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REGULAR CONVERGENCE AND FINITE ELEMENT METHODS FOR EIGENVALUE PROBLEMS*

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Abstract. Regular convergence, together with other types of convergence, have been studied since the 1970s for discrete approximations of linear operators. In this paper, we consider the eigenvalue approximation of a compact operator T that can be written as an eigenvalue problem of a holomorphic Fredholm operator function $F(\eta) = T - \frac{1}{\eta} I$. Focusing on finite element methods (conforming, discontinuous Galerkin, non-conforming, etc.), we show that the regular convergence of the discrete holomorphic operator functions F_n to F follows from the compact convergence of the discrete operators T_n to T. The convergence of the eigenvalues is then obtained using abstract approximation theory for the eigenvalue problems of holomorphic Fredholm operator functions. The result can be used to prove the convergence of various finite element methods for eigenvalue problems such as the Dirichlet eigenvalue problem and the biharmonic eigenvalue problem.

Key words. regular convergence, finite element methods, eigenvalue problems

AMS subject classifications. 65N25, 65N30

1. Introduction.

Eigenvalue problems of partial differential equations have many important applications in science and engineering, e.g., the design of solar cells for clean energy, the calculation of electronic structure in condensed matter, extraordinary optical transmission, non-destructive testing, photonic crystals, and biological sensing. Due to the flexibility in treating complex structures and a rigorous theoretical justification, finite element methods have been widely used to solve eigenvalue problems [2, 4, 7, 11, 20, 24].

The study of the finite element methods for eigenvalue problems started in 1970s and has been an active research area since then. Many results obtained before the 1990s can be found in the book chapter by Babuška and Osborn [2]. We refer the readers to [4, 24] for discussions of recent developments. The main functional analysis tool is the spectral perturbation theory for linear compact operators [10, 18]. Essentially, if there is uniform convergence of the finite element solution operators to the continuous solution operator, then the theory of Babuška and Osborn [2] can be employed to obtain the convergence of the eigenvalues and the associated eigenfunctions, i.e., all eigenpairs are approximated correctly, and there are no spurious modes.

While uniform convergence of discrete operators can be proved for some finite element methods such as conforming finite element methods, it is impossible or challenging for many other finite element methods, for instance, the discontinuous Galerkin methods. Various methods have been proposed in the literature to prove convergence when uniform convergence is not available [1, 11, 19]. In this paper, we consider the eigenvalue approximation of a compact operator T and reformulate it as an eigenvalue problem of the holomorphic Fredholm operator function. The regular convergence of the discrete holomorphic operator functions F_n follows from the compact convergence of the discrete operators. The convergence of eigenvalues and eigenfunctions is then obtained using the abstract approximation theory for the eigenvalue problems of holomorphic Fredholm operator functions in [16, 17]. This work extends the results in [15, 16, 25] and provides an alternative way to analyze the convergence of various finite element approximations for eigenvalue problems of partial differential equations [21, 26, 27, 28, 29].

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The rest of the paper is arranged as follows. In Section 2, we recall the discrete approximation scheme, different types of convergence for linear operators, and the abstract approximation theory for eigenvalue problems of holomorphic Fredholm operator functions. Section 3 contains the study of regular convergence related to the finite element approximation operators for partial differential equations in L^2-spaces. It turns out that compact convergence of the discrete solution operator guarantees regular convergence. Using the abstract approximation theory for holomorphic Fredholm operator functions, we show convergence of various finite element methods for the Dirichlet eigenvalue problem in Section 4 and the biharmonic eigenvalue problem in Section 5. We end the paper with some conclusions and future work in Section 6.

2. Preliminaries. We present a brief introduction on the discrete approximation scheme, different types of convergence, and the abstract approximation theory for eigenvalue problems of holomorphic Fredholm operator functions. The readers are referred to [10, 16, 17, 22, 23, 25] for more details.

2.1. Discrete approximation schemes. Let X and X_n, $n \in \mathbb{N}$, be Banach spaces, with $\{X_n\}_{n\in\mathbb{N}}$ a sequence of approximation spaces for X. Let $P = \{p_n\}_{n\in\mathbb{N}}$ be a sequence of bounded linear operators $p_n : X \to X_n$ such that

$$\|p_n x\|_{X_n} \longrightarrow \|x\|_X \qquad (n \in \mathbb{N}),$$

for all $x \in X$.

DEFINITION 2.1. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in X_n$ is said to P-converge to $x \in X$ if*

$$\|x_n - p_n x\|_{X_n} \longrightarrow 0 \qquad (n \in \mathbb{N}).$$

We write it as $x_n \xrightarrow{P} x\,(n \in \mathbb{N})$ or simply $x_n \to x\,(n \in \mathbb{N})$.

DEFINITION 2.2. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in X_n$ is called (discrete) P-compact if for every $\mathbb{N}' \subset \mathbb{N}$ there exists $\mathbb{N}'' \subset \mathbb{N}'$ and an $x \in X$ such that $x_n \xrightarrow{P} x\,(n \in \mathbb{N}'')$.*

Let Y be a Banach space. Denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y. We denote by $\mathcal{N}(A) = \{x \in X : Ax = 0\}$ and $\mathcal{R}(A) = \{y \in Y : y = Ax, x \in X\}$ the null space and the range of the operator $A \in \mathcal{L}(X, Y)$, respectively.

DEFINITION 2.3. *An operator $A \in \mathcal{L}(X, Y)$ is called semi-Fredholm if $\mathcal{R}(A) \subset Y$ is closed and additionally $\mathcal{N}(A)$ has a finite dimension or $\mathcal{R}(A)$ has a finite codimension. If $\mathcal{R}(A)$ is closed and in addition $\dim \mathcal{N}(A)$ and $\operatorname{codim} \mathcal{R}(A)$ are both finite, then A is called Fredholm. The index of a Fredholm operator is defined as*

$$\operatorname{ind} A = \dim \mathcal{N}(A) - \operatorname{codim} \mathcal{R}(A).$$

In the following, we define various notions of convergence for linear operators; see [10, 22, 25] for details. These are point convergence, stable convergence, compact convergence, and regular convergence. Let $\{Y_n\}_{n\in\mathbb{N}}$ be a sequence of Banach spaces that approximates Y, and let $Q = \{q_n\}_{n\in\mathbb{N}}$ be a sequence of bounded linear operators $q_n : Y \to Y_n$ such that $\|q_n y\|_{Y_n} \longrightarrow \|y\|_Y\ (n \in \mathbb{N})$ for all $y \in Y$.

DEFINITION 2.4. *A sequence $\{A_n\}_{n\in\mathbb{N}}$ of linear operators $A_n \in \mathcal{L}(X_n, Y_n)$ converges (or PQ-converges, or converges discretely) to $A \in \mathcal{L}(X, Y)$ if*

$$x_n \xrightarrow{P} x \quad (n \in \mathbb{N}) \quad \Longrightarrow \quad A_n x_n \xrightarrow{Q} Ax \quad (n \in \mathbb{N}).$$

We write this as $A_n \longrightarrow A\,(n \in \mathbb{N})$ or $A_n \xrightarrow{PQ} A\,(n \in \mathbb{N})$.

In Theorem 2-8 of [25], the equivalence of $A_n \stackrel{PQ}{\longrightarrow} A\,(n \in \mathbb{N})$ and

$$\|A_n\| \le \text{const}, \quad (n \in \mathbb{N}), \qquad \|A_n p_n x - q_n A x\|_{Y_n} \to 0 \quad (n \in \mathbb{N}), \quad \forall x \in X,$$

is claimed. The following theorem is a consequence of Theorem 2-8 and Theorem 2-9 in [25].

THEOREM 2.5. *If* $p_n \in \mathcal{L}(X, X_n)$, $p_n X = X_n$, *and* $b = \text{const}$ *with*

$$\inf_{x \in X, p_n x = x_n} \|x\|_X \le b \|x_n\|_{X_n} \qquad (n \in \mathbb{N}, \forall x_n \in X_n), \tag{2.1}$$

then $\|A_n p_n x - q_n A x\|_{Y_n} \to 0\,(n \in \mathbb{N}), \forall x \in X$, *implies* $\|A_n\| \le \text{const}\,(n \in \mathbb{N})$.

Proof. If the conditions are satisfied, then by Theorem 2-9 in [25], one has the equivalence of $\|A_n p_n x - q_n A x\|_{Y_n} \to 0\,(n \in \mathbb{N}), \forall x \in X$, and $A_n \stackrel{PQ}{\longrightarrow} A\,(n \in \mathbb{N})$. By Theorem 2-8 in [25], $\|A_n\| \le \text{const}\,(n \in \mathbb{N})$. ∎

DEFINITION 2.6. *A sequence* $\{A_n\}_{n \in \mathbb{N}}$ *of linear operators* $A_n \in \mathcal{L}(X_n, Y_n)$ *converges stably to* $A \in \mathcal{L}(X, Y)$ *if the following two conditions are met:*

1. $A_n \stackrel{PQ}{\longrightarrow} A\,(n \in \mathbb{N})$.
2. *There is some* $n_0 \in \mathbb{N}$ *such that the inverse operators* $A_n^{-1} \in \mathcal{L}(Y_n, X_n)$ *exist for all* $n \ge n_0$, *where*

$$\|A_n^{-1}\| \le C \quad (n \ge n_0).$$

We write in short form $A_n \to A$ *stably.*

DEFINITION 2.7. *A sequence* $\{A_n\}_{n \in \mathbb{N}}$ *of linear operators* $A_n \in \mathcal{L}(X_n, Y_n)$ *converges compactly to* $A \in \mathcal{L}(X, Y)$ *if the following conditions are met:*

1. $A_n \stackrel{PQ}{\longrightarrow} A\,(n \in \mathbb{N})$.
2. $\|x_n\|_{X_n} \le C\,(n \in \mathbb{N}) \implies \{A_n x_n\}_{n \in \mathbb{N}}$ *Q-compact.*

We write in short form $A_n \to A$ *compactly.*

DEFINITION 2.8. *A sequence* $\{A_n\}_{n \in \mathbb{N}}$ *of linear operators* $A_n \in \mathcal{L}(X_n, Y_n)$ *converges regularly to* $A \in \mathcal{L}(X, Y)$ *if the following two conditions are satisfied:*

1. $A_n \stackrel{PQ}{\longrightarrow} A\,(n \in \mathbb{N})$.
2. $\{A_n\}_{n \in \mathbb{N}}$ *is regular, which means*

$$\|x_n\|_{X_n} \le C\,(n \in \mathbb{N}), \quad \{A_n x_n\}_{n \in \mathbb{N}}\ Q\text{-compact} \implies \{x_n\}_{n \in \mathbb{N}}\ P\text{-compact}.$$

We write in short form $A_n \to A$ *regularly.*

The following theorem from [25] (Theorem 2-55 therein) states a sufficient condition for the regular convergence of A_n to A.

THEOREM 2.9. *Let*

$$\begin{aligned} B_n \to B &\quad \text{stably} &&(B_n \in \mathcal{L}(X_n, Y_n), B \in \mathcal{L}(X, Y)),\\ C_n \to C &\quad \text{compactly} &&(C_n \in \mathcal{L}(X_n, Y_n), C \in \mathcal{L}(X, Y)),\end{aligned}$$

where $\mathcal{R}(B) = Y$. *Then*

$$A_n := B_n + C_n \quad \implies \quad B + C =: A \quad \text{regularly}.$$

DEFINITION 2.10. *A sequence* $\{A_n\}_{n \in \mathbb{N}}$ *of linear operators* $A_n \in \mathcal{L}(X_n, Y_n)$ *converges uniformly to* $A \in \mathcal{L}(X, Y)$ *if* $\|A_n p_n - q_n A\| \to 0\,(n \in \mathbb{N})$. *We write in short form* $A_n \to A$ *uniformly.*

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2.2. Holomorphic Fredholm operator functions. We now introduce the abstract approximation theory for eigenvalue problems of holomorphic Fredholm operator functions [16, 17]. Let X and Y be complex Banach spaces. Let $\Omega \subseteq \mathbb{C}$ be a compact simply connected region.

Let $F : \Omega \to \mathcal{L}(X, Y)$ be a holomorphic operator function on Ω and, for each $\eta \in \Omega$, $F(\eta)$ be a Fredholm operator of index zero [14].

DEFINITION 2.11. *A complex number $\lambda \in \Omega$ is called an eigenvalue of F if there exists a nontrivial $x \in X$ such that $F(\lambda)x = 0$. The element x is called an eigenfunction of F associated with λ.*

The resolvent set $\rho(F)$ and the spectrum $\sigma(F)$ of $F(\cdot)$ are defined respectively as

$$\rho(F) = \{\eta \in \Omega : F(\eta)^{-1} \text{ exists and is bounded}\}$$

and

$$\sigma(F) = \Omega \setminus \rho(F).$$

If $\rho(F) \neq \emptyset$, then, since $F(\eta)$ is holomorphic, the spectrum $\sigma(F)$ has no cluster points in Ω and every $\lambda \in \sigma(F)$ is an eigenvalue for $F(\eta)$. Furthermore, the operator function $F^{-1}(\cdot)$ is meromorphic; see [16, Section 2.3]. The dimension of $\mathcal{N}(F(\lambda))$ is called the geometric multiplicity of an eigenvalue λ.

DEFINITION 2.12. *An ordered sequence of elements $x_0, x_1, \ldots, x_k$ in X is called a Jordan chain of F at an eigenvalue λ if*

$$F(\lambda)x_j + \frac{1}{1!}F^{(1)}(\lambda)x_{j-1} + \ldots + \frac{1}{j!}F^{(j)}(\lambda)x_0 = 0, \quad j = 0, 1, \ldots, k,$$

where $F^{(j)}$ denotes the jth derivative.

The length of any Jordan chain of an eigenvalue λ is finite. Denote by $m(F, \lambda, x_0)$ the length of a Jordan chain formed by an eigenfunction x_0. The maximal length of all Jordan chains of λ is denoted by $\kappa(F, \lambda)$, called the ascent of λ. Elements of any Jordan chain of an eigenvalue λ are called generalized eigenfunctions of λ.

DEFINITION 2.13. *The closed linear hull of all generalized eigenfunctions of an eigenvalue λ, denoted by $G(\lambda)$, is called the generalized eigenspace of λ.*

Let X_n, Y_n be Banach spaces, not necessarily subspaces of X, Y. Let $p_n \in \mathcal{L}(X, X_n)$ and $q_n \in \mathcal{L}(Y, Y_n)$ be such that

$$\lim_{n\to\infty} \|p_n x\|_{X_n} = \|x\|_X, \ \forall x \in X \qquad \text{and} \qquad \lim_{n\to\infty} \|q_n y\|_{Y_n} = \|y\|_Y, \ \forall y \in Y.$$

Consider a sequence of discrete operator functions

$$F_n : \Omega \to \mathcal{L}(X_n, Y_n), \quad n \in \mathbb{N}.$$

Assume that $\rho(F) \neq \emptyset$ and

(b1) For every $\eta \in \Omega$, $\{F_n(\eta)\}_{n\in\mathbb{N}}$ is a sequence of Fredholm operators with index zero.
(b2) For every $\eta \in \Omega$, $\{F_n(\eta)\}_{n\in\mathbb{N}}$ is equibounded on Ω, i.e., $\|F_n(\eta)\| \leq C$ $(n \in \mathbb{N})$.
(b3) For every $\eta \in \Omega$, $\{F_n(\eta)\}_{n\in\mathbb{N}}$ approximates $F(\eta)$, i.e.,

$$\|[F_n(\eta)p_n - p_n F(\eta)]x\|_{Y_n} \to 0.$$

(b4) For every $\eta \in \Omega$, $\{F_n(\eta)\}_{n\in\mathbb{N}}$ is regular.

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Under these assumptions, the following theorem states that all eigenvalues are approximated correctly; see [16, 17] or [3].

THEOREM 2.14. *For any $\lambda \in \sigma(F)$ there exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associate eigenfunctions $v_n^0 \in \mathcal{N}(F_n(\lambda_n))$, $\|v_n^0\|_{X_n} = 1$, one has that*

$$|\lambda_n - \lambda| \leq C\epsilon_n^{1/\kappa},$$
$$\inf_{v \in \mathcal{N}(F(\lambda))} \|v_n^0 - p_n v\|_{X_n} \leq C\epsilon_n^{1/\kappa},$$

where

$$\epsilon_n = \max_{|\eta-\lambda|\leq\delta} \max_{\substack{v \in G(\lambda) \\ \|v\|_X = 1}} \|F_n(\eta)p_n v - q_n F(\eta)v\|_{Y_n}$$

for sufficiently small $\delta > 0$.

The theorem above states that all eigenvalues and eigenfunctions are approximated correctly. Note that condition (b3) (pointwise convergence) itself is not sufficient to rule out spurious discrete eigenvalues. We refer the readers to [5] for some (pointwise convergent) mixed finite element methods producing spurious eigenvalues.

3. Finite element approximations. To analyze finite element methods for eigenvalue problems of partial differential equations, we choose $X = Y = L^2(D)$ for the rest of the paper, where $D \subset \mathbb{R}^2$ is a Lipschitz polygonal domain. The result in this paper holds for higher-dimensional cases if the corresponding approximation properties of the finite element methods for the source problem are available. Denote by $\|\cdot\|$ the usual L^2-norm. Let T be the solution operator for the source problem associated to the eigenvalue problem. For example, the Poisson equation with homogeneous Dirichlet boundary condition is the source problem associated with the Dirichlet eigenvalue problem. In this section, we present some general results that can be used to prove the convergence of a large class of finite element methods for eigenvalue problems. It turns out that compact convergence of the finite element solution operators T_n in the L^2-norm is crucial.

Assume that $T \in \mathcal{L}(X, X)$ is compact, and let $I : X \to X$ be the identity operator. The eigenvalue problem for T is to find $\lambda \neq 0$ and nontrivial $x \in X$ such that

$$Tx = \frac{1}{\lambda}x. \tag{3.1}$$

Define $F : \Omega \to \mathcal{L}(X, X)$ such that

$$F(\eta) := T - \frac{1}{\eta}I, \quad \eta \in \Omega, \tag{3.2}$$

where Ω is a compact subset of $\mathbb{C}$ such that $0 \notin \Omega$. It is clear that $F(\eta)$ is a holomorphic Fredholm operator for every $\eta \in \Omega$ and $\rho(F) \neq \emptyset$. The eigenvalue problem of $F(\cdot)$ is to find $\lambda \in \Omega$ and $x \in X$ such that

$$F(\lambda)x = 0. \tag{3.3}$$

Clearly, (3.3) is equivalent to the eigenvalue problem (3.1) for T.

Let $\mathcal{T}_n := \mathcal{T}_{h_n}$ be a regular triangular mesh for D with mesh size $h_n \to 0^+, n \to \infty$. Let $X_n \subset X$ be the associated finite element space endowed with the L^2-norm and $P = \{p_n\}_{n\in\mathbb{N}}$, $p_n : X \to X_n$ be the L^2-projection, i.e., for $f \in X$, $p_n f \in X_n$ is such that

$$(p_n f, x_n) = (f, x_n) \qquad \text{for all } x_n \in X_n.$$

Let $I_n : X_n \to X_n$ be the identity operator and $q_n = p_n$. Assume that there exists a series of finite element approximation operators $T_n : X_n \to X_n$ for T. Define the discrete approximation operators

$$F_n(\eta) := T_n - \frac{1}{\eta} I_n, \quad \eta \in \Omega. \tag{3.4}$$

Since X_n is finite-dimensional, $F_n(\eta)$ is Fredholm with index zero for $\eta \in \Omega$ and $n \in \mathbb{N}$. For the convergence analysis of the discrete eigenvalues of $F_n(\cdot)$ to those of $F(\cdot)$, we shall see that

$$T_n \to T \quad \text{compactly}$$

is sufficient. The stable convergence of I_n to I holds.

LEMMA 3.1. *$I_n \to I$ stably.*

Proof. If $p_n I - I_n p_n = 0$, then $I_n \to I$. Furthermore, $I_n^{-1} : X_n \to X_n$ exists and is bounded. Hence $I_n \to I$ stably. ∎

The following lemmas establish sufficient conditions for compact convergence.

LEMMA 3.2. *Assume that $T_n \to T$ discretely. If there exists a space X' compactly embedded in X with $X_n \subset X'$ and $\{T_n\}_{n\in\mathbb{N}}$ as operators from X_n to X' are uniformly bounded, then $T_n \to T$ compactly.*

Proof. Let $\|x_n\| \leq C$ $(n \in \mathbb{N})$. Then $\{T_n x_n\}_{n\in\mathbb{N}}$ is bounded in X'. For any subsequence $\{T_n x_n\}_{n\in\mathbb{N}'}$, $\mathbb{N}' \subset \mathbb{N}$, due to the compact embedding of X' into X, there exists a convergent subsequence of $\{T_n x_n\}_{n\in\mathbb{N}''}$, $\mathbb{N}'' \subset \mathbb{N}'$, such that $T_n x_n \to y \in X$ $(n \in \mathbb{N}'')$. Hence,

$$\|T_n x_n - p_n y\| = \|p_n T_n x_n - p_n y\| \leqslant \|T_n x_n - y\| \to 0 \quad (n \in \mathbb{N}'').$$

The proof is complete. ∎

REMARK 3.3. For a finite element approximation of the source problem, one usually has a discrete solution operator $T_n' : X \to X_n$. In general, the discrete operator $T_n : X_n \to X_n$ is such that $T_n p_n = T_n'$.

LEMMA 3.4. *Let $T : X \to X$ be a compact operator. If $T_n \to T$ uniformly, then $T_n \to T$ compactly.*

Proof. Let $\|x_n\| \leq C$ $(n \in \mathbb{N})$. Since T is compact, for any subsequence $\{Tx_n\}_{n\in\mathbb{N}'}$, $\mathbb{N}' \subset \mathbb{N}$, there exits a convergent subsequence $\{Tx_n\}_{n\in\mathbb{N}''}$, $\mathbb{N}'' \subset \mathbb{N}'$, such that $Tx_n \to y \in X$ as $\mathbb{N}'' \ni n \to \infty$. For $n \in \mathbb{N}''$,

$$\begin{aligned}\|T_n x_n - p_n y\| &= \|T_n x_n - p_n T x_n + p_n T x_n - p_n y\| \\ &\leq \|T_n p_n x_n - p_n T x_n\| + \|p_n T x_n - p_n y\| \to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

Hence $\{T_n x_n\}_{n\in\mathbb{N}}$ is discretely compact and $T_n \to T$ compactly. ∎

THEOREM 3.5. *Let $T : X \to X$ be compact. Assume that $T_n \to T$ compactly. Then:*

1. *$\|F_n(\eta)\| \leq C$ for every $\eta \in \Omega$ and $n \in \mathbb{N}$.*
2. *For every $\eta \in \Omega$ and $x \in X$, $\|[F_n(\eta)p_n - p_n F(\eta)]x\| \to 0$.*
3. *$F_n(\eta) \to F(\eta)$ regularly for each $\eta \in \Omega$.*
4. *For an eigenvalue λ of F, there exist n_0 and a sequence of eigenvalues λ_n of F_n, $n > n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence λ_n with this convergence property and the associated eigenfunctions $x_n \in \mathcal{N}(F_n(\lambda_n))$, $\|x_n\| = 1$, it holds that*

$$|\lambda_n - \lambda| \leq C\epsilon_n^{1/\kappa},$$
$$\inf_{x \in \mathcal{N}(F(\lambda))} \|x_n - p_n x\| \leq C\epsilon_n^{1/\kappa},$$

where $\epsilon_n = \left\|(T_n p_n - p_n T)|_{G(\lambda)}\right\|$ and κ is the ascent of λ.

Proof.

1. Since $X_n \subset X$ and the p_n are L^2-projections, we clearly have that $p_n \in \mathcal{L}(X, X_n)$ and $p_n X = X_n$. Furthermore, since p_n are orthogonal projections, (2.1) is satisfied by taking $x = x_n$ and $b = 1$. Since $T_n \to T$ discretely, the T_n are uniformly bounded in n due to Theorem 2.5. It is clear that I_n's are uniformly bounded. Then, $\|F_n(\eta)\| \le C$ due to the fact that Ω is compact.
2. For a fixed $\eta \in \Omega$ and $x \in X$,

$$\begin{aligned}\|[F_n(\eta)p_n - p_n F(\eta)]x\| &= \left\|(T_n p_n - p_n T)x - \frac{1}{\eta}(I_n p_n - p_n I)x\right\| \\ &= \|(T_n p_n - p_n T)x\| \to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

3. Since $T_n \to T$ compactly, $F_n(\eta) \to F(\eta)$ regularly for each $\eta \in \Omega$ due to Theorem 2.9.
4. We have that

$$\max_{|\eta-\lambda|\le\delta} \max_{\substack{v\in G(\lambda)\\ \|v\|_X=1}} \|F_n(\eta)p_n x - p_n F(\eta)x\| = \max_{\substack{v\in G(\lambda)\\ \|v\|_X=1}} \|(T_n p_n - p_n T)x\|.$$

The proof is completed by applying Theorem 2.14. ∎

REMARK 3.6. In view of Remark 3.3, one has that

$$\|(T_n' - p_n T)x\| = \|p_n(T_n' - T)x\| \le \|(T_n' - T)x\|.$$

Hence, the consistency error ϵ_n is bounded by the error of the finite element solution operators.

4. The Dirichlet eigenvalue problem.

In this section, we analyze the convergence of several finite element methods for the Dirichlet eigenvalue problem. Let $D \subset \mathbb{R}^2$ be a bounded Lipschitz polygonal domain. The Dirichlet eigenvalue problem is to find $\lambda \in \mathbb{R}$ and $u \neq 0$ such that

$$\begin{aligned} -\Delta u &= \lambda u \qquad \text{in } D, \\ u &= 0 \qquad \text{on } \partial D. \end{aligned}$$

The associated source problem is, given f, to find u such that

$$\begin{aligned} -\Delta u &= f \qquad \text{in } D, \\ u &= 0 \qquad \text{on } \partial D. \end{aligned} \tag{4.1}$$

For $f \in L^2(D)$, the weak formulation of (4.1) is to find $u \in H_0^1(D)$ such that

$$a(u, v) = (f, v) \qquad \text{for all } v \in H_0^1(D), \tag{4.2}$$

where

$$a(u, v) = \int_D \nabla u \cdot \nabla v \, dx, \qquad (f, v) = \int_D f v \, dx.$$

The variational formulation of the eigenvalue problem is: Find $\lambda \in \mathbb{R}$ and $u \in H_0^1(D)$ such that

$$a(u, v) = \lambda(u, v) \qquad \text{for all } v \in H_0^1(D).$$

There exists a unique solution $u \in H_0^1(D)$ to (4.2). Furthermore, $u \in H^{1+\alpha}(D)$, where $\alpha \in (\frac{1}{2}, 1]$ ($\alpha = 1$ if D is convex) is the elliptic regularity index; see, e.g., [24, Section 3.2].

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Due to the wellposedness of (4.2) and the compact embedding of $H_0^1(D)$ into $L^2(D)$, the solution operator to (4.2)

$$T : L^2(D) \to L^2(D) \quad \text{such that} \quad Tf = u$$

is compact. The Dirichlet eigenvalue problem is equivalent to the operator eigenvalue problem of finding $\lambda \in \mathbb{R}$ and $u \in L^2(D)$ such that

$$T(\lambda u) = u.$$

LEMMA 4.1. *Let $\Omega \subset \mathbb{C}\backslash\{0\}$ be a compact set and $F(\cdot)$ be defined in (3.2). Then $F : \Omega \to \mathcal{L}(L^2(D), L^2(D))$ is a holomorphic Fredholm operator function.*

Proof. It is clear that $F(\cdot)$ is holomorphic in Ω. Since T is compact and I is the identity operator, $F(\eta)$ is a Fredholm operator of index zero for every $\eta \in \Omega$. ∎

Assume that there exists a finite element solution operator $T_n : X_n \to X_n$ such that $T_n f_n = u_n$. In general, one has convergence of the finite element method for the source problem, i.e.,

$$\lim_{n\to\infty} \|T_n p_n f - p_n T f\| = 0 \qquad \text{for all } f \in X.$$

Next we investigate several finite element methods for the Dirichlet eigenvalue problem using the results in Section 3.

4.1. The conforming finite element method. Let X_n be the Lagrange element space equipped with the L^2-norm. The discrete formulation for the source problem (4.2) is as follows. For $f \in L^2(D)$, find $u_n \in X_n$ such that

$$a(u_n, v_n) = (f_n, v_n) \qquad \text{for all } v_n \in X_n, \tag{4.3}$$

where $a(u_n, v_n) = (\nabla u_n, \nabla v_n)$ and $f_n = p_n f$.

The discrete problem (4.3) has a unique solution u_n such that $\|u_n\|_{H^1(D)} \le C\|f_n\|$. Let u and u_n be the solutions of (4.2) and (4.3), respectively. The classical finite element error analysis gives that (see, e.g., [9])

$$\|u - u_n\| \le C h_n^{2\alpha} \|f\|. \tag{4.4}$$

Let $T_n : X_n \to X_n$, $T_n f_n = u_n$, be the finite element solution operator of (4.3). Since

$$\|u_n - p_n u\| = \|p_n u_n - p_n u\| \le \|u_n - u\|,$$

using (4.4), one obtains that

$$\lim_{n\to\infty} \|T_n p_n f - p_n T f\| = 0 \qquad \text{for all } f \in X.$$

LEMMA 4.2. *Let $F_n(\cdot)$ be defined in (3.4). Then $F_n \to F$ regularly.*

Proof. Due to Lemma 3.2, $T_n \to T$ compactly. Lemma 2.9 then implies that $F_n \to F$ regularly, completing the proof. ∎

The convergence for the Lagrange finite element method of finding λ_n and u_n such that

$$a(u_n, v_n) = \lambda_n (u_n, v_n) \qquad \text{for all } v_n \in X_n$$

follows from (4.4) and Theorem 3.5.

THEOREM 4.3. *Let $\lambda \in \sigma(F)$. There exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associated eigenfunction u_n, $\|u_n\| = 1$, one has that*

$$|\lambda_n - \lambda| \leq C h_n^{2\alpha} \quad \text{and} \quad \|u_n - u\| \leq C h_n^{2\alpha}, \tag{4.5}$$

where u is some eigenfunction associated to λ with $\|u\| = 1$.

Proof. Since T is self-adjoint, each eigenvalue has ascent $\kappa = 1$. From (4.4), it holds that $\epsilon_n \leq C h_n^{2\alpha}$. Hence (4.5) holds due to Theorem 3.5. ∎

REMARK 4.4. For convex domains, one has that $\alpha = 1$ and thus obtains second-order convergence for the eigenvalues and eigenfunctions using the linear Lagrange element.

4.2. Interior penalty discontinuous Galerkin methods. We consider the interior penalty discontinuous Galerkin methods for the Dirichlet eigenvalue problem [1, 26]. Denote by $\mathcal{F}_n^I$ and $\mathcal{F}_n^B$ the sets of the interior edges and boundary edges of the triangulation $\mathcal{T}_n$, respectively. Let $\mathcal{F}_n := \mathcal{F}_n^I \cup \mathcal{F}_n^B$. Let $\boldsymbol{w}$ and v be piecewise smooth vector-valued and scalar-valued functions. Let $F \in \mathcal{F}_n^I$ be an interior face shared by two elements K^+ and K^- with outward normal $\boldsymbol{\nu}^\pm$. Denote by $\boldsymbol{w}^\pm$ and $v^\pm$ the value of $\boldsymbol{w}$ and v on $\partial K^\pm$ taken from within $K^\pm$, respectively. The jumps across F are defined by

$$[\![\boldsymbol{w}]\!] = \boldsymbol{w}^+ \cdot \boldsymbol{\nu}^+ + \boldsymbol{w}^- \cdot \boldsymbol{\nu}^-, \qquad [\![v]\!] = v^+ \boldsymbol{\nu}^+ + v^- \boldsymbol{\nu}^-,$$

and the averages are defined by

$$\{\!\{\boldsymbol{w}\}\!\} = \frac{1}{2}\left(\boldsymbol{w}^+ + \boldsymbol{w}^-\right), \qquad \{\!\{v\}\!\} = \frac{1}{2}\left(v^+ + v^-\right).$$

For $F \in \mathcal{F}_n^B$, one simply defines

$$[\![\boldsymbol{w}]\!] = \boldsymbol{w} \cdot \boldsymbol{\nu}, \qquad [\![v]\!] = v\boldsymbol{\nu}, \qquad \{\!\{\boldsymbol{w}\}\!\} = \boldsymbol{w}, \qquad \{\!\{v\}\!\} = v.$$

Define the discontinuous Galerkin space X_n by

$$X_n := \{v \in L^2(D) : v|_K \in P^\ell(K), K \in \mathcal{T}_n\},$$

where $P^\ell(K)$ is the space of polynomials of degree at most $\ell \geq 1$ on K. Let p_n be the L^2-projection of $f \in L^2(D)$ to X_n such that

$$(p_n f, v_n) = (f, v_n) \qquad \text{for all } v_n \in X_n.$$

We consider the DG methods in primal form for the Poisson equation. Find $u_n \in X_n$ such that

$$a_n(u_n, v_n) = (f_n, v_n) \qquad \text{for all } v_n \in X_n, \tag{4.6}$$

where $a_n : X_n \times X_n \to \mathbb{R}$ is defined by

$$\begin{aligned} a_n(u_n, v_n) = (\nabla_n u_n, \nabla_n v_n) &- \int_{\mathcal{F}_n} \{\!\{\nabla_n u_n\}\!\} \cdot [\![v_n]\!] ds \\ &- r \int_{\mathcal{F}_n} \{\!\{\nabla_n v_n\}\!\} \cdot [\![u_n]\!] ds - s_n(u_n, v_n) \qquad \text{for } u_n, v_n \in X_n, \end{aligned}$$

where ∇_n is the element-wise gradient operator. The stabilization form $s_n(\cdot,\cdot)$ is defined as

$$s_n(u_n, v_n) = \int_{\mathcal{F}_n} \gamma h_n^{-1} [\![u_n]\!][\![v_n]\!] ds,$$

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for $\gamma > 0$ independent of the mesh size. In particular, one obtains the symmetric interior penalty (SIP) method by setting $r = 1$, the non-symmetric interior penalty (NIP) method by setting $r = -1$, and the incomplete interior penalty (IIP) by setting $r = 0$.

The finite element space X_n can be endowed with a second norm $\|\cdot\|_n$:

$$\|v_n\|_n = \|\nabla_n v_n\| + \|h^{-1/2}[\![v_n]\!]\|^2_{\mathcal{F}_n}.$$

For $f \in H^1_0(D)$ or $f \in X_n$, the Poincaré inequality holds (Property 1 in [1])

$$\|f\| \leq C\|f\|_n, \tag{4.7}$$

where C is a constant depending on D but not on the mesh.

For γ large enough, it is well-known that there exists a unique solution u_n to (4.6). Denote the discrete solution operator by $T_n : X_n \to X_n$. Let $T_n \to T$, and let $\|x_n\| \leq C$ $(n \in \mathbb{N})$. Since T_n's are uniformly bounded as operators from $(X_n, \|\cdot\|)$ to $(X_n, \|\cdot\|_n)$, $\{\|T_n x_n\|_n\}_{n\in\mathbb{N}}$ is uniformly bounded. Due to Theorem 5.7 in [12], there exists a convergent subsequence $T_n x_n$ in the L^2-norm. Thus, $T_n \to T$ compactly.

According to Property 2 in [1], for the discrete solution u_n to (4.6) and the exact solution u to (4.2), it holds that

$$\|u - u_n\|_n \leq Ch^\alpha \|f\| \qquad \text{for } f \in L^2(D).$$

Using the Poincaré inequality (4.7), for $f \in L^2(D)$, one has that

$$\|T_n p_n f - p_n T f\| = \|u_n - p_n u\| \leq \|u_n - u\| \leq \|u_n - u\|_n \leq Ch^\alpha \|f\|.$$

THEOREM 4.5. *Let $\lambda \in \sigma(F)$. There exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associated eigenfunction u_n, $\|u_n\| = 1$, one has that*

$$|\lambda_n - \lambda| \leq Ch_n^\alpha \quad \textit{and} \quad \|u_n - u\| \leq Ch_n^\alpha, \tag{4.8}$$

where u is some eigenfunction associated to λ with $\|u\| = 1$.

Proof. Since $T_n \to T$ compactly, $F_n(\eta)$ converges to $F(\eta)$ regularly due to Lemma 3.4. Then (4.8) follows from Theorem 3.5. ∎

REMARK 4.6. For the symmetric DG method, that is, when $r = 1$ in (4.6), the convergence order in Theorem 4.5 can be improved using the duality argument.

4.3. The nonconforming Crouzeix-Raviart method. We consider the nonconforming piecewise linear finite element space of Crouzeix-Raviart [6]:

$$\begin{aligned} X_n := \{v : v|_K \in \mathcal{P}^1(K) &\text{ is continuous at the midpoints of the edges of } K \\ &\text{and } v = 0 \text{ at the midpoints on } \partial D\}, \end{aligned}$$

where $\mathcal{P}^1(K)$ denotes the space of polynomials on K of degree less or equal to 1.

Define the bilinear form on X_n

$$a_n(u_n, v_n) := (\nabla_n u_n, \nabla_n v_n), \qquad u_n, v_n \in X_n.$$

The discrete problem is to find $u_n \in X_n$ such that

$$a_n(u_n, v_n) = (f_n, v_n) \qquad \text{for all } v_n \in X_n, \tag{4.9}$$

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where $f_n = p_n f$. There exists a unique solution u_n to (4.9). Denote the discrete solution operator to be $T_n : X_n \to X_n$ such that

$$u_n = T_n f_n.$$

Using the error estimate in the L^2-norm (Theorem 1.5 of Ch. III in [6]) and the property of the L^2-projection, one has that

$$\|p_n T f - T_n p_n f\| \le C h^{2\alpha} \|f\|,$$

which implies $T_n \to T$ uniformly.

Using Theorem 3.5, we obtain the convergence of the non-conforming Crouzeix-Raviart method. The proof is similar to the previous ones and thus omitted.

THEOREM 4.7. *Let $\lambda \in \sigma(F)$. There exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \ge n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associated eigenfunction u_n, $\|u_n\| = 1$, one has that*

$$|\lambda_n - \lambda| \le C h_n^{2\alpha} \quad \textit{and} \quad \|u_n - u\| \le C h_n^{2\alpha},$$

where u is some eigenfunction associated to λ with $\|u\| = 1$.

REMARK 4.8. The Crouzeix-Raviart element is non-conforming in the sense that the finite element space is not a subspace of $H_0^1(D)$, which is the solution space for (4.2). Note here, however, that we use the space $L^2(D)$ instead of $H_0^1(D)$ so that the discrete space becomes a subspace. A similar comment applies to the Morley element method for the biharmonic eigenvalue problem in the next section.

5. The biharmonic eigenvalue problem.

We now consider the biharmonic eigenvalue problem. Let D denote a bounded Lipschitz polygonal domain in $\mathbb{R}^2$ with boundary ∂D. Let ν denote the unit outward normal to ∂D. The biharmonic eigenvalue problem with clamped plate boundary condition is to find $\lambda \in \mathbb{R}$ and $u \ne 0$ such that

$$\begin{aligned} \Delta^2 u &= \lambda u && \text{in } D, \\ u = \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial D. \end{aligned} \tag{5.1}$$

The associated source problem is as follows. Given a function f, find a function u such that

$$\begin{aligned} \Delta^2 u &= f && \text{in } D, \\ u = \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial D. \end{aligned} \tag{5.2}$$

Define

$$H_0^2(D) := \left\{ v \in H^2(D) : v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial D \right\}$$

and a bilinear form $a : H_0^2(D) \times H_0^2(D)$ such that

$$a(u, v) := (\Delta u, \Delta v).$$

The weak formulation of (5.2) is, for $f \in L^2(D)$, to find $u \in H_0^2(D)$ such that

$$a(u, v) = (f, v) \qquad \text{for all } v \in H_0^2(D).$$

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The weak formulation of (5.1) is to find $\lambda \in \mathbb{R}$ and $u \in H_0^2(D), u \neq 0$, such that

$$a(u,v) = \lambda(u,v) \qquad \text{for all } v \in H_0^2(D). \tag{5.3}$$

There exists a unique solution u to (5.3) belonging to $H^{2+\alpha}(D)$ for some $\alpha \in (1/2, 1]$ such that

$$\|u\|_{H^{2+\alpha}(D)} \leq C\|f\|,$$

where the constant C depends only on D. When D is convex, $\alpha = 1$. The parameter α is referred to as the index of elliptic regularity for the biharmonic equation.

Consequently, there exists a solution operator $T : L^2(D) \to L^2(D)$ such that, given $f \in L^2(D)$,

$$a(Tf, v) = (f, v) \qquad \text{for all } v \in H_0^2(D).$$

It is obvious that T is self-adjoint due to the symmetry of $a(\cdot,\cdot)$ and compact due to the compact embedding of $H_0^2(D)$ into $L^2(D)$.

5.1. The Argyris element method. We consider the Argyris element (see, e.g., [24, Section 4.2]), which is H^2-conforming for triangular meshes $\mathcal{T}_{h_n}$. Denote the associated finite element space by X_n. The discrete problem for the source problem (5.2) can be stated as follows. For $f \in L^2(D)$, find $u_n \in X_n \subset H_0^2(D)$ such that

$$a(u_n, v_n) = (p_n f, v_n) \qquad \text{for all } v_n \in X_n. \tag{5.4}$$

There exists a unique solution u_n to (5.4) such that

$$\|u - u_n\|_{H^2(D)} \leq C h_n^{\alpha} \|f\|. \tag{5.5}$$

The discrete formulation for the eigenvalue problem (5.1) is to find $\lambda_n \in \mathbb{R}$ and $u_n \in X_n$, $u_n \neq 0$, such that

$$a(u_n, v_n) = \lambda_n(u_n, v_n) \qquad \text{for all } v_n \in X_n.$$

Using a duality argument (Ch. II of [6]) and (5.5), one has that

$$\|u - u_n\| \leq C h_n^{2\alpha} \|f\|.$$

Consequently, the discrete solution operator $T_n : X_n \to X_n$ is such that

$$\|T_n p_n f - p_n T f\| \leq C h_n^{2\alpha} \|f\| \qquad \text{for } f \in L^2(D).$$

Since $X_n \subset H_0^2(D), n \in \mathbb{N}$, and the embedding of $H_0^2(D)$ into X is compact, one has that $T_n \to T$ compactly due to Lemma 3.2.

Let $F(\cdot)$ and $F_n(\cdot)$ be defined as in (3.2) and (3.4), respectively, and $F_n \to F$ regularly. Using a similar argument as for the Dirichlet eigenvalue problem, we obtain the following convergence theorem of the Argyris element method for the biharmonic eigenvalue problem.

THEOREM 5.1. *Let $\lambda \in \sigma(F)$. There exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associated eigenfunction u_n, $\|u_n\| = 1$, one has that*

$$|\lambda_n - \lambda| \leq C h_n^{2\alpha} \quad \textit{and} \quad \|u_n - u\| \leq C h_n^{2\alpha},$$

where u is some eigenfunction associated to λ with $\|u\| = 1$.

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5.2. The C^0 interior penalty discontinuous Galerkin method. We consider the C^0 interior penalty discontinuous Galerkin method (C^0 IPG) for the biharmonic equation [8]; see also [27] using the holomorphic operator approach but for a different proof. Let $X_n \subset H^1(\Omega)$ be the Lagrange finite element space of order $k \geq 2$ associated with $\mathcal{T}_{h_n}$. Let $\mathcal{E}_{h_n}$ be the set of the edges in $\mathcal{T}_{h_n}$. For edges $e \in \mathcal{E}_{h_n}$ that are the common edge of two adjacent triangles $K_\pm \in \mathcal{T}_{h_n}$ and for $v \in X_n$, we define the jump of the flux to be

$$[\![\partial v/\partial n_e]\!] = \frac{\partial v_{K_+}}{\partial n_e}\Big|_e - \frac{\partial v_{K_-}}{\partial n_e}\Big|_e,$$

where n_e is the unit normal pointing from K_- to K_+. Let

$$\frac{\partial^2 v}{\partial n_e^2} = n_e \cdot (\triangle v) n_e$$

and define the average normal-normal derivative to be

$$\left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v_{K_+}}{\partial n_e^2} + \frac{\partial^2 v_{K_-}}{\partial n_e^2} \right).$$

For $e \subset \partial D$, we take n_e to be the unit outward normal and define

$$[\![\partial v/\partial n_e]\!] = -\frac{\partial v}{\partial n_e} \qquad \text{and} \qquad \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} = \frac{\partial^2 v}{\partial n_e^2}.$$

Given $f_n \in X_n$, $f_n = p_n f$, the corresponding C^0 IPG method for the source problem is to find $u_h \in X_n$ such that

$$a_n(u_n, v_n) = (f_n, v_n) \qquad \text{for all } v \in X_n, \tag{5.6}$$

where

$$\begin{aligned} a_n(w,v) = &\sum_{K \in \mathcal{T}_{h_n}} \int_K D^2 w : D^2 v \, \mathrm{d}x \\ &+ \sum_{e \in \mathcal{E}_{h_n}} \int_e \left\{\!\!\left\{ \frac{\partial^2 w}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n_e} \right]\!\!\right] + \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n_e} \right]\!\!\right] \mathrm{d}s \\ &+ \sigma \sum_{e \in \mathcal{E}_{h_n}} \frac{1}{|e|} \int_e \left[\!\!\left[\frac{\partial w}{\partial n_e} \right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n_e} \right]\!\!\right] \mathrm{d}s. \end{aligned}$$

In the above equation, $D^2 w : D^2 v = \sum_{i,j=1}^2 w_{x_i x_j} v_{x_i x_j}$ is the Frobenius inner product of the Hessian matrices of w and v, and $\sigma > 0$ is a (sufficiently large) penalty parameter. There exist discrete solution operators $T_n : X_n \to X_n$ to (5.6) such that $T_n \to T$.

The C^0 IPG method for the biharmonic eigenvalue problem is to find $\lambda_n \in \mathbb{R}$ and $u_n \neq 0$ such that

$$a_n(u_n, v_n) = \lambda_n (u_n, v_n) \qquad \text{for all } v_n \in X_n. \tag{5.7}$$

Let $X' = H_0^1(D)$. Since the $T_n : X_n \to X', n \in \mathbb{N}$, are uniformly bounded, due to the compact embedding of X' into X and Lemma 3.2, $T_n \to T$ compactly. Lemma 2.9 implies that $F_n \to F$ regularly. Then the following convergence theorem holds for the C^0 IPG method.

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THEOREM 5.2. *Let $\lambda \in \sigma(F)$. There exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associated eigenfunction u_n, $\|u_n\| = 1$, one has that*

$$|\lambda_n - \lambda| \leq Ch_n^{2\alpha} \quad \textit{and} \quad \|u_n - u\| \leq Ch_n^{2\alpha},$$

where u is some eigenfunction associated to λ with $\|u\| = 1$.

Proof. Due to Lemma 1 in [8], there exists a unique discrete solution u_n to (5.7) such that

$$\|u - u_n\| \leq Ch_n^{2\alpha}\|f\| \qquad \text{for } f \in L^2(D).$$

One has that

$$\|T_n p_n f - p_n T f\| \leq Ch_n^{2\alpha}\|f\| \qquad \text{for } f \in L^2(D).$$

Then Theorem 3.5 applies. ∎

5.3. The Morley element method. We consider the Morley element space X_n. Let p_n be the L^2-projection from X onto X_n. For $f \in X$, the Morley finite element method for the biharmonic equation is to find $u_n \in X_n$ such that

$$a_n(u_n, v_n) = (p_n f, v_n) \qquad \text{for all } \ v_n \in X_n, \tag{5.8}$$

where

$$a_n(u_n, v_n) := \sum_{K \in \mathcal{T}_n} \int_K D^2 u_n : D^2 v_n dK, \quad u_n, v_n \in X_n.$$

There exists a unique solution $u_n \in X_n$ such that (see, e.g., [24, Section 4.4.2])

$$\|u - u_n\|_{2,h_n} \leq Ch_n^{\alpha}\|f\|, \tag{5.9}$$

where the mesh-dependent norm $\|\cdot\|_{2,h_n}$ is defined as

$$\|u\|_{2,h_n} = \sum_{K \in \mathcal{T}_n} (u, u)_{H^2(K)}.$$

Let $T_n : X_n \to X_n$ be the discrete solution operator to (5.8). Since $\|u\| \leq \|u\|_{2,h_n}$, due to (5.9), it holds that

$$\|u - u_n\| \leq Ch_n^{\alpha}\|f\|.$$

As a consequence, one has that

$$\|T_n p_n f - p_n T f\| \leq Ch_n^{\alpha}\|f\| \qquad \text{for } f \in X.$$

We have the following convergence theorem for the Morley element method.

THEOREM 5.3. *Let $\lambda \in \sigma(F)$. There exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associated eigenfunction u_n, $\|u_n\| = 1$, one has that*

$$|\lambda_n - \lambda| \leq Ch_n^{\alpha} \quad \textit{and} \quad \|u_n - u\| \leq Ch_n^{\alpha},$$

where u is some eigenfunction associated to λ with $\|u\| = 1$.

REMARK 5.4. We would like to use this example to illustrate that when an order of convergence is available for the source problem, the same order of convergence holds for the eigenvalue problem. The convergence order can be improved using a sharper error estimate in the L^2-norm; see, e.g., [13].

6. Conclusions and future work. In this paper, we present a general approach to prove convergence of various finite element methods for eigenvalue problems, including the conforming methods, discontinuous Galerkin methods, and non-conforming methods. Using the abstract approximation theory for eigenvalue problems of holomorphic Fredholm operator functions, one needs to verify the compact convergence of the discrete solution operators. The result has the potential to prove convergence of many other finite element methods for eigenvalue problems such as the mixed finite element methods, virtual element methods, and weak Galerkin methods.

We use the space $L^2(D)$ and the L^2-projection for the finite element spaces. Thus, the convergence of the eigenfunctions is also in the L^2-norm. However, this framework also works if one chooses other spaces and projections, for example, $H^1(D)$ and H^1-projections for the finite element spaces for the Dirichlet eigenvalue problem. Then one can obtain convergence of the eigenfunctions in the H^1-norm.

As seen above, the convergence order obtained using Theorem 3.5 for certain methods is not optimal. However, if the optimal convergence order for the source problem is available in the L^2-norm, then the optimal convergence order for the eigenfunctions follows immediately.

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