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ON SYLVESTER'S LAW OF INERTIA FOR NONLINEAR EIGENVALUE PROBLEMS*

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Dedicated to Lothar Reichel on the occasion of his 60th birthday

Abstract. For Hermitian matrices and generalized definite eigenproblems, the LDL^H factorization provides an easy tool to slice the spectrum into two disjoint intervals. In this note we generalize this method to nonlinear eigenvalue problems allowing for a minmax characterization of (some of) their real eigenvalues. In particular we apply this approach to several classes of quadratic pencils.

Key words. eigenvalue, variational characterization, minmax principle, Sylvester's law of inertia

AMS subject classification. 15A18, 65F15

1. Introduction. The inertia of a Hermitian matrix A is the triplet of nonnegative integers $\text{In}(A) := (n_p, n_n, n_z)$, where n_p, n_n, and n_z are the number of positive, negative, and zero eigenvalues of A (counting multiplicities). Sylvester's classical law of inertia states that two Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$ are congruent (i.e., $A = S^H BS$ for some nonsingular matrix S) if and only if they have the same inertia $\text{In}(A) = \text{In}(B)$.

An obvious consequence of the law of inertia is the following corollary: if A has an LDL^H factorization $A = LDL^H$, then n_p and n_n equal the number of positive and negative entries of D, and if a block LDL^H factorization exists where D is a block diagonal matrix with 1×1 and indefinite 2×2 blocks on its diagonal, then one has to increase the number of positive and negative 1×1 blocks of D by the number of 2×2 blocks to get n_p and n_n, respectively. Hence, the inertia of A can be computed easily. This is particularly advantageous if the matrix is banded.

If $B \in \mathbb{C}^{n\times n}$ is positive definite, and $A - \sigma B = LDL^H$ is the block diagonal LDL^H factorization of $A - \sigma B$ for some $\sigma \in \mathbb{R}$, we get the inertia $\text{In}(A - \sigma B) = (n_p, n_n, n_z)$ as described in the previous paragraph. Then, the generalized eigenvalue problem $Ax = \lambda Bx$ has n_n eigenvalues smaller than σ. Hence, the law of inertia yields a tool to locate eigenvalues of Hermitian matrices or definite matrix pencils. Combining it with bisection or the secant method, one can determine all eigenvalues in a given interval or determine initial approximations for fast eigensolvers, and it can be used to check whether a method has found all eigenvalues in an interval of interest or not.

The law of inertia was first proved in 1858 by J. J. Sylvester [19], and several different proofs can be found in textbooks [3, 6, 11, 13, 15], one of which is based on the minmax characterization of eigenvalues of Hermitian matrices. In this note we discuss generalizations of the law of inertia to nonlinear eigenvalue problems allowing for a minmax characterization of their eigenvalues.

2. Minmax characterization. Our main tools in this paper are variational characterizations of eigenvalues of nonlinear eigenvalue problems generalizing the well known minmax characterization of Poincaré [16] or Courant [2] and Fischer [5] for linear eigenvalue problems.

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We consider the nonlinear eigenvalue problem

$$T(\lambda)x = 0, \tag{2.1}$$

where $T(\lambda) \in \mathbb{C}^{n\times n}$, $\lambda \in J$, is a family of Hermitian matrices depending continuously on the parameter $\lambda \in J$, and J is a real open interval which may be unbounded.

To generalize the variational characterization of eigenvalues, we need a generalization of the Rayleigh quotient. To this end we assume that

(A_1) for every fixed $x \in \mathbb{C}^n$, $x \neq 0$, the scalar real equation

$$f(\lambda; x) := x^H T(\lambda)x = 0 \tag{2.2}$$

has at most one solution $\lambda =: p(x) \in J$.

Then the equation $f(\lambda; x) = 0$ implicitly defines a functional p on some subset $\mathcal{D} \subset \mathbb{C}^n$, which is called the Rayleigh functional of (2.1), and which is exactly the Rayleigh quotient in case of a monic linear matrix function $T(\lambda) = \lambda I - A$.

Generalizing the definiteness requirement for linear pencils $T(\lambda) = \lambda B - A$, we further assume that

(A_2) for every $x \in \mathcal{D}$ and every $\lambda \in J$ with $\lambda \neq p(x)$ it holds that

$$(\lambda - p(x))f(\lambda; x) > 0.$$

If p is defined on $\mathcal{D} = \mathbb{C}^n \setminus \{0\}$, then the problem $T(\lambda)x = 0$ is called overdamped. This notion is motivated by the finite dimensional quadratic eigenvalue problem

$$T(\lambda)x = \lambda^2 Mx + \lambda Cx + Kx = 0,$$

where M, C, and K are Hermitian and positive definite matrices. If C is large enough such that $d(x) := (x^H Cx)^2 - 4(x^H Kx)(x^H Mx) > 0$ for every $x \neq 0$, then $T(\cdot)$ is overdamped. Generalizations of the minmax and maxmin characterizations of eigenvalues were proved by Duffin [4] for the quadratic case and by Rogers [17] for general overdamped problems.

For nonoverdamped eigenproblems, the natural ordering to call the smallest eigenvalue the first one, the second smallest the second one, etc., is not appropriate. This is obvious if we make a linear eigenvalue problem $T(\lambda)x := (\lambda I - A)x = 0$ nonlinear by restricting it to an interval J which does not contain the smallest eigenvalue of A. Then the conditions (A_1) and (A_2) are satisfied, p is the restriction of the Rayleigh quotient R_A to

$$\mathcal{D} := \{x \neq 0 \,:\, R_A(x) \in J\},$$

and $\inf_{x\in\mathcal{D}} p(x)$ will in general not be an eigenvalue.

If $\lambda \in J$ is an eigenvalue of $T(\cdot)$, then $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$, and therefore there exists an $\ell \in \mathbb{N}$ such that

$$0 = \max_{V\in H_\ell} \min_{v\in V\setminus\{0\}} \frac{v^H T(\lambda)v}{\|v\|^2},$$

where H_ℓ denotes the set of all ℓ–dimensional subspaces of $\mathbb{C}^n$. In this case, λ is called an ℓth eigenvalue of $T(\cdot)$.

With this enumeration the following minmax characterization for eigenvalues was proved in [20, 21].

THEOREM 2.1. *Let J be an open interval in $\mathbb{R}$, and let $T(\lambda) \in \mathbb{C}^{n\times n}$, $\lambda \in J$, be a family of Hermitian matrices depending continuously on the parameter $\lambda \in J$ such that the conditions (A_1) and (A_2) are satisfied. Then the following statements hold.*

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(i) For every $\ell \in \mathbb{N}$ there is at most one ℓth eigenvalue of $T(\cdot)$ which can be characterized by

(2.3) $$\lambda_\ell = \min_{V\in H_\ell,\ V\cap\mathcal{D}\neq\emptyset} \sup_{v\in V\cap\mathcal{D}} p(v).$$

(ii) If

$$\lambda_\ell := \inf_{V\in H_\ell,\ V\cap\mathcal{D}\neq\emptyset} \sup_{v\in V\cap\mathcal{D}} p(v) \in J$$

for some $\ell \in \mathbb{N}$, then λ_ℓ is the ℓth eigenvalue of $T(\cdot)$ in J, and (2.3) holds.

(iii) If there exist the kth and the ℓth eigenvalue λ_k and λ_ℓ in J ($k < \ell$), then J contains the jth eigenvalue λ_j ($k \le j \le \ell$) as well with $\lambda_k \le \lambda_j \le \lambda_\ell$.

(iv) Let $\lambda_1 = \inf_{x\in\mathcal{D}} p(x) \in J$ and $\lambda_\ell \in J$. If the minimum in (2.3) is attained for an ℓ-dimensional subspace V, then $V \subset \mathcal{D}\cup\{0\}$, and (2.3) can be replaced by

$$\lambda_\ell = \min_{V\in H_\ell,\ V\subset\mathcal{D}\cup\{0\}} \sup_{v\in V,\ v\neq 0} p(v).$$

(v) $\tilde\lambda$ is an ℓth eigenvalue if and only if $\mu = 0$ is the ℓth largest eigenvalue of the linear eigenproblem $T(\tilde\lambda)x = \mu x$.

(vi) The minimum in (2.3) is attained for the invariant subspace of $T(\lambda_\ell)$ corresponding to its ℓ largest eigenvalues.

3. Sylvester's law for nonlinear eigenvalue problems. We first consider the overdamped case. Then $T(\cdot)$ has exactly n eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ in J [17].

THEOREM 3.1. *Assume that $T : J \to \mathbb{C}^{n\times n}$ satisfies the conditions of the minmax characterization in Theorem 2.1, and assume that the nonlinear eigenvalue problem (2.1) is overdamped, i.e., for every $x \neq 0$ Equation (2.2) has a unique solution $p(x) \in J$.*

For $\sigma \in J$, let (n_p, n_n, n_z) be the inertia of $T(\sigma)$. Then the nonlinear eigenproblem $T(\lambda)x = 0$ has n eigenvalues in J, n_p of which are less than σ, n_n exceed σ, and for $n_z > 0$, σ is an eigenvalue of geometric multiplicity n_z.

Proof. The invariant subspace W of $T(\sigma)$ corresponding to its positive eigenvalues has dimension n_p, and it holds that $f(\sigma; x) = x^H T(\sigma)x > 0$ for every $x \in W$, $x \neq 0$. Hence $p(x) < \sigma$ by (A_2), and therefore the n_pth smallest eigenvalue of $T(\cdot)$ satisfies

$$\lambda_{n_p} = \min_{\dim V = n_p} \max_{x\in V, x\neq 0} p(x) \le \max_{x\in W, x\neq 0} p(x) < \sigma.$$

On the other hand for every subspace V of $\mathbb{C}^n$ of dimension $n_p + n_z + 1$, there exists a vector $x \in V$ such that $f(\sigma; x) < 0$. Thus $p(x) > \sigma$, and it holds that

$$\lambda_{n_p+n_z+1} = \min_{\dim V = n_p+n_z+1} \max_{x\in V, x\neq 0} p(x) > \sigma,$$

which completes the proof. ∎

Next we consider the case that an extreme eigenvalue, i.e., either $\lambda_1 := \inf_{x\in\mathcal{D}} p(x)$ or $\lambda_n := \sup_{x\in\mathcal{D}} p(x)$, is contained in J.

THEOREM 3.2. *Assume that $T : J \to \mathbb{C}^{n\times n}$ satisfies the conditions of the minmax characterization, and let (n_p, n_n, n_z) be the inertia of $T(\sigma)$ for some $\sigma \in J$.*

(i) If $\lambda_1 := \inf_{x\in\mathcal{D}} p(x) \in J$, then the nonlinear eigenproblem $T(\lambda)x = 0$ has exactly n_p eigenvalues $\lambda_1 \le \cdots \le \lambda_{n_p}$ in J which are smaller than σ.

(ii) If $\sup_{x\in\mathcal{D}} p(x) \in J$, then the nonlinear eigenproblem $T(\lambda)x = 0$ has exactly n_n eigenvalues $\lambda_{n-n_n+1} \le \cdots \le \lambda_n$ in J exceeding σ.

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Proof. (i): We first show that $f(\lambda;x) < 0$ for every $\lambda \in J$ with $\lambda < \lambda_1$ and for every vector $x \neq 0$. Assume that $f(\lambda;x) \geq 0$ for some $\lambda < \lambda_1$ and $x \neq 0$, let $\hat{x}$ be an eigenvector of (2.1) corresponding to λ_1, and let $w(t) := t\hat{x} + (1-t)x$, $0 \leq t \leq 1$. Then $\phi(t) := f(\lambda; w(t))$ is continuous in $[0,1]$, $\phi(0) = f(\lambda;x) \geq 0$, and $\phi(1) = f(\lambda;\hat{x}) < 0$ by (A_2). Hence, there exists a value $\hat{t} \in [0,1)$ such that $f(\lambda; w(\hat{t})) = 0$, i.e., $w(\hat{t}) \in \mathcal{D}$ and $p(w(\hat{t})) = \lambda < \lambda_1$ contradicting $\lambda_1 := \inf_{x\in\mathcal{D}} p(x)$.

For $n_p = 0$, the matrix $T(\sigma)$ is negative semidefinit, i.e., $x^H T(\sigma)x \leq 0$ for $x \neq 0$, and it follows from (A_2) that $p(x) \geq \sigma$ holds true for every $x \in \mathcal{D}$. Hence, there is no eigenvalue less than σ.

For $n_p > 0$, let W denote the invariant subspace of $T(\sigma)$ corresponding to its positive eigenvalues. Then $f(\sigma;x) = x^H T(\sigma)x > 0$ for $x \in W$, $x \neq 0$, and from $f(\lambda;x) < 0$ for $\lambda < \lambda_1$, it follows that $x \in \mathcal{D}$ and $p(x) < \sigma$. Hence, $W \subset \mathcal{D} \cup \{0\}$ and as in the proof of Theorem 3.1 we obtain

$$\lambda_{n_p} = \min_{\dim V = n_p,\ V\cap\mathcal{D}\neq\emptyset} \ \max_{x\in V\cap\mathcal{D}} p(x) \leq \max_{x\in W, x\neq 0} p(x) < \sigma,$$

i.e., $T(\cdot)$ has at least n_p eigenvalues less than σ.

Assume that there exists an (n_p+1)th eigenvalue $\lambda_{n_p+1} < \sigma$ of $T(\cdot)$, and let W be the invariant subspace of $T(\lambda_{n_p+1})$ corresponding to its nonnegative eigenvalues. Then this implies that $\dim W \geq n_p + 1$, $W \setminus \{0\} \subset \mathcal{D}$, and $p(x) \leq \lambda_{n_p+1} < \sigma$ for every $x \in W$ with $x \neq 0$, contradicting the fact that for every subspace V with $\dim V = n_p + 1$ there exists a vector $x \in V$ with $x^H T(\sigma)x \leq 0$, i.e., either $x \notin \mathcal{D}$ or $p(x) \geq \sigma$.

(ii) $S(\lambda) := -T(-\lambda)$ satisfies the conditions of Theorem 2.1 in the interval $-J$, and $-J$ contains the smallest eigenvalue $-\lambda_n$ of S. ☐

For the general case the law of inertia has the following form:

THEOREM 3.3. *Let $T : J \to \mathbb{C}^{n\times n}$ satisfy the conditions of the minmax characterization, and let $\sigma, \tau \in J$, $\sigma < \tau$.*

Let $(n_{p_\sigma}, n_{n_\sigma}, n_{z_\sigma})$ and $(n_{p_\tau}, n_{n_\tau}, n_{z_\tau})$ be the inertias of $T(\sigma)$ and $T(\tau)$, respectively. Then the inequality $n_{p_\sigma} \leq n_{p_\tau}$ holds, and the eigenvalue problem (2.1) has exactly $n_{p_\tau} - n_{p_\sigma}$ eigenvalues $\lambda_{n_{p_\sigma}+1} \leq \cdots \leq \lambda_{n_{p_\tau}}$ in (σ,τ).

Proof. Let W be the invariant subspace of $T(\sigma)$ corresponding to its positive eigenvalues. Then by the positive definiteness, $x^H T(\sigma)x > 0$, for every $x \in W$, $x \neq 0$, it follows from (A_2) that $x^H T(\tau)x > 0$ holds as well, hence, $n_{p_\sigma} \leq n_{p_\tau}$.

Let V be a subspace of $\mathbb{C}^n$ with $V \cap \mathcal{D} \neq \emptyset$ and $\dim V = n_{p_\sigma} + 1$. We first show that there exits a $x \in V \cap \mathcal{D}$ with $p(x) > \sigma$, from which we then obtain

$$\lambda_{n_{p_\sigma}+1} := \inf_{\dim V = n_{p_\sigma}+1,\ V\cap\mathcal{D}\neq\emptyset} \ \sup_{x\in V\cap\mathcal{D}} p(x) > \sigma.$$

From the hypothesis $\dim V > n_{p_\sigma}$, it follow that there exists a vector $x \in V$, $x \neq 0$ such that $x^H T(\sigma)x < 0$. If $x \in \mathcal{D}$, then it follows from (A_2) that we are done. Otherwise, we choose $y \in V \cap \mathcal{D}$ and $\omega > \min(p(y), \sigma)$. Then $x^H T(\omega)x < 0 < y^H T(\omega)y$, and with $w(t) := tx + (1-t)y$ it follows in the same way as in the proof of Theorem 3.2 that there exist a value $\hat{t} \in [0,1]$ such that $w(\hat{t}) \in V \cap \mathcal{D}$ and $p(w(\hat{t})) = \omega > \sigma$.

If U denotes the invariant subspace of $T(\tau)$ corresponding to its positive eigenvalues, then $x^H T(\tau)x > 0$ holds for every $x \in U$, $x \neq 0$.

If $U \cap \mathcal{D} = \emptyset$, then $x^H T(\lambda)x > 0$ holds for every $\lambda \in J$ and $x \in U$, $x \neq 0$ and in particular for $\lambda = \sigma$. Hence, $U \subset W$ and from $\sigma < \tau$ the equality $n_{p_\sigma} = n_{p_\tau}$ follows. In this case, $T(\cdot)$ has no eigenvalue in (σ,τ) because otherwise a corresponding eigenvector x would satisfy $x^H T(\sigma)x < 0 < x^H T(\tau)x$, i.e., $x \notin W$ and $x \in U$ contradicting $U \subset W$.

If $U \cap \mathcal{D} \neq \emptyset$, then $p(x) < \tau$ holds for every $x \in U \cap \mathcal{D}$ and therefore

$$\lambda_{n_{p_\tau}} = \inf_{\dim V = n_{p_\tau},\ V \cap \mathcal{D} \neq \emptyset} \ \sup_{x \in V \cap \mathcal{D}} p(x) \leq \sup_{x \in U \cap \mathcal{D}} p(x) < \tau.$$

Hence, $\lambda_{n_{p_\sigma}+1}$ and $\lambda_{n_{p_\tau}}$ are both contained in (σ, τ) and so are the eigenvalues λ_j for $j = n_{p_\sigma} + 1, \ldots, n_{p_\tau}$. □

REMARK 3.4. Without using the minmax characterization of eigenvalues, Neumaier [13] proved Theorem 3.3 for matrices $T : J \to \mathbb{C}^{n \times n}$ which are Hermitian and (elementwise) differentiable in J with positive definite derivative $T'(\lambda)$, $\lambda \in J$. Obviously, such $T(\cdot)$ satisfy the conditions of the minmax characterization.

EXAMPLE 3.5. Consider the rational eigenvalue problem

$$T(\lambda) := -K + \lambda M + \sum_{j=1}^{p} \frac{\lambda}{\sigma_j - \lambda} C_j C_j^T,$$

where $K, M \in \mathbb{R}^{n \times n}$ are symmetric and positive definite, the matrix $C_j \in \mathbb{R}^{n \times k_j}$ has rank k_j, and $0 < \sigma_1 < \cdots < \sigma_p$. This problem models the free vibrations of certain fluid–solid structures; cf. [1].

In each interval $J_\ell := (\sigma_\ell, \sigma_{\ell+1})$, $\ell = 0, \ldots, p$, $\sigma_0 = 0$, $\sigma_{p+1} = \infty$, the function $f_\ell(\lambda, x) := x^H T(\lambda) x$ is strictly monotonically increasing, and therefore all eigenvalues in J_ℓ are minmax values of the Rayleigh functional p_ℓ.

For the first interval J_0, Theorem 3.2 applies. Hence, if $\tau \in J_0$ and (n_p, n_n, n_z) is the inertia of $T(\tau)$, then there are exactly n_p eigenvalues in J_0 which are less than τ. Moreover, if $\tau_1 < \tau_2$ are contained in one interval J_j, then the number of eigenvalues in the interval (τ_1, τ_2) can be obtained from the inertias of $T(\tau_1)$ and $T(\tau_2)$ according to Theorem 3.3.

4. Quadratic eigenvalue problems.

We consider quadratic matrix pencils

$$Q(\lambda) := \lambda^2 A + \lambda B + C, \tag{4.1}$$

with Hermitian matrices $A, B, C \in \mathbb{C}^{n \times n}$ under several conditions that guarantee that (some of) the real eigenvalues allow for a variational characterization and hence for slicing of the spectrum using the inertia.

4.1. $\mathbf{C < 0}$ and $\mathbf{A \geq 0}$.

Let C be negative definite and A positive semidefinite. Multiplying $Q(\lambda)x = 0$ by λ^{-1}, one gets the equivalent nonlinear eigenvalue problem

$$\tilde{Q}(\lambda)x := \lambda A x + B x + \lambda^{-1} C x = 0.$$

Differentiating $f(\lambda; x) := x^H \tilde{Q}(\lambda) x$ with respect to λ yields

$$\frac{\partial}{\partial \lambda} f(\lambda; x) = x^H A x - \lambda^{-2} x^H C x > 0 \quad \text{for every } x \neq 0 \text{ and every } \lambda \neq 0.$$

Hence, $\tilde{Q}$ satisfies the conditions of the minmax characterization for both intervals $J_- := (-\infty, 0)$ and $J_+ := (0, \infty)$.

For the corresponding Rayleigh functional $p_\pm$ with domain $\mathcal{D}_\pm$, it holds that $\lambda_1^+ = \inf_{x \in \mathcal{D}_+} p_+(x) \in J_+$ and $\lambda_n^- = \sup_{x \in \mathcal{D}_-} p_-(x) \in J_-$, and therefore the following statement follows from Theorem 3.2.

THEOREM 4.1. *Let C be negative definite and A positive semidefinite.*

(i) For $\sigma > 0$ let $\mathrm{In}(\tilde{Q}(\sigma)) = (n_p, n_n, n_z)$ be the inertia of $\tilde{Q}(\sigma)$. Then the quadratic pencil (4.1) has n_p positive eigenvalues less than σ.

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(ii) For $\sigma < 0$ let $\mathrm{In}(\tilde{Q}(\sigma)) = (n_p, n_n, n_z)$ *be the inertia of* $\tilde{Q}(\sigma)$*. Then* (4.1) *has* n_n *negative eigenvalues exceeding* σ*.*

If A is positive definite, then $\tilde{Q}$ is overdamped with respect to J_+ and J_-, and therefore there exist exactly n positive and n negative eigenvalues. If $A \neq 0$ is positive semidefinite and $r = \mathrm{rank}(A)$, then ∞ is an infinite eigenvalue of multiplicity $n-r$, and there are only $n+r$ finite eigenvalues.

If B is positive definite, then the Rayleigh functional

$$p_+(x) = -2\frac{x^H Cx}{x^H Bx + \sqrt{(x^H Bx)^2 - 4(x^H Ax)(x^H Cx)}}$$

is defined on $\mathbb{C}^n \setminus \{0\}$. Hence, $(\tilde{Q}, J_+)$ is overdamped, and there exist n positive and r negative eigenvalues. Theorem 4.1 can be strengthened according to the following result.

THEOREM 4.2. *Assume that A is positive semidefinite, B is positive definite, and C is negative definite.*

(i) For $\sigma > 0$ let $\mathrm{In}(\tilde{Q}(\sigma)) = (n_p, n_n, n_z)$ *be the inertia of* $\tilde{Q}(\sigma)$*. Then the quadratic pencil* (4.1) *has n_p positive eigenvalues less than σ, n_n eigenvalues exceeding σ, and if $n_z \neq 0$, then σ is an eigenvalue of $Q(\cdot)$ with multiplicity n_z.*

(ii) For $\sigma < 0$ let $\mathrm{In}(\tilde{Q}(\sigma)) = (n_p, n_n, n_z)$ *be the inertia of* $\tilde{Q}(\sigma)$*. Then* (4.1) *has n_n negative eigenvalues exceeding σ, n_p-r finite eigenvalues less than σ, and if $n_z \neq 0$, then σ is an eigenvalue of $Q(\cdot)$ with multiplicity n_z.*

4.2. Hyperbolic problems. The quadratic pencil $Q(\cdot)$ defined by the Hermitian matrices $A, B, C \in \mathbb{C}^{n\times n}$ is called hyperbolic if A is positive definite and for every $x \in \mathbb{C}^n$, $x \neq 0$, the quadratic polynomial

$$f(\lambda; x) := \lambda^2 x^H Ax + \lambda x^H Bx + x^H Cx = 0$$

has two distinct real roots

$$p_\pm(x) = -\frac{x^H Bx}{2x^H Ax} \pm \sqrt{\left(\frac{x^H Bx}{2x^H Ax}\right)^2 - \frac{x^H Cx}{x^H Ax}}. \tag{4.2}$$

A hyperbolic quadratic matrix polynomial $Q(\cdot)$ has the following properties (cf. [12]): the ranges $\tilde{J}_\pm := p_\pm(\mathbb{C}^n \setminus \{0\})$ are disjoint real closed intervals with $\max \tilde{J}_- < \min \tilde{J}_+$, moreover $Q(\lambda)$ is positive definite for $\lambda < \min \tilde{J}_-$ and $\lambda > \max \tilde{J}_+$, and it is negative definite for $\lambda \in (\max \tilde{J}_-, \min \tilde{J}_+)$.

Let J_+ be an open interval with $\tilde{J}_+ \subset J_+$ and $\tilde{J}_- \cap J_+ = \emptyset$, and let J_- be an open interval with $\tilde{J}_- \subset J_-$ and $\tilde{J}_+ \cap J_- = \emptyset$. Then (Q, J_+) and $(-Q, J_-)$ satisfy the conditions of the variational characterization of eigenvalues and they are both overdamped. Hence, there exist $2n$ eigenvalues

$$\lambda_1 \leq \cdots \leq \lambda_n < \lambda_{n+1} \leq \cdots \leq \lambda_{2n}$$

and

$$\lambda_j = \min_{\dim V = j} \max_{x\in V, x\neq 0} p_-(x) \quad \text{and} \quad \lambda_{n+j} = \min_{\dim V = j} \max_{x\in V, x\neq 0} p_+(x), \quad j = 1, \ldots, n.$$

If $\mathrm{In}(Q(\sigma)) = (n_p, n_n, n_z)$ is the inertia of $Q(\sigma)$ and $n_n = n$, then $Q(\sigma)$ is negative definite and there are n eigenvalues smaller than σ and n eigenvalues exceeding σ. If $n_p = n$ holds, then $Q(\sigma)$ is positive definite, $f(\sigma; x) > 0$ is valid for every $x \neq 0$, and if $\frac{\partial}{\partial\lambda} f(\sigma; x) < 0$ holds, then it follows that $\sigma < \lambda_1$, and $\sigma > \lambda_{2n}$ otherwise.

If $n_p \neq n$ and $n_n \neq n$, then $\sigma \in J_- \cup J_+$ and Theorem 3.1 applies. We only have to find out which of these intervals σ is located in. To this end we determine $x \neq 0$ such that $f(\sigma; x) := x^H Q(\sigma) x > 0$ (this can be done by a few steps of the Lanczos method which is known to converge first to extreme eigenvalues). If $\frac{\partial}{\partial\lambda} f(\sigma; x) = 2\sigma x^H A x + x^H B x < 0$, then it follows that $p_-(x) > \sigma$, and therefore $\sigma < \lambda_n = \max_{x\neq 0} p_-(x)$. Similarly, the inequalites $f(\sigma; x) > 0$ and $2\sigma x^H A x + x^H B x > 0$ imply $\sigma > \lambda_{n+1} = \min_{x\neq 0} p_+(x)$. Hence we obtain the following slicing of the spectrum of $Q(\cdot)$.

THEOREM 4.3. *Let*

$$Q(\lambda) := \lambda^2 A + \lambda B + C$$

be hyperbolic, and let (n_p, n_n, n_z) be the inertia of $Q(\sigma)$ for $\sigma \in \mathbb{R}$.

(i) If $n_n = n$ then there are n eigenvalues smaller than σ and n eigenvalues greater than σ.

(ii) Let $n_p = n$.
If $2\sigma x^H A x + x^H B x < 0$ for an arbitrary $x \neq 0$, then there are $2n$ eigenvalues exceeding σ.
If $2\sigma x^H A x + x^H B x > 0$ for an arbitrary $x \neq 0$, then all $2n$ eigenvalues are less than σ.

(iii) For $n_p = 0$ and $n_z > 0$ let $x \neq 0$ be an element of the null space of $Q(\sigma)$.
If $2\sigma x^H A x + x^H B x < 0$, then $Q(\lambda)x = 0$ has $n - n_z$ eigenvalues in $(-\infty, \sigma)$, n eigenvalues in (σ, ∞) and $\sigma = \lambda_n$ with multiplicity n_z.
If $2\sigma x^H A x + x^H B x > 0$, then $Q(\cdot)$ has n eigenvalues in $(-\infty, \sigma)$, $n - n_z$ eigenvalues in (σ, ∞), and $\sigma = \lambda_{n+1}$ with multiplicity n_z.

(iv) For $n_p > 0$ and $n_z = 0$ let $x \neq 0$ be such that $f(\sigma; x) > 0$.
If $2\sigma x^H A x + x^H B x < 0$, then $Q(\cdot)$ has $n - n_p$ eigenvalues in $(-\infty, \sigma)$ and $n + n_p$ eigenvalues in (σ, ∞).
If $2\sigma x^H A x + x^H B x > 0$, then $Q(\cdot)$ has $n + n_p$ eigenvalues in $(-\infty, \sigma)$ and $n - n_p$ eigenvalues in (σ, ∞).

(v) For $n_p > 0$ and $n_z > 0$ let $x \neq 0$ be such that $f(\sigma; x) > 0$.
If $2\sigma x^H A x + x^H B x < 0$, then $Q(\cdot)$ has $n - n_p - n_z$ eigenvalues in $(-\infty, \sigma)$ and $n + n_p$ eigenvalues in (σ, ∞).
If $2\sigma x^H A x + x^H B x > 0$, then $Q(\cdot)$ has $n + n_p$ eigenvalues in $(-\infty, \sigma)$ and $n - n_p - n_z$ eigenvalues in (σ, ∞).
In either case σ is an eigenvalue with multiplicity n_z.

REMARK 4.4. These results on quadratic hyperbolic pencils can be generalized to a hyperbolic matrix polynomial of higher degree

$$P(\lambda) = \sum_{j=0}^{k} \lambda^j A_j, \quad A_j = A_j^H,\ j = 0, \dots, k, \quad A_k > 0,$$

which is hyperbolic if A_k is positive definite and for every $x \neq 0$ the corresponding polynomial $f(\lambda; x) := x^H P(\lambda) x$ has k real and distinct roots.

In this case there exist k disjoint open intervals $J_j \subset \mathbb{R}$, $j = 1, \dots, k$ such that $P(\cdot)$ has exactly n eigenvalues in each J_j, and these eigenvalues allow for a minmax characterization; cf. [12, 14]. To fix the numeration let $\sup J_{j+1} < \inf J_j$ for $j = 1, \dots, k-1$.

For $\sigma \in \mathbb{R}$, let (n_p, n_n, n_z) be the inertia of $P(\sigma)$, and let $x \in \mathbb{C}^n$ be a vector such that $x^H P(\sigma) x > 0$. If $f(\cdot; x)$ has exactly j roots which exceed σ, then it holds that

$$\sigma \in J_{j+1} \quad \text{or} \quad \sigma \in [\sup J_{j+1}, \inf J_j] \quad \text{or} \quad \sigma \in J_j.$$

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Which one of these situations occur can be deduced from the inertia ($n_n = n$ or $n_p = n$) and the derivative $\frac{\partial}{\partial \lambda} f(\sigma; x)$ as for the quadratic case.

4.3. Definite quadratic pencils. In a recent paper, Higham, Mackey, and Tisseur [10] generalized the concept of hyperbolic quadratic polynomials waiving the positive definiteness of the leading matrix A.

A quadratic pencil (4.1) is definite if A, B, and C are Hermitian, there exists a real value $\mu \in \mathbb{R} \cup \{\infty\}$ such that $Q(\mu)$ is positive definite, and for every $x \in \mathbb{C}^n$, $x \neq 0$ the scalar quadratic polynomial

$$f(\lambda; x) := \lambda^2 x^H A x + \lambda x^H B x + x^H C x = 0$$

has two distinct roots in $\mathbb{R} \cup \{\infty\}$.

The following theorem was proved in [10].

THEOREM 4.5. *The Hermitian matrix polynomial $Q(\lambda)$ is definite if and only if any two (and hence all) of the following properties hold:*

- $d(x) := (x^H B x)^2 - 4(x^H A x)(x^H C x) > 0$ *for every* $x \in \mathbb{C}^n \setminus \{0\}$,
- $Q(\eta) > 0$ *for some* $\eta \in \mathbb{R} \cup \{\infty\}$,
- $Q(\xi) < 0$ *for some* $\xi \in \mathbb{R} \cup \{\infty\}$.

For $\xi < \eta$ (otherwise consider $Q(-\lambda)$) it was shown in [14] that there are n eigenvalues in (ξ, η) which are minmax values of a Rayleigh functional, and the remaining n eigenvalues in $[-\infty, \xi)$ and $(\eta, \infty]$ are maxmin and minmax values of a second Rayleigh functional. Hence, if ξ and η are known, then the slicing of the spectrum using the LDL^H factorization follows similarly to the hyperbolic case. However, given σ and the LDL^H factorization of $Q(\sigma)$, we are not aware of an easy way to decide in which of the intervals $[-\infty, \xi)$, (ξ, η), or $(\eta, \infty]$ the parameter σ is located. The articles [7, 8, 14] contain methods to detect whether a quadratic pencil is definite and to compute the parameters ξ and η, however they are much more costly than computing an LDL^H factorization of a matrix. For the Examples 4.6 and 4.8 at least one of these parameters are known and the slicing can be given explicitly.

EXAMPLE 4.6. Duffin [4] called a quadratic eigenproblem (4.1) an overdamped network, if A, B, and C are positive semidefinite and the so called overdamping condition

$$d(x) = (x^H B x)^2 - 4(x^H A x)(x^H C x) > 0 \quad \text{for every } x \neq 0$$

is satisfied.

So, actually B has to be positive definite, and therefore $Q(\mu)$ is positive definite for every $\mu > 0$, and $Q(\cdot)$ is definite.

If r denotes the rank of A, then (4.1) has $n + r$ finite real eigenvalues, the largest n ones of which (called primary eigenvalues by Duffin) are minmax values of

$$p_+(x) := -2 \frac{x^H C x}{x^H B x + \sqrt{d(x)}},$$

and the smallest r ones (called secondary eigenvalues) are maxmin values

$$\lambda_{n+1-j} = \max_{\dim V = j,\ V \cap \mathcal{D}_- \neq \emptyset} \ \min_{x \in V \cap \mathcal{D}_-} \ p_-(x),$$

where $\mathcal{D}_- := \{x \in \mathbb{C}^n \ : \ x^H A x \neq 0\}$ and $p_-(x) = \left(-x^H B x - \sqrt{d(x)}\right) / (2 x^H A x)$ for $x \in \mathcal{D}_-$.

Hence, the following slicing of the spectrum can be derived.

THEOREM 4.7. *Let $A, B, C \in \mathbb{C}^{n \times n}$ be positive semidefinite and assume that $d(x) > 0$ for $x \neq 0$. Let r be the rank of A and $\operatorname{In}(Q(\sigma)) = (n_p, n_n, n_z)$ be the inertia of $Q(\sigma)$ for some $\sigma \in \mathbb{R}$. Then the following holds.*

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(i) *If $n_n = n$, then there are r eigenvalues smaller than σ and n eigenvalues greater than σ.*

(ii) *For $n_p = 0$ and $n_z > 0$ let $x \neq 0$ be an element of the null space of $Q(\sigma)$.*
If $2\sigma x^H Ax + x^H Bx < 0$, then $Q(\cdot)$ has $r - n_z$ eigenvalues in $(-\infty, \sigma)$ and n eigenvalues in $(\sigma, 0]$.
If $2\sigma x^H Ax + x^H Bx > 0$, then $Q(\cdot)$ has r eigenvalues in $(-\infty, \sigma)$, and $n - n_z$ eigenvalues in $(\sigma, 0]$. In either case, σ is an eigenvalue of $Q(\cdot)$ with multiplicity n_z.

(iii) *For $n_p > 0$ let $x \neq 0$ be such that $f(\sigma; x) > 0$.*
If $2\sigma x^H Ax + x^H Bx < 0$, then $Q(\lambda)x = 0$ has $r - n_p$ eigenvalues in $(-\infty, \sigma)$ and $n + n_p - n_z$ eigenvalues in $(\sigma, 0]$.
If $2\sigma x^H Ax + x^H Bx > 0$, then $Q(\lambda)x = 0$ has $r + n_p - n_z$ eigenvalues in $(-\infty, \sigma)$ and $n - n_p$ eigenvalues in $(\sigma, 0]$.

EXAMPLE 4.8. Free vibrations of fluid–solid structures are governed by the nonsymmetric eigenvalue problem [9, 18]

$$\begin{bmatrix} K_s & C \\ 0 & K_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} = \lambda \begin{bmatrix} M_s & 0 \\ -C^T & M_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} \tag{4.3}$$

where $K_s \in \mathbb{R}^{s\times s}$, $K_f \in \mathbb{R}^{f\times f}$ are the stiffness matrices, and $M_s \in \mathbb{R}^{s\times s}$, $M_f \in \mathbb{R}^{f\times f}$ are the mass matrices of the structure and the fluid, respectively, and $C \in \mathbb{R}^{s\times f}$ describes the coupling of structure and fluid. The vector x_s is the structure displacement vector, and x_f is the fluid pressure vector. K_s, M_s, K_f, and M_f are symmetric and positive definite.

Multiplying the first line of (4.3) by λ, one obtains the quadratic pencil

$$Q(\lambda) := \lambda^2 \begin{bmatrix} M_s & 0 \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} -K_s & -C \\ -C^T & M_f \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -K_f \end{bmatrix}.$$

It can easily be seen that for $x_s \neq 0$ the quadratic equation $[x_s^T, x_f^T] Q(\lambda) \begin{bmatrix} x_s \\ x_f \end{bmatrix} = 0$ has one positive solution $p_+(x_s, x_f)$ and one negative solution $p_-(x_s, x_f)$, and for $x_s = 0$ it has one positive solution $p_+(x_s, x_f) := x_f^T K_f x_f / (x_f^T M_f x_f)$ and the solution $p_-(x_s, x_f) := \infty$. Moreover the positive eigenvalues of (4.3) are minmax values of the Rayleigh functional p_+. Hence, one gets the following result for the (physically meaningful) positive eigenvalues: if $\mathrm{In}(Q(\sigma)) = (n_p, n_n, n_z)$ for $\sigma > 0$, then there are exactly n_p eigenvalues in $(0, \sigma)$, n_n eigenvalues in (σ, ∞), and if $n_z \neq 0$, then σ is an eigenvalue of multiplicity n_z.

4.4. Nonoverdamped quadratic pencils. We consider the quadratic pencil (4.1) where the matrices A, B, and C are positive definite. Then for $x \neq 0$, the two complex roots of $f(\lambda; x) := x^H Q(\lambda) x$ are given in (4.2).

Let us define

$$\begin{aligned} \delta_- &:= \sup\{p_-(x) \ :\ p_-(x) \in \mathbb{R}\}, \qquad & \delta_+ &:= \inf\{p_+(x) \ :\ p_+(x) \in \mathbb{R}\}, \\ J_- &:= (-\infty, \delta_+), & J_+ &= (\delta_-, 0), \qquad \text{and} \\ D_\pm &:= \{x \in \mathbb{C}^n \ :\ p_\pm(x) \in J_\pm\}. \end{aligned}$$

If $f(\lambda, x) > 0$ for $x \neq 0$ and $\lambda \in \mathbb{R}$, then it follows that $\delta_- = -\infty$ and $\delta_+ = \infty$. The eigenvalue problem $Q(\lambda)x = 0$ has no real eigenvalues, but this does not have to be known in advance. Theorem 4.9 applies to this case as well.

It is obvious that $-Q$ and Q satisfy the conditions of the minmax characterization of its eigenvalues in J_- and J_+, respectively. Hence, all eigenvalues in J_- are minmax-values of p_-:

$$\lambda_j^- = \min_{\dim V = j,\ V\cap\mathcal{D}_- \neq \emptyset}\ \max_{x\in V\cap\mathcal{D}_-} p_-(x),\ j = 1, 2, \ldots.$$

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Taking advantage of the minmax characterization of the eigenvalues of $\tilde{Q}(\lambda) := -Q(-\lambda)$ in $\tilde{J} := J_+$ with the Rayleigh functional $\tilde{p} := -p_+$, we obtain the following maxmin characterization

$$\lambda^+_{2n+1-j} = \max_{\dim V=j,\ V\cap\mathcal{D}_+\neq\emptyset}\ \min_{x\in V\cap\mathcal{D}_+} p_+(x),\ j=1,2,\ldots.$$

of all eigenvalues of Q in J_+.

Hence, for $\sigma < \delta_+$ and for $\sigma > \delta_-$, we obtain slicing results for the spectrum of $Q(\cdot)$ from Theorem 3.2. If $\mathrm{In}(Q(\sigma)) = (n_p, n_n, n_z)$ and $\sigma < \delta_+$, then there exist n_n eigenvalues of $Q(\cdot)$ in $(-\infty, \sigma)$, and if $\sigma \in (\delta_-, 0)$, then there are n_n eigenvalues in $(\sigma, 0)$. However, δ_+ and δ_- are usually not known. The following theorem contains upper bounds of δ_- and lower bounds of δ_+, thus yielding subintervals of $(-\infty, \delta_+)$ and $(\delta_-, 0)$ where the above slicing applies.

THEOREM 4.9. *Let $A, B, C \in \mathbb{C}^{n\times n}$ be positive definite, and let p_+ and p_- be defined in* (4.2)*. Then it holds that*

(i)

$$\tilde{\delta}_+ := -\sqrt{\max_{x\neq 0}\frac{x^HCx}{x^HAx}} \le \delta_+ = \inf\{p_+(x)\ :\ p_+(x)\in\mathbb{R}\}$$

and

$$\delta_- = \sup\{p_-(x)\ :\ p_-(x)\in\mathbb{R}\} \le -\sqrt{\min_{x\neq 0}\frac{x^HCx}{x^HAx}} =: \tilde{\delta}_-.$$

(ii)

$$\hat{\delta}_+ := -2\max_{x\neq 0}\frac{x^HCx}{x^HBx} \le \delta_+ \quad \textit{and} \quad \delta_- \le -2\min_{x\neq 0}\frac{x^HCx}{x^HBx} =: \hat{\delta}_-.$$

Proof: (i): The value $\tilde{\delta}_+$ is a lower bound of δ_+ if for every $x \neq 0$ such that $p_+(x)\in\mathbb{R}$ it holds that $p_+(x) \ge \tilde{\delta}_+$. The following proof takes advantage of the facts that $p_+(x) < 0$ and $\frac{\partial}{\partial\lambda} f(p_+(x);x) \ge 0$.

The equation $f(p_+(x);x) = x^HQ(p_+(x))x = 0$ is satisfied if and only if

$$x^HBx = -p_+(x)x^HAx - \frac{1}{p_+(x)}x^HCx.$$

Hence,

$$\frac{\partial}{\partial\lambda} f(p_+(x);x) = 2p_+(x)x^HAx + x^HBx = p_+(x)x^HAx - \frac{1}{p_+(x)}x^HCx \ge 0$$

if and only if

$$p_+(x)^2 \le \frac{x^HCx}{x^HAx}, \quad \text{i.e.,} \quad \delta_+ \ge -\sqrt{\max_{x\neq 0}\frac{x^HCx}{x^HAx}} = \tilde{\delta}_+,$$

and analogously we obtain

$$\delta_- \le -\sqrt{\min_{x\neq 0}\frac{x^HCx}{x^HAx}} = \tilde{\delta}_-.$$

(ii): Solving $f(p_+(x);x)=0$ for x^HAx, one gets from $\frac{\partial}{\partial\lambda}f(p_+(x);x)\geq 0$ that

$$p_+(x)\geq -2\frac{x^HCx}{x^HBx},\ \text{i.e., } \delta_+\geq -2\max_{x\neq 0}\frac{x^HCx}{x^HBx}=\hat{\delta}_+,$$

and analogously

$$\delta_-\leq -2\min_{x\neq 0}\frac{x^HCx}{x^HBx}=\hat{\delta}_-. \qquad \square$$

From Theorem 3.2 we obtain the following slicing of the spectrum of $Q(\cdot)$.

THEOREM 4.10. *Let A, B, and C be positive definite matrices, and for some $\sigma\in\mathbb{R}$ let* $\mathrm{In}(Q(\sigma))=(n_p,n_n,n_z)$.

(i) If

$$\sigma\leq\max\left\{-\sqrt{\max_{x\neq 0}\frac{x^HCx}{x^HAx}},-2\max_{x\neq 0}\frac{x^HCx}{x^HBx}\right\},$$

then there exist n_n eigenvalues of $Q(\lambda)x=0$ in $(-\infty,\sigma)$.

(ii) If

$$\sigma\geq\min\left\{-\sqrt{\min_{x\neq 0}\frac{x^HCx}{x^HAx}},-2\min_{x\neq 0}\frac{x^HCx}{x^HBx}\right\},$$

then there exist n_n eigenvalues of $Q(\lambda)x=0$ in $(\sigma,0)$.

EXAMPLE 4.11. In a numerical experiment, the matrices A, B, and C were generated by the following MATLAB statements:

```
randn('state',0);
A=eye(20); B=randn(20);B=B'*B; C=randn(20);C=C'*C;.
```

It was found that $Q(\lambda)x=0$ has 26 real eigenvalues, 13 in the domain of p_- and 13 in the domain of p_+. So, Sylvester's theorem can be applied to all of them. 12 eigenvalues are less than $-\sqrt{\max(\lambda(C,A))}$ and 6 eigenvalues exceed $-\sqrt{\min(\lambda(C,A))}$.

Conclusions. We have considered a given family of Hermitian matrices $T(\lambda)$ depending continuously on a parameter λ in an open real interval J which allows for a variational characterization of its eigenvalues. We proved slicing results for the spectrum of $T(\cdot)$, where at first general nonlinear eigenvalue problems are considered, which are then specialized to various types of quadratic eigenproblems.

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