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FRACTIONAL HERMITE INTERPOLATION FOR NON-SMOOTH FUNCTIONS*

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Abstract. The interpolation of functions plays a fundamental role in numerical analysis. The highly accurate approximation of non-smooth functions is a challenge in science and engineering as traditional polynomial interpolation cannot characterize the singular features of these functions. This paper aims at designing a fractional Hermite interpolation for non-smooth functions based on the local fractional Taylor expansion and at deriving the corresponding explicit formula and its error remainder. We also present a piecewise hybrid Hermite interpolation scheme, a combination of fractional Hermite interpolation and traditional Hermite interpolation. Some numerical examples are presented to show the high accuracy of the fractional Hermite interpolation method.

Key words. non-smooth function, local fractional Taylor expansion, fractional Hermite interpolation, error remainder

AMS subject classifications. 26A30, 41A05, 65D05, 97N50

1. Introduction.

We consider in this paper fractional Hermite interpolation of non-smooth function defined on a bounded interval, where the function is sufficiently smooth except at a finite set of points. We usually call these points singularities, where the function is discontinuous or its derivative is discontinuous. Traditional Hermite interpolation approximates a complicated function by a simple polynomial, where the values of the function and its first (or first few) derivative(s) are matched with the values of the polynomial and its derivatives at some prescribed nodes [8]. General descriptions of the Hermite interpolating polynomial in some more general cases may be found in [11, 28, 33]. The error remainder of the Hermite interpolating polynomial can be found by applying Rolle's theorem repetitively [21], which shows that the accuracy of the interpolation depends upon the smoothness of the function. Traditional Hermite interpolation for approximating non-smooth functions is not accurate since the values of non-smooth functions or their derivatives do not exist at their singularities.

In the field of numerical mathematics and approximation theory, many papers have been published on the constructions, error estimates, and applications of Hermite interpolation in one and several variables; see [2, 6, 7, 9, 14, 15, 16, 32]. For some more complex interpolation problems, many scholars have carried out thorough expositions. For example, Tachev [30] provided norm estimates for the approximation of continuous functions by piecewise linear interpolation with non-equidistant nodes. Arandiga [1] gave the approximation order for a class of nonlinear interpolation procedures with a uniform mesh. In [20], a new representation of Hermite osculatory interpolation was presented in order to construct weighted Hermite quadrature rules with arithmetic and geometric nodes. For $f(x) = |x|^\alpha$, $x \in [-1, 1]$, Revers [26], Lu [19], and Su [27] showed that the sequence of Lagrange interpolating polynomials with equidistant nodes is divergent everywhere in the interval except at zero and the endpoints, for $0 < \alpha \le 1$, $1 < \alpha \le 2$, and $2 < \alpha < 4$, respectively. For fractional smooth functions, Wang et al. [35] derived a general form for a local fractional Taylor expansion based on the local fractional derivative at the singular points and obtained the remainder expansions for

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linear and quadratic interpolants. The fractional Lagrange interpolation formula with its error remainder obtained in [10] is an effective approximation for non-smooth functions. Rapaić et al. presented an auxiliary result from the numerical evaluation of fractional-order integrals in [24], where they constructed Lagrange and Hermite quasi-polynomial interpolations by replacing $(\pm(x-x_0))^\beta$ ($\beta > 0$ is fractional) with a variable t. Besides, quasi-polynomial interpolants were considered for the approximation of solutions of integral equations with weakly singular kernels [3, 4, 25, 31]. In fact, these quasi-polynomials are a kind of fractional interpolation formulas. In recent years, the interpolation method was applied to approximate fractional derivatives in [12, 13, 29]. Hence, fractional interpolation methods play an increasingly important role in the approximation of non-smooth functions. However, there are few papers that discuss the construction and error analysis of Hermite interpolation for non-smooth functions in detail. In the present paper, we will construct an efficient fractional Hermite interpolation method to accurately approximate non-smooth functions.

As it is well known, sufficiently smooth functions have a Taylor series at every point in the interval. For non-smooth functions, a standard Taylor series does not exist at the singularities. However, there may exist a Puiseux series [5, 23] at the singularities in the interval. Puiseux series are generalizations of power series and may contain negative and fractional exponents and logarithms, and they were first introduced by Isaac Newton in 1676 and afterwards rediscovered by Victor Puiseux in 1850 [34]. Puiseux series are interpreted as local fractional Taylor series when they do not involve logarithmic factors. In this paper, we assume that $f(x)$ is sufficiently smooth in (a,b) except at $x=a$ or $x=b$ and $f(x)$ possesses the local fractional Taylor expansion

$$f(x) = \sum_{i=1}^{u} a_i (x-a)^{\alpha_i} + r_a(x), \qquad x > a, \tag{1.1}$$

or

$$f(x) = \sum_{i=1}^{v} a_i (b-x)^{\alpha_i} + r_b(x), \qquad x < b, \tag{1.2}$$

at $x=a$ or $x=b$ or both of them, where all the exponents α_i $(i=1,2,\ldots)$ are real numbers satisfying $\alpha_1 < \alpha_2 < \ldots$ We note that the numbers α_i $(i=1,2,\ldots)$ are called critical orders, $\pm\Gamma(1+\alpha_i)a_i$ $(i=1,2,\ldots)$ are called local fractional derivatives, and $\Gamma(\cdot)$ is the gamma function [17]. If all the α_i are positive integers, then (1.1) or (1.2) degenerates to a standard Taylor expansion. Liu [18] designed an extrapolation method to recover the first few critical orders and calculated the corresponding local fractional derivatives. In fact, the local fractional Taylor expansion of a function at a point can be easily obtained by symbolic computation. It is noted that the remainder $r_a(x) = o((x-a)^{\alpha_u})$ or $r_b(x) = o((b-x)^{\alpha_v})$ can be made sufficiently small on $[a,b]$ by choosing u and v suitably large [36]. Although the local fractional Taylor expansion can approximate a non-smooth function, its accuracy is confined by the local properties in the full interval, which is similar to the standard Taylor expansion approximating a smooth function. Therefore, interpolation is essential for approximating a function in order to obtain uniform accuracy in the full interval.

In this paper, we construct a fractional Hermite interpolation method based on the local fractional Taylor expansions for non-smooth functions such that the local approximation property of the Taylor expansion can be extended to the whole interval. To this end, we choose $(x-x_0)^{\alpha_i}$ $(i=1,2,\ldots,n \le u)$ in (1.1) or $(x_0-x)^{\alpha_i}$ $(i=1,2,\ldots,n\le v)$ in (1.2) as the basis functions to construct the fractional Hermite interpolation function. We will prove the existence and uniqueness of this function and give the corresponding explicit formula and its error remainder. The proposed fractional Hermite interpolation can achieve higher accuracy than traditional Hermite interpolation near singular points.

The rest of the paper is organized as follows. In Section 2, we prove the unique existence of a fractional Hermite interpolation function for non-smooth functions when suitable interpolation conditions are imposed and give the corresponding explicit form as well as its error remainder. In Section 3, a combination of fractional Hermite interpolation and traditional Hermite interpolation is developed. In Section 4, some numerical examples are given to show that fractional Hermite interpolation is superior to traditional Hermite interpolation when the functions are not sufficiently smooth at the endpoints, and it is illustrated that the convergence order of fractional Hermite interpolation is consistent with the theoretical result. We give a brief conclusion in the last section. By the way, we note that lightface Latin and Greek letters denote scalars and boldface uppercase Latin letters denote matrices throughout the paper.

2. Fractional Hermite interpolation. The goal of this paper is to construct an efficient fractional Hermite interpolation function $H_{\alpha_n}(x)$ for a non-smooth function $f(x)$ defined on the bounded interval (a, b). Without loss of generality, we suppose that $f(x)$, $x \in (a, b]$ (or $[a, b)$), is sufficiently smooth except at $x = a$ (or $x = b$), where $f(x)$ has the local fractional Taylor expansion (1.1) (or (1.2)). Otherwise, we can take the singularities of $f(x)$ as the nodes and split (a, b) into subintervals, on each of which $f(x)$ is singular at the left (or right) endpoint. In the following, we will discuss the case that $f(x)$ has the local fractional Taylor expansion (1.1) at the left endpoint $x = a$ in detail. The case of the right endpoint $x = b$ is treated in an analogous way.

At first, we give the following definition of fractional Hermite interpolation.

DEFINITION 2.1. *Suppose that $f(x)$, $x \in (a, b]$, has a local fractional Taylor expansion (1.1) at $x = a$, where the exponents α_i $(i = 1, 2, \dots, n \le u)$ and the coefficients a_i $(i = 1, 2, \dots, \sigma \le n)$ are known and some of the α_i may be negative. Then, the fractional Hermite interpolation function has the form*

$$H_{\alpha_n}(x) = \sum_{i=1}^{\sigma} a_i (x-a)^{\alpha_i} + \sum_{i=\sigma+1}^{n} b_i (x-a)^{\alpha_i} \tag{2.1}$$

satisfying

$$H_{\alpha_n}^{(j)}(b) = f^{(j)}(b), \qquad j = 0, 1, \dots, k, \quad k = n - \sigma - 1. \tag{2.2}$$

We next provide a fundamental lemma to show existence and uniqueness of the function $H_{\alpha_n}(x)$ in (2.1). We will use the Pochhammer k-symbol $(x)_{n,k}$, which is defined as

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k) = \prod_{i=1}^{n} \big(x + (i-1)k\big)$$

for $n \in \mathbb{N}_+$ with the initial setting $(x)_{0,k} = 1$.

LEMMA 2.2. *For $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, the following determinant satisfies*

$$D_n = \begin{vmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_n \\ \alpha_1(\alpha_1 - 1) & \cdots & \alpha_n(\alpha_n - 1) \\ \vdots & & \vdots \\ (\alpha_1)_{n-1,-1} & \cdots & (\alpha_n)_{n-1,-1} \end{vmatrix} = \prod_{i=2}^{n} \prod_{j=1}^{i-1} (\alpha_j - \alpha_i) \neq 0. \tag{2.3}$$

Proof. We prove this by mathematical induction. If $n = 1, 2$, then $D_n \neq 0$ is obviously true since $\alpha_1 < \alpha_2$. Now, suppose as induction hypothesis that $n \geq 2$ and that (2.3) holds for $n - 1$. We have

$$D_n = \begin{vmatrix} 1 & & & & \\ -\alpha_n & 1 & & & \\ & -(\alpha_n - 1) & 1 & & \\ & & \ddots & \ddots & \\ & & & -(\alpha_n - n + 2) & 1 \end{vmatrix} \begin{vmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_n \\ \alpha_1(\alpha_1 - 1) & \cdots & \alpha_n(\alpha_n - 1) \\ \vdots & & \vdots \\ (\alpha_1)_{n-1,-1} & \cdots & (\alpha_n)_{n-1,-1} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cdots & 1 & 1 \\ \alpha_1 - \alpha_n & \cdots & \alpha_{n-1} - \alpha_n & 0 \\ \alpha_1(\alpha_1 - \alpha_n) & \cdots & \alpha_{n-1}(\alpha_{n-1} - \alpha_n) & 0 \\ \vdots & & \vdots & 0 \\ (\alpha_1)_{n-2,-1}(\alpha_1 - \alpha_n) & \cdots & (\alpha_{n-1})_{n-2,-1}(\alpha_{n-1} - \alpha_n) & 0 \end{vmatrix}.$$

Expanding the determinant D_n at the last column and extracting the common factor $(\alpha_j - \alpha_n)$ from column j $(j = 1, 2, \ldots, n - 1)$, we can deduce that

$$D_n = \prod_{j=1}^{n-1} (\alpha_j - \alpha_n) D_{n-1} = \cdots = \prod_{i=2}^{n} \prod_{j=1}^{i-1} (\alpha_j - \alpha_i) \neq 0,$$

since $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. This completes the induction argument. ∎

With the above preparation, we prove the unique existence of the fractional Hermite interpolation function $H_{\alpha_n}(x)$ in (2.1), summarized in the following theorem.

THEOREM 2.3. *Supposed that $f(x)$, $x \in (a, b]$, is sufficiently smooth except at the endpoint $x = a$, where $f(x)$ has the local fractional Taylor expansion (1.1). Then, the fractional Hermite interpolation function $H_{\alpha_n}(x)$ in Definition 2.1 exists and is unique.*

Proof. Let

$$\begin{aligned} G(x) &:= H_{\alpha_n}(x) - \sum_{i=1}^{\sigma} a_i (x - a)^{\alpha_i} = \sum_{i=\sigma+1}^{n} b_i (x - a)^{\alpha_i}, \\ F(x) &:= f(x) - \sum_{i=1}^{\sigma} a_i (x - a)^{\alpha_i}. \end{aligned} \tag{2.4}$$

Then $G(x)$ satisfies the $k + 1$ interpolation conditions

$$G^{(j)}(b) = F^{(j)}(b), \qquad j = 0, 1, \ldots, k,$$

which means that

$$\begin{cases} \displaystyle\sum_{i=\sigma+1}^{n} (b - a)^{\alpha_i} b_i = F(b), \\ \displaystyle\sum_{i=\sigma+1}^{n} \alpha_i (b - a)^{\alpha_i - 1} b_i = F'(b), \\ \qquad \cdots \\ \displaystyle\sum_{i=\sigma+1}^{n} (\alpha_i)_{k,-1} (b - a)^{\alpha_i - k} b_i = F^{(k)}(b). \end{cases} \tag{2.5}$$

The coefficient matrix for the linear system (2.5) is

$$\boldsymbol{A} := \begin{bmatrix} (b-a)^{\alpha_{\sigma+1}} & \cdots & (b-a)^{\alpha_n} \\ \alpha_{\sigma+1}(b-a)^{\alpha_{\sigma+1}-1} & \cdots & \alpha_n(b-a)^{\alpha_n-1} \\ \vdots & & \vdots \\ \left(\alpha_{\sigma+1}\right)_{k,-1}(b-a)^{\alpha_{\sigma+1}-k} & \cdots & \left(\alpha_n\right)_{k,-1}(b-a)^{\alpha_n-k} \end{bmatrix} = \boldsymbol{C_1 D C_2}, \tag{2.6}$$

where

$$\begin{aligned} \boldsymbol{C_1} &:= \operatorname{diag}\big(1, (b-a)^{-1}, \cdots, (b-a)^{-k}\big), \\ \boldsymbol{C_2} &:= \operatorname{diag}\big((b-a)^{\alpha_{\sigma+1}}, (b-a)^{\alpha_{\sigma+2}}, \cdots, (b-a)^{\alpha_n}\big), \\ \boldsymbol{D} &:= \begin{bmatrix} 1 & \cdots & 1 \\ \alpha_{\sigma+1} & \cdots & \alpha_n \\ \vdots & & \vdots \\ \left(\alpha_{\sigma+1}\right)_{k,-1} & \cdots & \left(\alpha_n\right)_{k,-1} \end{bmatrix}. \end{aligned}$$

Since $n = \sigma + k + 1$, we know from Lemma 2.2 that

$$\det(\boldsymbol{D}) = \prod_{i=2}^{k+1}\prod_{j=1}^{i-1}(\alpha_{\sigma+j} - \alpha_{\sigma+i}) \neq 0.$$

Because $\det(\boldsymbol{C_1}) = (b-a)^{\frac{-k(k+1)}{2}} \neq 0$ and $\det(\boldsymbol{C_2}) = (b-a)^{\sum_{l=\sigma+1}^{n}\alpha_l} \neq 0$, we obtain

$$\begin{aligned} \det(\boldsymbol{A}) &= \det(\boldsymbol{C_1})\det(\boldsymbol{D})\det(\boldsymbol{C_2}) \\ &= (b-a)^{\sum_{l=\sigma+1}^{n}\alpha_l - \frac{k(k+1)}{2}} \prod_{i=2}^{k+1}\prod_{j=1}^{i-1}(\alpha_{\sigma+j} - \alpha_{\sigma+i}) \neq 0. \end{aligned} \tag{2.7}$$

Thus, the solution $(b_{\sigma+1}, \cdots, b_n)^{\mathrm{T}}$ of the linear system (2.5) exists and is unique, from which we conclude that the fractional Hermite interpolation function $H_{\alpha_n}(x)$ in (2.1) is uniquely determined by the interpolation conditions (2.2). The theorem is proved. □

By denoting $A = \det(\boldsymbol{A})$, we deduce via Cramer's rule that

$$\begin{aligned} b_{\sigma+j} =& \frac{1}{A}\begin{vmatrix} (b-a)^{\alpha_{\sigma+1}} & \cdots & F(b) & \cdots & (b-a)^{\alpha_n} \\ \alpha_{\sigma+1}(b-a)^{\alpha_{\sigma+1}-1} & \cdots & F'(b) & \cdots & \alpha_n(b-a)^{\alpha_n-1} \\ \vdots & & \vdots & & \vdots \\ (\alpha_{\sigma+1})_{k,-1}(b-a)^{\alpha_{\sigma+1}-k} & \cdots & F^{(k)}(b) & \cdots & (\alpha_n)_{k,-1}(b-a)^{\alpha_n-k} \end{vmatrix} \\ =& \frac{1}{A}\sum_{i=1}^{k+1} F^{(i-1)}(b) A_{ij}, \qquad j = 1, 2, \ldots, n-\sigma, \end{aligned}$$

where A_{ij} is the algebraic complement of the entry $a_{ij} = (\alpha_{\sigma+j})_{i-1,-1}(b-a)^{\alpha_{\sigma+j}-(i-1)}$ in

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the determinant of the matrix $\boldsymbol{A}$ defined in (2.6). Substituting $b_{\sigma+j}$ into (2.1), we have

$$
\begin{aligned}
H_{\alpha_n}(x) &= \sum_{j=1}^{\sigma} a_j (x-a)^{\alpha_j} + \frac{1}{A}\sum_{j=1}^{k+1}\Big[\sum_{i=1}^{k+1} F^{(i-1)}(b) A_{ij}\Big](x-a)^{\alpha_{\sigma+j}} \\
&= \sum_{j=1}^{\sigma} a_j (x-a)^{\alpha_j} + \frac{1}{A}\sum_{i=1}^{k+1} F^{(i-1)}(b)\Big[\sum_{j=1}^{k+1} A_{ij}(x-a)^{\alpha_{\sigma+j}}\Big] \qquad (2.8)\\
&= \sum_{j=1}^{\sigma} a_j (x-a)^{\alpha_j} + \frac{1}{A}\sum_{i=1}^{k+1} F^{(i-1)}(b)(-1)^{i-1}A_i(x),
\end{aligned}
$$

where

$$
\begin{aligned}
&\sum_{j=1}^{k+1} A_{ij}(x-a)^{\alpha_{\sigma+j}} \\
&= \begin{vmatrix}
(b-a)^{\alpha_{\sigma+1}} & \cdots & (b-a)^{\alpha_n} \\
\alpha_{\sigma+1}(b-a)^{\alpha_{\sigma+1}-1} & \cdots & \alpha_n(b-a)^{\alpha_n-1} \\
\vdots & & \vdots \\
(\alpha_{\sigma+1})_{i-2,-1}(b-a)^{\alpha_{\sigma+1}-(i-2)} & \cdots & (\alpha_n)_{i-2,-1}(b-a)^{\alpha_n-(i-2)} \\
(x-a)^{\alpha_{\sigma+1}} & \cdots & (x-a)^{\alpha_n} \\
(\alpha_{\sigma+1})_{i,-1}(b-a)^{\alpha_{\sigma+1}-i} & \cdots & (\alpha_n)_{i,-1}(b-a)^{\alpha_n-i} \\
\vdots & & \vdots \\
(\alpha_{\sigma+1})_{k,-1}(b-a)^{\alpha_{\sigma+1}-k} & \cdots & (\alpha_n)_{k,-1}(b-a)^{\alpha_n-k}
\end{vmatrix} \qquad (2.9)\\
&= (-1)^{i-1}\begin{vmatrix}
(x-a)^{\alpha_{\sigma+1}} & \cdots & (x-a)^{\alpha_n} \\
(b-a)^{\alpha_{\sigma+1}} & \cdots & (b-a)^{\alpha_n} \\
\alpha_{\sigma+1}(b-a)^{\alpha_{\sigma+1}-1} & \cdots & \alpha_n(b-a)^{\alpha_n-1} \\
\vdots & & \vdots \\
(\alpha_{\sigma+1})_{i-2,-1}(b-a)^{\alpha_{\sigma+1}-(i-2)} & \cdots & (\alpha_n)_{i-2,-1}(b-a)^{\alpha_n-(i-2)} \\
(\alpha_{\sigma+1})_{i,-1}(b-a)^{\alpha_{\sigma+1}-i} & \cdots & (\alpha_n)_{i,-1}(b-a)^{\alpha_n-i} \\
\vdots & & \vdots \\
(\alpha_{\sigma+1})_{k,-1}(b-a)^{\alpha_{\sigma+1}-k} & \cdots & (\alpha_n)_{k,-1}(b-a)^{\alpha_n-k}
\end{vmatrix} \\
&:= (-1)^{i-1}A_i(x).
\end{aligned}
$$

Substituting $F(b)$ defined in (2.4) into (2.8), we obtain an explicit formula for the fractional Hermite interpolant:

(2.10)

$$
\begin{aligned}
H_{\alpha_n}(x) &= \sum_{j=1}^{\sigma} a_j (x-a)^{\alpha_j} \\
&\quad + \frac{1}{A}\sum_{i=1}^{k+1}\Big[f^{(i-1)}(b) - \sum_{j=1}^{\sigma} a_j(\alpha_j)_{i-1,-1}(b-a)^{\alpha_j-(i-1)}\Big](-1)^{i-1}A_i(x) \\
&= \sum_{j=1}^{\sigma}\Big[(x-a)^{\alpha_j} - \frac{1}{A}\sum_{i=1}^{k+1}(\alpha_j)_{i-1,-1}(b-a)^{\alpha_j-(i-1)}(-1)^{i-1}A_i(x)\Big]a_j \\
&\quad + \frac{1}{A}\sum_{i=1}^{k+1}(-1)^{i-1}A_i(x) f^{(i-1)}(b).
\end{aligned}
$$

The above arguments can be summarized in the following theorem.

THEOREM 2.4. *Under the conditions of Theorem 2.3, the fractional Hermite interpolant (2.1) is given by (2.10).*

We further discuss the error remainder for the fractional Hermite interpolant (2.1) with (2.2). Let us begin with an important lemma about the corresponding basis functions.

LEMMA 2.5. *For* $f(x) = (x-a)^{\alpha_i}$, $x \in (a,b]$,

$$(x-a)^{\alpha_i} - H_{\alpha_n}(x) = \begin{cases} 0, & i = 1,2,\dots,\sigma, \\ \frac{(-1)^k C_i(x)}{A}, & i = \sigma+1, \sigma+2, \dots, \end{cases} \tag{2.11}$$

where

$$C_i(x) = \begin{vmatrix} (x-a)^{\alpha_{\sigma+1}} & \cdots & (x-a)^{\alpha_n} & (x-a)^{\alpha_i} \\ (b-a)^{\alpha_{\sigma+1}} & \cdots & (b-a)^{\alpha_n} & (b-a)^{\alpha_i} \\ \alpha_{\sigma+1}(b-a)^{\alpha_{\sigma+1}-1} & \cdots & \alpha_n(b-a)^{\alpha_n-1} & \alpha_n(b-a)^{\alpha_i-1} \\ \vdots & & \vdots & \vdots \\ (\alpha_{\sigma+1})_{k,-1}(b-a)^{\alpha_{\sigma+1}-k} & \cdots & (\alpha_n)_{k,-1}(b-a)^{\alpha_n-k} & (\alpha_n)_{k,-1}(b-a)^{\alpha_i-k} \end{vmatrix}. \tag{2.12}$$

Proof. We note that the local Taylor expansion of $f(x) = (x-a)^{\alpha_i}$ is identical to this function itself and $f^{(l-1)}(x) = (\alpha_i)_{l-1,-1}(x-a)^{\alpha_i-(l-1)}$, $l \geq 1$.

When $i \in \{1,2,\dots,\sigma\}$, we use (2.10) and obtain the fractional Hermite interpolant of $f(x) = (x-a)^{\alpha_i}$ as

$$\begin{aligned} H_{\alpha_n}(x) =& \Big[(x-a)^{\alpha_i} - \frac{1}{A}\sum_{l=1}^{k+1}(\alpha_i)_{l-1,-1}(b-a)^{\alpha_i-(l-1)}(-1)^{l-1}A_l(x)\Big] \\ & + \frac{1}{A}\sum_{l=1}^{k+1}(-1)^{l-1}A_l(x)f^{(l-1)}(b) \\ =& (x-a)^{\alpha_i}. \end{aligned}$$

Therefore, $(x-a)^{\alpha_i} - H_{\alpha_n}(x) = 0$, for $i \in \{1,2,\dots,\sigma\}$.

Likewise, when $i \in \{\sigma+1, \sigma+2, \dots\}$, we also obtain

$$\begin{aligned} H_{\alpha_n}(x) &= \frac{1}{A}\sum_{l=1}^{k+1}(-1)^{l-1}A_l(x)f^{(l-1)}(b) \\ &= \frac{1}{A}\sum_{l=1}^{k+1}(-1)^{l-1}A_l(x)(\alpha_i)_{l-1,-1}(b-a)^{\alpha_i-(l-1)} \end{aligned}$$

and

$$(x-a)^{\alpha_i} - H_{\alpha_n}(x) = \frac{1}{A}\Big[A(x-a)^{\alpha_i} + \sum_{l=1}^{k+1}(-1)^l A_l(x)(\alpha_i)_{l-1,-1}(b-a)^{\alpha_i-(l-1)}\Big].$$

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On the other hand, expanding $C_i(x)$ with respect to the last column yields

$$\begin{aligned} C_i(x) &= (-1)^{k+2}A(x-a)^{\alpha_i} + \sum_{l=1}^{k+1}(-1)^{k+2+l}A_l(x)(\alpha_i)_{l-1,-1}(b-a)^{\alpha_i-(l-1)} \\ &= (-1)^k\Big[A(x-a)^{\alpha_i} + \sum_{l=1}^{k+1}(-1)^{l}A_l(x)(\alpha_i)_{l-1,-1}(b-a)^{\alpha_i-(l-1)}\Big]. \end{aligned} \tag{2.13}$$

Comparing the above two formulas, we deduce that

$$(x-a)^{\alpha_i} - H_{\alpha_n}(x) = \frac{(-1)^k C_i(x)}{A}, \qquad i \in \{\sigma+1, \sigma+2, \ldots\}.$$

Hence, formula (2.11) holds, and the lemma is proved. ∎

Noting that $C_i(x) = 0$ in (2.12) when $\sigma + 1 \le i \le n$, we have

$$(x-a)^{\alpha_i} - H_{\alpha_n}(x) = 0, \qquad 1 \le i \le n.$$

Hence, we obtain the following error remainder:

THEOREM 2.6. *Under the conditions of Theorem 2.3, the error remainder of the fractional Hermite interpolant $H_{\alpha_n}(x)$ in (2.1) is*

$$\begin{aligned} R_{\alpha_n}(x) &= f(x) - H_{\alpha_n}(x) \\ &= \frac{(-1)^k}{A}\sum_{i=n+1}^{u} a_i C_i(x) + \Big[r_a(x) - \frac{1}{A}\sum_{j=1}^{k+1}(-1)^{j-1}A_j(x) r_a^{(j-1)}(b)\Big], \quad x \in (a,b], \end{aligned} \tag{2.14}$$

where $A = \det(\boldsymbol{A})$, $C_i(x)$ is the determinant (2.12), and $r_a(x)$ and $A_j(x)$ are defined in (1.1) and (2.9), respectively.

Proof. From (1.1), (2.10), and (2.11) we have

$$\begin{aligned} R_{\alpha_n}(x) &= f(x) - H_{\alpha_n}(x) \\ &= \sum_{i=\sigma+1}^{u} a_i(x-a)^{\alpha_i} + r_a(x) + \sum_{i=1}^{\sigma}\frac{a_i}{A}\sum_{j=1}^{k+1}(-1)^{j-1}A_j(x)(\alpha_i)_{j-1,-1}(b-a)^{\alpha_i-(j-1)} \\ &\qquad - \frac{1}{A}\sum_{j=1}^{k+1}(-1)^{j-1}A_j(x)f^{(j-1)}(b) \\ &= \sum_{i=\sigma+1}^{u} a_i(x-a)^{\alpha_i} + r_a(x) + \sum_{i=1}^{\sigma}\frac{a_i}{A}\sum_{j=1}^{k+1}(-1)^{j-1}A_j(x)(\alpha_i)_{j-1,-1}(b-a)^{\alpha_i-(j-1)} \\ &\qquad - \frac{1}{A}\sum_{j=1}^{k+1}(-1)^{j-1}A_j(x)\Big[\sum_{i=1}^{u} a_i(\alpha_i)_{j-1,-1}(b-a)^{\alpha_i-(j-1)} + r_a^{(j-1)}(b)\Big] \\ &= \sum_{i=\sigma+1}^{u} a_i(x-a)^{\alpha_i} + r_a(x) - \sum_{i=\sigma+1}^{u}\frac{a_i}{A}\sum_{j=1}^{k+1}(-1)^{j-1}A_j(x)(\alpha_i)_{j-1,-1}(b-a)^{\alpha_i-(j-1)} \\ &\qquad - \frac{1}{A}\sum_{j=1}^{k+1}(-1)^{j-1}A_j(x)r_a^{(j-1)}(b) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=\sigma+1}^{u} \frac{a_i}{A}\Big[A(x-a)^{\alpha_i} - \sum_{j=1}^{k+1}(-1)^{j-1}A_j(x)(\alpha_i)_{j-1,-1}(b-a)^{\alpha_i-(j-1)}\Big] + r_a(x) \\
&\qquad - \frac{1}{A}\sum_{j=1}^{k+1}(-1)^{j-1}A_j(x) r_a^{(j-1)}(b) \\
&= \frac{(-1)^k}{A}\sum_{i=\sigma+1}^{u} a_i C_i(x) + \Big[r_a(x) - \frac{1}{A}\sum_{j=1}^{k+1}(-1)^{j-1}A_j(x) r_a^{(j-1)}(b)\Big].
\end{aligned}
$$

Here we used equation (2.13) to obtain the last equality. The proof is complete. □

REMARK 2.7. More generally, we can divide the interval $[a, b]$ into a mesh $\mathcal{T}_h$ with nodes $a = x_0 < x_1 < x_2 < \cdots < x_m = b$. By analogy with the above process, we can also construct the fractional Hermite interpolant

$$
H_{\alpha_n}(x) = \sum_{i=1}^{\sigma} a_i (x-a)^{\alpha_i} + \sum_{i=\sigma+1}^{n} b_i (x-a)^{\alpha_i}
$$

satisfying the interpolation conditions

$$
H_{\alpha_n}^{(j)}(x_l) = f^{(j)}(x_l), \qquad j = 0, 1, 2, \ldots, k, \quad l = 1, 2, \ldots, m, \quad n = m(k+1) + \sigma.
$$

As it is well known, the local fractional Taylor expansion has a local approximation property just as the standard one, which means that the fractional Taylor expansion may not be accurate enough when the variable is far away from the expansion point. This phenomenon is clearly illustrated in Example 4.1, which shows that this problem can be effectively overcome by fractional Hermite interpolation.

In addition, by observing Theorem 2.6, we expand $C_i(x)$ in (2.12) with respect to the first row

$$
C_i(x) = \sum_{j=1}^{k+1}(-1)^{j+1}(x-a)^{\alpha_{\sigma+j}} C_{1j} + (-1)^{k+2}(x-a)^{\alpha_i} A,
$$

where C_{1j} is the algebraic complement of the entry $c_{1j} = (x-a)^{\alpha_{\sigma+j}}$ in $C_i(x)$. Substituting the above equation and (2.9) into (2.14) gives

$$
\begin{aligned}
R_{\alpha_n}(x) = \sum_{i=n+1}^{u} a_i \Big[\sum_{j=1}^{k+1}(-1)^{k+j+1}\frac{C_{1j}}{A}(x-a)^{\alpha_{\sigma+j}} + (x-a)^{\alpha_i}\Big] \\
+ \Big[r_a(x) - \frac{1}{A}\sum_{j=1}^{k+1}(-1)^{j-1}A_j(x) r_a^{(j-1)}(b)\Big], \qquad x \in (a, b].
\end{aligned}
$$

According to (2.6), (2.7), and (2.12), we have

$$
\frac{C_{1j}}{A} = (b-a)^{\alpha_i - \alpha_{\sigma+j}} \frac{\prod\limits_{l=1, l\neq j}^{k+1}(\alpha_{\sigma+l} - \alpha_i)}{\prod\limits_{l=1}^{j-1}(\alpha_{\sigma+l} - \alpha_j)\prod\limits_{l=j+1}^{k+1}(\alpha_j - \alpha_{\sigma+l})}, \qquad i = 1, 2, \ldots, u.
$$

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Let us assume for a moment that $b-a<1$, and let

$$\delta := \max\left\{1,\ \left|\frac{\prod\limits_{l=1,l\neq j}^{k+1}(\alpha_{\sigma+l}-\alpha_i)}{\prod\limits_{l=1}^{j-1}(\alpha_{\sigma+l}-\alpha_j)\prod\limits_{l=j+1}^{k+1}(\alpha_j-\alpha_{\sigma+l})}\right|,\ i=1,2,\ldots,u\right\}.$$

We have

$$\begin{aligned}&\left|\sum_{i=n+1}^{u} a_i\Big[\sum_{j=1}^{k+1}(-1)^{k+j+1}\frac{C_{1j}}{A}(x-a)^{\alpha_{\sigma+j}}+(x-a)^{\alpha_i}\Big]\right|\\ &\quad\le \sum_{i=n+1}^{u}|a_i|\delta\Big[\sum_{j=1}^{k+1}(b-a)^{\alpha_i-\alpha_{\sigma+j}}(x-a)^{\alpha_{\sigma+j}}+(x-a)^{\alpha_i}\Big],\\ &\quad\le \sum_{i=n+1}^{u}|a_i|\delta(k+1)(b-a)^{\alpha_i}.\end{aligned}$$

Since $r_a(x)=o((x-a)^{\alpha_u})$, we choose u suitably large such that $r_a(x)=C(x-a)^{\alpha_{u+1}}$ holds, where C is a constant. A similar analysis gives

$$\left|r_a(x)-\frac{1}{A}\sum_{i=1}^{k+1}(-1)^{i-1}A_i(x)r_a^{(i-1)}(b)\right|\le|C|\widetilde{\delta}(k+1)(b-a)^{\alpha_{u+1}}=o((b-a)^{\alpha_u}),$$

where $\widetilde{\delta}$ is a constant depending on $\alpha_{\sigma+j}$ $(j=1,2,\ldots,k+1)$ and α_{u+1}. It follows from the above analysis that the error remainder can be written as

$$|R_{\alpha_n}(x)|\le\sum_{i=n+1}^{u}|a_i|\delta(k+1)(b-a)^{\alpha_i}+o\big((b-a)^{\alpha_u}\big) \tag{2.15}$$

with the leading error term $|a_{n+1}|\mathcal{O}((b-a)^{\alpha_{n+1}})$.

By summarizing the above analysis, we have the following result for the fractional Hermite interpolation function $H_{\alpha_n}(x)$:

THEOREM 2.8. *Assume that $f(x)$, $x\in(a,b]$, is sufficiently smooth and has a local fractional Taylor expansion (1.1) at $x=a$ which is absolutely convergent as $x\to a$. Then the fractional Hermite interpolant $H_{\alpha_n}(x)$ is convergent to the non-smooth function $f(x)$ as $b\to a$ in the interval $(a,b]$, and the convergence order is α_{n+1}.*

It is noted that the precision of the fractional Hermite interpolant (2.1) may deteriorate when the length of the interval is larger than one. Simultaneously, because the coefficients $|a_i|$ in the local fractional Taylor expansion (1.1) are not always monotonically decreasing (see Example 4.2), in practical approximation we usually use low-degree piecewise hybrid Hermite interpolation of non-smooth functions, which is introduced in the next section.

3. Piecewise hybrid Hermite interpolation. In this section, we discuss piecewise hybrid Hermite interpolation by combining fractional Hermite interpolation with traditional Hermite interpolation. Generally speaking, if a non-smooth function $f(x)$ has a local fractional Taylor expansion at some points of $[a,b]$, we should apply fractional Hermite interpolation in the subintervals that contain these singularities. At other subintervals, we use traditional Hermite interpolation.

Here is a practical case to illustrate the use of piecewise hybrid Hermite interpolation. Suppose that $f(x)$, $x \in (a, b]$, is sufficiently smooth except at $x = a$, where at this point the local fractional Taylor expansion (1.1) holds. Generate a mesh with nodes $a = x_0 < x_1 < x_2 < \cdots < x_N = b$. Let $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$, and $h = \max_{1\le i\le N} h_i$. We suppose that $h < 1$ and that the values of $f(x)$ and $f'(x)$ are given at the nodes x_i, $i = 1, 2, \ldots, N$.

In the subintervals $[x_{i-1}, x_i]$, $i = 2, 3, \ldots, N$, we use the cubic Hermite interpolating polynomial (cf. [22])

$$h_3(x) = \lambda_{i-1}(x)f(x_{i-1}) + \mu_{i-1}(x)f'(x_{i-1}) + f(x_i)\lambda_i(x) + f'(x_i)\mu_i(x), \tag{3.1}$$

where

$$\lambda_{i-1}(x) = \frac{1}{h_i^3}\big(h_i + 2(x - x_{i-1})\big)(x - x_i)^2, \qquad \mu_{i-1}(x) = \frac{1}{h_i^2}(x - x_{i-1})(x - x_i)^2,$$
$$\lambda_i(x) = \frac{1}{h_i^3}\big(h_i - 2(x - x_i)\big)(x - x_{i-1})^2, \qquad \mu_i(x) = \frac{1}{h_i^2}(x - x_{i-1})^2(x - x_i).$$

In the first subinterval $(x_0, x_1]$, using the values of $f(x_1)$ and $f'(x_1)$, we can construct the fractional Hermite interpolant $H_{\alpha_{\sigma+2}}(x)$ via Theorem 2.3,

$$\begin{aligned} H_{\alpha_{\sigma+2}}(x) = &\sum_{j=1}^{\sigma}\Big[(x - x_0)^{\alpha_j} - \frac{h_1^{\alpha_j}A_1(x)}{A} + \frac{\alpha_j h_1^{\alpha_j-1}A_2(x)}{A}\Big]a_j \\ &+ \frac{A_1(x)}{A}f(x_1) - \frac{A_2(x)}{A}f'(x_1), \end{aligned} \tag{3.2}$$

where

$$A = \begin{vmatrix} h_1^{\alpha_{\sigma+1}} & h_1^{\alpha_{\sigma+2}} \\ \alpha_{\sigma+1}h_1^{\alpha_{\sigma+1}-1} & \alpha_{\sigma+2}h_1^{\alpha_{\sigma+2}-1} \end{vmatrix},$$
$$A_1(x) = \begin{vmatrix} (x-x_0)^{\alpha_{\sigma+1}} & (x-x_0)^{\alpha_{\sigma+2}} \\ \alpha_{\sigma+1}h_1^{\alpha_{\sigma+1}-1} & \alpha_{\sigma+2}h_1^{\alpha_{\sigma+2}-1} \end{vmatrix},$$
$$A_2(x) = \begin{vmatrix} (x-x_0)^{\alpha_{\sigma+1}} & (x-x_0)^{\alpha_{\sigma+2}} \\ h_1^{\alpha_{\sigma+1}} & h_1^{\alpha_{\sigma+2}} \end{vmatrix}.$$

This leads to the result that the piecewise hybrid Hermite interpolation function for the non-smooth function $f(x)$ is given by

$$H(x) = \begin{cases} H_{\alpha_{\sigma+2}}(x), & x \in (x_0, x_1], \\ h_3(x), & x \in [x_{i-1}, x_i], \quad i = 2, 3, \ldots, N. \end{cases}$$

The remainder of $h_3(x)$ defined in (3.1) is [22]

$$\begin{aligned} R_3(x) = f(x) - h_3(x) &= \frac{1}{4!}f^{(4)}(\xi_i)(x - x_{i-1})^2(x - x_i)^2 \\ &= \mathcal{O}((h_i)^4), \qquad\qquad x, \xi_i(x) \in (x_{i-1}, x_i). \end{aligned} \tag{3.3}$$

From Theorem 2.6, the remainder of (3.2) is

$$\begin{aligned} R_{\alpha_{\sigma+2}}(x) &= -\frac{1}{A}\sum_{i=\sigma+3}^{u} a_i \begin{vmatrix} (x-x_0)^{\alpha_{\sigma+1}} & (x-x_0)^{\alpha_{\sigma+2}} & (x-x_0)^{\alpha_i} \\ h_1^{\alpha_{\sigma+1}} & h_1^{\alpha_{\sigma+2}} & h_1^{\alpha_i} \\ \alpha_{\sigma+1}h_1^{\alpha_{\sigma+1}-1} & \alpha_{\sigma+2}h_1^{\alpha_{\sigma+2}-1} & \alpha_i h_1^{\alpha_i-1} \end{vmatrix} + o(h_1^{\alpha_u}) \\ &= \sum_{i=\sigma+3}^{u} a_i\mathcal{O}(h_1^{\alpha_{\sigma+3}}) + o(h_1^{\alpha_u}). \end{aligned}$$

REMARK 3.1. In order to obtain uniform accuracy over the whole interval $(a, b]$, we should choose α_σ such that the truncation error of the fractional Hermite interpolation is the same as the maximum error of (3.1). The detailed choice of α_σ will be discussed in Section 4.

4. Numerical examples. In this section, some typical examples are provided to illustrate that fractional Hermite interpolation is more powerful than traditional Hermite interpolation for approximating non-smooth functions. We also show that it is necessary to use the piecewise hybrid Hermite interpolation method with a nonuniform mesh in practical computations. Because the exact values of numerous integrals cannot be obtained by analytic methods, it is extremely important to obtain the approximate values with high enough accuracy in practical simulations. As an application of fractional Hermite interpolation, we use interpolation to compute an integral and get a highly accurate result. The following examples are implemented in Mathematica 10.1.

We start with the construction of the fractional Hermite interpolation function for a singular function leading to high precision results.

EXAMPLE 4.1. Construct fractional Hermite interpolation for the singular function

$$f(x) = \frac{1}{e^{\sqrt{x}}\sin(x^{1/3})}, \qquad x \in (0, 0.5].$$

Since $f(x)$ is singular at the point $x = 0$, we can not apply traditional Hermite interpolation in the interval $(0, 0.5]$. It is easy to find the local fractional Taylor expansion of $f(x)$ at $x = 0$ using Mathematica, which gives

$$\begin{aligned} f(x) =& \frac{1}{x^{1/3}} + \frac{x^{1/3}}{6} - \frac{x^{2/3}}{2} + \frac{7x}{360} - \frac{x^{4/3}}{12} + \frac{1921x^{5/3}}{15120} - \frac{7x^2}{720} + \frac{12727x^{7/3}}{604800} \\ & - \frac{661x^{8/3}}{30240} + \frac{8389x^3}{3421440} + \cdots, \qquad x \to 0^+. \end{aligned} \tag{4.1}$$

According to Theorem 2.3 and the above expansion, we use the interpolation conditions with $f(0.5)$, $f'(0.5)$, and $f''(0.5)$ to construct fractional Hermite interpolants $H_{\alpha_\sigma+3}(x)$. Here we simply take $\alpha_\sigma = 4/3$ and obtain

$$\begin{aligned} H_{7/3}(x) = & \frac{1}{x^{1/3}} + \frac{x^{1/3}}{6} - \frac{x^{2/3}}{2} + \frac{7x}{360x} - \frac{x^{4/3}}{12} + 0.115691x^{5/3} \\ & + 0.0304425x^2 - 0.0281134x^{7/3}. \end{aligned} \tag{4.2}$$

We take the first eight terms of the fractional Taylor expansion in (4.1) to approximate the non-smooth function $f(x)$, denoted by $s_{7/3}(x)$. We display the errors of the truncated fractional Taylor expansion $s_{7/3}(x)$ and the fractional Hermite interpolation $H_{7/3}(x)$ in Figure 4.1. Obviously, $H_{7/3}(x)$ is superior to $s_{7/3}(x)$ when they are used to approximate the singular function $f(x)$. In addition, the error of the truncated Taylor expansion $s_{7/3}(x)$ increases fast when x is away from zero, which verifies the local property of the fractional Taylor expansion.

In addition, we choose some values of h and compute the maximum error

$$\varepsilon_h = \max_{0 \le x \le h} |f(x) - H_{7/3}(x)|$$

and the convergence order $\mathcal{O}_h = \log_2(\varepsilon_h/\varepsilon_{h/2})$ of the fractional Hermite interpolation function $H_{7/3}(x)$ in the interval $(0, h]$. We present the results in Table 4.1. Note that the theoretical convergence order of $H_{7/3}(x)$ is $\mathcal{O}_h = 8/3 \approx 2.666\ldots$ from Theorem 2.8.

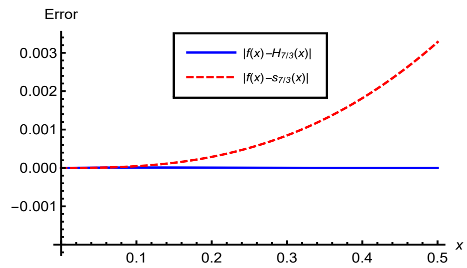

FIG. 4.1. *The errors of* $H_{7/3}(x)$ *and* $s_{7/3}(x)$.

TABLE 4.1
The maximum error and convergence order of $H_{7/3}(x)$ *near* $x = 0$.

h	ε_h	$\mathcal{O}_h$
0.5	1.10872E-5	
$0.5/2$	2.24189E-6	2.3061
$0.5/2^2$	3.87358E-7	2.53297
$0.5/2^3$	6.31746E-8	2.61625
$0.5/2^4$	1.00855E-8	2.64706
$0.5/2^5$	1.59871E-9	2.6573
$0.5/2^6$	2.52973E-10	2.65985

Table 4.1 shows that the convergence order of the fractional Hermite interpolant $H_{7/3}(x)$ is consistent with the theoretical result as $h \to 0$.

We finally take some values of α_σ and construct a series of fractional Hermite interpolants in different intervals. We also obtain their maximum absolute errors and display these errors in Table 4.2, where $\varepsilon_h = \max\limits_{0<x\leq h} |f(x) - H_{\alpha_\sigma+3}(x)|$. From Table 4.2, it is easy to see that the accuracy improves if a larger σ_6 or a smaller h are used.

TABLE 4.2
The maximum absolute errors of $H_{\sigma+3}(x)$ *in different intervals.*

α_σ \ h (ε_h)	1.0	0.5	0.1
0	1.80356E-2	7.65776E-3	1.17603E-3
1/3	6.11154E-4	7.66619E-4	8.78003E-5
2/3	1.09545E-3	1.80134E-4	1.64608E-6
1	2.67558E-4	5.09412E-5	5.19636E-7
4/3	3.16056E-5	1.10872E-5	2.16896E-7
5/3	2.85861E-5	2.21069E-6	4.03282E-9

EXAMPLE 4.2. Construct the piecewise hybrid Hermite interpolation for the function

$$f(x) = \ln(1 + \arcsin(x^{1/3})), \qquad x \in [0, 1].$$

A straightforward computation shows that

$$f'(x) = \frac{1}{3x^{2/3}\sqrt{1 - x^{2/3}}(1 + \arcsin x^{1/3})}, \qquad x \in (0, 1).$$

Hence, the derivatives of $f(x)$ do not exist at $x = 0$ and $x = 1$. The local fractional Taylor expansions at $x = 0$ and $x = 1$ are denoted by $f_l(x)$ and $f_r(x)$, respectively, which read

$$\begin{aligned} f_l(x) =& x^{1/3} - 0.5x^{2/3} + 0.5x - 0.416667x^{4/3} + 0.441667x^{5/3} - 0.422222x^2 \\ &+ 0.456944x^{7/3} - 0.465476x^{8/3} + \cdots, \qquad x \to 0^+; \\ f_r(x) =& 0.944216 - 0.317605\sqrt{1-x} - 0.0504363(1-x) - 0.0724356(1-x)^{3/2} \\ &- 0.0221579(1-x)^2 - 0.0369454(1-x)^{5/2} + \cdots, \qquad x \to 1^-. \end{aligned}$$

We first construct the piecewise hybrid Hermite interpolant $H(x)$ with a uniform mesh

$$\{x_i = i/10 : i = 0, 1, 2, \ldots, 10\} \tag{4.3}$$

over $[0, 1]$. We can now use (3.1) to obtain the traditional cubic Hermite interpolating polynomial $h_3^{c_i}(x)$ with the interpolation conditions $f(x_{i-1})$, $f'(x_{i-1})$, $f(x_i)$, and $f'(x_i)$ on $[x_{i-1}, x_i]$, $i = 2, 3, \ldots, 9$. The maximum absolute error of the piecewise cubic Hermite polynomial $h_3^c(x)$ is computed as

$$\max_{x_1 \le x \le x_9} |f(x) - h_3^c(x)| = 2.59228 \times 10^{-4}. \tag{4.4}$$

In order to make the error uniformly distributed over $[0, 1]$, the values $\alpha_{\sigma_l} = 5/3$ and $\alpha_{\sigma_r} = 1/2$ are chosen by comparing (4.4) with the truncation errors of $f_l(x)$ and $f_r(x)$, respectively.

By means of Theorem 2.3 with $f(x_1)$, $f'(x_1)$, and the expansion $f_l(x)$, we have

$$\begin{aligned} H^l_{7/3}(x) =& x^{1/3} - 0.5x^{2/3} + 0.5x - 0.416667x^{4/3} + 0.441667x^{5/3} \\ &- 0.380615x^2 + 0.225788x^{7/3}, \qquad x \in [x_0, x_1]. \end{aligned} \tag{4.5}$$

Similarly, with $f(x_9)$, $f'(x_9)$, and the expansion $f_r(x)$, we have

$$\begin{aligned} H^r_{3/2}(x) =& 0.944216 - 0.317605\sqrt{1-x} - 0.0450847(1-x) \\ &- 0.317605(1-x)^{3/2}, \qquad x \in [x_9, x_{10}]. \end{aligned} \tag{4.6}$$

Then, the piecewise hybrid Hermite interpolation function $H(x)$ is

$$H(x) = \begin{cases} H^l_{7/3}(x), & x \in [x_0, x_1], \\ h_3^{c_i}(x), & x \in [x_{i-1}, x_i], \quad i = 2, 3, \ldots, 9, \\ H^r_{3/2}(x), & x \in [x_9, x_{10}]. \end{cases}$$

A straightforward computation shows that

$$\begin{aligned} \max_{x_0 \le x \le x_1} |f(x) - H^l_{7/3}(x)| &= 3.07971 \times 10^{-6}, \\ \max_{x_9 \le x \le x_{10}} |f(x) - H^r_{3/2}(x)| &= 3.96730 \times 10^{-5}. \end{aligned} \tag{4.7}$$

We also plot the error in Figure 4.2. It can be seen from (4.7) and Figure 4.2 that the fractional Hermite interpolation functions $H^l_{7/3}(x)$ and $H^r_{3/2}(x)$ are more accurate near the left and right endpoints, respectively, which proves our treatment for the singularity successful. It can also be seen that the maximum error of the traditional cubic Hermite interpolant $h_3^c(x)$ is relatively

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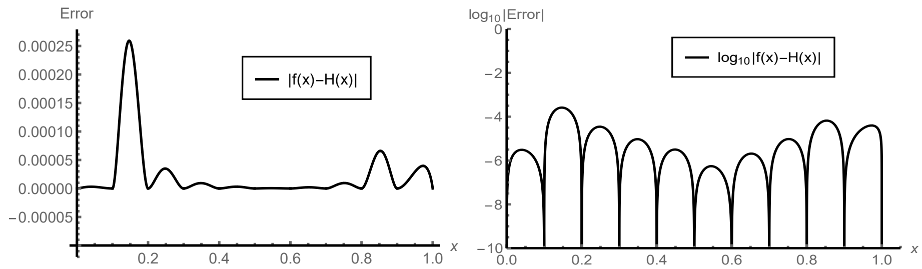

FIG. 4.2. *The error (left) and the absolute error on a logarithmic scale (right) of $H(x)$.*

large near $x = 0.15$. So we try to generate a non-uniform mesh to arrive at uniform accuracy in the whole interval. To the end, the simple function

$$x(t) = 0.1 + \frac{21 - 21^{1-2t}}{25(1 + 21^{1-2t})}, \qquad t \in [0, 1]$$

transforms the nodes $\{x_i = i/10 : i = 1, 2, \ldots, 9\}$ to a nonuniform mesh

$$\{x'_i : i = 0, 1, 2, \ldots, 10\} = \{x_0, x(t_1), x(t_2), \cdots, x(t_9), x_{10}\}, \qquad t_j = \frac{j-1}{8}. \tag{4.8}$$

We construct the piecewise hybrid Hermite interpolation function $\widetilde{H}(x)$ at these nonuniform points in a similar manner as before. Since the maximum absolute error of the piecewise cubic Hermite polynomial $\widetilde{h}_3^c(x)$ is

$$\max_{x_1 \le x \le x_9} |f(x) - \widetilde{h}_3^c(x)| = 5.27115 \times 10^{-5},$$

we still choose the functions $H^l_{7/3}(x)$ from (4.5) and $H^r_{3/2}(x)$ from (4.6) on the intervals $[x_0, x_1]$ and $[x_9, x_{10}]$, respectively. The errors are displayed in Figure 4.3. It can be seen from Figure 4.3 that the error of the piecewise hybrid Hermite interpolation function $\widetilde{H}(x)$ is clearly reduced, and the error distributions of $\widetilde{H}(x)$ is more uniform than the one for $H(x)$. If we further refine the mesh, the result will be more conspicuous.

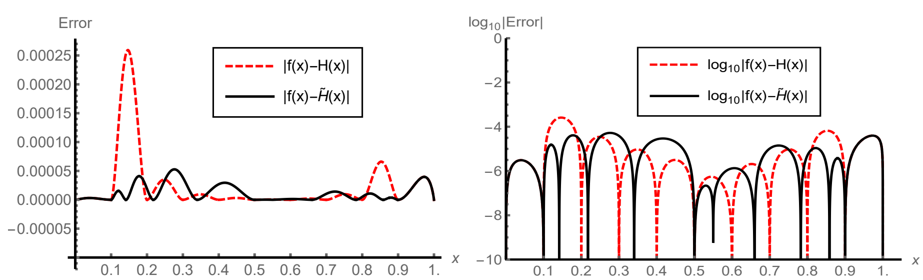

FIG. 4.3. *The error (left) and the absolute error on a logarithmic scale (right) of $\widetilde{H}(x)$ and $H(x)$.*

In addition, the convergence orders for fractional Hermite interpolants are calculated for this example. In the interval $[0, h_1] \subseteq [0, 1]$, we compute the maximum error

$$\varepsilon_{h_1} = \max_{0 \le x \le h_1} |f(x) - H^l_{7/3}(x)|$$

and the convergence order $\mathcal{O}_{h_1} = \log_2(\varepsilon_{h_1}/\varepsilon_{h_1/2})$ of $H^l_{7/3}(x)$. We choose some values of h_1 and present the results in Table 4.3. Note that the theoretical convergence order of $H^l_{7/3}(x)$ is $\mathcal{O}_l = 8/3 \approx 2.666\ldots$ from (2.15). Similarly, in the interval $[1-h_n, 1] \subseteq [0, 1]$, we also choose some values of h_n with the results of $H^r_{3/2}(x)$ presented in Table 4.4. The theoretical convergence order of $H^r_{3/2}(x)$ is $\mathcal{O}_r = 2$. Table 4.3 and Table 4.4 show that the convergence orders are consistent with the theoretical results of Theorem 2.8 as $h_1 \to 0$ and $h_n \to 0$, respectively.

TABLE 4.3
The maximum error and convergence order of $H^l_{7/3}(x)$.

h_1	ε_{h_1}	O_{h_1}
0.1	3.0797E-6	
$0.1/2$	6.41581E-7	2.26309
$0.1/2^2$	1.23817E-7	2.37342
$0.1/2^3$	2.28515E-8	2.43785
$0.1/2^4$	4.08555E-9	2.48368
$0.1/2^5$	7.12701E-10	2.51916
$0.1/2^6$	1.21902E-10	2.54757

TABLE 4.4
The maximum error and convergence order of $H^l_{3/2}(x)$.

h_1	ε_{h_n}	O_{h_n}
0.1	3.96730E-5	
$0.1/2$	7.45828E-6	2.41125
$0.1/2^2$	1.5191E-6	2.29562
$0.1/2^3$	2.39009E-7	2.66808
$0.1/2^4$	5.36471E-8	2.15549
$0.1/2^5$	1.23975E-8	2.11345
$0.1/2^6$	2.92689E-9	2.08261

Finally, we consider the convergence of the piecewise hybrid Hermite interpolant. In order to guarantee the precision of the cubic Hermite interpolating polynomial $h^c_3(x)$ in $[0.1, 0.9]$, we choose suitably large values α_{σ_l} and α_{σ_r} such that the maximum errors of the fractional Hermite interpolants $H^l_{\alpha_{\sigma_l}+3}(x)$, $x \in [0, 0.1]$, and $H^r_{\alpha_{\sigma_r}+3}(x)$, $x \in [0.9, 1.0]$, are of the order of machine precision. We take different stepsizes $h = \max\limits_{2\le i\le n-1}\{h_i = x_i - x_{i-1}\}$ and display the convergence order of $h^c_3(x)$ in Table 4.5. It is clearly seen that $h^c_3(x)$ converges to $f(x)$ with $\mathcal{O}_c = 4$ as $h \to 0$, theoretically from (3.3) and numerically from Table 4.5. So the piecewise hybrid Hermite interpolation converges to the original function $f(x)$ fast.

EXAMPLE 4.3. We give an effective application of fractional Hermite interpolation to the computation of the integral

$$\int_0^1 f(x)dx = \int_0^1 \ln(1 + \arcsin(x^{1/3}))\, dx.$$

Apparently, the integral cannot be computed analytically, so we must evaluate it numerically. Here, we compute approximate values with high enough accuracy by the "NIntegrate" com-

TABLE 4.5
The maximum error and convergence order of $h_3^c(x)$.

h_1	ε_h	O_h
0.1	2.59228E-4	
$0.1/2$	2.90655E-5	3.15684
$0.1/2^2$	2.59609E-6	3.48489
$0.1/2^3$	1.98492E-7	3.70919
$0.1/2^4$	1.38197E-8	3.84428
$0.1/2^5$	9.09039E-10	3.92624
$0.1/2^6$	5.87373E-11	3.95199

mand of Mathematica using 20-digits precision, from which we obtain the true errors. The fractional Hermite interpolation functions $H(x)$ and $\widetilde{H}(x)$ of $f(x)$ can be integrated analytically. The results are compared with the composite trapezoidal rule for the corresponding meshes (4.3) and (4.8), respectively. We provide the errors in Table 4.6, where FHI-error and CTR-error represent the absolute errors computed by fractional Hermite interpolation and the composite trapezoidal rule, respectively. It can be seen form Table 4.6 that the results of fractional Hermite interpolation are far superior to the composite trapezoidal rule with the same mesh. This is attributed to the property that fractional Hermite interpolation can accurately characterize the singular features of non-smooth functions.

TABLE 4.6
The absolute error of the numerical integral $\int_0^1 f(x)dx$.

	Mesh (4.3)	Mesh (4.8)
FHI-error	1.46014E-5	8.77327E-6
CTR-error	9.41746E-3	9.43063E-3

5. Conclusion. In this paper, we develop a fractional Hermite interpolation method for non-smooth functions. The corresponding explicit formula and the error remainder are presented and its convergence order is verified. A piecewise hybrid Hermite interpolant is developed. The proposed methods have the following features.

- The basis functions of the fractional Hermite interpolation method are adaptively chosen from the Puiseux series of the function at its singularity.
- The proposed fractional Hermite interpolant extends the local property of the Taylor expansion to the full interval such that the precision of the interpolant significantly increases away from the singularities.
- In practical computation, we usually apply piecewise hybrid Hermite interpolation with low degree to the whole interval and also use a non-uniform mesh to arrive at uniform accuracy on the whole interval.

Typical numerical examples are implemented, and accurate results are obtained for non-smooth functions with singular points. The methods can be used to efficiently solve a broad class of integral equations with singular kernels, which will be discussed in the near future.

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