Electronic Transactions on Numerical Analysis.
Volume 55, pp. 568–584, 2022.

ISSN 1068–9613.
DOI: 10.1553/etna_vol55s568

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ANALYSIS OF STABILITY AND CONVERGENCE FOR L-TYPE FORMULAS COMBINED WITH A SPATIAL FINITE ELEMENT METHOD FOR SOLVING SUBDIFFUSION PROBLEMS*

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Abstract. A time-fractional diffusion equation with the Caputo fractional derivative of order $\alpha \in (0,1)$ is considered on a bounded polygonal domain. Some numerical methods are presented based on the finite element method (FEM) in space on a quasi-uniform mesh and L-type discretizations (i.e., L1, L1-2, and L1-2-3 formulas) to approximate the Caputo derivative. Stability and convergence of the L1-2-3 FEM as well as L1-2 FEM are proved rigorously. The lack of positivity of the coefficients of these formulas is the main difficulty in the analysis of the proposed methods. This has hampered the analysis of methods using finite elements mixed with L1-2 and L1-2-3 discretizations. Our proofs are based on the concept of a special kind of discrete Grönwall's inequality and the energy method. Numerical examples confirm the theoretical analysis.

Key words. subdiffusion equation, finite element method, Caputo derivative, L1 formula, L1-2 formula, L1-2-3 formula, Grönwall's inequality, stability analysis, convergence analysis

AMS subject classifications. 65M12, 65M60

1. Introduction.

Time-fractional diffusion equations have been increasingly used to model physical processes where anomalous diffusion may occur, i.e., processes whose mean-squared displacement of the spreading particles grow non-linearly in time $\langle x^2 \rangle \sim K_\alpha t^\alpha$ with the anomalous diffusion exponent $0 < \alpha < 2$ and the generalized diffusion coefficient K_α. If $\alpha \in (0,1)$, then the process is slower than Brownian diffusion and is called subdiffusion [14].

Our aim in this paper is to investigate stability and convergence of the finite element method (FEM) combined with an L-type (L1-2 and L1-2-3) formula for solving the following subdiffusion problem

$$
\begin{cases}
\mathrm{D}_t^\alpha u(\mathbf{x},t) - \nabla \cdot (\kappa \nabla u(\mathbf{x},t)) = f(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times (0,T], \\
u(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \partial\Omega \times [0,T], \\
u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x} \in \Omega,
\end{cases}
\tag{1.1}
$$

where $T > 0$ denotes the final time. Here, the given functions κ, f, and u_0 are bounded and smooth enough, κ is the diffusion coefficient, which satisfies $0 < \kappa_0 \leqslant \kappa$, and $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain. The Caputo time-fractional derivative of order α, denoted by D_t^α, is defined as

$$
\mathrm{D}_t^\alpha u(\cdot,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_t(\cdot,s)}{(t-s)^\alpha} ds,
\tag{1.2}
$$

with Γ the Euler gamma function.

Finding an analytic solution of (1.1) is non-trivial due to the non-locality of the fractional derivative (1.2) [32], and therefore various numerical methods have been proposed to approximate the Caputo fractional derivative and solve such equations.

Many authors use the L1 formula [9, 17] for approximating the Caputo fractional derivative; however, other L-type formulas such as L1-2 [6] and L1-2-3 [15] with $3-\alpha$ and

*Received June 3, 2021. Accepted April 21, 2022. Published online on June 10, 2022. Recommended by Stefan Vanderwalle.
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third-order accuracy, respectively, L2-1_σ with second-order accuracy [1], the L2 formula with $3-\alpha$ order of accuracy [25], S-type formulas (Sl discretizations, $l=1,2,3$) with $(l+1-\alpha)$-order accuracy [18], and other high-order schemes can be employed for the time discretization by means of the Caputo derivative. Efficient numerical methods such as finite difference methods (FDMs) [13, 16, 30], FEMs [5, 8, 12, 24, 33], and local discontinuous Galerkin (LDG) methods [2, 3, 10, 11, 28] have been used for the spatial direction. Recently, Erfani et al. [4] solved and analyzed some fractional differential equations using the fractional pseudo-spectral integration and differentiation matrices. Ford et al. [5] introduced a fully discrete scheme based on a quadrature formula and the FEM for time-fractional partial differential equations and obtained optimal convergence error estimates. Zeng et al. [30, 31] considered numerical algorithms for time-fractional subdiffusion equations using fractional linear multistep methods for the time discretization and the FEM for the space direction, and proved unconditionally stability and convergence of the methods. Two methods based on the piecewise linear finite element method and convolution quadrature were developed in [8] for the subdiffusion and diffusion-wave equations with first- and second-order accuracy in time for both smooth and non-smooth data.

Zhao et al. [36] established an unconditionally stable fully-discrete scheme for 2D time-fractional diffusion equations using nonconforming finite element and the L1 formula. A Crank-Nicolson scheme in time and linear triangular finite element method in space have been used for 2D multi-term time-fractional diffusion-wave equations [21]. Ren et al. [20] derived highly accurate error estimates of the FEM for the nonlinear subdiffusion equation. In [24], a weak Galerkin finite element method (WG-FEM) is derived for the time-fractional diffusion equation with the L1 formula, and the convergence order of this scheme is obtained. Recently, a new scheme for the subdiffusion equation has been provided based on the fractional Crank-Nicolson convolution quadrature in time and the FEM in spatial direction [23].

Recently, a method based on the L1 formula and the FEM has been established in [33] with a new error bound of order β for the L1 formula, where β $(1-\alpha \le \beta \le 2-\alpha)$ is related to the smoothness of the function u. Stability and superconvergence of the proposed scheme was proven with respect to the H^1-norm. In [29], an LDG method was constructed for 2D time-fractional diffusion equations utilizing L1 and L1-2 formulas for the time discretization. Recently, Wang et al. [26] changed two steps of the classic L1-2 formula and proved the convergence of the FEM mixed with the corrected L1-2 scheme. Although many alternative schemes have been proposed for the time discretization, they are not as simple as the L1 formula, and they are more complicated to implement. The main advantage of L-type formulas consists in their simplicity as well as in their high-order accuracy. To the best of our knowledge, no attempt has been made to analyze the stability and convergence of the FEM mixed with the classic L1-2 and L1-2-3 formulas.

This gap in the literature is filled by this paper. It is worth pointing out that the property of strictly monotone decrease does not hold for all coefficients of the L1-2 and L1-2-3 formulas, causing technical difficulties in comparison to the L1 formula when proving stability and convergence. In this case, the method of induction breaks down. Hence, we use a special kind of Grönwall's inequality to achieve these goals. Recently, a stability and convergence analysis of finite difference methods based on L-type formulas have been investigated in [16]. The novelty of the present paper consists in using FEMs instead of FDMs and that proofs and results are generated for a two-dimensional spatial domain. The obtained results related to the time discretization remain valid for one-dimensional subdiffusion equations and can be easily extended to 3D problems where meshing is tedious.

A brief outline of the paper follows. In Section 2, we present some preliminaries related to the FEM and L-type formulas. Section 3 is devoted to the analysis of stability and convergence

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of the FEM combined with L-type formulas. Some numerical results are demonstrated in Section 4 to verify the correctness of the theoretical analysis. The paper is finished with a short conclusion and Appendix A.

2. Preliminaries. In this section, we recall some well-known statements from functional analysis and then present a method based on L-type schemes in time and the FEM in space resulting in a fully discrete scheme for solving the time-fractional diffusion, i.e., the subdiffusion equation. Let us consider $\mathcal{T}_h$, a shape-regular family of triangulations over a domain Ω such that $\overline{\Omega} = \bigcup_{\tau\in\mathcal{T}_h} \tau$. Throughout this paper, the mesh parameter h represents the largest edge of all triangles in $\mathcal{T}_h$ and $H^q(\Omega)$ denotes the standard Sobolev space with its associated norm $\|\cdot\|_q$ and seminorm $|\cdot|_q$. Furthermore, the $L^2(\Omega)$-inner product is defined as

$$(v,w) := \int_\Omega vw \,\mathrm{d}\mathbf{x}, \qquad (\nabla v, \nabla w) := \int_\Omega \nabla v\cdot\nabla w \,\mathrm{d}\mathbf{x},$$

with the corresponding L^2-norm $\|\cdot\|$.

LEMMA 2.1 (Young's inequality [27]). *For all $\epsilon > 0$ and v, w, $(v,w) \leqslant \frac{1}{4\epsilon}\|v\|^2 + \epsilon\|w\|^2$.*

LEMMA 2.2 (Special kind of Grönwall's inequality [7, p. 4]). *Let $\{v_k\}$ and $\{w_k\}$ be non-negative sequences and c a non-negative constant. If*

$$v_k \leqslant c + \sum_{j=0}^{k-1} w_j v_j, \qquad \textit{for all } k \geq 1,$$

then $v_k \leqslant c\exp(\sum_{j=0}^{k-1} w_j)$ for all $k \geq 1$.

2.1. Space discretization. The first step to discretize equation (1.1) is to consider its weak formulation: Find $u \in C^1(0,T;H_0^1(\Omega))$ such that

$$(\mathrm{D}_t^\alpha u, v) + (\kappa\nabla u, \nabla v) = (f,v), \qquad \forall v \in H_0^1(\Omega). \tag{2.1}$$

The standard finite element space $V_h^0 \subset V = H_0^1(\Omega)$ is defined as

$$V_h^0 = \{v \in C(\Omega), v|_{\partial\Omega} = 0 \text{ and } v|_\tau \in \mathbb{P}_1(\tau) \text{ for all } \tau \in \mathcal{T}_h\},$$

where $\mathbb{P}_1(\tau)$ denotes the space of polynomial functions $p(\tau)$ with $\deg p \leq 1$. Corresponding to the finite element space V_h^0, a projection operator $\mathcal{P}_h : V \to V_h^0$

$$(\mathcal{P}_h w, v) = (w,v), \qquad \forall v \in V_h^0,$$

is defined for any $w \in V$.

LEMMA 2.3 ([19, Lemma 3.1]). *Let $w \in H^3(\Omega)$. Then*

$$|(\nabla(w - P_h w), \nabla v)| \leqslant Ch^2\|w\|_3\|v\|.$$

LEMMA 2.4 (Special kind of approximation property, [37]). *If $w \in H^2(\Omega)\cap H_0^1(\Omega)$, then there exists a constant C depending only on Ω such that*

$$\|w - \mathcal{P}_h w\| \leqslant Ch^2\|w\|_2.$$

2.2. Time discretization. Let n be a positive integer. We define the time step size $\Delta t = T/n$, as well as $\mu = \Delta t^\alpha \Gamma(2-\alpha)$, and we introduce u_h^k as the approximation to $u^k := u(\cdot, k\Delta t)$. In the fully discrete FEM corresponding to (2.1), we aim to find $u_h^n \in V_h^0$ such that for $k = 1, \dots, n$

$$(\mathbb{D}_t^{\alpha,l} u_h^k, v) + (\kappa \nabla u_h^k, \nabla v) = (f^k, v), \qquad \forall v \in V_h^0. \tag{2.2}$$

Therein, $\mathbb{D}_t^{\alpha,l}$ denotes the L1 (A.1), L1-2 (A.2), and L1-2-3 (A.4) formula, for $l = 1, 2, 3$, respectively, to approximate the time-fractional derivative in the Caputo-sense. For more details about these formulas, see Appendix A.

Let us denote $u^k - u_h^k = e_p^k + e_n^k$, where $e_p^k = u^k - \mathcal{P}_h u^k$ and $e_n^k = \mathcal{P}_h u^k - u_h^k$. Subtracting (2.2) from (2.1) for $t = t^k$ leads to the following error equation

$$(\mathbb{D}_t^{\alpha,l} e_n^k, v) + (\kappa \nabla e_n^k, \nabla v) = -(\mathbb{D}_t^{\alpha,l} e_p^k, v) - (\kappa \nabla e_p^k, \nabla v) - (R_l^k, v), \qquad \forall v \in V_h^0, \tag{2.3}$$

where $R_l^k = \mathrm{D}_t^\alpha u^k - \mathbb{D}_t^{\alpha,l} u^k$, for $l = 1, 2, 3$. The quantity $\|\mathbb{D}_t^{\alpha,l} e_p^k\|$ can be estimated via

$$\begin{aligned} \|\mathbb{D}_t^{\alpha,l} e_p^k\| &= \|\mathbb{D}_t^{\alpha,l} e_p^k - \mathrm{D}_t^\alpha e_p^k + \mathrm{D}_t^\alpha e_p^k\| \\ &\leqslant \|\mathbb{D}_t^{\alpha,l} u^k - \mathrm{D}_t^\alpha u^k - \mathcal{P}_h(\mathbb{D}_t^{\alpha,l} u^k - \mathrm{D}_t^\alpha u^k)\| + \|\mathrm{D}_t^\alpha e_p^k\| \\ &\leqslant C\|R_l^k\| + Ch^2 \|\mathrm{D}_t^\alpha u^k\|_2, \end{aligned} \tag{2.4}$$

wherein Lemma 2.4 has been used.

3. Analysis of the method. Throughout this section, we aim to investigate the numerical stability as well as the convergence of the scheme (2.2) assuming $f = 0$ for the simplicity of expressions in the stability analysis. Afterwards, error estimates are extracted for $l = 3, 2, 1$ requiring $u \in C^{l+1}(0, T; H_0^1(\Omega) \cap H^3(\Omega))$, respectively.

3.1. L1-2-3 discretization. Using the discrete Grönwall's inequality, we prove that the scheme (2.2) is unconditionally stable in the L^2- and in the H^1-norm for $l = 3$ followed by the error estimates for this method.

THEOREM 3.1 (L^2-stability). *The numerical solution of* (2.2) *with* $l = 3$ *satisfies*

$$\|u_h^k\| \leqslant \|u_h^0\|, \qquad \text{for } k = 1, \dots, n.$$

Proof. Choosing $v = u_h^k$, for $k = 3, \dots, n$, in (2.2) and using the assumption $\kappa_0 \leq \kappa$, we have

$$(\mathbb{D}_t^{\alpha,3} u_h^k, u_h^k) \leqslant -\kappa_0 (\nabla u_h^k, \nabla u_h^k).$$

Applying the L1-2-3 discretization defined in Appendix A allows us to reformulate the previous inequality as

$$d_0^\alpha \|u_h^k\|^2 \leqslant \sum_{j=1}^{k-1} \left(d_{k-j-1}^\alpha - d_{k-j}^\alpha\right)(u_h^j, u_h^k) + d_{k-1}^\alpha (u_h^0, u_h^k) - \mu\kappa_0 \|\nabla u_h^k\|^2 \leqslant \mathrm{I} + \mathrm{II}, \tag{3.1}$$

with $\mathrm{I} = \sum_{j=1}^{k-1} (d_{k-j-1}^\alpha - d_{k-j}^\alpha)(u_h^j, u_h^k)$ and $\mathrm{II} = d_{k-1}^\alpha (u_h^0, u_h^k)$. We have to consider two cases.

Case 1: If $d_2^\alpha < d_1^\alpha$, then we obtain by using Lemma 2.1 with $\epsilon = \frac{1}{2}$ and Lemma A.11,

$$\mathrm{I} \leqslant \frac{1}{2}\left(d_0^\alpha - d_{k-1}^\alpha\right)\|u_h^k\|^2 + \frac{1}{2}\sum_{j=1}^{k-1}\left(d_{k-j-1}^\alpha - d_{k-j}^\alpha\right)\|u_h^j\|^2, \tag{3.2}$$

$$\mathrm{II} \leqslant \frac{1}{2} d_{k-1}^\alpha \left(\|u_h^k\|^2 + \|u_h^0\|^2\right). \tag{3.3}$$

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Combining (3.1), (3.2), and (3.3) yields

$$\|u_h^k\|^2 \leqslant \sum_{j=1}^{k-1} \frac{d_{k-j-1}^\alpha - d_{k-j}^\alpha}{d_0^\alpha} \|u_h^j\|^2 + \frac{d_{k-1}^\alpha}{d_0^\alpha} \|u_h^0\|^2, \quad k = 3, \ldots, n.$$

Assuming $\varepsilon \geq 0$ results in

$$\|u_h^k\|^2 \leqslant \varepsilon \|u_h^0\|^2 + \sum_{j=0}^{k-1} w_j \, \|u_h^j\|^2, \quad k = 3, \ldots, n,$$

where $w_0 = d_{k-1}^\alpha / d_0^\alpha$ and $w_j = \left(d_{k-j-1}^\alpha - d_{k-j}^\alpha\right)/d_0^\alpha$, for $j = 1, \ldots, k-1$. The w_j's are non-negative according to Lemma A.11 which allows us to use Lemma 2.2 resulting in

$$\|u_h^k\|^2 \leqslant \varepsilon \exp\left(\sum_{j=0}^{k-1} w_j\right) \|u_h^0\|^2 = \varepsilon \exp(1) \|u_h^0\|^2, \quad k = 3, \ldots, n.$$

Choosing $\varepsilon \leq 1/\exp(1)$ gives $\|u_h^k\| \leqslant \|u_h^0\|$, for $k = 3, \ldots, n$, which implies that in this case the fully discrete scheme (2.2) is unconditionally stable in the L^2-norm for $l = 3$.

Case 2: If $d_1^\alpha < d_2^\alpha$, then using Lemma 2.1 with $\epsilon = 1$ yields

$$\text{II} \leqslant \frac{1}{4} d_{k-1}^\alpha \|u_h^k\|^2 + d_{k-1}^\alpha \|u_h^0\|^2. \tag{3.4}$$

On the other hand,

$$\text{I} = \sum_{\substack{j=1 \\ j \neq k-2}}^{k-1} \left(d_{k-j-1}^\alpha - d_{k-j}^\alpha\right) (u_h^j, u_h^k) - \left(d_2^\alpha - d_1^\alpha\right) (u_h^{k-2}, u_h^k),$$

and so

$$\begin{aligned} \text{I} \leqslant \frac{1}{4} \left(d_0^\alpha + 2d_2^\alpha - 2d_1^\alpha - d_{k-1}^\alpha\right) \|u_h^k\|^2 + \sum_{\substack{j=1 \\ j \neq k-2}}^{k-1} \left(d_{k-j-1}^\alpha - d_{k-j}^\alpha\right) \|u_h^j\|^2 \\ + \left(d_2^\alpha - d_1^\alpha\right) \|u_h^{k-2}\|^2, \end{aligned} \tag{3.5}$$

where Lemma 2.1 with $\epsilon = 1$ and Lemma A.11 were used. Using Lemma A.12, we get $\dfrac{4}{3d_0^\alpha + 2d_1^\alpha - 2d_2^\alpha} \leqslant 2$. Therefore, (3.1), (3.4), and (3.5) leads to

$$\|u_h^k\|^2 \leqslant 2 \sum_{\substack{j=1 \\ j \neq k-2}}^{k-1} \left(d_{k-j-1}^\alpha - d_{k-j}^\alpha\right) \|u_h^j\|^2 + 2d_{k-1}^\alpha \|u_h^0\|^2 + 2\left(d_2^\alpha - d_1^\alpha\right) \|u_h^{k-2}\|^2.$$

We get for $\varepsilon \geq 0$

$$\|u_h^k\|^2 \leqslant \varepsilon \|u_h^0\|^2 + \sum_{j=0}^{k-1} w_j \, \|u_h^j\|^2, \quad k = 3, \ldots, n,$$

where

$$w_j = 2\begin{cases} d_{k-1}^{\alpha}, & j=0,\\ d_2^{\alpha}-d_1^{\alpha}, & j=k-2,\\ d_{k-j-1}^{\alpha}-d_{k-j}^{\alpha}, & \text{otherwise.}\end{cases}$$

By Lemma A.11, the w_j's are non-negative and hence Lemma 2.2 gives us

$$\|u_h^k\|^2 \leqslant \varepsilon \exp\Big(\sum_{j=0}^{k-1} w_j\Big)\|u_h^0\|^2 = \varepsilon \exp(C)\|u_h^0\|^2, \quad k=3,\dots,n,$$

where $C = 4(d_0^{\alpha} - 2d_1^{\alpha} + 2d_2^{\alpha})$. Following Lemma A.11 with $C > 0$ and choosing $\varepsilon \le 1/\exp(C)$ gives $\|u_h^k\| \leqslant \|u_h^0\|$, for $k = 3,\dots,n$, which implies that in this case the fully discrete scheme (2.2) is unconditionally stable in the L^2-norm for $l = 3$.

This is completed by showing that $\|u_h^k\| \leqslant \|u_h^0\|$, for $k = 1, 2$. The proof is straightforward for $k = 1$. For $k = 2$, taking $v = u_h^2$ in (2.2) and using the boundedness of κ, we have

$$(\mathbb{D}_t^{\alpha,3} u_h^2, u_h^2) \leqslant -\kappa_0(\nabla u_h^2, \nabla u_h^2),$$

that is,

$$d_0^{\alpha}\|u_h^2\|^2 \leqslant (d_0^{\alpha}-d_1^{\alpha})\,(u_h^1,u_h^2) + d_1^{\alpha}(u_h^0,u_h^2) - \mu\kappa_0\|\nabla u_h^2\|^2 \leqslant \mathrm{I}+\mathrm{II}, \tag{3.6}$$

where

$$\mathrm{I} = (d_0^{\alpha}-d_1^{\alpha})\,(u_h^1,u_h^2) \leqslant \frac{1}{4}\,(d_0^{\alpha}-d_1^{\alpha})\,\|u_h^2\|^2 + (d_0^{\alpha}-d_1^{\alpha})\,\|u_h^1\|^2, \tag{3.7}$$

$$\mathrm{II} = d_1^{\alpha}(u_h^0,u_h^2) \leqslant |d_1^{\alpha}|\left(\frac{1}{4}\|u_h^2\|^2 + \|u_h^0\|^2\right) \leqslant d_0^{\alpha}\left(\frac{1}{4}\|u_h^2\|^2 + \|u_h^0\|^2\right), \tag{3.8}$$

in which Lemma 2.1 with $\epsilon = 1$ and Lemma A.11 were used. (3.6), (3.7), and (3.8) lead to

$$\frac{1}{2}\|u_h^2\|^2 \leqslant \frac{2d_0^{\alpha}+d_1^{\alpha}}{4}\|u_h^2\|^2 \leqslant (d_0^{\alpha}-d_1^{\alpha})\|u_h^1\|^2 + d_0^{\alpha}\|u_h^0\|^2,$$

where Lemma A.11 and Lemma A.12 have been applied. For $\varepsilon \geq 0$, we get now

$$\|u_h^2\|^2 \leqslant \varepsilon\|u_h^0\|^2 + \sum_{j=0}^{1} w_j\|u_h^j\|^2,$$

in which $w_0 = 2d_0^{\alpha}$ and $w_1 = 2\big(d_0^{\alpha} - d_1^{\alpha}\big)$. Using the non-negativity of the w_j's and Lemma A.11, we have

$$\|u_h^2\|^2 \leqslant \varepsilon \exp\Big(\sum_{j=0}^{1} w_j\Big)\|u_h^0\|^2 = \varepsilon\exp(C)\|u_h^0\|^2,$$

where $C = 2(2d_0^{\alpha} - d_1^{\alpha}) > 0$, obviously. Choosing $\varepsilon \le 1/\exp(C)$ yields $\|u_h^2\| \leqslant \|u_h^0\|$. ☐

THEOREM 3.2 (H^1-stability). *The numerical solution of* (2.2) *satisfies*

$$\|\nabla u_h^k\| \leqslant \|\nabla u_h^0\|, \qquad \textit{for } k=1,\dots,n \textit{ and } l=3.$$

Proof. Set $v = \mathbb{D}_t^{\alpha,3} u_h^k$ in (2.2). Then the proof is similar to the proof of Theorem 3.1. □

Now, we aim to establish some error estimates for the scheme (2.2) with $l = 3$.

THEOREM 3.3. *The scheme (2.2) is convergent for $l = 3$, and it holds that*

$$\|e_n^k\|_1 \leqslant C(\Delta t^3 + \Delta t^{4-\alpha} + \Delta t^4 + h^2), \qquad k = 3, \ldots, n. \tag{3.9}$$

Proof. First of all, we show that $\|e_n^k\| \leqslant C(\Delta t^3 + \Delta t^{4-\alpha} + \Delta t^4 + h^2)$, for $k = 3, \ldots, n$. Choosing $v = e_n^k$ in (2.3), we have

$$(\mathbb{D}_t^{\alpha,3} e_n^k, e_n^k) + (\kappa \nabla e_n^k, \nabla e_n^k) = -(\mathbb{D}_t^{\alpha,3} e_p^k, e_n^k) - (\kappa \nabla e_p^k, \nabla e_n^k) - (R_3^k, e_n^k).$$

Applying the Cauchy-Schwarz inequality and Lemma 2.3, we obtain

$$(\mathbb{D}_t^{\alpha,3} e_n^k, e_n^k) + (\kappa \nabla e_n^k, \nabla e_n^k) \leqslant \left(\|\mathbb{D}_t^{\alpha,3} e_p^k\| + Ch^2 \|u^k\|_3 + \|R_3^n\|\right) \|e_n^k\|,$$

in which $-(\kappa \nabla e_p^k, \nabla e_n^k) \leqslant \kappa |(\nabla(u^k - \mathcal{P}_h u^k), \nabla e_n^k))| \leqslant Ch^2 \|u^k\|_3 \|e_n^k\|$. Therefore,

$$\begin{aligned} d_0^\alpha \|e_n^k\|^2 &\leqslant \sum_{j=1}^{k-1} (d_{k-j-1}^\alpha - d_{k-j}^\alpha)(e_n^j, e_n^k) + d_{k-1}^\alpha (e_n^0, e_n^k) - \mu \kappa_0 \|\nabla e_n^k\|^2 \\ &\quad + \mu \left(\|\mathbb{D}_t^{\alpha,3} e_p^k\| + Ch^2 \|u^k\|_3 + \|R_3^k\|\right) \|e_n^k\| \\ &\leqslant \mathrm{I} + \mathrm{II} + \mathrm{III}, \end{aligned} \tag{3.10}$$

where $\mathrm{I} = \sum_{j=1}^{k-1} (d_{k-j-1}^\alpha - d_{k-j}^\alpha)(e_n^j, e_n^k)$ and

$$\mathrm{II} = d_{k-1}^\alpha (e_n^0, e_n^k) \leqslant d_{k-1}^\alpha \|e_n^0\|^2 + \frac{1}{4} d_{k-1}^\alpha \|e_n^k\|^2, \tag{3.11}$$

$$\mathrm{III} \leqslant \mu^2 \left(\|\mathbb{D}_t^{\alpha,3} e_p^k\| + Ch^2 \|u^k\|_3 + \|R_3^k\|\right)^2 + \frac{1}{4} \|e_n^k\|^2, \tag{3.12}$$

in which Lemma 2.1 with $\epsilon = 1$ has been used. We distinguish the two following cases.

Case 1: If $d_2^\alpha < d_1^\alpha$, then we obtain by using Lemma 2.1 with $\epsilon = 1$ and Lemma A.11

$$\mathrm{I} \leqslant \sum_{j=1}^{k-1} (d_{k-j-1}^\alpha - d_{k-j}^\alpha) \|e_n^j\|^2 + \frac{1}{4} (d_0^\alpha - d_{k-1}^\alpha) \|e_n^k\|^2. \tag{3.13}$$

Substituting (3.11), (3.12), and (3.13) into (3.10) leads to

$$\begin{aligned} \frac{1}{2} \|e_n^k\|^2 \leqslant \frac{3 d_0^\alpha - 1}{4} \|e_n^k\|^2 &\leqslant \sum_{j=1}^{k-1} (d_{k-j-1}^\alpha - d_{k-j}^\alpha) \|e_n^j\|^2 + d_{k-1}^\alpha \|e_n^0\|^2 \\ &\quad + \mu^2 \left(\|\mathbb{D}_t^{\alpha,3} e_p^k\| + Ch^2 \|u^k\|_3 + \|R_3^k\|\right)^2, \end{aligned}$$

where Lemma A.12 has been used. It follows that

$$\|e_n^k\|^2 \leqslant \sum_{j=0}^{k-1} w_j \|e_n^j\|^2 + 2\mu^2 \left(\|\mathbb{D}_t^{\alpha,3} e_p^k\| + Ch^2 \|u^k\|_3 + \|R_3^k\|\right)^2,$$

with $w_0 = 2d_{k-1}^\alpha$ and $w_j = 2(d_{k-j-1}^\alpha - d_{k-j}^\alpha)$, for $j = 1, \ldots, k-1$. Using Lemma 2.2 results in

$$\|e_n^k\|^2 \leqslant 2\mu^2 \left(\|\mathbb{D}_t^{\alpha,3} e_p^k\| + Ch^2\|u^k\|_3 + \|R_3^k\|\right)^2 \exp(\sum_{j=0}^{k-1} w_j).$$

Eventually, using Theorem A.13 and (2.4), we obtain

$$\|e_n^k\| \leqslant C_1\sqrt{2}\mu \exp(d_0^\alpha) \left(\|R_3^k\| + h^2\|\mathrm{D}_t^\alpha u^k\|_2 + h^2\|u^k\|_3\right).$$

It therefore holds that $\|e_n^k\| \leqslant C(\Delta t^3 + \Delta t^{4-\alpha} + \Delta t^4 + h^2)$.

Case 2: If $d_1^\alpha < d_2^\alpha$, then

$$\mathrm{I} = \sum_{\substack{j=1 \\ j \neq k-2}}^{k-1} \left(d_{k-j-1}^\alpha - d_{k-j}^\alpha\right)(e_n^j, e_n^k) - (d_2^\alpha - d_1^\alpha)(e_n^{k-2}, e_n^k)$$

and

$$\begin{aligned} \mathrm{I} \leqslant\ & \frac{1}{4}(d_0^\alpha + 2d_2^\alpha - 2d_1^\alpha - d_{k-1}^\alpha)\|e_n^k\|^2 \\ & + \sum_{\substack{j=1 \\ j \neq k-2}}^{k-1} \left(d_{k-j-1}^\alpha - d_{k-j}^\alpha\right)\|e_n^j\|^2 + (d_2^\alpha - d_1^\alpha)\|e_n^{k-2}\|^2, \end{aligned} \tag{3.14}$$

where Lemma 2.1 with $\epsilon = 1$ has been used. Substituting (3.11), (3.12), and (3.14) into (3.10) leads to

$$\begin{aligned} \frac{1}{4}\|e_n^k\|^2 \leqslant\ & \frac{3d_0^\alpha + 2d_1^\alpha - 2d_2^\alpha - 1}{4}\|e_n^k\|^2 \\ \leqslant\ & \sum_{\substack{j=1 \\ j \neq k-2}}^{k-1} \left(d_{k-j-1}^\alpha - d_{k-j}^\alpha\right)\|e_n^j\|^2 + (d_2^\alpha - d_1^\alpha)\|e_n^{k-2}\|^2 \\ & + d_{k-1}^\alpha\|e_n^0\|^2 + \mu^2\left(\|\mathbb{D}_t^{\alpha,3} e_p^k\| + Ch^2\|u^k\|_3 + \|R_3^k\|\right)^2, \end{aligned}$$

in which Lemma A.12 has been applied. Therefore,

$$\|e_n^k\|^2 \leqslant \sum_{j=0}^{k-1} w_j\|e_n^j\|^2 + 4\mu^2\left(\|\mathbb{D}_t^{\alpha,3} e_p^k\| + Ch^2\|u^k\|_3 + \|R_3^k\|\right)^2$$

holds using the definition

$$w_j = 4\begin{cases} d_{k-1}^\alpha, & j = 0, \\ d_2^\alpha - d_1^\alpha, & j = k-2, \\ d_{k-j-1}^\alpha - d_{k-j}^\alpha, & \text{otherwise}. \end{cases}$$

Using Lemma 2.2 we get

$$\|e_n^k\|^2 \leqslant 4\mu^2 \left(\|\mathbb{D}_t^{\alpha,3} e_p^k\| + Ch^2\|u^k\|_3 + \|R_3^k\|\right)^2 \exp(\sum_{j=0}^{k-1} w_j).$$

Eventually, using Theorem A.13 and (2.4), we obtain

$$\begin{aligned}\|e_n^k\| &\leqslant C_1 2\mu\, \exp\left(2(d_0^\alpha + 2d_2^\alpha - 2d_1^\alpha)\right)\left(\|R_3^k\| + h^2\|\mathrm{D}_t^\alpha u^k\|_2 + h^2\|u^k\|_3\right)\\ &\leqslant C(\Delta t^3 + \Delta t^{4-\alpha} + \Delta t^4 + h^2).\end{aligned}$$

Similarly, we can show that $\|\nabla e_n^k\| \leqslant C(\Delta t^3 + \Delta t^{4-\alpha} + \Delta t^4 + h^2)$ for $k = 3, \ldots, n$, and we then obtain (3.9) by applying the Poincaré inequality.

In order to complete the first part of this proof, it remains to show that $\|e_n^k\|$ tends to zero as $\Delta t, h \to 0$, for $k = 1, 2$. The proof is straightforward for $k = 1$. For $k = 2$ and choosing $v = e_n^2$ in (2.3) we have

$$(\mathbb{D}_t^{\alpha,3} e_n^2, e_n^2) + (\kappa\nabla e_n^2, \nabla e_n^2) = -(\mathbb{D}_t^{\alpha,3} e_p^2, e_n^2) - (\kappa\nabla e_p^2, \nabla e_n^2) - (R_3^2, e_n^2).$$

Applying the Cauchy-Schwarz inequality and Lemma 2.3 we obtain

$$(\mathbb{D}_t^{\alpha,3} e_n^2, e_n^2) + (\kappa\nabla e_n^2, \nabla e_n^2) \leqslant \left(\|\mathbb{D}_t^{\alpha,3} e_p^2\| + Ch^2\|u^2\|_3 + \|R_3^2\|\right)\|e_n^2\|,$$

resulting in

$$\begin{aligned}d_0\|e_n^2\|^2 &\leqslant (d_0^\alpha - d_1^\alpha)(e_n^1, e_n^2) + d_1^\alpha(e_n^0, e_n^2) - \mu\kappa_0\|\nabla e_n^2\|^2\\ &\quad + \mu\left(\|\mathbb{D}_t^{\alpha,3} e_p^2\| + Ch^2\|u^2\|_3 + \|R_3^2\|\right)\|e_n^2\|\\ &\leqslant \mathrm{I} + \mathrm{II} + \mathrm{III},\end{aligned} \tag{3.15}$$

by using Lemma 2.1 with $\epsilon = 1$ as well as Lemma A.11 and by introducing

$$\mathrm{I} = (d_0^\alpha - d_1^\alpha)(e_n^1, e_n^2) \leqslant (d_0^\alpha - d_1^\alpha)\|e_n^1\|^2 + \frac{1}{4}(d_0^\alpha - d_1^\alpha)\|e_n^2\|^2, \tag{3.16}$$

$$\mathrm{II} = d_1^\alpha(e_n^0, e_n^2) \leqslant |d_1^\alpha|\left(\|e_n^0\|^2 + \frac{1}{4}\|e_n^2\|^2\right) \leqslant d_0^\alpha\left(\|e_n^0\|^2 + \frac{1}{4}\|e_n^2\|^2\right), \tag{3.17}$$

$$\mathrm{III} \leqslant \mu^2\left(\|\mathbb{D}_t^{\alpha,3} e_p^2\| + Ch^2\|u^2\|_3 + \|R_3^2\|\right)^2 + \frac{1}{4}\|e_n^2\|^2. \tag{3.18}$$

Substituting (3.16), (3.17), and (3.18) into (3.15) yields

$$\begin{aligned}\frac{1}{4}\|e_n^2\|^2 &\leqslant \frac{2d_0^\alpha + d_1^\alpha - 1}{4}\|e_n^2\|^2\\ &\leqslant (d_0^\alpha - d_1^\alpha)\|e_n^1\|^2 + d_0^\alpha\|e_n^0\|^2 + \mu^2\left(\|\mathbb{D}_t^{\alpha,3} e_p^2\| + Ch^2\|u^2\|_3 + \|R_3^2\|\right)^2,\end{aligned}$$

where Lemma A.12 has been used. Hence

$$\|e_n^2\|^2 \leqslant \sum_{j=0}^{1} w_j\|e_n^j\|^2 + 4\mu^2\left(\|\mathbb{D}_t^{\alpha,3} e_p^2\| + Ch^2\|u^2\|_3 + \|R_3^2\|\right)^2,$$

holds with $w_0 = 4d_0^\alpha$ and $w_1 = 4(d_0^\alpha - d_1^\alpha)$. Applying Lemma 2.2 results in

$$\|e_n^2\|^2 \leqslant 4\mu^2\left(\|\mathbb{D}_t^{\alpha,3} e_p^2\| + Ch^2\|u^2\|_3 + \|R_3^2\|\right)^2 \exp(4(4d_0^\alpha - d_1^\alpha)).$$

Eventually, by using Theorem A.13 and (2.4) we obtain

$$\|e_n^2\| \leqslant C_1 2\mu \exp(2(4d_0 - d_1))\left(\|R_3^2\| + h^2\|\mathrm{D}_t^\alpha u^2\|_2 + h^2\|u^2\|_3\right).$$

So $\|e_n^2\|$ tends to zero as $\Delta t, h \to 0$, which completes the proof. $\square$

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3.2. L1-2 discretization. The proofs of Theorems 3.1-3.3 can be reformulated in case of the L1-2 formula. Our scheme and our proofs are quite different from the strategy applied in [26] wherein the convergence of the FEM mixed with the corrected L1-2 formula using the discrete Laplace transform has been proven by correcting some starting steps of the classic L1-2 scheme. Let us consider the FEM mixed with the classic L1-2 formula and prove stability and convergence utilizing Lemma 2.2.

THEOREM 3.4 (L^2-stability). *The numerical solution of* (2.2) *satisfies*

$$\|u_h^k\| \leqslant \|u_h^0\|, \qquad \text{for } k = 1, \dots, n \text{ and } l = 2.$$

Proof. If we choose $v = u_h^k$ in (2.2) then the analysis is similar to the proof of Theorem 3.1. ∎

THEOREM 3.5 (H^1-stability). *The numerical solution of* (2.2) *for $l = 2$ satisfies*

$$\|\nabla u_h^k\| \leqslant \|\nabla u_h^0\|, \qquad \text{for } k = 1, \dots, n.$$

Proof. By taking $v_h = \mathbb{D}_t^{\alpha,2} u_h^k$ in (2.2), the rest of proof is similar to the proof of Theorem 3.4. ∎

Now, we aim to establish an error estimates for the scheme (2.2) with $l = 2$.

THEOREM 3.6. *The scheme* (2.2) *is convergent for $l = 2$, and it holds that*

$$\|e_n^k\|_1 \leqslant C(\Delta t^{3-\alpha} + \Delta t^3 + h^2), \qquad k = 2, \dots, n.$$

Proof. Choosing $v = e_n^k$ in (2.3), the proof proceeds along the same lines as the proof of Theorem 3.3. ∎

3.3. L1 discretization. The analysis of the discretization based on the FEM and the L1 formula has been already widely studied in the literature [34, 35, 36]. The ideas introduced in this paper behind the analysis of the L1-2-3 and L1-2 methods can also be used for the L1 case, leading to a new analysis approach for that discretization.

THEOREM 3.7 (L^2-stability). *The numerical solution of* (2.2) *for $l = 1$ satisfy*

$$\|u_h^k\| \leqslant \|u_h^0\| \qquad \text{for } k = 1, \dots, n.$$

THEOREM 3.8 (H^1-stability). *The numerical solution of* (2.2) *for $l = 1$ satisfies*

$$\|\nabla u_h^k\| \leqslant \|\nabla u_h^0\| \qquad \text{for } k = 1, \dots, n.$$

THEOREM 3.9. *The scheme* (2.2) *for $l = 1$ is convergent and for $k = 1, \dots, n$ it holds that*

$$\|e_n^k\|_1 \leqslant C\left(h^2 + \Delta t^{2-\alpha}\right).$$

4. Numerical results. In this section, we present numerical examples to demonstrate the effectiveness of the theoretical results. The convergence orders with respect to space and time are defined as

$$\mathcal{O}_h = \log_2\left(\frac{E_2(2h, \Delta t)}{E_2(h, \Delta t)}\right), \qquad \mathcal{O}_{\Delta t}^p = \log_2\left(\frac{E_p(h, 2\Delta t)}{E_p(h, \Delta t)}\right), \quad p = 1, 2,$$

where E_1 and E_2 denote the error estimates in the H^1- and L^2-norms, respectively. The problems below are discretized in space using linear finite elements on quasi-uniform Delaunay triangulations of Ω. The expected temporal accuracy indicated by the fixed ratio r_l, $l = 1, 2, 3$,

$$r_1 = \frac{\Delta t^{2-\alpha}}{h^2}, \quad r_2 = \frac{\Delta t^{3-\alpha}}{h^2}, \quad r_3 = \frac{\Delta t^3}{h^2}, \tag{4.1}$$

expresses the achieved numerical accuracy of the FEM mixed with L1, L1-2, and L1-2-3 schemes, respectively.

EXAMPLE 4.1. Let $\kappa = 1$ and $f = \sin(2\pi x)\sin(2\pi y)t^4(\frac{\Gamma(5+\alpha)}{24} + 8\pi^2 t^\alpha)$ in problem (1.1). Then, the exact solution is

$$u(\mathbf{x}, t) = \sin(2\pi x)\sin(2\pi y)t^{4+\alpha}.$$

First, we consider the problem on $\Omega = (0, 1)\times(0, 1)$. Taking different values $\alpha = 0.1, 0.5, 0.9$, we compute the numerical results using the fully discrete schemes (2.2). In order to confirm the error estimates and temporal convergence orders, including the L^2-norm and H^1-norm, we choose different uniform time steps (Δt) and Delaunay meshes in the spatial direction. The corresponding results are listed in Tables 4.1-4.2, which illustrate the expected temporal order of convergence. The results displayed in Tables 4.1-4.2 indicate that the order of convergence of the FEM with L1-2-3 discretization in the temporal direction is 3, which is superior to the results of the $2 - \alpha$ and $3 - \alpha$-order of accuracy of the FEM with L1 and L1-2 discretizations. To confirm the spatial convergence rate of the proposed methods, we take $\alpha = 0.5$ and $\Delta t = 1/10000$ and observe the expected second-order accuracy as shown in Figure 4.1. Then, we consider an L-shaped domain; see Figure 4.2. The results of the L^2-norm error estimates and the temporal convergence orders of the fully discrete schemes (2.2) with different $\alpha = 0.1, 0.5, 0.9$ are presented in Table 4.3, which confirm the theoretical results. Noticing Table 4.3, we can say that the computed errors of the L1-2-3 FEM are smaller than that of the L1 FEM and L1-2 FEM.

EXAMPLE 4.2. We consider problem (1.1) with $\Omega = (-1, 1) \times (-1, 1)$, $f = 0$, and the following non-smooth initial condition

$$u(\mathbf{x}, 0) = \begin{cases} 0.1, & |x - 0.25| \leqslant 0.2 \ \& \ |y - 0.25| \leqslant 0.2, \\ 0.1 \exp\left(-\left(\frac{(x-x_0)^2}{2\delta_x} + \frac{(y-y_0)^2}{2\delta_y}\right)\right), & \text{otherwise}. \end{cases}$$

with $x_0 = y_0 = -0.5$ and $\delta_x = \delta_y = 10^{-1}$; see Figure 4.3. The effect of subdiffusion can be seen in Figure 4.4, where the numerical solutions for different values $\alpha = 0.1, 0.5, 0.9$ have been plotted using the L1-2-3 FEM.

5. Conclusion. In this paper, we have considered the time-fractional diffusion equation with the fractional derivative of order $\alpha \in (0, 1)$ in the Caputo sense. We have performed a stability and convergence analysis of the L-type FEM, especially L1-2-3 and classic L1-2 formulas, where the lack of positivity of their coefficients made it hard to carry out the analysis. We have applied the L-type schemes to approximate the Caputo derivative and the finite element method on a quasi-uniform mesh to discretize the spatial variable. We then proved that the proposed schemes are unconditionally stable. Furthermore, the convergence order of accuracy of the methods has been established using a special kind of discrete Grönwall's inequality. Numerical examples demonstrate the validity of the theoretical results. It can be concluded that the L1-2-3 FEM may be an efficient method to solve the subdiffusion equation.

Appendix A. L-type formula. In the appendix, we briefly introduce the L-type formula and present some lemmas and theorems used in this work. Here, we define

$$R_l(v(t_k)) = \mathrm{D}_t^\alpha v(t)|_{t=t^k} - \mathbb{D}_t^{\alpha,l} v(t^k), \qquad l = 1, 2, 3.$$

TABLE 4.1
(Example 4.1) The L^2-norm errors and temporal numerical order of convergence with the fixed ratio (4.1) on the square domain.

α	Δt	E$_2$(L1)	$\mathcal{O}^2_{\Delta t}$(L1)	E$_2$(L1-2)	$\mathcal{O}^2_{\Delta t}$(L1-2)	E$_2$(L1-2-3)	$\mathcal{O}^2_{\Delta t}$(L1-2-3)
0.1	$1/16$	2.5615e-2	1.84	2.5627e-2	2.91	2.5627e-2	3.02
	$1/32$	7.1651e-3	1.93	3.4185e-3	2.90	3.1657e-3	3.03
	$1/64$	1.8822e-3	1.99	4.5743e-4	2.92	3.8877e-4	3.00
	$1/128$	4.7535e-4		6.0562e-5		4.8553e-5	
0.5	$1/16$	2.5146e-2	1.39	2.5388e-2	2.42	2.5411e-2	3.02
	$1/32$	9.5702e-3	1.63	4.7564e-3	2.61	3.1368e-3	3.03
	$1/64$	3.0971e-3	1.50	7.8029e-4	2.51	3.8523e-4	3.00
	$1/128$	1.0922e-3		1.3689e-4		4.8114e-5	
0.9	$1/16$	2.2975e-2	1.02	2.4758e-2	1.98	2.4969e-2	3.02
	$1/32$	1.1302e-2	1.14	6.2560e-3	2.19	3.0781e-3	3.03
	$1/64$	5.1181e-3	1.14	1.3721e-3	2.16	3.7802e-4	3.00
	$1/128$	2.3183e-3		3.0717e-4		4.7221e-5	

TABLE 4.2
(Example 4.1) The H^1-norm errors and temporal numerical order of convergence with the fixed ratio (4.1) on the square domain.

α	Δt	E$_2$(L1)	$\mathcal{O}^2_{\Delta t}$(L1)	E$_2$(L1-2)	$\mathcal{O}^2_{\Delta t}$(L1-2)	E$_2$(L1-2-3)	$\mathcal{O}^2_{\Delta t}$(L1-2-3)
0.1	$1/4$	9.8081e-1	1.80	9.8098e-1	2.82	9.8103e-1	2.96
	$1/8$	2.8189e-1	1.96	1.3864e-1	2.96	1.2614e-1	3.03
	$1/16$	7.2518e-2		1.7851e-2		1.5495e-2	
0.5	$1/4$	9.7650e-1	1.44	9.7842e-1	2.44	9.7915e-1	2.96
	$1/8$	3.5984e-1	1.48	1.8032e-1	2.53	1.2613e-1	3.03
	$1/16$	1.2893e-1		3.1284e-2		1.5495e-2	
0.9	$1/4$	9.6488e-1	0.95	9.7128e-1	1.99	9.7449e-1	2.95
	$1/8$	5.0077e-1	1.08	2.4370e-1	2.15	1.2610e-1	3.02
	$1/16$	2.3618e-1		5.4936e-2		1.5496e-2	

DEFINITION A.1 ([9, 17]). *The* L1 *formula is defined as follows:*

$$\mathbb{D}_t^{\alpha,1} v(t^k) = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\left(v^k - \sum_{j=1}^{k-1}\left(a^\alpha_{k-j-1} - a^\alpha_{k-j}\right) v^j - a^\alpha_{k-1} v^0\right), \tag{A.1}$$

where $a_j^\alpha = (j+1)^{1-\alpha} - j^{1-\alpha}$ *for* $j = 0, 1, \ldots, n$.

LEMMA A.2 ([9, 17]). *We have* $1 = a_0^\alpha > a_1^\alpha > \cdots > a_n^\alpha > 0$.

THEOREM A.3 ([22]). *Let* $\partial^s v/\partial t^s \in \mathrm{C}[0, t^k]$ *for* $s = 0, 1, 2$ *and* $M_{tt} = \max\limits_{0\le t\le t^k} |v''(t)|$. *Then*

$$|R_1(v(t^k))| \leqslant \frac{M_{tt}}{2}\Delta t^{2-\alpha}.$$

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TABLE 4.3
(Example 4.1) The L^2-norm errors and temporal numerical order of convergence with the fixed ratio (4.1) on the square domain.

α	Δt	E$_2$(L1)	$\mathcal{O}^2_{\Delta t}(L1)$	E$_2$(L1-2)	$\mathcal{O}^2_{\Delta t}$(L1-2)	E$_2$(L1-2-3)	$\mathcal{O}^2_{\Delta t}$(L1-2-3)
0.1	1/16	2.5615e-2	1.84	2.5627e-2	2.91	2.5627e-2	3.02
	1/32	7.1651e-3	1.93	3.4185e-3	2.90	3.1657e-3	3.03
	1/64	1.8822e-3	1.99	4.5743e-4	2.92	3.8877e-4	3.00
	1/128	4.7535e-4		6.0562e-5		4.8553e-5	
0.5	1/16	2.5146e-2	1.39	2.5388e-2	2.42	2.5411e-2	3.02
	1/32	9.5702e-3	1.63	4.7564e-3	2.61	3.1368e-3	3.03
	1/64	3.0971e-3	1.50	7.8029e-4	2.51	3.8523e-4	3.00
	1/128	1.0922e-3		1.3689e-4		4.8114e-5	
0.9	1/16	2.2975e-2	1.02	2.4758e-2	1.98	2.4969e-2	3.02
	1/32	1.1302e-2	1.14	6.2560e-3	2.19	3.0781e-3	3.03
	1/64	5.1181e-3	1.14	1.3721e-3	2.16	3.7802e-4	3.00
	1/128	2.3183e-3		3.0717e-4		4.7221e-5	

DEFINITION A.4 ([6]). *The* L1-2 *formula is defined as follows:*

$$\mathbb{D}_t^{\alpha,2} v(t^k) = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \left(c_0^\alpha v^k - \sum_{j=1}^{k-1} \left(c_{k-j-1}^\alpha - c_{k-j}^\alpha \right) v^j - c_{k-1}^\alpha v^0 \right), \tag{A.2}$$

where $c_0^\alpha = 1$, for $k = 1$, and for $k \geq 2$, we have

$$c_j^\alpha = \begin{cases} a_0^\alpha + b_0^\alpha, & j = 0, \\ a_j^\alpha + b_j^\alpha - b_{j-1}^\alpha, & 1 \leq j \leq k-2, \\ a_j^\alpha - b_{j-1}^\alpha, & j = k-1, \end{cases} \tag{A.3}$$

in which

$$b_j^\alpha = \frac{(j+1)^{2-\alpha} - j^{2-\alpha}}{2-\alpha} - \frac{(j+1)^{1-\alpha} + j^{1-\alpha}}{2}, \qquad j = 0, 1, \ldots, n.$$

LEMMA A.5 ([6]). *$\{b_j^\alpha\}$ is strictly monotone decreasing with respect to j and $b_j^\alpha > 0$.*

LEMMA A.6 ([6]). *For $k \geq 3$, we get $c_0^\alpha > |c_1^\alpha|$ and*

$$c_0^\alpha > c_2^\alpha \geq c_3^\alpha \geq \cdots \geq c_{k-1}^\alpha > 0.$$

LEMMA A.7 ([16]). *For c_j^α's defined in (A.3), it holds that*

$$c_0^\alpha > 1, \qquad c_0^\alpha + c_1^\alpha > 1, \qquad c_0^\alpha + 2c_1^\alpha - 2c_2^\alpha > \frac{1}{2}.$$

THEOREM A.8 ([6]). *Let $\partial^s v / \partial t^s \in \mathrm{C}[0, t^k]$, for $s = 0, \ldots, 3$. Then*

$$|R_2(v(t^1))| \leqslant \frac{M_{tt}}{2} \Delta t^{2-\alpha},$$
$$|R_2(v(t^k))| \leqslant \frac{M_{tt}}{40} \Delta t^3 + \frac{M_{ttt}}{3} \Delta t^{3-\alpha}, \quad k \geq 2,$$

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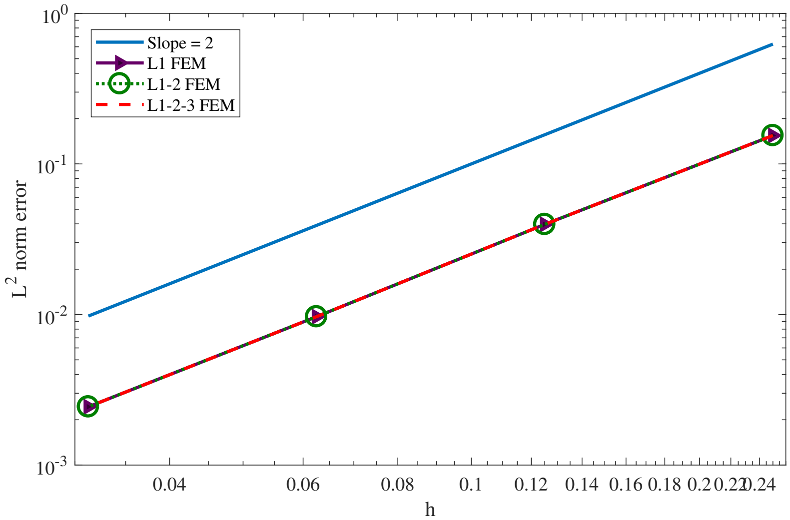

FIG. 4.1. *(Example 4.1) Spatial convergence orders with* $\alpha = 0.5$ *and* $\Delta t = 1/10000$.

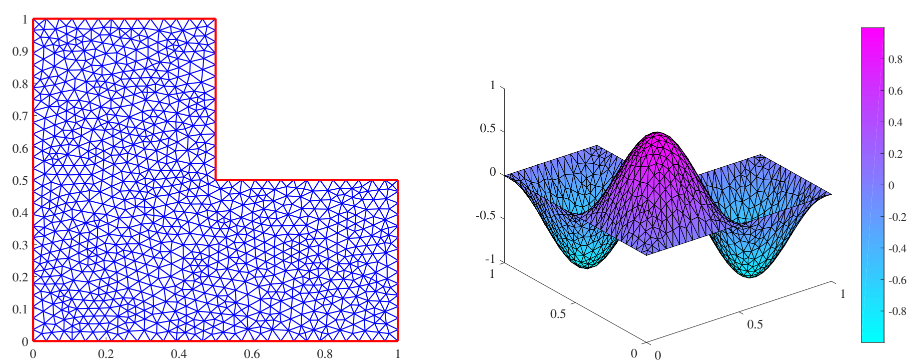

FIG. 4.2. *(Example 4.1) Delaunay mesh on L-shaped domain(left) and numerical solution using the L1-2-3 FEM with* $\alpha = 0.5$ *and* $\Delta t = 2^{-5}$*(right).*

where $M_{tt} = \max_{0 \le t \le t^1} |v''(t)|$ *and* $M_{ttt} = \max_{0 \le t \le t^k} |v'''(t)|$.

DEFINITION A.9 ([15]). *The* L1-2-3 *formula is defined as follows:*

$$\mathbb{D}_t^{\alpha,3} v(t^k) = \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \left(d_0^\alpha v^k - \sum_{j=1}^{k-1} \left(d_{k-j-1}^\alpha - d_{k-j}^\alpha \right) v^j - d_{k-1}^\alpha v^0 \right) \tag{A.4}$$

where $d_0^\alpha = 1$, *for* $k = 1$, $d_0^\alpha = a_0^\alpha + b_0^\alpha$ *and* $d_1^\alpha = a_1^\alpha - b_0^\alpha$, *for* $k = 2$, *and*

$$d_l^\alpha = \begin{cases} a_l^\alpha + b_l^\alpha + f_l^\alpha, & l = 0, \\ a_l^\alpha + b_l^\alpha - b_{l-1}^\alpha - 2 f_{l-1}^\alpha, & l = 1, \\ a_l^\alpha - b_{l-1}^\alpha + f_{l-2}^\alpha, & l = 2, \end{cases} \tag{A.5}$$

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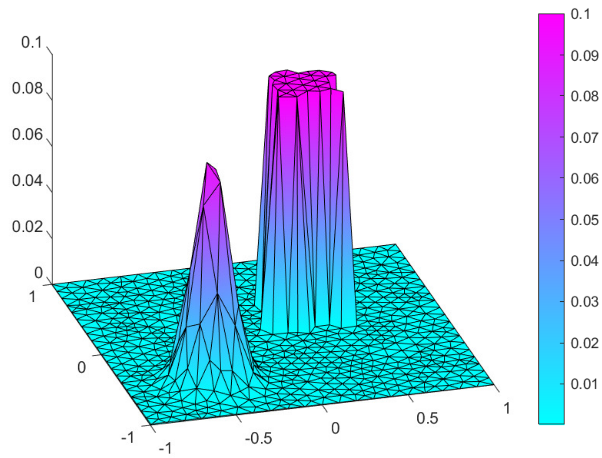

FIG. 4.3. *(Example 4.2) A non-smooth initial data.*

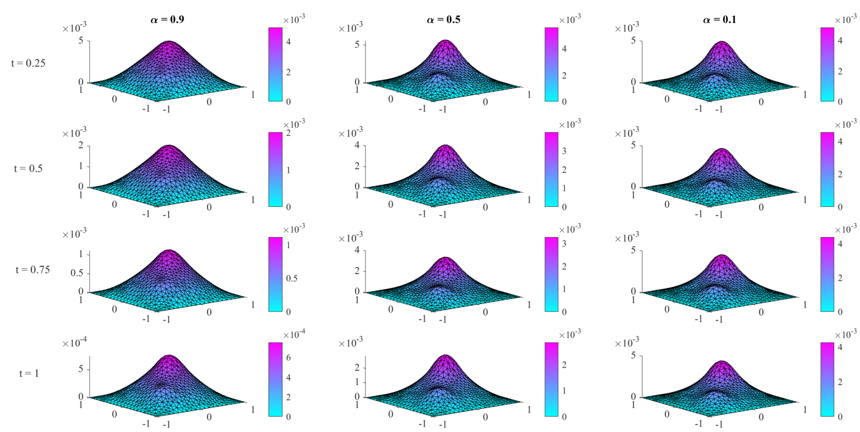

FIG. 4.4. *(Example 4.2) Numerical solutions of L1-2-3 FEM with different values* $\alpha = 0.1, 0.5, 0.9$ *in various times with* $n = 100, m = 20$.

for $k = 3$*, and for* $k \geq 4$*, we have*

$$d_j^\alpha = \begin{cases} a_j^\alpha + b_j^\alpha + f_j^\alpha, & j = 0, \\ a_j^\alpha + b_j^\alpha - b_{j-1}^\alpha + f_j^\alpha - 2f_{j-1}^\alpha, & j = 1, \\ a_j^\alpha + b_j^\alpha - b_{j-1}^\alpha + f_j^\alpha - 2f_{j-1}^\alpha + f_{j-2}^\alpha, & 2 \leq j \leq k-3, \\ a_j^\alpha + b_j^\alpha - b_{j-1}^\alpha - 2f_{j-1}^\alpha + f_{j-2}^\alpha, & j = k-2, \\ a_j^\alpha - b_{j-1}^\alpha + f_{j-2}^\alpha, & j = k-1, \end{cases} \tag{A.6}$$

in which for $j \geq 0$*,*

$$f_j^\alpha = \frac{1}{(2-\alpha)(3-\alpha)}\left((j+1)^{3-\alpha} - j^{3-\alpha}\right) - \frac{1}{6}\left((j+1)^{1-\alpha} + 2j^{1-\alpha}\right) - \frac{1}{2-\alpha} j^{2-\alpha}.$$

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LEMMA A.10 ([15]). *$\{f_j^\alpha\}$ is strictly monotone decreasing with respect to j and $f_j^\alpha > 0$.*

LEMMA A.11 ([15]). *For $k \geq 4$, we get $d_0^\alpha > |d_1^\alpha|$ and*

$$d_0^\alpha > d_2^\alpha \geq d_3^\alpha \geq \cdots \geq d_{k-1}^\alpha > 0.$$

LEMMA A.12 ([16]). *For d_j^α's defined in (A.5) and (A.6), it holds that*

$$d_0^\alpha > 1, \qquad 3d_0^\alpha + 2d_1^\alpha - 2d_2^\alpha > 2, \qquad d_0^\alpha + d_1^\alpha - d_2^\alpha > \frac{1}{3}.$$

THEOREM A.13 ([15]). *Let $\partial^s v/\partial t^s \in \mathrm{C}[0, t^k]$, for $s = 0, \ldots, 4$. Then*

$$\begin{aligned}
|R_3(v(t^1))| &\leqslant \frac{M_{tt}}{2}\Delta t^{2-\alpha},\\
|R_3(v(t^2))| &\leqslant \frac{M_{tt}}{40}\Delta t^3 + \frac{M_{ttt}}{3}\Delta t^{3-\alpha},\\
|R_3(v(t^k))| &\leqslant \frac{7M_{tt}}{2}\Delta t^3 + \frac{M_{ttt}}{25}\Delta t^4 + \frac{M_{tttt}}{4}\Delta t^{4-\alpha}, \quad k \geq 3,
\end{aligned}$$

where $M_{tt} = \max_{0\leq t\leq t^1} |v''(t)|$, $M_{ttt} = \max_{0\leq t\leq t^2} |v'''(t)|$, and $M_{tttt} = \max_{0\leq t\leq t^k} |v^{(4)}(t)|$.

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