerance for the linear solves with \mathbf{A}_+ and \mathbf{A} .

The results are shown in Table 5.6. As predicted theoretically, with the AWG preconditioner the condition number hardly increases with the number of subdomains, and this points

towards weak scalability. For all values of N , the subdomains are identical (with a difference between ones that are at the edge of ω and others), so the coarse space grows almost linearly with the number of subdomains. This can be viewed as a first weak scalability result.

6. Conclusion. In this article new preconditioners, called AWG for Algebraic Woodbury-GenEO, have been introduced. Combined with PCG, they are algebraic domain decomposition methods with two coarse spaces. Convergence in a small number of iterations can be guaranteed by adjusting some user-chosen thresholds and enriching one of the coarse spaces accordingly. Numerical results have been presented as a proof of concept and to illustrate the behavior of the AWG preconditioners on some simple test cases. Further numerical simulations must be performed to assess the overall efficiency of the AWG preconditioners. Some possible improvements of the AWG preconditioners are still under investigation. This includes decreasing the size n_- of the second coarse space by proposing other choices of \mathbf{B}^s , finding sparser approximations for the second coarse space and coarse solve, and considering multilevel versions of AWG.

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