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ORTHOGONALITY ON THE SEMICIRCLE: OLD AND NEW RESULTS*

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Dedicated to Professor Lothar Reichel on the Occasion of His 70th Anniversary.

Abstract. Orthogonal polynomials on the semicircle were introduced by Gautschi and Milovanović in [Rend. Sem. Mat. Univ. Politec. Torino, Special Issue (July 1985), pp. 179-185] and [J. Approx. Theory, 46 (1986), pp. 230-250]. In this paper we give an account of this kind of orthogonality, weighted generalizations mainly oriented to Chebyshev weights of the first and second kinds, including several interesting properties of such polynomials. Moreover, we also present a number of new results including those for Laurent polynomials (rational functions) orthogonal on the semicircle. In particular, we give their recurrence relations and study special cases for the Legendre weight and for the Chebyshev weights of the first and second kind. Explicit expressions for such orthogonal systems with Chebyshev weights are presented, as well as the corresponding zero distributions.

Key words. complex orthogonal systems, recurrence relations, zeros, weight function, orthogonal Laurent polynomials

AMS subject classifications. 30C10, 30C15, 33C45, 33C47, 42C05

1. Introduction. Let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree at most n, $\mathcal{P}$ be a set of all polynomials, and $\{\pi_k\}$ be a system of monic orthogonal polynomials with respect to a given inner product $(\,\cdot\,,\,\cdot\,)$.

Two standard types of orthogonal polynomials are *polynomials orthogonal on the real line* (cf. Szegő [30], Gautschi [6]), when the inner product is given by

$$(1.1)\qquad (p,q)=\int_{\mathbb{R}} p(x)q(x)\mathrm{d}\sigma(x)\quad (p,q\in\mathcal{P}),$$

where $\mathrm{d}\sigma(x)$ is a given nonnegative measure with finite or unbounded support, and *polynomials orthogonal on the unit circle* (cf. Simon [25, 26]), when

$$(1.2)\qquad (p,q)=\int_{-\pi}^{\pi} p(\mathrm{e}^{\mathrm{i}\theta})\overline{q(\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{d}\sigma(\theta)\quad (p,q\in\mathcal{P}),$$

where $\mathrm{d}\sigma(\theta)$ is a positive measure on the interval $[-\pi,\pi]$ whose support is an infinity set. The last class of polynomials on a unit circle was introduced by Szegő; see [28, 29].

If $x\mapsto\sigma(x)$ is an absolutely continuous function in (1.1), then $\mathrm{d}\sigma(x)=w(x)\mathrm{d}x$, where $w(x)=\sigma'(x)$ is the weight function, i.e., a nonnegative function measurable in the Lebesgue sense, for which all moments μ_k $(k=0,1,\ldots)$ exist and $\mu_0>0$. The famous examples are classical (Jacobi, generalized Laguerre and Hermite) weight functions.

Orthogonal polynomials on curves and domains have also appeared occasionally; cf. Mastroianni and Milovanović [14, p. 80].

In this paper we are interested in orthogonal polynomials on the semicircle [7, 8, 9], with the inner product given by

$$(1.3)\qquad \langle p,q\rangle=\int_{0}^{\pi} p(\mathrm{e}^{\mathrm{i}\theta})q(\mathrm{e}^{\mathrm{i}\theta})w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta\quad (p,q\in\mathcal{P}),$$

where the second factor is not conjugated as in (1.2), so that the inner product (1.3) is not Hermitian. Beside an account of old results for this type of orthogonality, we present some new results.

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In [24] Lothar Reichel wrote: "In 1985, before the publication of the first paper by Gautschi and Milovanović [GA95] on orthogonal polynomials on the semicircle, only polynomials orthogonal with respect to an inner product on an interval or on a circle were known to satisfy recurrence relations with few terms and not to suffer from the possibility of break down. The results of this paper and of the more complete investigations [GA97, GA104] therefore were quite surprising. The uncovering of the many nice properties of orthogonal polynomials on the semicircle was very important for analysis, approximation theory, and computational mathematics, and has spurred related work."[1]

This paper is organized as follows. In Section 2 we give a short account of first results on orthogonal polynomials on the semicircle and give same of their important properties, including considerations for the Gegenbauer weight function. The so-called Laurent polynomials on the semicircle and their numerator polynomials, with their recurrence relations, are studied in Section 3, while special classes of these orthogonal systems with respect to Legendre and Chebyshev weights of the first and second kind are treated in Section 4. Finally, Section 5 is devoted to zero distribution of these classes of orthogonal systems.

2. Orthogonal polynomials on the semicircle. In the non-Hermitian case (1.3), Gautschi and Milovanović firstly treated the simplest (Legendre) case $w(z) = 1$ by using an approach based on a *complex moment functional* (cf. Chihara [2, pp. 6–10]) given by

$$\mathcal{L}[z^k] = \mu_k, \quad \mu_k = \langle 1, z^k\rangle = \int_0^\pi \mathrm{e}^{\mathrm{i}k\theta}\mathrm{d}\theta = \begin{cases} \pi, & k = 0, \\ 2\mathrm{i}/k, & k \text{ is odd}, \\ 0, & k \text{ is even}, \ k \neq 0, \end{cases}$$

and proved that the related orthogonal polynomials exist uniquely, because the moment sequence $\{\mu_k\}$ is quasi-definite, i.e.,

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} \neq 0, \quad n = 1, 2, \ldots$$

Precisely, after a long calculation, they proved that

$$\Delta_n = \begin{cases} \dfrac{2^{(n-1)^2}\pi}{[((n-1)/2)!]^4} \left\{ \displaystyle\prod_{k=1}^{n-1} k! \prod_{k=1}^{(n-1)/2} \frac{(2k)!}{(n+2k-1)!} \right\}^2, & n \text{ (odd)} \geq 1, \\ \dfrac{2^{n^2}}{[n!]^2} \left\{ \displaystyle\prod_{k=1}^{n-1} k! \prod_{k=1}^{n/2} \frac{(2k)!}{(n+2k-2)!} \right\}^2, & n \text{ (even)} \geq 2, \end{cases}$$

i.e., $\Delta_n > 0$ for each $n \geq 1$, as well as the three-term recurrence relation (because of $\langle zp, q\rangle = \langle p, zq\rangle$)

$$\pi_{k+1}(z) = (z - \mathrm{i}\alpha_k)\pi_k(z) - \beta_k\pi_{k-1}(z), \quad k = 0, 1, \ldots, \tag{2.1}$$

with starting polynomials $\pi_{-1}(z) = 0$, $\pi_0(z) = 1$, where

$$\alpha_0 = \theta_0, \quad \alpha_k = \theta_k - \theta_{k-1}, \quad \beta_k = \theta_{k-1}^2, \ k \geq 1,$$

[1] References [GA95, GA97, GA104] are respectively [7, 8, 9] in this paper.

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and

$$\theta_k = \frac{2}{2k+1}\left[\frac{\Gamma((k+2)/2)}{\Gamma((k+1)/2)}\right]^2, \quad k \geq 0.$$

The case with a weight function w in (1.3) was considered in [9]. The function w is positive and integrable on $(-1,1)$, with possible singularities at ± 1, and which can be extended to a holomorphic function $w(z)$ in the half disc $D_+ = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$.

To connect two monic polynomial systems $\{\pi_k\}$ and $\{p_k\}$, the first orthogonal with respect to the non-Hermitian product (1.3) and the second one orthogonal to the standard inner product (1.1), we take a contour C_ε, with small circular parts of radius ε and centers at ± 1 (see Figure 2.1), and consider the weighted integral of an arbitrary polynomial $g \in \mathcal{P}$ over C_ε. Then, by Cauchy's theorem, we have $\int_{C_\varepsilon} g(z)w(z)\mathrm{d}z = 0$. Supposing that the weight function w is such that integrals over $\gamma_{\varepsilon,\pm 1}$ tend to zero when $\varepsilon \to 0$, we obtain the following connection

$$(\forall g \in \mathcal{P}) \qquad \int_\Gamma g(z)w(z)\mathrm{d}z + \int_{-1}^{1} g(z)w(z)\mathrm{d}z = 0.$$

It enables us to express the polynomials π_k in terms of the real polynomials $\{p_k\}$. Under the

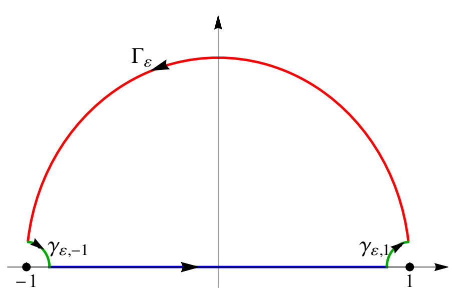

FIG. 2.1. *The contour* $C_\varepsilon = [-1+\varepsilon, 1-\varepsilon] \cup \gamma_{\varepsilon,1} \cup \Gamma_\varepsilon \cup \gamma_{\varepsilon,-1}$.

mild restriction

$$\operatorname{Re}\langle 1,1\rangle = \operatorname{Re}\left\{\int_0^\pi w\big(\mathrm{e}^{\mathrm{i}\theta}\big)\mathrm{d}\theta\right\} \neq 0, \tag{2.2}$$

we proved that the orthogonal polynomials $\{\pi_k\}$ exist and can be expressed in terms of the real orthogonal polynomials $\{p_k\}$

$$\pi_k(z) = p_k(z) - \mathrm{i}\theta_{k-1}p_{k-1}(z), \quad k = 0, 1, \ldots,$$

where

$$\theta_{k-1} = \frac{\mu_0 p_k(0) + \mathrm{i}q_k(0)}{\mathrm{i}\mu_0 p_{k-1}(0) - q_{k-1}(0)}, \quad k = 0, 1, \ldots,$$

$\mu_0 = \langle 1,1\rangle = \int_0^\pi w\big(\mathrm{e}^{\mathrm{i}\theta}\big)\mathrm{d}\theta$, and $\{q_k\}$ are the associated polynomials, defined by (cf. [14, pp. 111–114])

$$q_k(z) = \int_{-1}^{1} \frac{p_k(z) - p_k(x)}{z - x} w(x)\mathrm{d}x, \quad k = 0, 1, \ldots$$

Also, the polynomials π_k satisfy the three-term recurrence relation of the form (2.1), where the coefficients α_k and β_k are given by

$$\alpha_0 = \frac{b_0}{\mu_0}, \quad \alpha_k = -\theta_{k-1} + \frac{b_k}{\theta_{k-1}} \quad (k \geq 1)$$

and

$$\beta_k = \frac{\theta_{k-1}}{\theta_{k-2}} b_{k-1} = \theta_{k-1}(\theta_{k-1} - \mathrm{i}a_{k-1}),$$

where a_k and b_k are coefficients in the corresponding three-term relation for the real orthogonal polynomials $\{p_k\}$; for details see [9].

It follows from the three-term recurrence relation (2.1) that the zeros of the polynomial $\pi_n(z)$ orthogonal on the semicircle are the eigenvalues of the (complex, tridiagonal) matrix

$$J_n = \begin{bmatrix} \mathrm{i}\alpha_0 & 1 & & & & \mathbf{0} \\ \beta_1 & \mathrm{i}\alpha_1 & 1 & & & \\ & \beta_2 & \mathrm{i}\alpha_2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & 1 \\ \mathbf{0} & & & & \beta_{n-1} & \mathrm{i}\alpha_{n-1} \end{bmatrix}.$$

Under certain conditions, all zeros of polynomials π_n orthogonal on the semicircle are in D_+. Some additional results on the zeros of such orthogonal polynomials are given in [5].

Polynomials orthogonal on a circular arc were considered by de Bruin [3] and Milovanović and Rajković [21], as well as the Geronimus concept of orthogonality for such polynomials [20].

Some applications of polynomials $\{\pi_k\}$ orthogonal on the semicircle in numerical differentiation and numerical integration were given in [8], [1], [15] and [17], including error bounds for the Gauss quadrature rules on the semicircle. The so-called s-orthogonal polynomials and multiple orthogonal polynomials on the semicircle, as well as the corresponding quadratures of Gaussian type, were considered in [27], [23], and [22].

2.1. Gegenbauer weight function. In this subsection we pay special attention to some results related to the Gegenbauer weight function $w(z) = w^\lambda(z) = (1-z^2)^{\lambda-1/2}$, with the parameter $\lambda > -1/2$. Since the assumption (2.2) is satisfied, $\mu_0 = \langle 1, 1 \rangle = \pi \neq 0$, the corresponding orthogonal polynomials $\{\pi_k^\lambda\}$ on the semicircle Γ, in this case, exist uniquely and can be expressed in terms of monic Gegenbauer polynomials $\widehat{C}_k(z)$ as

$$\pi_k^\lambda(z) = \widehat{C}_k^\lambda(z) - \mathrm{i}\theta_{k-1}\widehat{C}_{k-1}^\lambda(z),$$

where the sequence $\{\theta_k\}$ is given recursively by

$$\theta_0 = \frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\lambda+1)}, \quad \theta_k = \frac{k(k+2\lambda-1)}{4(k+\lambda)(k+\lambda-1)} \cdot \frac{1}{\theta_{k-1}}, \quad k = 1, 2, \ldots,$$

wherefrom we can obtain an explicit form in terms of the gamma function,

$$\theta_k = \frac{1}{\lambda+k} \cdot \frac{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\lambda + \frac{k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\lambda + \frac{k}{2}\right)}, \quad k \geq 0. \tag{2.3}$$

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These polynomials $\{\pi_k\}$ satisfy the three-term recurrence relation (2.1), with the coefficients (see [9] and [16])

$$\alpha_0 = \theta_0\,, \quad \alpha_k = \theta_k - \theta_{k-1}\,, \quad \beta_k = \theta_{k-1}^2\,, \quad k \geq 1.$$

Using Stirling's formula (cf. [18, p. 111])

$$\Gamma(z) = \sqrt{2\pi} z^{z-1/2} \mathrm{e}^{-z} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} + O\left(\frac{1}{z^3}\right)\right]$$

in (2.3), we find that $\theta_k \to 1/2$, i.e., $\alpha_k \to 0$ and $\beta_k \to 1/4$, when $k \to +\infty$.

If $\lambda > -1/2$, then all zeros of the orthogonal polynomials $\pi_k^\lambda(z)$, $k \geq 2$, are simple, distributed symmetrically with respect to the imaginary axis, and contained in the open upper unit half disk $D_+ = \{z \in \mathbb{C} : |z| < 1,\ \mathrm{Im}\, z > 0\}$; see [9, Thms. 6.5 & 6.7].

The zero distribution of $\pi_k^\lambda(z)$ for $\lambda = 0, 1/2, 1$ is presented in Figure 2.2, for $k = 3, 6, 10$. For example, the zeros of $\pi_3^\lambda(z)$, i.e.,

$$\pi_3^0(z) = z^3 - \frac{\mathrm{i}z^2}{2} - \frac{3z}{4} + \frac{\mathrm{i}}{4}, \quad \pi_3^{1/2}(z) = z^3 - \frac{8\mathrm{i}z^2}{5\pi} - \frac{3z}{5} + \frac{8\mathrm{i}}{15\pi}, \quad \pi_3^1(z) = z^3 - \frac{\mathrm{i}z^2}{2} - \frac{z}{2} + \frac{\mathrm{i}}{8}$$

are $\{\mathrm{i}\,0.3576126,\ \pm 0.8330738 + \mathrm{i}\,0.0711937\}$, $\{\mathrm{i}\,0.3150768,\ \pm 0.7275822 + \mathrm{i}\,0.0971095\}$, $\{\mathrm{i}\,0.2849201,\ \pm 0.6535706 + \mathrm{i}\,0.1075399\}$, respectively. With increasing k the zeros tend to fall to the interval $[-1, 1]$.

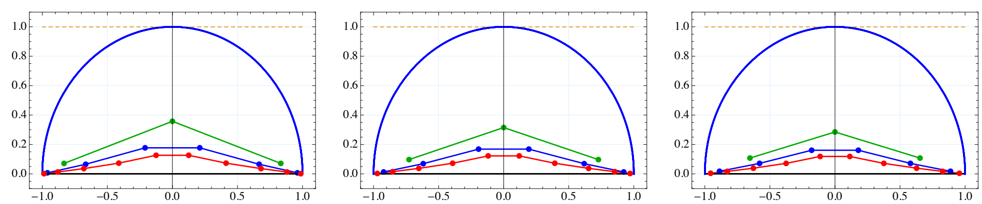

FIG. 2.2. *Zeros of $\pi_k^\lambda(z)$ for $k = 3$ (green), $k = 6$ (blue), $k = 10$ (red), when $\lambda = 0$ (left), $\lambda = 1/2$ (middle) and $\lambda = 1$ (right).*

In particular, two Gegenbauer cases are very interesting (see [19]).

1. For $\lambda = 0$ (*Chebyshev weight of the first kind* $w^0(z) = 1/\sqrt{1-z^2}$), we have

$$\pi_0^0(z) = 1, \quad \pi_k^0(z) = \frac{1}{2^{k-1}}\left(T_k(z) - \mathrm{i}T_{k-1}(z)\right), \quad k \geq 1,$$

or in the explicit form

$$\pi_k^0(z) = \frac{1}{2^k}\left\{\left[1 - \mathrm{i}\left(z - \sqrt{z^2-1}\right)\right]\left(z + \sqrt{z^2-1}\right)^k + \left[1 - \mathrm{i}\left(z + \sqrt{z^2-1}\right)\right]\left(z - \sqrt{z^2-1}\right)^k\right\}.$$

2. For $\lambda = 1$ (*Chebyshev weight of the second kind* $w^1(z) = \sqrt{1-z^2}$), we have

$$\pi_0^1(z) = 1, \quad \pi_k^1(z) = \frac{1}{2^k}\left(U_k(z) - \mathrm{i}U_{k-1}(z)\right), \quad k \geq 1,$$

or in the explicit form

$$\pi_k^1(z) = \frac{1}{2^{k+1}\sqrt{z^2-1}} \left\{ \left[\left(z+\sqrt{z^2-1}\right) - \mathrm{i} \right] \left(z+\sqrt{z^2-1}\right)^k \right.$$
$$\left. - \left[\left(z-\sqrt{z^2-1}\right) - \mathrm{i} \right] \left(z-\sqrt{z^2-1}\right)^k \right\}.$$

In all cases, $|z+\sqrt{z^2-1}| > 1$, when $z \in \mathbb{C} \setminus [-1,1]$.

3. Laurent polynomials on the semicircle. As before we use the non-Hermitian inner product (1.3), with an appropriate weight function w, and again the concept of orthogonality with respect to the complex moment functional

$$\mathcal{L}[z^k] = \mu_k = \langle 1, z^k \rangle, \quad k = 0, \pm 1, \pm 2, \dots \, , \tag{3.1}$$

which includes positive and negative exponents.

3.1. The linear space $\boldsymbol{\Lambda_{p,q}}$ and orthogonal Laurent polynomials. To begin with, let $\Lambda_{p,q}$ be a linear space, over the complex numbers $\mathbb{C}$, of polynomials generated by the basis $\mathcal{B}_{p,q} = \{z^p, z^{p+1}, \dots, z^q\}$, where $p \le q$ and $p, q \in \mathbb{Z}$.

Note that the standard space of polynomials of degree at most n $(\in \mathbb{N})$ of dimension $n+1$, i.e., $\mathcal{P}_n$ is just $\Lambda_{0,n}$. For $p = q = 0$, the space $\Lambda_{p,q}$ reduces to the space of complex constants

$$\Lambda_{0,0} = \Lambda_0 = \mathcal{P}_0 = \{c \in \mathbb{C} : c \neq 0\}.$$

For each $n \in \mathbb{N}$, we usually take $p = -n+1$ and $q = n$, so that the dimension of the space $\Lambda_{-n+1,n}$ is equal to $2n$. Such a space is appropriate, for example, for considering Gaussian quadrature formulas. Its extension when $n \to \infty$ will be denoted by $\Lambda_{\pm}$. We can also consider other spaces, e.g., $\Lambda_{-n,n-1}$, $\Lambda_{-n,n}$, etc.

Let $\{R_m(z)\}$ be a system of orthogonal elements in $\Lambda_{\pm}$, i.e., *orthogonal Laurent polynomials*, obtained by applying the well-known Gram-Schmidt orthogonalization process to the sequence of monomials $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$.

REMARK 3.1. An alternative system $\{\widetilde{R}_m(z)\}$ can be generated by the following sequence $\{1, z^{-1}, z, z^{-2}, z^2, \dots\}$ and it can be expressed in terms of the polynomials $R_m(z)$.

By the construction orthogonal Laurent polynomials from the sequence of monomials $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$, we conclude that

$$R_m(z) \in \Lambda_{-[m/2],[(m+1)/2]} \qquad (m \in \mathbb{N}_0),$$

so that the orthogonal element (Laurent's polynomial) $R_m(z)$ can be represented as

$$R_m(z) = \sum_{\nu=0}^{m} c_\nu^{(m)} z^{\nu-[m/2]}, \quad m \in \mathbb{N}_0, \tag{3.2}$$

for some constants $c_\nu^{(m)}$, $\nu = 0, 1, \dots, m$. The coefficients $c_m^{(m)}$ and $c_0^{(m)}$ are called the *leading* and *trailing coefficient* of $R_m(z)$, respectively. In our consideration these coefficients $c_m^{(m)}$ and $c_0^{(m)}$ are different from zero, and we always take the leading coefficient to be $c_m^{(m)} = 1$, so that for orthogonal Laurent's polynomials of even and odd indices we have

$$R_{2k}(z) = c_0^{(2k)} z^{-k} + c_1^{(2k)} z^{-k+1} + \cdots + c_{2k-1}^{(2k)} z^{k-1} + z^k, \tag{3.3}$$

$$R_{2k+1}(z) = c_0^{(2k+1)} z^{-k} + c_1^{(2k+1)} z^{-k+1} + \cdots + c_{2k}^{(2k+1)} z^k + z^{k+1}. \tag{3.4}$$

For such an orthogonal system $\{R_m(z)\}$, with respect to the non-Hermitian inner product (1.3), we have

$$\langle R_n, R_m\rangle = \int_0^\pi R_n(\mathrm{e}^{\mathrm{i}\theta})R_m(\mathrm{e}^{\mathrm{i}\theta})w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \|R_n\|^2\delta_{n,m} \quad (n,m\in\mathbb{N}_0), \tag{3.5}$$

where $\delta_{n,m}$ is Kronecker's delta and the quasi-norm

$$\|R_n\|^2 = \langle R_n, R_n\rangle = \int_0^\pi R_n(\mathrm{e}^{\mathrm{i}\theta})^2 w(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \qquad (n\in\mathbb{N}_0)$$

is, in general, a complex number.

3.2. Recurrence relations for the orthogonal Laurent polynomials. Supposing that orthogonal Laurent polynomials exist for a given weight function w, we can prove the following result. Otherwise, all weight functions used in this paper ensure the existence of these Laurent orthogonal polynomials.

THEOREM 3.2. *Let the complex moment functional be given by* (3.1) *and the corresponding Laurent polynomials $R_m(z)$ be orthogonal in $\Lambda_\pm$ satisfying* (3.5)*. Then the following two three-term recurrence relations*

$$R_{2k+1}(z) = (z-a_{2k})R_{2k}(z) + b_{2k}R_{2k-1}(z) \tag{3.6}$$

and

$$R_{2k+2}(z) = \left(1-\frac{a_{2k+1}}{z}\right)R_{2k+1}(z) + b_{2k+1}R_{2k}(z) \tag{3.7}$$

hold, where $R_0(z)=1$ and $R_{-1}(z)=0$, and $\{a_k\}$ and $\{b_k\}$ are sequences of complex numbers depending only on the weight function $w(z)$.

Proof. Evidently, $R_0(z)=1$ and $R_1(z) = z-\langle 1,z\rangle/\langle 1,z\rangle = z-\mu_1/\mu_0$. Using the previous expansions (3.3) and (3.4), we analyze the difference $R_{2k+1}(z)-zR_{2k}(z)$ for $k\ge 1$,

$$\begin{aligned} R_{2k+1}(z)-zR_{2k}(z) &= \sum_{\nu=0}^{2k+1} c_\nu^{(2k+1)}z^{\nu-k} - \sum_{\nu=0}^{2k} c_\nu^{(2k)}z^{\nu+1-k} \\ &= c_0^{(2k+1)}z^{-k} + \sum_{\nu=1}^{2k}\big(c_\nu^{(2k+1)} - c_{\nu-1}^{(2k)}\big)z^{\nu-k}. \end{aligned}$$

Since it belongs to the space $\Lambda_{-k,k}$, this difference can be expressed as a linear combination of the orthogonal Laurent polynomials

$$R_{2k+1}(z)-zR_{2k}(z) = \sum_{\nu=0}^{2k} d_\nu^{(k)}R_\nu(z), \tag{3.8}$$

so that, because of the orthogonality (3.5),

$$\langle R_{2k+1}, R_j\rangle - \langle zR_{2k}, R_j\rangle = \sum_{\nu=0}^{2k} d_\nu^{(k)}\langle R_\nu, R_j\rangle = d_j^{(k)}\langle R_j, R_j\rangle, \quad 0\le j\le 2k.$$

Note that for the non-Hermitian inner product (3.5) we have $\langle zR_{2k}, R_j\rangle = \langle R_{2k}, zR_j\rangle$. Taking $j\le 2k-2$, we conclude that $d_j^{(k)}=0$, while for $j=2k-1$ and $j=2k$, we get the following equalities

$$-\langle R_{2k}, zR_{2k-1}\rangle = d_{2k-1}^{(k)}\langle R_{2k-1}, R_{2k-1}\rangle \quad \text{and} \quad -\langle R_{2k}, zR_{2k}\rangle = d_{2k}^{(k)}\langle R_{2k}, R_{2k}\rangle.$$

Using the abbreviations for the coefficients $d_{2k}^{(k)} = -a_{2k}$ and $d_{2k-1}^{(k)} = b_{2k}$, according to (3.8), we prove the first recurrence relation (3.6), with

$$a_{2k} = \frac{\langle R_{2k}, zR_{2k}\rangle}{\langle R_{2k}, R_{2k}\rangle} \quad \text{and} \quad b_{2k} = -\frac{\langle R_{2k}, zR_{2k-1}\rangle}{\langle R_{2k-1}, R_{2k-1}\rangle}. \tag{3.9}$$

To prove the second recurrence relation (3.7), we again start with (3.3) and (3.4) in the expression $E_k(z;\gamma_k) := R_{2k+2}(z) - R_{2k+1}(z) + \gamma_k z^{-1} R_{2k+1}(z)$, with a free parameter γ_k. Since

$$\begin{aligned} E_k(z;\gamma_k) &= \sum_{\nu=0}^{2k+2} c_\nu^{(2k+2)} z^{\nu-k-1} - \sum_{\nu=0}^{2k+1} c_\nu^{(2k+1)} z^{\nu-k} + \gamma_k \sum_{\nu=0}^{2k+1} c_\nu^{(2k+1)} z^{\nu-k-1} \\ &= \big(c_0^{(2k+2)} + \gamma_k c_0^{(2k+2)}\big) z^{-k-1} + \sum_{\nu=0}^{2k} \big(c_{\nu+1}^{(2k+2)} - c_\nu^{(2k+1)} + \gamma_k c_{\nu+1}^{(2k+1)}\big) z^{\nu-k}, \end{aligned}$$

because $c_{2k+2}^{(2k+2)} = c_{2k+1}^{(2k+1)} = 1$, we see that $E_k(z;\gamma_k)$ belongs to the space $\Lambda_{-k-1,k}$. On the other side, by hypothesis, the trailing coefficients $c_0^{(2k+1)}$ and $c_0^{(2k+2)}$ are different from zero, and we can take $\gamma_k = a_{2k+1} = -c_0^{(2k+1)}/c_0^{(2k+2)}$, so that $E_k(z;a_{2k+1}) \in \Lambda_{-k,k}$. Therefore we can use the expansion

$$E_k(z;a_{2k+1}) = R_{2k+2}(z) - R_{2k+1}(z) + a_{2k+1} z^{-1} R_{2k+1}(z) = \sum_{\nu=0}^{2k} e_\nu^{(k)} R_\nu(z), \tag{3.10}$$

with some coefficients $e_\nu^{(k)}$, $\nu = 0, 1, \dots, 2k$. As before, from (3.10), we have

$$\langle R_{2k+2}, R_j\rangle - \langle R_{2k+1}, R_j\rangle + a_{2k+1}\langle R_{2k+1}, z^{-1}R_j\rangle = \sum_{\nu=0}^{2k} e_\nu^{(k)} \langle R_\nu, R_j\rangle, \quad 0 \le j \le 2k+1.$$

For $j = 0, 1, \dots, 2k-1$ in the last equality, we conclude that $e_j^{(k)} = 0$, while for $j = 2k+1$ and $j = 2k$, we get

$$-\langle R_{2k+1}, R_{2k+1}\rangle + a_{2k+1}\langle R_{2k+1}, z^{-1}R_{2k+1}\rangle = 0$$

and

$$a_{2k+1}\langle R_{2k+1}, z^{-1}R_{2k}\rangle = e_{2k}^{(k)}\langle R_{2k}, R_{2k}\rangle,$$

respectively. This proves the second recurrence relation (3.7), including the parameters

$$a_{2k+1} = \frac{\langle R_{2k+1}, R_{2k+1}\rangle}{\langle R_{2k+1}, z^{-1}R_{2k+1}\rangle} \quad \text{and} \quad b_{2k+1} = a_{2k+1}\frac{\langle R_{2k+1}, z^{-1}R_{2k}\rangle}{\langle R_{2k}, R_{2k}\rangle}, \tag{3.11}$$

where we put $e_{2k}^{(k)} = b_{2k+1}$. □

REMARK 3.3. Similar approaches for certain formal Laurent polynomials were used in [4], [10], [11], [12], and [13].

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3.3. Numerators of the Laurent orthogonal polynomials. According to the explicit expression (3.2), i.e.,

$$R_m(z) = \frac{1}{z^{[m/2]}} \sum_{\nu=0}^{m} c_\nu^{(m)} z^\nu, \quad m \in \mathbb{N}_0, \tag{3.12}$$

we can consider the sequence of the numerators $\{Q_m(z)\}$, given by $Q_m(z) = z^{[m/2]} R_m(z)$, as a monic polynomial sequence, because $c_m^{(m)} = 1$. Note that $Q_{2k}(z) = z^k R_{2k}(z)$ and $Q_{2k+1}(z) = z^k R_{2k+1}(z)$ for each $k \in \mathbb{N}_0$.

The following proposition is an immediate consequence of recurrence relations (3.6) and (3.7).

PROPOSITION 3.4. *We have*

$$Q_{k+1}(z) = (z - a_k) Q_k(z) + b_k z\, Q_{k-1}(z), \quad k = 0, 1, \ldots, \tag{3.13}$$

with $Q_0(z) = 1$ and $Q_{-1}(z) = 0$, where the recurrence coefficients are given in (3.9) and (3.11).

REMARK 3.5. The coefficient b_0 which is multiplied by $Q_{-1}(z) = 0$ in the three-term recurrence relation (3.13) may be arbitrary, but it is sometimes convenient to define it as $b_0 = \int_0^\pi w(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta$.

The three-term recurrence relation (3.13) appears in several papers (e.g., [12], [11], [31]), where the polynomials $Q_m(z)$ are defined by the determinantal formula

$$Q_m(z) = \frac{1}{\Delta_m} \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_m \\ \mu_{-1} & \mu_0 & \mu_1 & \cdots & \mu_{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{1-m} & \mu_{2-m} & \mu_{3-m} & \cdots & \mu_1 \\ 1 & z & z^2 & \cdots & z^m \end{vmatrix} \quad (m \geq 1), \tag{3.14}$$

where

$$\Delta_0 = 1, \quad \Delta_m = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{m-1} \\ \mu_{-1} & \mu_0 & \cdots & \mu_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{1-m} & \mu_{2-m} & \cdots & \mu_0 \end{vmatrix} \quad (m \geq 1) \tag{3.15}$$

and $\mu_k = \mathcal{L}[z^k]$, $k = 0, \pm 1, \pm 2, \ldots$ (in general, the μ_k may be arbitrary complex numbers).

According to (3.12) and (3.14) we can conclude that for the existence of the polynomials $Q_m(z)$, and also $R_m(z)$, we need $\Delta_m \neq 0$ for each $m \in \mathbb{N}$, as well as

$$\Delta_0^{(1)} = 1, \quad \Delta_m^{(1)} = \begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_m \\ \mu_0 & \mu_1 & \cdots & \mu_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{2-m} & \mu_{3-m} & \cdots & \mu_1 \end{vmatrix} \neq 0 \quad (m \geq 1), \tag{3.16}$$

because the trailing coefficient of $R_m(z)$ is exactly $c_0^{(m)} = \Delta_m^{(1)}/\Delta_m$. Then the coefficients in the recurrence relation (3.13) are given by (cf. [31, Proposition 1.1])

$$a_k = -\frac{Q_{k+1}(0)}{Q_k(0)} = \frac{\Delta_k}{\Delta_{k+1}} \cdot \frac{\Delta_{k+1}^{(1)}}{\Delta_k^{(1)}} \quad (k \in \mathbb{N}_0) \tag{3.17}$$

and

$$b_k = \frac{\Delta_{k-1}}{\Delta_k} \cdot \frac{\Delta_{k+1}^{(1)}}{\Delta_k^{(1)}} \quad (k \in \mathbb{N}). \tag{3.18}$$

Since

$$\mathcal{L}[Q_m(z)z^{-k}] = \frac{1}{\Delta_m} \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_m \\ \mu_{-1} & \mu_0 & \mu_1 & \cdots & \mu_{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{1-m} & \mu_{2-m} & \mu_{3-m} & \cdots & \mu_1 \\ \mu_{-k} & \mu_{1-k} & \mu_{2-k} & \cdots & \mu_{n-k} \end{vmatrix} = 0$$

for each $k = 0, 1, \ldots, m-1$, it is clear that the following result holds.

PROPOSITION 3.6. *The polynomials $Q_m(z)$ are characterized by the following orthogonality relations*

$$\int_0^\pi \mathrm{e}^{-\mathrm{i}k\theta} Q_m(\mathrm{e}^{\mathrm{i}\theta}) w(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\theta = 0, \quad k = 0, 1, \ldots, m-1.$$

4. Special cases of Laurent orthogonal polynomials on the semicircle. In this section we consider three interesting weight functions:

- Legendre weight $w(z) = 1$;
- Chebyshev weight of the first kind $w(z) = (1 - z^2)^{-1/2}$;
- Chebyshev weight of the second kind $w(z) = (1 - z^2)^{1/2}$.

In all cases we start with analytic expressions for the moments μ_k $(k \in \mathbb{Z})$. In Chebyshev's cases we are able to find analytic expressions for the determinants Δ_m and $\Delta_m^{(1)}$, analytic expressions for the coefficients a_k and b_k in the recurrence relation (3.13), as well as for the corresponding polynomials $Q_m(z)$, in notations $Q_m^T(z)$ and $Q_m^U(z)$. Also, the zeros of $Q_m^T(z)$ can be found in analytic form.

4.1. Legendre weight. In this simplest case for the weight function $w(z) = 1$, the moment functional (3.1) is given by

$$\mathcal{L}[z^k] = \mu_k = \int_0^\pi \mathrm{e}^{\mathrm{i}k\theta} \mathrm{d}\theta = \begin{cases} \pi, & k = 0, \\ 2\mathrm{i}/k, & k \text{ odd}, \\ 0, & k \text{ even},\ k \neq 0. \end{cases}$$

Expanding the determinants (3.15) and (3.16) we see that they are values of algebraic polynomials with integer coefficients at π, i.e.,

$$\Delta_1 = \pi, \quad \Delta_2 = \pi^2 - 4, \quad \Delta_3 = \pi^3 - 8\pi, \quad \Delta_4 = \frac{1}{9}\left(256 - 112\pi^2 + 9\pi^4\right),$$

$$\Delta_5 = \frac{1}{81}\left(5632\pi - 1368\pi^3 + 81\pi^5\right),$$

$$\Delta_6 = \frac{1}{18225}\left(-4194304 + 2515968\pi^2 - 391716\pi^4 + 18225\pi^6\right), \text{ etc.}$$

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and

$$\Delta_1^{(1)} = 2\mathrm{i}, \quad \Delta_2^{(1)} = -4, \quad \Delta_3^{(1)} = \frac{2\mathrm{i}}{3}\left(-16+\pi^2\right), \quad \Delta_4^{(1)} = \frac{8}{9}\left(32-3\pi^2\right),$$

$$\Delta_5^{(1)} = \frac{2\mathrm{i}}{405}\left(16384 - 2448\pi^2 + 81\pi^4\right),$$

$$\Delta_6^{(1)} = -\frac{4}{18225}\left(1048576 - 198144\pi^2 + 9315\pi^4\right), \text{ etc.,}$$

so that we can conclude that Δ_m and $\Delta_m^{(1)}$ are different from zero for all $m \in \mathbb{N}$, because the transcendental number π cannot be a zero of any polynomial with integer coefficients (all its zeros must be algebraic numbers!). Notice that the determinants $\Delta_k^{(1)}$ of odd order have pure imaginary values, while the values of other determinants are real.

According to (3.17) and (3.18) the corresponding coefficients a_k and b_k in the recurrence relation (3.13) have pure imaginary values and they are represented as rational functions of π:

$$a_0 = \frac{2\mathrm{i}}{\pi}, \quad a_1 = \frac{2\mathrm{i}\pi}{\pi^2-4}, \quad a_2 = \frac{\mathrm{i}\left(16-\pi^2\right)\left(\pi^2-4\right)}{6\pi\left(\pi^2-8\right)},$$

$$a_3 = \frac{12\mathrm{i}\pi\left(\pi^2-8\right)\left(32-3\pi^2\right)}{4096-2048\pi^2+256\pi^4-9\pi^6},$$

$$a_4 = \frac{\mathrm{i}\left(256-112\pi^2+9\pi^4\right)\left(16384-2448\pi^2+81\pi^4\right)}{20\pi\left(180224-60672\pi^2+6696\pi^4-243\pi^6\right)}, \ \ldots$$

and

$$b_1 = \frac{2\mathrm{i}}{\pi}, \quad b_2 = \frac{\mathrm{i}\pi\left(16-\pi^2\right)}{6\left(\pi^2-4\right)}, \quad b_3 = \frac{4\mathrm{i}\left(\pi^2-4\right)\left(3\pi^2-32\right)}{3\pi\left(\pi^2-16\right)\left(\pi^2-8\right)},$$

$$b_4 = \frac{\mathrm{i}\pi\left(\pi^2-8\right)\left(16384-2448\pi^2+81\pi^4\right)}{20\left(32-3\pi^2\right)\left(256-112\pi^2+9\pi^4\right)}, \ \ldots$$

This case has been recently considered in [19], including some applications in numerical integration.

4.2. Chebyshev weight of the first kind. In this case the weight function is given by $w(z) = (1-z^2)^{-1/2}$, and the corresponding moments are

$$\mu_k = \int_0^\pi \frac{\mathrm{e}^{\mathrm{i}k\theta}}{\sqrt{1-\mathrm{e}^{2\mathrm{i}\theta}}}\,\mathrm{d}\theta = \begin{cases} 2^k\pi\binom{-k}{-k/2}, & k \text{ (even)} \le 0, \\ \mathrm{i}\frac{(k+1)\pi}{2^{k+1}k}\binom{k+1}{(k+1)/2}, & k \text{ (odd)} \ge 1, \\ 0, & k \text{ (odd)} \le -1 \text{ or } k \text{ (even)} \ge 2. \end{cases}$$

For the determinants (3.15) and (3.16) we obtain explicit expressions

$$\Delta_n = \frac{\pi^n}{2^{\frac{1}{2}(n-2)(n-1)}} > 0, \quad \Delta_n^{(1)} = \frac{\mathrm{i}^n\pi^n}{2^{\frac{1}{2}(n-2)(n-1)}} \ne 0, \quad n \in \mathbb{N},$$

so that, using (3.17) and (3.18), we have

$$a_k = \mathrm{i} \quad (k \ge 0), \quad b_1 = \mathrm{i}, \quad b_k = \frac{\mathrm{i}}{2} \quad (k \ge 2),$$

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and $b_0 = \pi$ (by definition). In this way, for the recurrence relation (3.13), we have

$$\begin{aligned} Q_1^T(z) &= (z-\mathrm{i})Q_0^T(z),\\ Q_2^T(z) &= (z-\mathrm{i})Q_1^T(z) + \mathrm{i}\, z\, Q_0^T(z),\\ Q_{k+1}^T(z) &= (z-\mathrm{i})Q_k^T(z) + \frac{\mathrm{i}}{2}\, z\, Q_0^T(z) \quad (k \ge 2), \end{aligned}$$

with $Q_0^T(z) = 1$. Solving the last (difference) equation (for a fixed z), with starting polynomials $Q_0^T(z) = 1$ and $Q_1^T(z) = z - \mathrm{i}$, we get an explicit formula for the polynomials $Q_k^T(z)$.

PROPOSITION 4.1. *We have*

$$Q_k^T(z) = \frac{1}{2^k}\left[\left(z + \sqrt{z^2-1} - \mathrm{i}\right)^k + \left(z - \sqrt{z^2-1} - \mathrm{i}\right)^k\right] \quad (k \ge 0).$$

Here, $|z + \sqrt{z^2-1}| = r > 1$*, whenever* $z \in \mathbb{C} \setminus [-1,1]$.

A few of the first polynomials are

$$\begin{aligned} &Q_0^T(z) = 1, \quad Q_1^T(z) = z - \mathrm{i}, \quad Q_2^T(z) = z^2 - \mathrm{i}z - 1, \quad Q_3^T(z) = z^3 - \frac{3\mathrm{i}}{2}z^2 - \frac{3}{2}z + \mathrm{i},\\ &Q_4^T(z) = z^4 - 2\mathrm{i}z^3 - \frac{5}{2}z^2 + 2\mathrm{i}z + 1, \quad Q_5^T(z) = z^5 - \frac{5\mathrm{i}}{2}z^4 - \frac{15}{4}z^3 + \frac{15\mathrm{i}}{4}z^2 + \frac{5}{2}z - \mathrm{i},\\ &Q_6^T(z) = z^6 - 3\mathrm{i}z^5 - \frac{21}{4}z^4 + \frac{25\mathrm{i}}{4}z^3 + \frac{21}{4}z^2 - 3\mathrm{i}z - 1,\\ &Q_7^T(z) = z^7 - \frac{7\mathrm{i}}{2}z^6 - 7z^5 + \frac{77\mathrm{i}}{8}z^4 + \frac{77}{8}z^3 - 7\mathrm{i}z^2 - \frac{7}{2}z + \mathrm{i}, \quad \text{etc.} \end{aligned}$$

The corresponding Laurent-Chebyshev polynomials (exact rational functions) of the first kind are $R_m^T(z) = Q_m^T(z)/z^{[m/2]}$ $(m \ge 0)$.

PROPOSITION 4.2. *The (quasi)norms of the Laurent-Chebyshev polynomials of the first kind are given by*

$$\|R_m^T\|^2 = \int_0^\pi \frac{R_m^T(\mathrm{e}^{\mathrm{i}\theta})^2}{\sqrt{1-\mathrm{e}^{2\mathrm{i}\theta}}}\,\mathrm{d}\theta = \begin{cases} \pi, & m = 0,\\ (-1)^{[m/2]}\dfrac{\pi}{2^{m-1}}, & m \ge 1. \end{cases}$$

An explicit formula for zeros of the polynomial $Q_n^T(z)$ is given in the following theorem.

THEOREM 4.3. *For a given* $n \in \mathbb{N}$ *let*

$$\varphi_k = \frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \ldots, n,$$

and

$$\theta_k = \begin{cases} \arcsin\left(\sin^2\varphi_k\right), & k = 1, 2, \ldots, \left[\frac{n+1}{2}\right],\\ \pi - \arcsin\left(\sin^2\varphi_k\right), & k = \left[\frac{n+3}{2}\right], \ldots, n. \end{cases} \tag{4.1}$$

For each $n \in \mathbb{N}$ *the zeros of the polynomial* $Q_n^T(z)$ *are given by*

$$\zeta_k = \cos\varphi_k\sqrt{1+\sin^2\varphi_k} + \mathrm{i}\sin^2\varphi_k = \mathrm{e}^{\mathrm{i}\theta_k}, \quad k = 1, 2, \ldots, n.$$

All zeros are mutually different and symmetrically distributed on the upper semicircle, i.e., $\theta_{n-k+1} = \pi - \theta_k$, $k = 1, 2, \ldots, [(n+1)/2]$.

Proof. Using the explicit form of the polynomial $Q_n^T(z)$ (see Proposition 4.1) we should solve the equations

$$\frac{z-\mathrm{i}-\sqrt{z^2-1}}{z-\mathrm{i}+\sqrt{z^2-1}} = \mathrm{e}^{\mathrm{i}(2k-1)\pi/n} = \mathrm{e}^{2\mathrm{i}\varphi_k} = q_k, \quad k=1,\dots,n,$$

i.e.,

$$z^2 + \mathrm{i}\,\frac{(1-q_k)^2}{2q_k}\,z - 1 = 0, \quad k=1,\dots,n,$$

where $\varphi_k = (2k-1)/\pi/(2n)$. Since $(1-q_k)^2/(2q_k) = -2\sin^2\varphi_k$ we get the solutions of the last quadratic equations in the form

$$\zeta_k = \mathrm{i}\sin^2\varphi_k + \cos\varphi_k\sqrt{1+\sin^2\varphi_k} = \mathrm{e}^{\mathrm{i}\theta_k}, \quad k=1,\dots,n,$$

where

$$\tan\theta_k = \frac{\sin^2\varphi_k}{\cos\varphi_k\sqrt{1+\sin^2\varphi_k}}, \quad \sin\theta_k = \sin^2\varphi_k, \quad k=1,\dots,n.$$

Note that $|\zeta_k| = 1$ and $\operatorname{Im}\zeta_k > 0$ for each $k = 1,2,\dots,n$, as well as that $\theta_{n-k+1} = \pi - \theta_k$, $k = 1,2,\dots,[(n+1)/2]$. ◻

In the sequel we use the notation $\theta_k^{(n)}$ instead of θ_k.

REMARK 4.4. According to (4.1) the sequences of angles $\Theta^{(n)} = \{\theta_k^{(n)}\}_{k=1}^{[(n+1)/2]}$ in (4.1), for $n = 1,\dots,6$ are

$$\Theta^{(1)} = \left\{\frac{\pi}{2}\right\}; \qquad \Theta^{(2)} = \left\{\frac{\pi}{6}\right\}; \qquad \Theta^{(3)} = \left\{\arcsin\frac{1}{4},\ \frac{\pi}{2}\right\};$$
$$\Theta^{(4)} = \left\{\arcsin\frac{2-\sqrt{2}}{4},\ \arcsin\frac{2+\sqrt{2}}{4}\right\};$$
$$\Theta^{(5)} = \left\{\arcsin\frac{3-\sqrt{5}}{8},\ \arcsin\frac{3+\sqrt{5}}{8},\ \frac{\pi}{2}\right\},$$
$$\Theta^{(6)} = \left\{\arcsin\frac{2-\sqrt{3}}{4},\ \frac{\pi}{6},\ \arcsin\frac{2+\sqrt{3}}{4}\right\},$$

respectively.

Zeros of polynomials $Q_n^T(z)$ are presented in Figure 4.1 for $n = 5$ and $n = 50$.

According to (4.1) it is easy to see that $0 < \theta_1^{(n)} < \theta_2^{(n)} < \cdots < \theta_n^{(n)} < \pi$. The following result represents an analogue of the well-know theorem on interlacing property of real orthogonal polynomials on the real line; cf. et al. [14, p. 99].

THEOREM 4.5. *The zeros of $Q_n^T(z)$ and $Q_{n+1}^T(z)$ interlace on the upper semicircle, i.e., for their arguments the following inequalities*

$$0 < \theta_1^{(n+1)} < \theta_1^{(n)} < \theta_2^{(n+1)} < \theta_2^{(n)} < \cdots < \theta_n^{(n)} < \theta_{n+1}^{(n+1)} < \pi \tag{4.2}$$

hold.

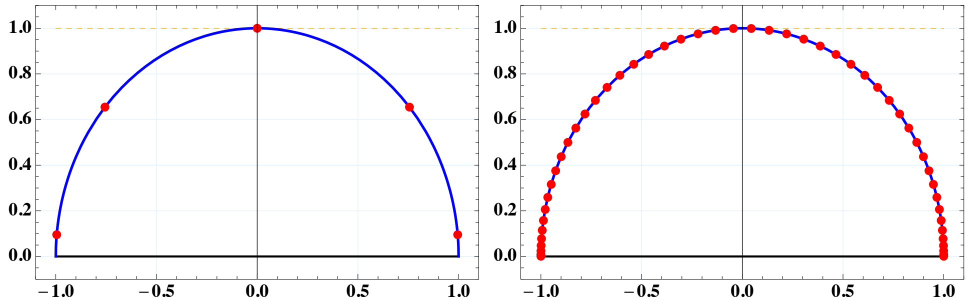

FIG. 4.1. *Zeros of the polynomials $Q_n^T(z)$ for $n=5$ (left) and $n=50$ (right).*

Proof. It is enough to prove (4.2) for arguments $\le \pi/2$, because the sinus function is increasing in that domain. Then

$$\sin\theta_k^{(n)} - \sin\theta_k^{(n+1)} = \sin^2\frac{(2k-1)\pi}{2n} - \sin^2\frac{(2k-1)\pi}{2(n+1)} > 0$$

and

$$\sin\theta_{k+1}^{(n+1)} - \sin\theta_k^{(n)} = \sin^2\frac{(2k+1)\pi}{2(n+1)} - \sin^2\frac{(2k-1)\pi}{2n} > 0,$$

i.e., $\theta_k^{(n+1)} < \theta_k^{(n)} < \theta_{k+1}^{(n+1)}$. ∎

An interesting illustration of the interlacing property of zeros of polynomials on the semicircle is presented in Figure 4.2.

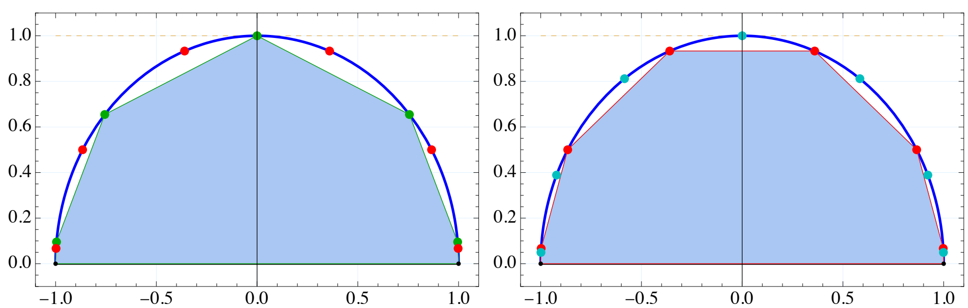

FIG. 4.2. *Zeros of the polynomials $Q_n^T(z)$ for $n=5$ and $n=6$ (left) and for $n=6$ and $n=7$ (right).*

4.3. Chebyshev weight of the second kind. In this case the weight function is given by $w(z)=(1-z^2)^{1/2}$, and the corresponding moments are

$$\mu_k = \int_0^\pi e^{ik\theta}\sqrt{1-e^{2i\theta}}\,d\theta = \begin{cases} \frac{2^k\pi}{k+1}\binom{-k}{-k/2}, & k\ (\text{even}) \le 0,\\ i\frac{\pi}{2^k k}\binom{k}{(k-1)/2}, & k\ (\text{odd}) \ge 1,\\ -i\pi, & k=-1,\\ 0, & k\ (\text{odd}) \le -3\ \text{ or }\ k\ (\text{even}) \ge 2, \end{cases}$$

For the determinants (3.15) and (3.16), in this case, we also obtain explicit expressions

$$\Delta_n = \frac{\pi^n}{2^{\frac{1}{2}(n-1)n}} > 0, \quad \Delta_n^{(1)} = \frac{\mathrm{i}^n \pi^n}{2^{\frac{1}{2}(n^2-n+2)}} \neq 0, \quad n \in \mathbb{N}.$$

Using (3.17) and (3.18), we get

$$a_0 = \frac{\mathrm{i}}{2}, \quad a_k = \mathrm{i} \quad (k \geq 1), \quad b_k = \frac{\mathrm{i}}{2} \quad (k \geq 1),$$

and $b_0 = \pi$ (by definition).

PROPOSITION 4.6. *We have*

$$Q_k^U(z) = \frac{1}{2^{k+1}\sqrt{z^2-1}} \left[\left(z + \sqrt{z^2-1}\right) \left(z + \sqrt{z^2-1} - \mathrm{i}\right)^k - \left(z - \sqrt{z^2-1}\right) \left(z - \sqrt{z^2-1} - \mathrm{i}\right)^k \right].$$

Here, $|z + \sqrt{z^2-1}| = r > 1$, *whenever* $z \in \mathbb{C} \setminus [-1, 1]$.

PROPOSITION 4.7. *The (quasi)norms of the Laurent-Chebyshev polynomials of the second kind are given by*

$$\|R_m^U\|^2 = \int_0^\pi R_m^U(\mathrm{e}^{\mathrm{i}\theta})^2 \sqrt{1-\mathrm{e}^{2\mathrm{i}\theta}}\, \mathrm{d}\theta = \begin{cases} \pi, & m = 0, \\ \dfrac{3\pi}{4}, & m = 1, \\ \dfrac{\pi}{2^{m+1}} \left(\cos\frac{m\pi}{2} + \sin\frac{m\pi}{2}\right), & m \geq 2. \end{cases}$$

5. Zero distribution. As we have seen all the zeros of the polynomials $Q_n^T(z)$, in the case of Chebyshev weight of the first kind, are on the upper (unit) semicircle, including the interlacing property (Theorem 4.5).

Zeros of the polynomials $Q_n^U(z)$ $\left(w(z) = \sqrt{1-z^2}\,\right)$ for $n = 5$, $n = 20$, $n = 50$ and $n = 100$ are displayed in Figure 5.1. As n increases, the zeros of the polynomials $Q_n^U(z)$ tend to reach the upper semicircle. A very similar behaviour is shown by the zeros of the polynomials $Q_n^P(z)$ (Legendre's case).

Finally, based on a large number of numerical experiments, we can state a conjecture about zero distribution of the polynomials $Q_n(z)$ (some kind of the "interlacing property") for the Legendre weight $w(z) = 1$ and the Chebyshev weight of the second kind $w(z) = \sqrt{1-z^2}$. Thus, let $S_n = \{\zeta_1, \zeta_2, \ldots, \zeta_n\}$ be the set of all zeros of the polynomial $Q_n(z)$. With C_n we denote the convex hull of the set $S_n \cup \{-1, 1\}$, i.e.,

$$C_n = \mathcal{C}o\left(S_n \cup \{-1, 1\}\right) = \mathcal{C}o\left(\{-1, \zeta_1, \zeta_2, \ldots, \zeta_n, 1\}\right).$$

CONJECTURE 5.1. *All zeros of the polynomial* $Q_n(z)$ *are contained in the domain*

$$\{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\} \setminus C_{n-1},$$

i.e., $S_n \subset D_+ \setminus C_{n-1}$.

In Figure 5.2 we present the cases with the Legendre weight $(w(z) = 1)$ and the Chebyshev weight of the second kind $(w(z) = \sqrt{1-z^2}\,)$. Zeros of the polynomial of degree $n = 7$ are contained in $S_7 \subset D_+ \setminus C_6$, where the convex hull $C_6 = \mathcal{C}o\left(S_6 \cup \{-1, 1\}\right) = \mathcal{C}o\left(\{-1, \zeta_1, \zeta_2, \ldots, \zeta_6, 1\}\right)$.

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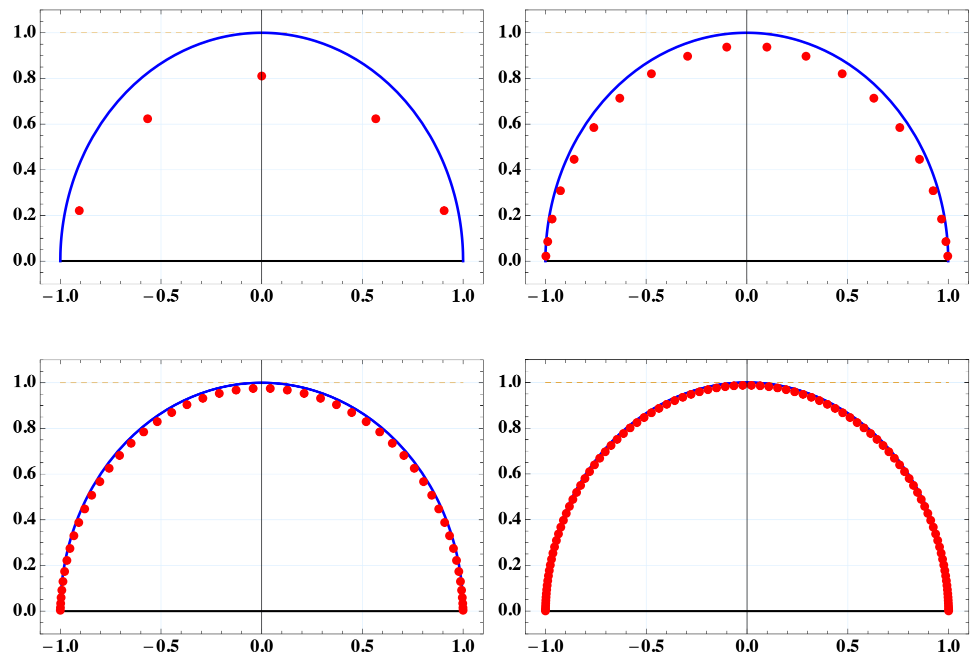

FIG. 5.1. *Zeros of the polynomials* $Q_n^U(z)$ *for* $n = 5, 20, 50, 100$.

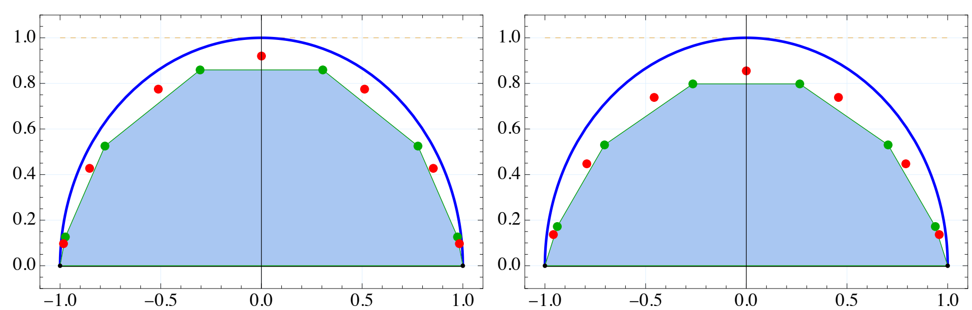

FIG. 5.2. *Zeros of the polynomials* $Q_n^P(z)$ *(left) and* $Q_n^U(z)$ *(right) for* $n = 6$ *and* $n = 7$.

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