ANALYSIS OF THE FINITE ELEMENT METHOD FOR TRANSMISSION/MIXED BOUNDARY VALUE PROBLEMS ON GENERAL POLYGONAL DOMAINS*

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Abstract. We study theoretical and practical issues arising in the implementation of the Finite Element Method for a strongly elliptic second order equation with jump discontinuities in its coefficients on a polygonal domain Ω that may have cracks or vertices that touch the boundary. We consider in particular the equation $-\operatorname{div}(A\nabla u) = f \in H^{m-1}(\Omega)$ with mixed boundary conditions, where the matrix A has variable, piecewise smooth coefficients. We establish regularity and Fredholm results and, under some additional conditions, we also establish well-posedness in weighted Sobolev spaces. When Neumann boundary conditions are imposed on adjacent sides of the polygonal domain, we obtain the decomposition $u = u_{\text{reg}} + \sigma$, into a function u_{reg} with better decay at the vertices and a function σ that is locally constant near the vertices, thus proving well-posedness in an augmented space. The theoretical analysis yields interpolation estimates that are then used to construct improved graded meshes recovering the (quasi-)optimal rate of convergence for piecewise polynomials of degree $m \geq 1$. Several numerical tests are included.

Key words. Neumann-Neumann vertex, transmission problem, augmented weighted Sobolev space, finite element method, graded mesh, optimal rate of convergence

AMS subject classifications. 65N30, 35J25, 46E35, 65N12

1. Introduction. In this paper we study second-order, strongly elliptic operators in divergence form $P = -\operatorname{div} A\nabla$ on generalized polygonal domains in the plane, where the coefficients are piecewise smooth with possibly jump discontinuities across a finite number of curves, collectively called the *interface*.

Let Ω be a bounded polygonal domain that may have curved boundaries, cracks, or vertices touching the boundary. We refer to such domains as *domains with polygonal structure* (see Figure 2.1 for a typical example). We assume that $\bar{\Omega} = \cup\bar{\Omega}_j$, where Ω_j are disjoint domains with a polygonal structure such that the interface $\Gamma := \cup\partial\Omega_j \smallsetminus \partial\Omega$ is a union of disjoint, piecewise smooth curves Γ_k. The curves Γ_k are allowed to intersect transversely. We are interested in the *non-homogeneous transmission/mixed boundary value problem*

$$(1.1) \quad \begin{aligned} Pu = f \text{ in } \Omega, \quad D_\nu^P u = g_N \text{ on } \partial_N\Omega, \quad u = g_D \text{ on } \partial_D\Omega, \\ u_+ = u_- \quad \text{and} \quad D_\nu^{P+}u = D_\nu^{P-}u \quad \text{on } \Gamma, \end{aligned}$$

and the convergence properties of its Finite Element discretizations. Here, $A = (A_{ij})$ is the symmetric matrix of coefficients of P, $D_\nu^P := \sum_{ij} \nu^i A_{ij}\partial_j$ is the conormal derivative associated to P, and the boundary $\partial\Omega$ is partitioned into two disjoints sets $\partial_D\Omega$, $\partial_N\Omega$ with $\partial_D\Omega$ a union of closed sides of $\partial\Omega$.

Transmission problems of the form in Equation (1.1) (also called "interface problems" or "inclusion problems" in the engineering literature) appear in many practical applications, in particular they are likely to appear any time that more than one type of material (or medium) is used. Therefore, they have been studied in a very large number of papers devoted to applications. Among those, let us mention the paper by Peskin [68], LeVeque and Li [51], Li and Lubkin [55], Yu, Zhou, and Wei [80]. See also the references therein. By contrast,

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relatively fewer papers were devoted to these problems from the point of view of qualitative properties of Partial Differential Equations. Let us nevertheless mention here the papers of Kellogg [44], Kellogg and Aziz [6], Mitrea, Mitrea, and Shi [61], Li and Nirenberg [53], Li and Vogelius [54], Roitberg and Sheftel [71, 72], and Schechter [75]. Our paper starts with some theoretical results for transmission problems and then provides applications to numerical methods. See also the papers of Kellogg [43] and Nicaise and Sändig [67], and the books of Nicaise [66] and Harutyunyan and Schulze [40].

The equation $Pu = f$ in Ω has to be interpreted in a weak sense and then the discontinuity of the coefficients A^{ij} leads to "transmission conditions" at the interface Γ. Since Γ is a union of piecewise smooth curves, we can locally choose a labeling of the non-tangential limits u_+ and u_- of u at the smooth points of the interface Γ. We can label similarly D_ν^{P+} and D_ν^{P-} the two conormal derivatives associated to P at the two sides of the interface. Then the usual transmission conditions $u^+ = u^-$ and $D_\nu^{P+}u = D_\nu^{P-}u$ at the two sides of the smooth points of the interface are a consequence of the weak formulation, and will always be considered as part of Equation (1.1). This equation does not change if we switch "+" to "−," so our choice of labeling is not essential. At the *non-smooth* points of Γ, we assign no meaning to the interface condition $D_\nu^{P+}u = D_\nu^{P-}u$. The more general conditions $u^+ - u^- = h_0$ and $D_\nu^{P+}u - D_\nu^{P-}u = h_1$ can be treated with only minor modifications. We also allow the cracks to ramify as part of $\partial\Omega$.

It is well-known that when $\partial\Omega$ is not smooth there is a loss of regularity in elliptic boundary-value problems. Because of this loss of regularity, a quasi-uniform sequence of triangulations on Ω does *not* give optimal rates of convergence for the Galerkin approximations u_h of the solution of (1.1) [78]. One needs to consider *graded* meshes instead (see for example [7, 12, 70]). We approach the problem (1.1) using higher regularity in weighted Sobolev spaces. For transmission problems, these results are new (see Theorems 3.1–3.3).

We therefore begin by establishing regularity results for (1.1) in the weighted Sobolev spaces $\mathcal{K}_{\vec{a}}^m(\Omega)$, where the weight may depend on each vertex of Ω (see Definition (2.7)). We identify the weights that make P Fredholm following the results of Kondratiev [46] and Nicaise [66]. If no two adjacent sides are assigned Neumann boundary conditions (i. e., when there are no Neumann–Neumann vertices), we also obtain a well-posedness result for the weight parameter $\vec{a}$ close to 1. In the general case, we first compute the Fredholm index of P, and then we use this computation to obtain a decomposition $u = u_{\text{reg}} + \sigma$ of the solution of u of (1.1) into a function with good decay at the vertices and a function that is locally constant near the vertices. This decomposition leads to a new well-posedness result if there are Neumann–Neumann vertices.

Our main focus is the analysis of the Finite Element Method for Equation (1.1). We are especially interested in obtaining a sequence of meshes that provides quasi-optimal rates of convergence. For this reason, in this paper we restrict to domains in the plane. However, Theorems 3.1, 3.2, and 3.3 extend to 3D (see [58] for proofs in the absence of interfaces and [16] for a proof of the regularity in the presence of interfaces in n-dimensions). We assume that Ω has straight faces and consider a sequence $\mathcal{T}_n$ of triangulations of Ω. We let

$$S_n \subset H_D^1(\Omega) := H^1(\Omega) \cap \{u = 0 \text{ on } \partial_D\Omega\}$$

be the finite element space of continuous functions on Ω that restrict to a polynomial of degree $m \geq 1$ on each triangle of $\mathcal{T}_n$, and let $u_n \in S_n$ be the Finite Element approximation of u, defined by equation (5.1). We then say that S_n provides *quasi-optimal rates of convergence* for $f \in H^{m-1}(\Omega)$ if there exists $C > 0$ such that

$$(1.2) \qquad \|u - u_n\|_{H^1} \leq C \dim(S_n)^{-m/2}\|f\|_{H^{m-1}},$$

for all $f \in H^{m-1}(\Omega)$. We do *not* assume $u \in H^{m+1}(\Omega)$. (In three dimensions, the power $m/2$ has to be replaced with $m/3$.) Hence the sequence S_n provides a quasi-optimal rate of convergence if it recovers the asymptotic order of convergence that is expected if $u \in H^{m+1}(\Omega)$ and if quasi-uniform meshes are used. See the papers of Brenner, Cui, and Sung [26], Brannick, Li, and Zikatanov [24], and Guzmán [38] for other applications of graded meshes. Corner singularities and discontinuous coefficients have been studied also using "least squares methods" [21, 22, 29, 50, 49]. Here we concentrate on improving the convergence rate of the usual Galerkin Finite Element Method, to approximate singular solutions in the transmission problem (1.1). The *new* a priori estimates in augmented weighted Sobolev spaces developed in Section 4 play a crucial role in our analysis of the numerical method.

The problem of constructing sequences of meshes that provide quasi-optimal rates of convergence has received much attention in the literature – we mention in particular the work of Apel [2], Babuška and collaborators [7, 11, 12, 13, 37], Bacuta, Nistor, and Zikatanov [17], Bacuta, Bramble, and Xu [14], Costabel and Dauge [33], Dauge [34], Grisvard [36], Lubuma and Nicaise [56], Schatz, Sloan, and Wahlbin [74]. Let us mention the related approach of adaptive mesh refinements, which also leads to quasi-optimal rates of convergence in two dimensions [23, 59, 63]. Similar results are needed for the study of stress-intensity factors [25, 28]. However, the case of hyperbolic equations is more difficult [60]. Cracks are important in Engineering applications, see [35] and the references therein. Transmission problems are important in optics and acoustics [30]

We exploit the theoretical analysis of the operator P to obtain an *a priori* bound and interpolation inequalities. These in turn allow us to verify that the sequence of graded meshes we explicitly construct yields quasi-optimal rates of convergence. For transmission problems, we recover quasi-optimal rates of convergence if the data is in $H^{m-1}(\Omega_j)$ for each j. To account for the pathologies in Ω, we work in weighted Sobolev spaces with weights that depend on a particular vertex a more general setting than the one considered in [18]. The use of inhomogeneous norms allows us to theoretically justify the use of different grading parameters at different vertices when constructing graded meshes. A priori estimates are a well-established tool in Numerical Analysis; see e.g., [4, 5, 8, 10, 20, 27, 31, 39, 45, 62, 77].

At the same time, we address several issues that are of interest in concrete applications, but have received little attention. For instance, we consider cracks and higher regularity for transmission problems. Regularity and numerical issues for transmission problems were studied before by several authors; see for example Nicaise [66] and Nicaise and Sändig [67] and references therein. As in these papers, we use weighted Sobolev spaces, but our emphasis is not on singular functions, but rather on well-posedness results. This approach leads to a unified way to treat mixed boundary conditions and interface transmission conditions. In particular, there is no additional computational complexity in treating Neumann–Neumann vertices. Thus, although the theoretical results we establish are different in the case of Neumann–Neumann corners than in the case of Dirichlet–Neumann or simply Dirichlet boundary conditions, the numerical method that results is *the same* in all these cases, which should be an advantage in implementation.

The paper is organized as follows. In Section 2, we introduce the notion of domain with polygonal structure and discuss the precise formulation of the transmission/boundary value problem (1.1) in the weighted Sobolev space $\mathcal{K}^m_{\vec{a}}(\Omega)$. In Section 3, we state and prove preliminary results concerning regularity and solvability of the problem (1.1) when the interface is smooth and no two adjacent sides of Ω are given Neumann boundary conditions (Theorems 3.1, 3.2, 3.3). In Section 4, we consider the more difficult case of Neumann-Neumann vertices and non-smooth interfaces. We exploit these results and spectral analysis to obtain
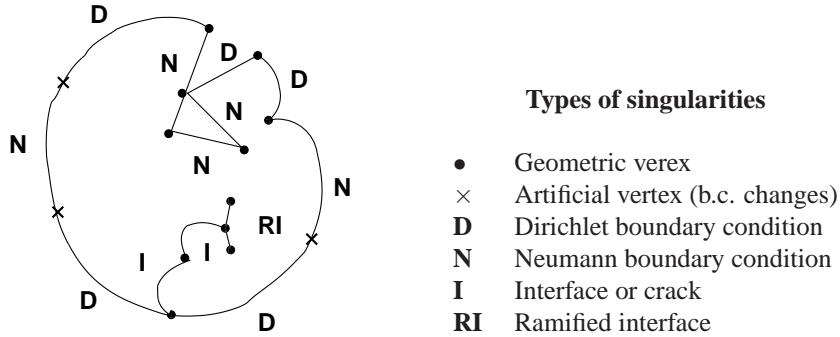
Types of singularities

-    Geometric verex
- ×   Artificial vertex (b.c. changes)
- **D**   Dirichlet boundary condition
- **N**   Neumann boundary condition
- **I**   Interface or crack
- **RI**   Ramified interface

FIG. 2.1. *A domain with a polygonal structure.*

a new well-posedness result in a properly augmented space $\mathcal{K}^1_{\vec{\mathbf{a}}+1}(\Omega) + W_s$, and *arbitrarily high* regularity of the weak solution u in each subdomain Ω_j (Theorems 4.5 and 4.7). For simplicity, we state and prove these results for the model example of $P = \mathrm{div}(A\nabla u)$, A a piecewise constant function, which will be used for numerical tests. By contrast, when interfaces cross, compatibility conditions on the coefficients need be imposed to obtain higher regularity in $H^s(\Omega)$, $1 < s < 3/2$ [69]. In Section 5, we tackle the explicit construction of graded meshes giving quasi-optimal rates of convergence for the FEM solution of the mixed boundary/transmission problem (1.1) in the case of a piecewise linear domain, and derive the necessary interpolation estimates (Theorems 5.11 and 5.12). In Section 6, we test our methods and results on several examples and verify the optimal rate of convergence.

We hope to extend our results to three dimensional polyhedral domains. The regularity results are known to extend to that case [16]. The problem is that the space of singular functions is infinite dimensional in the three dimensional case. Further ideas will therefore be needed to handle the case of three dimensions.

2. Formulation of the problem . We start by describing informally the class of "domains with a polygonal structure" Ω, a class of domains introduced (with different names and slightly different definitions) by many authors. Here we follow most closely [34]. Next we describe in more detail the formulation of the transmission/mixed boundary value problem (1.1) associated to P and interface Γ. The coefficients of P may have jumps at Γ.

2.1. The domain. The purpose of this section is to provide an informal description of the domains under consideration, emphasizing their rich structure and their suitability for transmission/mixed boundary value problems. In Figure 2.1, we exemplify the various types of singularities, some of geometric nature, others stemming from solving the transmission/mixed boundary value problem (1.1). These singularities are discussed in more detail below.

We consider bounded polygonal-like domains Ω that may have cracks or vertices that touch a smooth part of the boundary. Recall that polygonal domains are not always Lipschitz domains, however, the outer normal to the boundary is well-defined except at the vertices. If cracks are present, then the outer normal is not well-defined since $\partial\bar{\Omega} \neq \partial\Omega$. In order to study cracks, we model each smooth part of a crack as a double covering of a smooth curve. We then distinguish the two normal directions in which we approach the boundary. This distinction is also needed when we study vertices that touch the boundary. When cracks ramify, we need further to differentiate from which direction we approach the point of ramification. This distinction will be achieved by considering the connected neighborhoods of $B(x,r)\cap\Omega$, when x is on the boundary, as in Dauge [34]. More precisely, we will distinguish for each point

of the boundary the side from which we approach it. This defines, informally, the "unfolded boundary" $\partial^u\Omega$ of Ω. What is most important for us in this concept, is that each smooth crack point p of Ω will be replaced in $\partial^u\Omega$ by two points, corresponding to the two sides of the crack and the two possible non-tangential limits at p of functions defined on Ω.

We really need the distinction between the usual boundary $\partial\Omega$ and the unfolded boundary $\partial^u\Omega$, since it plays a role in the implementations. Moreover, we can define the "inner-pointing normal" vector consistently $\tilde{\nu}$ at every smooth point of $\partial^u\Omega$, even at crack points (but not at vertices). The *outer normal* to $\partial^u\Omega$ is defined by $\nu = -\tilde{\nu}$. Similarly, we defined the "unfolded closure" $^u\overline{\Omega} := \Omega \cup \partial^u\Omega$. The test functions used in our implementation will be defined on $^u\overline{\Omega}$ and *not* on $\overline{\Omega}$ (this point is especially relevant for the difficult and important case of cracks that are assigned Neumann boundary conditions on each side). More details will be included in a forthcoming paper [52].

When considering mixed boundary conditions, it is well known that singularities appear at the points where the boundary conditions change (from Dirichlet to Neumann). These singularities are very similar in structure to the singularities that appear at geometric vertices. We therefore view "vertices" simply as points on the boundary with special properties, the geometric vertices being "true vertices" and all others being "artificial vertices." The set of artificial vertices includes, in particular, all points where the type of boundary conditions change, but may include other points as well (coming from the interface for example). This choice allows for a greater generality, which is convenient in studying operators with singular coefficients.

We therefore fix a finite set $\mathcal{V} \subset \partial^u\Omega$, which will serve as the set where we allow singularities in the solution of our equation. We shall call the set $\mathcal{V}$ *the set of vertices* of Ω. The set of vertices $\mathcal{V}$ will contain at a minimum all non-smooth points of the boundary or of the interface, all points where the boundary conditions change, and all points where the boundary intersects the interface, but there could be other points in $\mathcal{V}$ as well. In particular, $\mathcal{V}$ is such that all connected components of $\partial^u\Omega \smallsetminus \mathcal{V}$ consist of smooth curves on which a unique type of boundary condition (Dirichlet or Neumann) is given. In particular, the structure on Ω determined by $\mathcal{V}$ is not entirely given by the geometry and depends also on the specifics of the transmission/boundary value problem. This structure, in turns, when combined with the introduction of the unfolded boundary, gives rise to the concept of a *domain with a polygonal structure*, introduced in [34] and discussed at length in [58] (except the case of a vertex touching a smooth side).

2.2. The equation. We consider a second order *scalar* differential operator with real coefficients $P : \mathcal{C}_c^\infty(\Omega) \to \mathcal{C}_c^\infty(\Omega)$

$$(2.1) \qquad\qquad Pu := -\operatorname{div}\big(A\nabla u\big) = -\sum_{i,j=1}^{2} \partial_j A^{ij} \partial_i u.$$

We assume, for simplicity, that $A^{ij} = A^{ji}$. The model example, especially for the numerical implementation, is the operator $P = \operatorname{div} A\nabla$, where A is a piece-wise constant function. Under some mild assumptions on the lower-order coefficients, the results in the paper extend also to operators of the form $P = -\sum_{i,j=1}^{2} \partial_j A^{ij} \partial_i + \sum_{i=1}^{2} b^i \partial_i + c$. Our methods apply as well to systems and complex-valued operators, but we restrict to the scalar case for the sake of clarity of presentation. In [58], we studied the system of anisotropic elasticity $P = -\operatorname{div} \circ \mathsf{C} \circ \nabla$ in 3 dimensions (in the notation above (2.1), $A^{ij} = [\mathsf{C}_{pq}]^{ij}$).

We assume throughout the paper that P is *uniformly strongly elliptic*, i.e.,

$$\sum_{i,j=1}^{2} A^{ij}(x)\xi_i\xi_j \geq C\|\xi\|^2, \tag{2.2}$$

for some constant $C > 0$ independent of $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^2$.

We also assume that we are given a decomposition

$$\overline{\Omega} = \cup_{j=1}^{N}\overline{\Omega}_j, \tag{2.3}$$

where Ω_j are disjoint domains with a polygonal structure, and define the *interface*

$$\Gamma := \left(\cup_{j=1}^{N}\partial\Omega_j\right) \smallsetminus \partial\Omega, \tag{2.4}$$

which *we assume to be the union of finitely many piecewise smooth curves* Γ_k. We allow the curves Γ_k to intersect, but we require these intersections to be transverse, i. e., not tangent. We take the coefficients of the differential operator P to be *piecewise smooth* in Ω with possible jumps only along Γ, that is, the coefficients of P on Ω_j extend to smooth functions on $\overline{\Omega}_j$. *Also, we assume that all the vertices of the domains with a polygonal structure Ω_j that are on the boundary of Ω are already included in the set $\mathcal{V}$ of vertices of Ω.*

To formulate our problem, we introduce *inhomogeneous* weighted Sobolev spaces, where the weight depends on the vertex, considered before in [57]. Let $d(x, Q)$ be the distance from x to $Q \in \mathcal{V}$, computed using paths in ${}^u\overline{\Omega}$ and let

$$\vartheta(x) = \prod_{Q\in\mathcal{V}} d(x,Q). \tag{2.5}$$

Let $\overrightarrow{\mathbf{a}} = (a_Q)$ be a vector with real components indexed by $Q \in \mathcal{V}$. We denote $t + \overrightarrow{\mathbf{a}} = (t + a_Q)$, but write t instead of $\overrightarrow{\mathbf{a}}$ if all the components of $\overrightarrow{\mathbf{a}}$ are equal to t. We then set

$$\vartheta^{t+\overrightarrow{\mathbf{a}}}(x) := \prod_{Q\in\mathcal{V}} d(x,Q)^{t+a_Q} = \vartheta^t(x)\vartheta^{\overrightarrow{\mathbf{a}}}(x), \tag{2.6}$$

and define the *mth weighted Sobolev space with weight* $\overrightarrow{\mathbf{a}}$ by

$$\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m}(\Omega) := \{f : \Omega \to \mathbb{C},\ \vartheta^{|\alpha|-\overrightarrow{\mathbf{a}}}\partial^\alpha f \in L^2(\Omega),\ \text{for all } |\alpha| \leq m\}. \tag{2.7}$$

The distance function ϑ is continuous on ${}^u\overline{\Omega}$ but it is not smooth at the vertices. Whenever derivatives of ϑ are involved, *we implicitly assume that ϑ has been replaced by a more regular weight function r_Ω.* This weight function is comparable to ϑ and induces an equivalent norm on $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^m$. One can describe the spaces $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^m(\Omega)$ also using certain dyadic partitions of unity. See [1, 33, 47, 58] for example. Such partitions of unity allow also to define spaces on the (unfolded) boundary of Ω, $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^s(\partial^u\Omega)$, $s \in \mathbb{R}$, for which the usual interpolation, duality, and trace properties still apply.

Our first goal is to study solvability of the problem (1.1) in $\mathcal{K}_{\overrightarrow{\mathbf{a}}}^{m+1}(\Omega)$, $m \geq 0$. The boundary conditions are given on *each side* in the unfolded boundary $\overset{\rightarrow}{\partial}{}^u\Omega$, where we assume that

$$\partial^u\Omega = \partial_N\Omega \cup \partial_D\Omega, \quad \partial_D\Omega \cap \partial_N\Omega = \emptyset,$$

such that $\partial_D\Omega$ a union of closed sides of Ω. We impose Neumann data $g_N \in \mathcal{K}^{m-1/2}_{\vec{\mathbf{a}}-1/2}(\partial_N\Omega)$ and Dirichlet data $g_D \in \mathcal{K}^{m+1/2}_{\vec{\mathbf{a}}+1/2}(\partial_D\Omega)$, $m \geq 0$. By the surjectivity of the trace map, we can reduce to the case $g_D = 0$ (in trace sense).

For $m = 0$, the problem (1.1) must be interpreted in an appropriate weak (or variational) sense, which we now discuss. For each $u, v \in H^1(\Omega)$, we define the bilinear form $B_P(u,v)$

$$(2.8) \qquad B_P(u,v) := \sum_{ij} \int_\Omega A^{ij} \partial_i u \, \partial_j v \, dx, \qquad 1 \leq i, j \leq 2,$$

and denote by D_ν^P the *conormal derivative* operator associated to P, given by

$$(2.9) \qquad (D_\nu^P u) := \sum_{ij} \nu_i A^{ij} \partial_j.$$

The definition of $D_\nu^P u$ is understood in the sense of the trace at the boundary. In particular, when u is regular enough $D_\nu^P u$ is defined almost everywhere as a non-tangential limit, consistently with ν being defined only almost everywhere on $\partial^u\Omega$. We recall that ν is defined on $\partial^u\Omega$ except at the vertices because the smooth crack points of $\partial\Omega$ are doubled in $\partial^u\Omega$.

Since Ω is a finite union of Lipschitz domains, Green's Lemma holds for functions in $H^2(\Omega)$ [36], that is,

$$(2.10) \qquad (Pu,v)_{L^2(\Omega)} = B_P(u,v) - (D_\nu^P u, v)_{L^2(\partial^u\Omega)}, \qquad u, v \in H^2(\Omega).$$

Hence, we let

$$(2.11) \qquad \mathcal{H}_{\vec{\mathbf{a}}} := \{u \in \mathcal{K}^1_{1+\vec{\mathbf{a}}}(\Omega), \ u = 0 \text{ on } \partial_D\Omega\},$$

and we define the weak solution u of equation (1.1) with $g_D = 0$ as the unique $u \in \mathcal{H}_{\vec{\mathbf{a}}}$ satisfying

$$(2.12) \qquad B_P(u,v) = \Phi(v) \quad \text{ for all } \ v \in \mathcal{H}_{-\vec{\mathbf{a}}}.$$

where $\Phi \in (\mathcal{H}_{-\vec{\mathbf{a}}})^*$ is defined by $\Phi(u) = \int_\Omega f u \, dx + \int_{\partial_N\Omega} g_N u \, dS(x)$, the integrals being duality pairings between distributions and (suitable) functions.

When u is regular enough, problem (1.1) is equivalent to the following mixed boundary value/interface problem

$$(2.13) \qquad \begin{cases} Pu = f & \text{in } \Omega, \\ u = g_D = 0 & \text{on } \partial_D\Omega \subset \partial^u\Omega, \\ D_\nu^P u = g_N & \text{on } \partial_N\Omega \subset \partial^u\Omega, \\ u^+ = u^- \text{ and } D_\nu^{P+}u = D_\nu^{P-}u & \text{on } \Gamma, \end{cases}$$

where it is crucial that $\partial_N\Omega$ and $\partial_D\Omega$ are subsets of the *unfolded boundary*. (Recall that the unfolded boundary is defined by doubling the smooth points of the crack. In particular, one can have Dirichlet boundary conditions on one side of the crack and Neumann boundary conditions on the other side of the crack.) In (2.13), u^+ and u^- denote the two non-tangential limits of u at the two sides of the interface Γ. This choice can be done consistently at each smooth point of Γ. Similarly, D_ν^{P+} and D_ν^{P-} denote the two conormal derivatives associated to P and the two sides of Γ. Note that the singularities in the coefficients of A are taken into account in the definitions of D_ν^{P+} and D_ν^{P-}. If u is only in $\mathcal{K}^1_{\vec{\mathbf{a}}+1}(\Omega)$ and satisfy (2.12), then

the difference $D_\nu^{P+}u - D_\nu^{P-}u$ may be non-zero (so (2.13) is not strictly satisfied), but may be included as a distributional term in f.

Thus the usual transmission conditions $u^+ = u^-$ and $D_\nu^{P+}u = D_\nu^{P-}u$ at the two sides of the interface are a consequence of the weak formulation, and will always be considered as part of equation (1.1). The slightly more general conditions $u^+ - u^- = h_0$ and $D_\nu^{P+}u - D_\nu^{P-}u = h_1$ can be treated with only minor modifications, as explained in [67]. More precisely, the term h_0 can be treated using extensions similarly to the term g_D. The term h_1 can be treated by introducing in the the weak formulation the term $\int_\Gamma h_1 u ds$, where ds is arc length on Γ.

In order to establish regularity and solvability of (2.13), under the hypothesis that P is uniformly strongly elliptic, we shall use coercive estimates. We say that P is *coercive on* $\mathcal{H}_0 := \mathcal{H}_{\vec{\mathbf{a}}=0}$ if there exists $\theta > 0$ and $\gamma \in \mathbb{R}$ such that

$$(2.14) \qquad B_P(u,u) \geq \theta(\nabla u, \nabla v)_{L^2(\Omega)} - \gamma(\vartheta^{-2}u, v)_{L^2(\Omega)}, \quad \text{for all } u, v \in \mathcal{H}_0.$$

If this inequality holds for some $\gamma < 0$, we say that P is *strictly coercive on* $\mathcal{H}_0$ (or *strictly positive*) and write $P > 0$. The operator P in (2.1) is always coercive on $\mathcal{H}_0$. If there are no Neumann–Neumann vertices and the interface Γ is smooth, then P is strictly coercive on $\mathcal{H}_0$, as it will be discussed in the next section.

3. Preliminary results. Our approach in studying singularities for problem (2.13) is based on solvability in weighted spaces rather than on singular functions expansions. We begin with three results on regularity and well-posedness for the boundary-value problem (2.13), which we first state and then prove. See [17, 18, 42, 43, 44, 6, 46, 47, 65, 66, 67] for related results. In particular our result should be compared with [66], especially Theorem 3.12. By "well-posedness" we mean "existence and uniqueness of solutions and continuous dependence on the data." Recall that for transmission problems we assume that all the vertices of the domains with a polygonal structure Ω_j that are on the boundary of Ω are included in the set of vertices of Ω. Below, if no interface is given, we take $\Omega = \Omega_1$. When $\Omega \neq \Omega_1 \neq \emptyset$, we have a *proper transmission problem*.

We first deal with the general case of an interface that is the union of finitely many *piecewise smooth curves with transverse intersections*, and establish that the transmission/mixed boundary problem (1.1) satisfies a regularity property. We assume that the non-smooth points of the interface Γ are included in the vertices of the adjacent domains Ω_j (the self-intersection points, which are assumed to be transverse, are also included in the set of vertices). This regularity result is crucial in obtaining the necessary *a priori* estimates for quasi-optimal rates of convergence in Section 5 for transmission problems.

We first state our main results on regularity and well-posedness and then we prove them.

THEOREM 3.1. *Assume that* $P = -\operatorname{div} A\nabla$ *is a uniformly strongly elliptic, scalar operator in divergence form on* Ω *with piecewise smooth coefficients. Also, assume that* $u : \Omega \to \mathbb{R}$ *with* $u \in \mathcal{K}^1_{\vec{\mathbf{a}}+1}(\Omega)$ *is a solution of the transmission/mixed boundary problem* (1.1). *Let* $m \geq 0$, *and suppose that* $g_N \in \mathcal{K}^{m-1/2}_{\vec{\mathbf{a}}-1/2}(\partial_N\Omega)$, $g_D \in \mathcal{K}^{m+1/2}_{\vec{\mathbf{a}}+1/2}(\partial_D\Omega)$, *and* $f : \Omega \to \mathbb{R}$ *is such that* $f|_{\Omega_j} \in \mathcal{K}^{m-1}_{\vec{\mathbf{a}}-1}(\Omega_j)$. *Then* $u|_{\Omega_j} \in \mathcal{K}^{m+1}_{\vec{\mathbf{a}}+1}(\Omega_j)$, *for each* j, *and we have the estimate*

$$\|u\|_{\mathcal{K}^1_{\vec{\mathbf{a}}+1}(\Omega)} + \|u\|_{\mathcal{K}^{m+1}_{\vec{\mathbf{a}}+1}(\Omega_j)} \leq C\Big(\sum_{k=1}^N \|f\|_{\mathcal{K}^{m-1}_{\vec{\mathbf{a}}-1}(\Omega_k)} + \|g_N\|_{\mathcal{K}^{m-1/2}_{\vec{\mathbf{a}}-1/2}(\partial_N\Omega)} +$$
$$\|g_D\|_{\mathcal{K}^{m+1/2}_{\vec{\mathbf{a}}+1/2}(\partial_D\Omega)} + \|u\|_{\mathcal{K}^0_{\vec{\mathbf{a}}+1}(\Omega)}\Big)$$

for a constant C that is independent of u and the data f, g_N, and g_D.

Note that, in the above result, the spaces $\mathcal{K}^{m-1}_{\vec{\mathbf{a}}-1}(\Omega_k)$ are defined intrinsically, i. e., without reference to $\mathcal{K}^{m-1}_{\vec{\mathbf{a}}-1}(\Omega)$, using as weight the distance to the set of vertices of Ω_k, which includes also the points of $\overline{\Omega}_j$ where Γ is not smooth or where it ramifies.

The next two results deal with solvability of the problem (1.1), in the case of a smooth interface and when $\partial^u\Omega$ contains no adjacent sides with Neumann boundary conditions. (The condition that Γ is smooth in particular implies that Γ is a disjoint union of smooth curves.) These results are also the basis for the analysis in Section 4 in the presence of Neumann-Neumann vertices and general interfaces (Theorems 4.5 and 4.7, where an augmented domain for the operator is required). Recall that the weak solution u is given in equation (2.12) with $\Phi = (f, g_N) \in \mathcal{H}^*_{-\vec{\mathbf{a}}}$ (because we take $g_D = 0$).

THEOREM 3.2. *Assume that P is a uniformly strongly elliptic, scalar operator on $\bar{\Omega}$. Assume also that no two adjacent sides of Ω are given Neumann boundary conditions and that the interface Γ is smooth. Then P is strongly coercive on $\mathcal{H}_0$ and for each vertex Q of Ω there exists a positive constant η_Q with the following property: for any $\Phi \in \mathcal{H}^*_{-\vec{\mathbf{a}}}$ with $|a_Q| < \eta_Q$, there exists a unique weak solution $u \in \mathcal{K}^1_{\vec{\mathbf{a}}+1}(\Omega)$, $u = 0$ on $\partial_D\Omega$ of equation (2.13), and we have the estimate*

$$\|u\|_{\mathcal{K}^1_{\vec{\mathbf{a}}+1}(\Omega)} \leq C\|\Phi\|$$

for a constant $C = C(\vec{\mathbf{a}})$ that is independent of Φ.

When the data is more regular, we can combine the above two theorems into a well-posedness result for the transmission/mixed boundary problem. We note that continuous dependence of the solution on the data immediately follows from the estimate below since the boundary-value problem is linear.

THEOREM 3.3. *Let $m \geq 1$. In addition to the assumptions of Theorem 3.2, assume that $g_N \in \mathcal{K}^{m-1/2}_{\vec{\mathbf{a}}-1/2}(\partial_N\Omega)$, $g_D \in \mathcal{K}^{m+1/2}_{\vec{\mathbf{a}}+1/2}(\partial_D\Omega)$, and that $f : \Omega \to \mathbb{R}$ is such that $f|_{\Omega_j} \in \mathcal{K}^{m-1}_{\vec{\mathbf{a}}-1}(\Omega_j)$. Then the solution $u \in \mathcal{K}^1_{\vec{\mathbf{a}}+1}(\Omega)$ of equation (2.13) satisfies $u|_{\Omega_j} \in \mathcal{K}^{m+1}_{\vec{\mathbf{a}}+1}(\Omega_j)$, for all j, and we have the estimate*

$$\|u\|_{\mathcal{K}^{m+1}_{\vec{\mathbf{a}}+1}(\Omega_j)} \leq C\Big(\sum_k \|f\|_{\mathcal{K}^{m-1}_{\vec{\mathbf{a}}-1}(\Omega_k)} + \|g_N\|_{\mathcal{K}^{m-1/2}_{\vec{\mathbf{a}}-1/2}(\partial_N\Omega)} + \|g_D\|_{\mathcal{K}^{m+1/2}_{\vec{\mathbf{a}}+1/2}(\partial_D\Omega)} \Big).$$

If $P = -\sum_{i,j=1}^2 \partial_j A^{ij}\partial_i + \sum_{i=1}^2 b^i\partial_i + c$, that is, if lower order coefficients are included, our results extend to the case when $2c - \nabla\cdot\vec{\mathbf{b}} \geq 0$ in Ω and $\nu\cdot\vec{\mathbf{b}} \geq 0$ on $\partial_N\Omega$, where $\vec{\mathbf{b}} = (b^i)$.

Let us denote by $\tilde{P}v = (\oplus P|_{\Omega_j}, D_\nu^P)v := (Pv|_{\Omega_1}, \ldots, Pv|_{\Omega_N}, D_\nu^P v)$, decorated with various indices. As a corollary to the theorem, we establish the following isomorphism.

COROLLARY 3.4. *We proceed as in [58]. Let $m \geq 1$. Under the assumptions of Theorem 3.3, the map*

$$\tilde{P}_{m,\vec{\mathbf{a}}} := (\oplus P|_{\Omega_j}, D_\nu^P) : \{u \in \mathcal{K}^1_{\vec{\mathbf{a}}+1}(\Omega), \ u|_{\Omega_j} \in \mathcal{K}^{m+1}_{\vec{\mathbf{a}}+1}(\Omega_j), \ u = 0 \text{ on } \partial_D\Omega,$$

$$u^+ = u^- \text{ and } D_\nu^{P+}u = D_\nu^{P-}u \text{ on } \Gamma\} \to \oplus_j\mathcal{K}^{m-1}_{\vec{\mathbf{a}}-1}(\Omega_j) \oplus \mathcal{K}^{m-1/2}_{\vec{\mathbf{a}}-1/2}(\partial_N\Omega)$$

is an isomorphism for $|a_Q| < \eta_Q$. See [58] for more details of this method.

We next turn to the proofs of Theorems 3.1, 3.2, and 3.3. We will only sketch proofs and concentrate on the new issues raised by the presence of interfaces, referring for more details

to [1, 17, 58], where similar results were established for mixed boundary value problems in homogeneous $\mathcal{K}_a^m$ spaces.

Proof of Theorem 3.1. Using a partition of unity, it is enough to prove the result on the model problem (2.13) with $\Omega = \mathbb{R}^n$ and $\Gamma = \{x_n = 0\}$, that is, no boundary and one interface. We can assume without loss of generality that u has compact support on a fixed ball B centered at the origin. Then by known regularity results [71] (see also [66] and references therein), if $u \in H_0^1(B)$ and $Pu|_{\mathbb{R}_\pm} \in H^{m-1}(\mathbb{R}_\pm^n)$, $u|_{\mathbb{R}_\pm} \in H^{m+1}(\mathbb{R}_\pm^n)$. $\quad\square$

We next turn to the proof of well-posedness for the transmission/mixed boundary problem (2.13), namely, to the proofs of Theorems 3.2 and 3.3. As before, we denote $\mathcal{H}_{\vec{\mathbf{a}}} := \{u \in \mathcal{K}^1_{1+\vec{\mathbf{a}}}(\Omega), u = 0 \text{ on } \partial_D\Omega\}$, where $\partial_D\Omega$ is assumed non empty, and we set $\mathcal{H}_0 = \mathcal{H}_{\vec{\mathbf{0}}}$. Strict coercivity of P on $\mathcal{H}_0$ then ensues in the standard fashion from a weighted form of Poincaré inequality, which we now recall.

LEMMA 3.5. *Let $\Omega \subset \mathbb{R}^2$ be a domain with a polygonal structure. Let $\vartheta(z)$ be the canonical weight function on Ω and let $\partial_D\Omega$ be a non-empty closed subset of the unfolded boundary $\partial^u\Omega$ such that $\partial_N\Omega = \partial^u\Omega \setminus \partial_D\Omega$ is a union of oriented open sides of Ω, no two of which are adjacent. Then there exists a constant $C_\Omega > 0$ such that*

$$\|u\|^2_{\mathcal{K}_1^0(\Omega)} := \int_\Omega \frac{|u(z)|^2}{\vartheta(z)^2} \, dz \leq C_\Omega \int_\Omega |\nabla u(z)|^2 dz$$

for any $u \in H^1(\Omega)$ satisfying $u = 0$ on $\partial_D\Omega$.

In particular, any $u \in H^1(\Omega)$ satisfying the assumptions of the above theorem will be automatically in $\mathcal{K}_1^0(\Omega)$. This estimate is a consequence of the corresponding estimate on a sector, which can be proved in the usual way, given that are only finitely many vertices and that near each vertex Q, ${}^u\overline{\Omega}$ is diffeomorphic to a sector of angle $0 < \alpha \leq 2\pi$ [17, 65] (the angle is 2π at crack tips).

Proof of Theorems 3.2 and 3.3. We first observe that

$$B_P(u,u) = \int_\Omega \sum_{ij} A^{ij}(x)\partial_i u(x)\,\partial_j u(x)\,dx \geq C_P \int_\Omega |\nabla u(x)|^2\,dx, \qquad u \in \mathcal{H}_0,$$

using the strong ellipticity condition, equation (2.2). By Lemma 3.5, $-\Delta$ is strictly coercive on $\mathcal{H}_0$, given the hypotheses on $\partial^u\Omega$. Therefore, if $u \in \mathcal{H}_0$

$$B_P(u,u) = \int_\Omega \sum_{ij} A^{ij}(x)\partial_i u(x)\,\partial_j u(x)\,dx \geq C_P \int_\Omega |\nabla u(x)|^2\,dx \geq C_{P,\Omega}\,\|u\|^2_{\mathcal{K}_1^0(\Omega)}.$$

The first part of Theorem 3.2 is proved.

Next, we employ the maps

$$\tilde{P}_{m,\vec{\mathbf{a}}} := (\oplus P|_{\Omega_j}, D_\nu^P) \,:\, \{u \in \mathcal{K}^1_{\vec{\mathbf{a}}+1}(\Omega),\ u|_{\Omega_j} \in \mathcal{K}^{m+1}_{\vec{\mathbf{a}}+1}(\Omega_j),\ u = 0 \text{ on } \partial_D\Omega,$$

$$u^+ = u^- \text{ and } D_\nu^{P+}u = D_\nu^{P-}u \text{ on } \Gamma\} \rightarrow \oplus_j \mathcal{K}^{m-1}_{\vec{\mathbf{a}}-1}(\Omega_j) \oplus \mathcal{K}^{m-1/2}_{\vec{\mathbf{a}}-1/2}(\partial_N\Omega)$$

of Corollary 3.4. To prove the rest of the Theorems 3.2 and 3.3, we will show that $\tilde{P}_{m,\vec{\mathbf{a}}}$ is an isomorphism for $m \geq 0$ and $|a_Q| < \eta_Q$. Since B_P is strictly coercive on $\mathcal{H}_0$, it satisfies the assumptions of the Lax-Milgram lemma, and hence $B_P^* : \mathcal{H}_0 \to \mathcal{H}_0^*$ is an isomorphism, where $B_P^*(u)(v) = B_P(u,v)$. That is, $\tilde{P}_{0,0}$ is an isomorphism. Hence, Theorems 3.2 and 3.3 are established for $m = 0$ and $\vec{\mathbf{a}} = 0$.

To extend the results to the case $\vec{\mathbf{a}} \neq 0$ with $|a_Q| < \eta_Q$, we exploit continuity. Let r_Ω be a smoothing of ϑ outside the vertices. As in [1, 58], the family of operators $r_\Omega^{-\vec{\mathbf{a}}} P_{m,\vec{\mathbf{a}}} r_\Omega^{\vec{\mathbf{a}}}$ act on the same space and depend continuously on $\vec{\mathbf{a}}$. Since $P_{0,0}$ is an isomorphism, we obtain that $P_{0,\vec{\mathbf{a}}}$ is an isomorphism for $\vec{\mathbf{a}}$ close to 0. In particular, there exists $\eta_Q > 0$ such that for $|a_Q| < \eta_Q$, $P_{0,\vec{\mathbf{a}}}$ is an isomorphism. The proof of Theorems 3.2 and 3.3 are complete for $m = 0$.

It only remains to prove Theorem 3.3 for $m \geq 1$. Indeed, Theorem 3.1 gives that $\tilde{P}_{m,\vec{\mathbf{a}}}$ is surjective for $|a_Q| < \eta_Q$, since it is surjective for $m = 0$. This map is also continuous and injective (because it is injective for $m = 0$), hence it is an isomorphism. Consequently $\tilde{P}_{m,\vec{\mathbf{a}}}$, $|a_Q| < \eta_Q$, is an isomorphism by the open mapping theorem.

The above three theorems extend to the case of polyhedral domain in three dimensions using the methods of [58] and [16]. The case of three dimensions will be however treated separately, because the 3D Neumann problem is significantly more complex, especially when it comes to devising efficient numerical methods. The case of Neumann–Neumann adjacent faces in 3D cannot be treated by the methods of this paper alone, however.

4. Neumann-Neumann vertices and nonsmooth interfaces. In this section, we obtain a new type of well-posedness for the problem (1.1) in the spaces $\mathcal{K}_{\vec{\mathbf{a}}}^m$ that applies also to general interfaces and to Neumann-Neumann vertices. Our result combines the singular function decompositions with more typical well-posedness results. Singular function decompositions for interface problems have been discussed also in [43, 42, 66, 67] and more recently [79], to give just a few examples.

We restrict to a special class of operators P, for which the spectral analysis is amenable. Specifically, we consider the case of the Laplace operator Δ, when there are Neumann-Neumann vertices but no interface, and the case of $-\operatorname{div} A\nabla$, with A piecewise constant, when there are interfaces. In this last case, the operator is still a multiple of the Laplacian on each subdomain. Except for the explicit determination of the constants η_Q, our results extend to variable coefficients. In both cases, we can compute explicitly the values of the weight a_Q for which the operator P is Fredholm. These values will be used to construct the graded meshes in Section 5.

4.1. The Laplace operator. When $P = -\Delta$, the Laplace operator, it is possible to explicitly determine the values of the constants η_Q appearing in Theorems 3.2 and 3.3. *In this subsection, we therefore assume that $P = -\Delta$ and there are no interfaces, that is, $\Omega = \Omega_1$.*

Recall that to a Fredholm operator $T : X \to Y$ between Banach spaces is associated a unique number, called the *index*, defined by the formula $\operatorname{ind}(T) = \dim \ker(T) - \dim(Y/X)$. For a discussion of Fredholm operators, see e.g., [73].

For each vertex $Q \in \mathcal{V}$, we let α_Q be the *interior* angle of $\partial^u \Omega$ at Q. In particular, $\alpha_Q = 2\pi$ if Q is the tip of a crack, and $\alpha_Q = \pi$ if Q is an artificial vertex. We then define

$$(4.1) \qquad \Sigma_Q := \{\, k\pi/\alpha_Q \,\},$$

where $k \in \mathbb{Z}$ if $Q \in \mathcal{V}$ is a Neumann–Neumann vertex, $k \in \mathbb{Z} \smallsetminus \{0\}$ if $Q \in \mathcal{V}$ is a Dirichlet–Dirichlet vertex, and $k \in 1/2 + \mathbb{Z}$ otherwise. The operator pencil $P_Q(\tau)$ (or *indicial family*) associated to $-\Delta$ at Q is $P_Q(\tau) := (\tau - \imath\epsilon)^2 - \partial_\theta^2$, where (r, θ) are local polar coordinates at Q. The operator $P_Q(\tau)$ is defined on functions in $H^2([0, \alpha_Q])$ that satisfy the given boundary conditions, and is obtained by evaluating

$$(4.2) \qquad -\Delta(r^{\imath\tau+\epsilon}\phi(\theta)) = r^{\imath\tau+\epsilon-2}\Big((\tau - \imath\epsilon)^2 - \partial_\theta^2\Big)\phi(\theta).$$

$P(\tau)$ is invertible for all $\tau \in \mathbb{R}$, as long as $\epsilon \notin \Sigma_Q$.

We are again interested in the well-posedness of the problem (1.1) when Neumann–Neumann vertices exist. We therefore consider the operator

$$(4.3) \qquad \tilde{\Delta}_{\overrightarrow{\mathbf{a}}} := (\Delta, \partial_\nu) : \mathcal{K}^{m+1}_{\overrightarrow{\mathbf{a}}+1}(\Omega) \cap \{u|_{\partial_D \Omega} = 0\} \to \mathcal{K}^{m-1}_{\overrightarrow{\mathbf{a}}-1}(\Omega) \oplus \mathcal{K}^{m-1/2}_{\overrightarrow{\mathbf{a}}-1/2}(\partial_N \Omega),$$

which is well defined for $m \geq 1$. Recall that we can extend $\Delta_{\overrightarrow{\mathbf{a}}}$ to the case $m = 0$ as

$$(4.4) \qquad\qquad \tilde{\Delta}_{\overrightarrow{\mathbf{a}}} : \mathcal{H}_{\overrightarrow{\mathbf{a}}} \to \left(\mathcal{H}_{-\overrightarrow{\mathbf{a}}}\right)^*, \quad (\tilde{\Delta} u, v) := -(\nabla u, \nabla v),$$

where $u \in \mathcal{H}_{\overrightarrow{\mathbf{a}}}$ and $v \in \mathcal{H}_{-\overrightarrow{\mathbf{a}}}$ (recall that $\mathcal{H}_{\overrightarrow{\mathbf{a}}}$ is defined in (2.11)). For transmission problems, a similar formula allows to extend the operator (P, ∂^ν_P) to the case $m = 0$.

Following Kondratiev [46] and Nicaise (for the case of transmission problems) [66] we can prove the result below, using also the regularity theorem 3.1.

THEOREM 4.1. *Let $P = -\Delta$, $m \geq 0$, and $\overrightarrow{\mathbf{a}} = (a_Q)$. Also, let $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}$ be the operator defined in equations (4.3) and (4.4) for the case when there is no interface. Then $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}$ is Fredholm if, and only if, $a_Q \notin \Sigma_Q$. Moreover, its index is independent of m.*

Proof. The Fredholm criterion is well known [46, 48, 76]. (The case $m = 0$ was not treated explicitly, but it is proved in exactly the same way.) We prove that the index is independent of m. Indeed, if $u \in \mathcal{H}_{\overrightarrow{\mathbf{a}}}$ is such that $\Delta_{\overrightarrow{\mathbf{a}}} u = 0$, then the regularity theorem, Theorem 3.1, implies that $u \in \mathcal{K}^\infty_{\overrightarrow{\mathbf{a}}+1}(\Omega)$. The same observation for the adjoint problem shows that the index is independent of m. $\square$

See also [32, 48, 76] and references therein.

The case $m = 0$ is relevant because in that case

$$(4.5) \qquad\qquad \left(\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}\right)^* = \tilde{\Delta}_{-\overrightarrow{\mathbf{a}}},$$

an equation that does not make sense (in any obvious way) for other values of m. It is then possible to determine the index of the operators $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}$ by the following index calculation. Recall that in this subsection we assume the interface to be empty. Let $\overrightarrow{\mathbf{a}} = (a_Q)$ and $\overrightarrow{\mathbf{b}} = (b_Q)$ be two vectorial weights that correspond to Fredholm operators in Theorem 4.1. Let us assume that there exists a vertex Q such that $a_Q < b_Q$ but $a_R = b_R$ if $R \neq Q$. We count the number of values in the set $(a_Q, b_Q) \cap \Sigma_Q$, with the values corresponding to $k = 0$ in the definition of Σ_Q, equation (4.1), counted twice (because of multiplicity, which happens only in the case of Neumann–Neumann boundary conditions). Let N be the total number. The following result, which can be found in [66] (see also [32, 46, 47, 64, 65]), holds.

THEOREM 4.2. *Assume the conditions of Theorem 4.1 are satisfied. Also, let us assume that $a_Q < b_Q$ but $a_R = b_R$ if $R \neq Q$, and let N be defined as in the paragraph above. Then*

$$\text{ind}(\tilde{\Delta}_{\overrightarrow{\mathbf{b}}}) - \text{ind}(\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}) = -N.$$

This theorem allows to determine the index of $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}$. For simplicity, we compute the index only for $a_Q > 0$ and small. Let δ_Q be the minimum values of $s \in \Sigma_Q \cap (0, \infty)$. Then $\delta_Q = \pi/\alpha_Q$, if both sides meeting at Q are assigned the same type of boundary conditions, and by $2\delta_Q = \pi/\alpha_Q$ otherwise.

THEOREM 4.3. *Assume the conditions of Theorem 4.1 are satisfied and let N_0 be the number of vertices Q such that both sides adjacent to Q are assigned Neumann boundary conditions. We assume the interface to be empty. Then $\tilde{\Delta}_a$ is Fredholm for $0 < a_Q < \delta_Q$ with index*

$$\text{ind}(\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}) = -N_0.$$

Consequently, $\tilde{\Delta}_{-\overrightarrow{\mathbf{a}}}$ has index $-N_0$ for $0 < a_Q < \delta_Q$.

For transmission problems, we shall count in N_0 also the points where the interface Γ is not smooth. Each such point is counted exactly once. On the other hand, a point where a crack ramifies is counted as many times as it is covered in thick closure ${}^u\overline{\Omega}$, so in effect we are counting the vertices in ${}^u\overline{\Omega}$ and not in Ω.

Proof. Since the index is independent of $m \geq 0$, we can assume that $m = 0$. A repeated application of Theorem 4.2 (more precisely of its generalization for $m = 0$) for each weight a_Q gives that $\mathrm{ind}(\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}) - \mathrm{ind}(\tilde{\Delta}_{-\overrightarrow{\mathbf{a}}}) = -2N_0$ (each time when we change an index from $-a_Q$ to a_Q we lose a 2 in the index, because the value $k = 0$ is counted twice). Since $\tilde{\Delta}_{-\overrightarrow{\mathbf{a}}} = \tilde{\Delta}_{\overrightarrow{\mathbf{a}}}^*$, we have $\mathrm{ind}(\tilde{\Delta}_{-\overrightarrow{\mathbf{a}}}) = -\mathrm{ind}(\tilde{\Delta}_{\overrightarrow{\mathbf{a}}})$, and therefore the desired result. $\square$

We now proceed to a more careful study of the invertibility properties of $\tilde{\Delta}_{\overrightarrow{\mathbf{a}}}$. In particular, we will determine the constants η_Q appearing in Theorems 3.2 and 3.3.

For each vertex $Q \in \mathcal{V}$ we choose a function $\chi_Q \in \mathcal{C}^\infty(\overline{\Omega})$ that is constant equal to 1 in a neighborhood of Q and satisfies $\partial_\nu \chi_Q = 0$ on the boundary. We can choose these functions to have disjoint supports.

Let W_s be the linear span of the functions χ_Q that correspond to Neumann–Neumann vertices Q. (For transmission problems, we have to take into account also the points where the interface Γ is not smooth. This is achieved by including a function of the form χ_Q for each point Q of the interface where the interface is not smooth. The condition $\partial_\nu \chi_Q = 0$ on the boundary becomes, of course, unnecessary.) We shall need the following version of Green's formula.

LEMMA 4.4. *Assume all $a_Q \geq 0$ and $u, v \in \mathcal{K}^2_{\overrightarrow{\mathbf{a}}+1}(\Omega) + W_s$. Then*

$$(\Delta u, v) + (\nabla u, \nabla v) = (\partial_\nu u, v)_{\partial\Omega}.$$

Proof. Assume first u and v are constant close to the vertices, then we can apply the usual Green's formula after smoothing the vertices without changing the terms in the formula. In general, we notice that $C(u, v) := (\Delta u, v) + (\nabla u, \nabla v) = (\partial_\nu u, v)_{\partial\Omega}$ depends continuously on u and v (since by hypothesis $a_Q \geq 0 \ \forall Q$) and we can then use a density argument. $\square$

Recall that we assume the interface to be empty. Then we have the following solvability (or well-posedness) result.

THEOREM 4.5. *Let $\overrightarrow{\mathbf{a}} = (a_Q)$ with $0 < a_Q < \delta_Q$ and $m \geq 1$. Assume $\partial_D\Omega \neq \emptyset$. Then for any $f \in \mathcal{K}^{m-1}_{\overrightarrow{\mathbf{a}}-1}(\Omega)$ and any $g_N \in \mathcal{K}^{m-1/2}_{\overrightarrow{\mathbf{a}}-1/2}(\Omega)$, there exists a unique $u = u_{\mathrm{reg}} + w_s$, $u_{\mathrm{reg}} \in \mathcal{K}^{m+1}_{\overrightarrow{\mathbf{a}}+1}(\Omega)$, $w_s \in W_s$ satisfying $-\Delta u = f$, $u = 0$ on $\partial_D\Omega$, and $\partial_\nu u = g_N$ on $\partial_N\Omega$. Moreover,*

$$\|u_{\mathrm{reg}}\|_{\mathcal{K}^{m+1}_{\overrightarrow{\mathbf{a}}+1}(\Omega)} + \|w_s\| \leq C\big(\|f\|_{\mathcal{K}^{m-1}_{\overrightarrow{\mathbf{a}}-1}(\Omega)} + \|g_N\|_{\mathcal{K}^{m-1/2}_{\overrightarrow{\mathbf{a}}-1/2}(\Omega)}\big),$$

for a constant $C > 0$ independent of f and g_N. When $\partial_D\Omega = \emptyset$ (the pure Neumann problem), the same conclusions hold if constant functions are factored out.

Proof. Using the surjectivity of the trace map, we can reduce to the case $g_D = 0$ and $g_N = 0$. Let $V = \{u \in \mathcal{K}^{m+1}_{\overrightarrow{\mathbf{a}}+1}(\Omega), \ u|_{\partial_D\Omega} = 0, \ \partial_\nu u|_{\partial_N\Omega} = 0\} + W_s$. Since $m \geq 1$, the map

$$(4.6) \qquad\qquad\qquad \Delta : V \to \mathcal{K}^{m-1}_{\overrightarrow{\mathbf{a}}-1}(\Omega)$$

is well defined and continuous. Then Theorem 4.3 implies that the map of equation (4.6) has index zero, given that the dimension of W_s is N_0. When there is at least a side in $\partial_D\Omega$, this map is in fact an isomorphism. Indeed, it is enough to show it is injective. This is seen as follows. Let $u \in V$ be such that $\Delta u = 0$. By Green's formula (Lemma 4.4), we have

$(\nabla u, \nabla u) = (-\Delta u, u) + (\partial_\nu u, u)_{\partial\Omega} = 0$. Therefore u is a constant. If there is at least one Dirichlet side, the constant must be zero, i.e, $u = 0$. In the pure Neumann case, the kernel of the map of equation (4.6) consists of constants. Another application of Green's formula shows that $(\Delta u, 1) = 0$, which identifies the range of Δ in this case as the functions with mean zero. $\square$

The same argument as in the above proof gives that $\tilde{\Delta}_{\vec{\mathbf{a}}}$ is injective, provided all components of $\vec{\mathbf{a}}$ are non-negative, a condition that we shall write as $\vec{\mathbf{a}} \geq 0$). From equation (4.5), it then follows that $\tilde{\Delta}_{-\vec{\mathbf{a}}}$ is *surjective* whenever it is Fredholm. This observation implies Theorem 3.2 for $\vec{\mathbf{a}} = 0$. Note that $\tilde{\Delta}_0$ is Fredholm precisely when there are no Neumann–Neumann faces. For operators of the form $- \operatorname{div} A\nabla$ with A piecewise smooth, we have to assume also that the interface Γ is smooth, otherwise the Fredholm property for the critical weight $\vec{\mathbf{a}} = 0$ is lost.

We can now determine the constants η_Q in Theorems 3.2 and 3.3.

THEOREM 4.6. *Assume $P = -\Delta$. Then we can take $\eta_Q = \delta_Q$ in Theorem 3.2.*

Proof. Assume that $|a_Q| < \eta_Q$. Then $\tilde{\Delta}_{\vec{\mathbf{a}}}$ is Fredholm of index zero, since $\tilde{\Delta}_{\vec{\mathbf{a}}}$ depends continuously on $\vec{\mathbf{a}}$ and it is of invertible for $\vec{\mathbf{a}} = 0$ as observed above in the context of Theorem 3.2. Assume then $\tilde{\Delta}_{\vec{\mathbf{a}}} u = 0$ for some $u \in \mathcal{H}_{\vec{\mathbf{a}}}$. The singular function expansion of u close to each vertex implies $u \in \mathcal{H}_{\vec{\mathbf{b}}}$ for all $\vec{\mathbf{b}} = (b_Q)$ with $0 < b_Q < \eta_Q$ [47, 66], where η_Q is the exponent s of the first singular function $r^s\phi(\theta)$, in polar coordinates centered at Q. Since $\tilde{\Delta}_{\vec{\mathbf{b}}}$ is injective for $b_Q > 0$, $\tilde{\Delta}_{\vec{\mathbf{a}}}$ is injective for $|a_Q| < \eta_Q$. Hence it must be an isomorphism, as it is Fredholm of index zero. $\square$

4.2. Transmission problems. The results of the previous section remain valid for general operators and transmission problems with $\bar{\Omega} = \cup\bar{\Omega}_j$, with a different (more complicated) definition of the sets Σ_Q. We consider only the case $P = - \operatorname{div} A\nabla u = \Delta_A$, where A is a *piecewise constant function*. Then, on each subdomain Ω_j, Δ_A is a constant multiple of the Laplacian and the associated conormal derivative is a constant multiple of ∂_ν, ν the unit outer normal. We assume all singular points on $\partial\Omega_j$ on the boundary of Ω are in the set of vertices of the adjacent domains Ω_j. Moreover, we assume that the points where the interfaces intersect are also among the vertices of some Ω_j.

Then for each vertex Q, the set Σ_Q is determined by $\{\pm\sqrt{\lambda}\}$, where λ ranges through the set of eigenvalues of $-\partial_\theta A\partial_\theta$ on $H^2([0, \alpha_Q])$ with suitable boundary conditions. When Q an *internal singular point*, we consider the operator $-\partial_\theta A\partial_\theta$ on $H^2([0, 2\pi])$ with *periodic* boundary conditions. We still take $\eta_Q > 0$ to be the least value in $\Sigma_Q \cap (0, \infty)$.

We define again $\tilde{\Delta}_{\vec{\mathbf{a}}} = (\Delta_A, \partial_\nu)$ but only for $m = 0$ or 1. For $m = 0$, it is given as in equation (4.4) with $(\tilde{\Delta}_A u, v) = -(A\nabla u, \nabla v)$. For $m = 1$, the transmission conditions $u^+ = u^-$ and $A^+\partial_\nu^+ u = A^-\partial_\nu^- u$ must be incorporated. Here A^+ and A^- are the limit values of A at the two sides of the interface Γ (notice that A is only locally constant on Γ). In view of Corollary 3.4, we set

$$(4.7) \quad \tilde{\Delta}_{\vec{\mathbf{a}}} := (\Delta, \partial_\nu) : \mathcal{D}_{\vec{\mathbf{a}}} \to \mathcal{K}^0_{\vec{\mathbf{a}}-1}(\Omega) \oplus \mathcal{K}^{1/2}_{\vec{\mathbf{a}}-1/2}(\partial_N\Omega),$$
$$\mathcal{D}_{\vec{\mathbf{a}}} := \{u : \Omega \to \mathbb{R}, \, u|_{\Omega_j} \in \mathcal{K}^2_{\vec{\mathbf{a}}+1}(\Omega_j), \, u|_{\partial_D\Omega} = 0,$$
$$u^+ = u^-, \text{ and } A^+\partial_\nu^+ u = A^-\partial_\nu^- u\}.$$

For higher values of m, additional conditions at the interface are needed. (These conditions are not included in (2.13).) We will however obtain higher regularity on each subdomain.

The theorems of the previous section then remain true for the transmission problem with the following changes. In Theorem 4.1, we take only $m = 0$ or $m = 1$. In Theorem 4.3, we again assume only $m = 0$ or $m = 1$ and in N_0 we also count the number of internal vertices (that is, the vertices on the interface that are not on the boundary). The proofs are as in Kondratiev's paper [46]. Theorem 4.2 is essentially unchanged. In particular, we continue to count twice $0 \in (a_Q, b_Q) \cap \Sigma_Q$, so that N_0 is the number of Neumann–Neumann vertices plus the number of internal vertices. The points where the boundary conditions change (Dirichlet-Neumann points) *are not* included in the calculation of N_0.

Let us state explicitly the form of Theorem 4.5, which will be needed in applications. In the following statement, W_s is the linear span of the functions χ_Q with Q corresponding to Neumann–Neumann vertices *and* internal vertices. We require that all the functions χ_Q have disjoint supports. Also, recall that for each Neumann–Neumann vertex Q, the function χ_Q satisfies $\chi_Q = 0$ on $\partial_D \Omega$ and $\partial_\nu \chi_Q = 0$ on $\partial_N \Omega$. However, the functions χ_Q corresponding to internal vertices Q need not satisfy any boundary conditions.

THEOREM 4.7. *Let* $\vec{\mathbf{a}} = (a_Q)$ *with* $0 < a_Q < \delta_Q$ *and* $m \geq 1$. *Assume that* $\partial_D \Omega \neq \emptyset$. *Then for any* $f : \Omega \rightarrow \mathbb{R}$ *such that* $f|_{\Omega_j} \in \mathcal{K}^{m-1}_{\vec{\mathbf{a}}-1}(\Omega_j)$, *for all* j, *and any* $g_N \in \mathcal{K}^{m-1/2}_{\vec{\mathbf{a}}-1/2}(\partial_N \Omega)$, *we can find a unique* $u = u_{\mathrm{reg}} + w_s$, $u_{\mathrm{reg}} : \Omega \rightarrow \mathbb{R}$, $u_{\mathrm{reg}}|_{\Omega_j} \in \mathcal{K}^{m+1}_{\vec{\mathbf{a}}+1}(\Omega_j)$, $w_s \in W_s$ *satisfying* $- \mathrm{div}\, A \nabla u = f$, $u = 0$ *on* $\partial_D \Omega$, $\partial_\nu u = g_N$ *on* $\partial_N \Omega$, *and the transmission conditions* $u^+ = u^-$ *and* $A^+ \partial_\nu^+ u = A^- \partial_\nu^- u$ *on the interface* Γ. *Moreover,*

$$\|u_{\mathrm{reg}}\|_{\mathcal{K}^1_{\vec{\mathbf{a}}+1}(\Omega)} + \sum_j \|u_{\mathrm{reg}}\|_{\mathcal{K}^{m+1}_{\vec{\mathbf{a}}+1}(\Omega_j)} + \|w_s\| \leq C \Big(\sum_j \|f\|_{\mathcal{K}^{m-1}_{\vec{\mathbf{a}}-1}(\Omega_j)} + \|g_N\|_{\mathcal{K}^{m-1/2}_{\vec{\mathbf{a}}-1/2}(\partial_N \Omega)} \Big),$$

for a constant $C > 0$ *independent of* f *and* g_N. *The same conclusions hold for the pure Neumann problem if constant functions are factored.*

Proof. Assume first $m = 1$. Then the same proof as that of Theorem 4.5 applies, since in this case we can restrict to the boundary and apply Green's formula. For the other values of m we use the case $m = 1$ to show the existence of a solution and then use the regularity result of Theorem 3.1 in each Ω_j.   ☐

We conclude this section with a few simple observations. First of all, any norm can be used on the finite-dimensional space W_s, as they are all equivalent. Secondly, $W_s \cap \mathcal{K}^1_{\vec{\mathbf{a}}+1}(\Omega) = 0$, whenever $a_Q > 0$ for any Neumann–Neumann vertex Q or internal Q. Finally, the condition $a_Q \in (0, \eta_Q)$ can be relaxed to $|a_Q| < \eta_Q$ for the vertices that are either Dirichlet-Dirichlet or Dirichlet-Neumann. We can also increase a_Q, provided that we include more singular functions. Most importantly, since $W_s \cap \mathcal{K}^1_{\vec{\mathbf{a}}+1}(\Omega) \subset H^1(\Omega)$, it follows that the solution provided by the Theorem 4.7 is the same as the weak solution of the Neumann problem provided by the coercivity of the form B_P on $H^1(\Omega)$.

5. Estimates for the Finite Element Method. The purpose of this section is to construct a sequence of (graded) triangular meshes $\mathcal{T}_n$ in the domain Ω that give the quasi-optimal rate of convergence for the Finite Element approximation of the mixed boundary value/interface problem (2.13).

For this and next section we make the following conventions. *We assume that the boundary of* Ω *and the interface* Γ *are piecewise linear and we fix a constant* $m \in \mathbb{N}$ *corresponding to the degree of approximation. For simplicity, we also assume for the theoretical analysis that there are no cracks or vertices touching the boundary, that is that* $\overline{\Omega} = {}^u\overline{\Omega}$. The case when $\overline{\Omega} \neq {}^u\overline{\Omega}$ can be addressed by using neighborhoods and distances in ${}^u\overline{\Omega}$.

5.1. A note on implementation. We include a numerical test on a domain with a crack in Section 6. In these tests, the "right" space of approximation functions consists of functions

defined on $^u\overline{\Omega}$, and not on $\overline{\Omega}$ (we need different limits according to the connected component from which we approach a crack point). Therefore the nodes used in the implementation will include the vertices of $^u\overline{\Omega}$, counted *as many times as they appear in that set.* The same remark applies to ramifying cracks, where even more points have to be considered where the crack ramifies.

5.2. Approximation away from the vertices. We start by discussing the simpler approximation of the solution u *far* from the singular points. We recall that all estimates in the spaces $\mathcal{K}_{\vec{a}}^m$ localize to subsets of Ω.

Let $\tilde{\mathcal{T}}$ be a mesh of Ω. By a mesh or a triangulation of Ω we shall mean the same thing. We denote by $\tilde{S}(\mathcal{T}, m)$ the Finite Element space associated to the mesh $\mathcal{T}$. That is, $\tilde{S}(\mathcal{T}, m)$ consists of all continuous functions $\chi : \bar{\Omega} \to \mathbb{R}$ such that χ coincides with a polynomial of degree $\leq m$ on each triangle $T \in \mathcal{T}$. Eventually, we will restrict ourselves to the smaller subspace $S(\mathcal{T}, m) \subset \tilde{S}(\mathcal{T}, m)$ of functions that are zero on the Dirichlet part of the boundary $\partial_D \Omega$. To simplify our presentation, we assume $g_N = 0$ in this section although our results extend to the case $g_N \neq 0$. Then, the Finite Element solution $u_S \in S(\mathcal{T}, m)$ for equation (2.13) is given by

$$(5.1) \qquad a(u_S, v_S) := \sum_{i,j=1}^{2} \int_{\Omega} A^{ij} \partial_i u_n \partial_j v_n dx = (f, v_S), \qquad \forall v_S \in S(\mathcal{T}, m).$$

We denote by $u_I = u_{I,\mathcal{T},m} \in \tilde{S}(\mathcal{T}, m)$ the Lagrange interpolant of $u \in \mathcal{C}(\Omega)$. We recall its definition for the benefit of the reader. First, given a triangle T, let $[t_0, t_1, t_2]$ be the barycentric coordinates on T. The nodes of the degree m Lagrange triangle T are the points of T whose barycentric coordinates $[t_0, t_1, t_2]$ satisfy $mt_j \in \mathbb{Z}$. The *degree m Lagrange interpolant* $u_{I,\mathcal{T},m}$ of u is the unique function $u_{I,\mathcal{T},m} \in \tilde{S}(\mathcal{T}, m)$ such that $u = u_{I,\mathcal{T},m}$ at the nodes of each triangle $T_i \in \mathcal{T}$. The shorter notation u_I will be used when only one mesh is understood in the discussion (recall that m is fixed). The interpolant u_I has the following approximation property [8, 27, 31, 77].

THEOREM 5.1. *Let $\mathcal{T}$ be a triangulation of Ω. Assume that all triangles T_i in $\mathcal{T}$ have angles $\geq \alpha$ and sides of length $\leq h$. Let $u \in H^{m+1}(\Omega)$ and let $u_I := u_{I,\mathcal{T},m} \in \tilde{S}(\mathcal{T}, m)$ be the degree m Lagrange interpolant of u. Then, there exist a constants $C(\alpha, m) > 0$ independent of u such that*

$$\|u - u_I\|_{H^1(\Omega)} \leq C(\alpha, m) h^m \|u\|_{H^{m+1}(\Omega)}.$$

The following estimate for the interpolation error on a proper subdomain of Ω then follows from the equivalence of the $H^m(\Omega)$-norm and the $\mathcal{K}_{\vec{a}}^m(\Omega)$-norm on proper subsets Ω. Recall the modified distance function ϑ defined in equation (2.5). If G is an open subset of Ω, we define

$$(5.2) \qquad \mathcal{K}_{\vec{a}}^m(G; \vartheta) := \{f : \Omega \to \mathbb{C}, \ \vartheta^{|\alpha| - \vec{a}} \partial^\alpha f \in L^2(G), \ \text{for all } |\alpha| \leq m\}.$$

and we let $\|u\|_{\mathcal{K}_{\vec{a}}^m(G;\vartheta)}$ denote the corresponding norm.

PROPOSITION 5.2. *Fix $\alpha > 0$ and $0 < \xi < \tilde{l}$. Let $G \subset \Omega$ be an open subset such that $\vartheta > \xi$ on G. Let $\mathcal{T} = (T_j)$ be a triangulation of Ω with angles $\geq \alpha$ and sides $\leq h$. Then for each given weight $\vec{a}$, there exists $C = C(\alpha, \xi, m, \vec{a}) > 0$ such that*

$$\|u - u_I\|_{\mathcal{K}_1^1(G;\vartheta)} \leq C \, h^m \|u\|_{\mathcal{K}_{\vec{a}+1}^{m+1}(G;\vartheta)}, \qquad \forall u \in \mathcal{K}_{\vec{a}+1}^{m+1}(\Omega).$$

The next step is to extend the above estimates to hold near the vertices. To this end, we consider the behavior of the $\mathcal{K}_{\vec{\mathbf{a}}}^m$ under appropriate dilations. Let us denote by $B(Q, \tilde{\ell})$ the ball centered at a vertex Q with radius $\tilde{\ell}$. We choose a positive number $\tilde{\ell}$ such that

(i) the sets $\mathcal{V}_i := \Omega \cap B(Q_i, \tilde{\ell})$ are disjoint,

(ii) $\vartheta(x) = |x - Q_i|$ on $\mathcal{V}_i$,

(iii) $\vartheta(x) \geq \tilde{\ell}/2$ outside the set $\mathcal{V} := \cup \mathcal{V}_i$.

We note that the space $\mathcal{K}_{\vec{\mathbf{a}}}^m(\mathcal{V}_i; \vartheta)$ depends only on the weight a_{Q_i}. Hence we will denote it simply by $\mathcal{K}_a^m(\mathcal{V}_i; \vartheta)$ with $a = a_{Q_i}$.

For the rest of this subsection, we fix a vertex $Q = Q_i$, and with abuse of notation we set $\mathcal{V} := \mathcal{V}_i = \Omega \cap B(Q, \tilde{l})$. We then study the local behavior with respect to dilations of a function $v \in \mathcal{K}_{\vec{\mathbf{a}}}^m(\Omega)$ with support in the neighborhood $\mathcal{V}$ of a vertex Q. Therefore, we translate the origin to agree with $\mathbf{Q}$ and call again (x, y) the new coordinates. Let G be a subset of $\mathcal{V}$ such that $\xi \leq \vartheta(x) \leq \tilde{l}$ on G. For any fixed $0 < \lambda < 1$, we set $G' := \lambda G = \{\lambda x \mid x \in G\}$. Then, we define the dilated function $v_\lambda(x) := v(\lambda x)$, for all $(x, y) \in G$. We observe that since $\mathcal{V}$ is a (straight) sector, if $G \subset \mathcal{V}$ then $G' \subset \mathcal{V}$. The following simple dilation lemma can be proved by direct calculation.

LEMMA 5.3. *Let $G \subset \mathcal{V}$ and $G' = \lambda G$, $0 < \lambda < 1$. If $u_\lambda(x) := u(\lambda x)$, then $\|u_\lambda\|_{\mathcal{K}_a^m(G; \vartheta)} = \lambda^{a-1} \|u\|_{\mathcal{K}_a^m(G'; \vartheta)}$ for any $u \in \mathcal{K}_a^m(\mathcal{V}; \vartheta)$.*

Lemma 5.3 and Proposition 5.2 easily give the following interpolation estimate near a vertex Q.

LEMMA 5.4. *Let $G' \subset \mathcal{V}$ be a subset such that $\vartheta > \xi > 0$ on G'. Let $\mathcal{T}$ be triangulation of G' with angles $\geq \alpha$ and sides $\leq h$. Given $u \in \mathcal{K}_{a+1}^{m+1}(\mathcal{V}, \theta)$, $a \geq 0$, the degree m Lagrange interpolant $u_{I,\mathcal{T}}$ of u satisfies*

$$\|u - u_{I,\mathcal{T}}\|_{\mathcal{K}_1^1(G'; \vartheta)} \leq C(\kappa, \alpha, m) \xi^a (h/\xi)^m \|u\|_{\mathcal{K}_{a+1}^{m+1}(G'; \vartheta)}$$

with $C(\kappa, \alpha, m)$ independent of ξ, h, a, and u.

This lemma will be used for $\xi \to 0$, while Proposition 5.2 will be used with a fixed ξ.

5.3. Approximation near the vertices. We are now ready to address approximation near the singular points. To this extent, we work with the smaller Finite Element Space $S(\mathcal{T}, m)$ defined for any mesh $\mathcal{T}$ of Ω as

(5.3) $\qquad S(\mathcal{T}, m) := \tilde{S}(\mathcal{T}, m) \cap \mathcal{H}_{\vec{\mathbf{a}}} = \{\chi \in \tilde{S}(\mathcal{T}, m), \ \chi = 0 \text{ on } \partial_D \Omega\},$

where $\mathcal{H}_{\vec{\mathbf{a}}} = \{u \in \mathcal{K}_{1+\vec{\mathbf{a}}}^1(\Omega), \ u = 0 \text{ on } \partial_D \Omega\}$. This definition takes into account that the variational space associated to the mixed boundary value/interface problem (1.1) is $\mathcal{H}_{\vec{\mathbf{a}}}$.

REMARK 5.5. We recall that when the interface is not smooth or there are Neumman-Neumann vertices, by Theorem 3.2 for any $|a_Q| < \eta_Q$ the variational solution u of (1.1) can be written $u = u_{\mathrm{reg}} + w_s$ with $u_{\mathrm{reg}} : \Omega \to \mathbb{R}$, $u_{\mathrm{reg}}|_{\Omega_j} \in \mathcal{K}_{\vec{\mathbf{a}}+1}^{m+1}(\Omega_j)$, and $w_s \in W_s$. The space W_s is the linear span of functions $\chi_i \in C_c^\infty(\mathcal{V}_i)$, one for each Neumann-Neumann or interface vertex Q_i, such that χ_i equals 1 on $\mathcal{V}_i$ and satisfied $\partial^\nu \chi_i = 0$ on $\partial \Omega$. For each vertex Q, we therefore fix $a_Q \in (0, \eta_Q)$, and we let $\epsilon = \min\{a_Q\}$. With this choice, we have that $u_{\mathrm{reg}} \in H^{1+\epsilon}(\Omega) \subset \mathcal{C}(\overline{\Omega})$, so that the interpolants of u can be defined directly, since W_s consists of smooth functions. Moreover, the condition that $\vartheta^{-(1+\epsilon)} u_{\mathrm{reg}}$ be integrable in a neighborhood of each vertex shows that u_{reg} must vanish at each vertex. Therefore $u(Q) = w(Q)$ for each Neumann-Neumann or interface vertex Q.
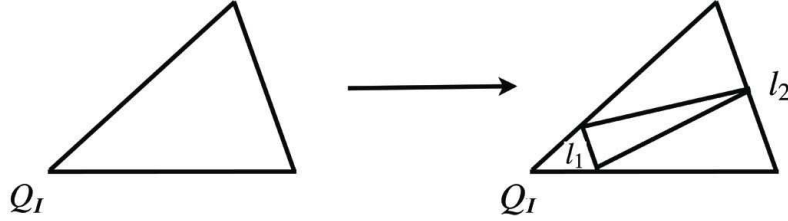
FIG. 5.1. *One refinement of the triangle T with vertex Q, $\kappa = l_1/l_2$.*

We now ready to introduce the mesh refinement procedure. For each vertex Q, we choose a number $\kappa_Q \in (0, 1/2]$ and set $\kappa = (\kappa_Q)$.

DEFINITION 5.6. *Let $\mathcal{T}$ be a triangulation of Ω such that no two vertices of Ω belong to the same triangle of $\mathcal{T}$. The κ refinement of $\mathcal{T}$, denoted by $\kappa(\mathcal{T})$ is obtained by dividing each side AB of $\mathcal{T}$ in two parts as follows. If neither A nor B is a vertex, then we divide AB into two equal parts. Otherwise, if A is a vertex, we divide AB into AC and CB such that $|AC| = \kappa_Q|AB|$.*

This procedure will divide each triangle T into four triangles. (See Figure 5.1). Let us notice that the assumption that no two vertices of Ω belong to the same triangle of the mesh is not really needed. Any reasonable division of an initial triangulation will achieve this condition. For instance, we suggest that if two vertices of Ω belong to the same triangle of the mesh, then the corresponding edge should be divided into equal parts or in a ratio given by the ratio of the corresponding κ constants.

DEFINITION 5.7. *We define by induction $\mathcal{T}_{n+1} = \kappa(\mathcal{T}_n)$, where the initial mesh $\mathcal{T}_0$ is such that every vertex of Ω is a vertex of a triangle in $\mathcal{T}_0$ and all sides of the interface Γ coincide with sides in the mesh. In addition, we choose $\mathcal{T}_0$ such that there is no triangle that contains more than one vertex and each edge in the mesh has length $\leq \tilde{\ell}/2$ (with $\tilde{\ell}$ chosen as in Section 5.2).*

We observe that, near the vertices, this refinement coincides with the ones introduced in [3, 12, 17, 70] for the Dirichlet problem. One of the main results of this work is to show that the same type of mesh gives optimal rates of convergence for mixed boundary value and interface problems as well.

We denote by $u_{I,n} = u_{I,\mathcal{T}_n,m} \in S_n := S(\mathcal{T}_n, m)$ the degree m Lagrange interpolant associated to $u \in \mathcal{C}(\overline{\Omega})$ and the mesh $\mathcal{T}_n$ on Ω, and investigate the approximation properties afforded by the triangulation $\mathcal{T}_n$ close to a fixed vertex Q. The most interesting cases are when Q is either a Neumann-Neumann vertex or a vertex of the interface. We shall therefore assume that this is the case in what follows. With abuse of notation we let $a = a_Q$ and $\kappa = \kappa_Q$ with $\kappa_Q \in (0, 2^{-m/a_Q})$. We also fix a triangle $T \in \mathcal{T}_0$ that has Q as a vertex. Then Theorem 4.7 gives that the solution u of our interface problem decomposes as $u = u_{reg} + w_s$, with $u_{reg} \in \mathcal{K}_{a+1}^{m+1}(T; \vartheta)$ and $w_s \in W_s$, if $f \in \mathcal{K}_{\mathbf{a}-1}^{m-1}(\Omega_j)$ and $T \subset \Omega_j$.

We next let $T_{\kappa^n} = \kappa^n T \in \mathcal{T}_n$ be the triangle that is similar to T with ratio κ^n, has Q as a vertex, and has all sides parallel to the sides of T. Then $T_{\kappa^n} \subset T_{\kappa^{n-1}}$ for $n \geq 1$ (with $T_{\kappa^o} = T$). Furthermore, since $\kappa < 1/2$ and the diameter of T is $\leq \tilde{\ell}/2$, we have $T_{\kappa^j} \subset \mathcal{V} = B(O, \tilde{\ell}) \cap \Omega$ for all $n \geq 0$. Recall that we assume all functions in W_s are constant on neighborhoods of vertices. We continue to fix $T \in T_0$ with vertex Q. The following interpolation estimate holds.

LEMMA 5.8. *Let $0 < \kappa = \kappa_Q \leq 2^{-m/a_Q}$ and $0 < a = a_Q < \eta_Q$. Let us denote by $T_{\kappa^N} = \kappa^N T \subset T$ the triangle with vertex Q obtained from T after N refinements. Let $u_{I,N}$ be the degree m Lagrange interpolant of u associated to $\mathcal{T}_N$. Then, if $u \in \left(\mathcal{K}_{a+1}^{m+1}(\mathcal{V}; \vartheta) + \right.$*

$W_s) \cap \{u|_{\partial_D \Omega} = 0\}$ *on* $T_{\kappa^N} \in \mathcal{T}_N$, *we have*

$$\|u - u_{I,N}\|_{\mathcal{K}_1^1(T_{\kappa^N};\vartheta)} \leq C 2^{-mN} \|u_{\text{reg}}\|_{\mathcal{K}_{a+1}^{m+1}(T_{\kappa^N};\vartheta)},$$

where C depends on m and κ, but not on N.

   Proof. By hypothesis $u = u_{\text{reg}} + w$, with $u_{\text{reg}} \in \mathcal{K}_{a+1}^{m+1}(\Omega)$ and $w \in W_s$. To simplify the notation, we let $\phi = u_{\text{reg}}$. By Remark 5.5, if N is large enough we can assume that $w = u(Q)$ a constant on T_{κ^N}. We again denote the dilated function $\phi_\lambda(x,y) = \phi(\lambda x, \lambda y)$, where (x,y) are coordinates at Q and $0 < \lambda < 1$. We choose $\lambda = \kappa^{N-1}$. Then, $\phi_\lambda(x,y) \in \mathcal{K}_{a+1}^{m+1}(T_\kappa;\vartheta)$ by Lemma 5.3. We next introduce the auxiliary function $v = \chi\phi_\lambda$ on T_κ, where $\chi : T_\kappa \to [0,1]$ is a smooth function that depends only on ϑ and is equal to 0 in a neighborhood of Q, but is equal to 1 at all the nodal points different from Q. Consequently,

$$\|v\|_{\mathcal{K}_1^{m+1}(T_\kappa;\vartheta)}^2 = \|\chi\phi_\lambda\|_{\mathcal{K}_1^{m+1}(T_\kappa;\vartheta)}^2 \leq C\|\phi_\lambda\|_{\mathcal{K}_1^{m+1}(T_\kappa;\vartheta)}^2,$$

where C depends on m and the choice of the nodal points. Moreover, since $\phi(Q) = 0$ by Remark 5.5, the interpolant of v if given by $v_I = (\phi_\lambda)_I = (\phi_I)_\lambda$ on T_κ. We also observe that the interpolant of w on T_{κ^N} is equal to w, because they are both constants, and hence $u - u_I = \phi - \phi_I$. Therefore

$$\|u - u_I\|_{\mathcal{K}_1^1(T_{\kappa^N};\vartheta)} = \|\phi - \phi_I\|_{\mathcal{K}_1^1(T_{\kappa^N};\vartheta)} = \|\phi_\lambda - \phi_{\lambda I}\|_{\mathcal{K}_1^1(T_\kappa;\vartheta)}$$

$$= \|\phi_\lambda - v + v - \phi_{\lambda I}\|_{\mathcal{K}_1^1(T_\kappa;\vartheta)} \leq \|\phi_\lambda - v\|_{\mathcal{K}_1^1(T_\kappa)} + \|v - \phi_{\lambda I}\|_{\mathcal{K}_1^1(T_\kappa;\vartheta)}$$

$$= \|\phi_\lambda - v\|_{\mathcal{K}_1^1(T_\kappa;\vartheta)} + \|v - v_I\|_{\mathcal{K}_1^1(T_\kappa;\vartheta)} \leq C\|\phi_\lambda\|_{\mathcal{K}_1^1(T_\kappa;\vartheta)} + C\|v\|_{\mathcal{K}_1^{m+1}(T_\kappa;\vartheta)}$$

$$\leq C\|\phi_\lambda\|_{\mathcal{K}_1^1(T_\kappa;\vartheta)} + C\|\phi_\lambda\|_{\mathcal{K}_1^{m+1}(T_\kappa;\vartheta)} = C\|\phi\|_{\mathcal{K}_1^1(T_{\kappa^N};\vartheta)} + C\|\phi\|_{\mathcal{K}_1^{m+1}(T_{\kappa^N};\vartheta)}$$

$$\leq C\kappa^{Na}\|\phi\|_{\mathcal{K}_{a+1}^{m+1}(T_{\kappa^N};\vartheta)} \leq C 2^{-mN}\|\phi\|_{\mathcal{K}_{a+1}^{m+1}(T_{\kappa^N};\vartheta)} = C 2^{-mN}\|u_{\text{reg}}\|_{\mathcal{K}_{a+1}^{m+1}(T_{\kappa^N};\vartheta)},$$

which gives the desired inequality. The second and the eighth relations above are due to Lemma 5.3, and the sixth is due to Proposition 5.2.   □

   We now combine the bounds on T_{κ^N} of the previous lemma with the bounds on sets of the form $T_{\kappa^j} \setminus T_{\kappa^{j+1}}$ of Lemma 5.4 to obtain the following estimate on an arbitrary, but fixed, triangle $T \in \mathcal{T}_0$ that has a vertex Q in common with Ω (the more difficult case not handled by Proposition 5.2).

   PROPOSITION 5.9. *Let* $T \in \mathcal{T}_0$ *such that a vertex Q of T belongs to* $\mathcal{V}$. *Let* $0 < \kappa_Q \leq 2^{-m/a_Q}$, $0 < a_Q < \eta_Q$. *Then there exists a constant* $C > 0$, *such that*

$$\|u - u_{I,N}\|_{\mathcal{K}_1^1(T;\vartheta)} \leq C 2^{-mN}\|u_{\text{reg}}\|_{\mathcal{K}_{a+1}^{m+1}(T;\vartheta)},$$

for all $u = u_{\text{reg}} + w$, *where* $w \in W_s$ *and* $u_{\text{reg}} \in \mathcal{K}_{a+1}^1(\Omega)$ *is such that* $u_{\text{reg}} \in \mathcal{K}_{a+1}^{m+1}(\Omega_j)$, *for all* j.

   Proof. As before, we set $\kappa_Q = \kappa$ and $a_Q = a$. As in the proof of Lemma 5.8, we have $u - u_I = u_{\text{reg}} - u_{\text{reg},I}$. We may thus assume that $u = u_{\text{reg}}$. The rest is as in [17, 18].   □

   REMARK 5.10. If $\mathbb{T}$ denotes the union of all the initial triangles that contain vertices of Ω, then $\mathbb{T}$ is a neighborhood of the set of vertices in $\overline{\Omega}$. Furthermore, the interpolation error on $\mathbb{T}$ is obtained as $\|u - u_I\|_{\mathcal{K}_1^1(\mathbb{T};\vartheta)} \leq C 2^{-mN}\|u_{\text{reg}}\|_{\mathcal{K}_{a+1}^{m+1}(\mathbb{T};\vartheta)}$ by summing up the squares of the estimates in Proposition 5.9 over all the triangles, as long as κ_Q is chosen appropriately.

   We now combine all previous results to obtain a global interpolation error estimate on Ω.

   THEOREM 5.11. *Let* $m \geq 1$ *and for each vertex* $Q \in \mathcal{V}$ *fix* $0 < a_Q < \eta_Q$ *and* $0 < \kappa_Q < 2^{-m/a_Q}$. *Assume that the conditions of Theorem 4.7 are satisfied and let* u

be the corresponding solution problem (2.13) *with* $f : \Omega \to \mathbb{R}$ *such that* $f \in \mathcal{K}^{m-1}_{\overrightarrow{\mathbf{a}}-1}(\Omega_j)$ *for all* j. *Let* $\mathcal{T}_n$ *be the n-th refinement of an intial triangulation* $\mathcal{T}_o$ *as in Definition* 5.7. *Let* $S_n := S_n(\mathcal{T}_n, m)$ *be the associated Finite Element space given in equation* (5.3) *and let* $u_n = u_{S_n} \in S_n$ *be the Finite Element solution defined in* (5.1). *Then there exists* $C > 0$ *such that*

$$\|u - u_n\|_{\mathcal{K}^1_1(\Omega)} \le C 2^{-mn} \sum_j \|f\|_{\mathcal{K}^{m-1}_{\overrightarrow{\mathbf{a}}-1}(\Omega_j)}.$$

Proof. Let $\mathbb{T}_i$ be the union of initial triangles that contain a given vertex Q_i. Recall from Theorem 4.7 that the solution of problem (2.13) can be written as $u = u_{\mathrm{reg}} + w$ with $w \in W_s$ and $\|w\| + \sum_j \|u_{\mathrm{reg}}\|_{\mathcal{K}^{m+1}_{\overrightarrow{\mathbf{a}}+1}(\Omega_j)} \le C \sum_j \|f\|_{\mathcal{K}^{m-1}_{\overrightarrow{\mathbf{a}}-1}(\Omega_j)}$. Because $u - u_I = u_{\mathrm{reg}} - u_{\mathrm{reg},I}$ on $\mathcal{V}_i$, we use the previous estimates to obtain

$$\begin{aligned}
\|u - u_n\|_{\mathcal{K}^1_1(\Omega)} &\le C\|u - u_I\|_{\mathcal{K}^1_1(\Omega)} \\
&\le C \sum_j \left(\|u - u_I\|_{\mathcal{K}^1_1(\Omega_j \smallsetminus \cup \mathbb{T}_i; \vartheta)} + \sum \|u_{\mathrm{reg}} - u_{\mathrm{reg},I}\|_{\mathcal{K}^1_1(\Omega_j \cap \mathbb{T}_i; \vartheta)} \right) \\
&\le C 2^{-mn} \sum_j \left(\|u\|_{\mathcal{K}^{m+1}_1(\Omega_j \smallsetminus \cup \mathbb{T}_i; \vartheta)} + \sum_j \|u_{\mathrm{reg}}\|_{\mathcal{K}^{m+1}_{\overrightarrow{\mathbf{a}}+1}(\Omega_j \cap \mathbb{T}_i; \vartheta)} \right) \\
&\le C 2^{-mn} \left(\sum_j \|u_{\mathrm{reg}}\|_{\mathcal{K}^{m+1}_{\overrightarrow{\mathbf{a}}+1}(\Omega_j)} + \|w\| \right) \le C 2^{-mn} \sum_j \|f\|_{\mathcal{K}^{m-1}_{\overrightarrow{\mathbf{a}}-1}(\Omega_j)}.
\end{aligned}$$

The first inequality is based on Céa's Lemma and the second inequality follows from Propositions 5.2 and 5.9. ☐

We can finally state the main result of this section, namely the quasi-optimal convergence rate of the Finite Element solution computed using the meshes $\mathcal{T}_n$.

THEOREM 5.12. *Under the notation and assumptions of Theorem* 5.11, $u_n = u_{S_n} \in S_n := S(\mathcal{T}_n, m)$ *satisfies*

$$\|u - u_n\|_{\mathcal{K}^1_1(\Omega)} \le C \dim(S_n)^{-m/2} \sum_j \|f\|_{\mathcal{K}^{m-1}_{\overrightarrow{\mathbf{a}}-1}(\Omega_j)},$$

for a constant $C > 0$ *independent of* f *and* n.

Proof. Let again $\mathcal{T}_n$ be the triangulation of Ω after n refinements. Then, the number of triangles is $O(4^n)$ given the refinement procedure of Definition 5.6. Therefore $\dim(S_n) \simeq 4^n$ so that Theorem 5.11 gives

$$\|u - u_n\|_{\mathcal{K}^1_1(\Omega)} \le C 2^{-nm} \sum_j \|f\|_{\mathcal{K}^{m-1}_{a-1}(\Omega_j)} \le C \dim(S_n)^{-m/2} \sum_j \|f\|_{\mathcal{K}^{m-1}_{a-1}(\Omega_j)}.$$

The proof is complete. ☐

Using that $H^{m-1}(\Omega_j) \subset \mathcal{K}^{m-1}_{\overrightarrow{\mathbf{a}}-1}(\Omega_j)$ if $a_Q \in (0, 1)$ for all vertices q, we obtain the following corollary.

COROLLARY 5.13. *Let* $0 < a_Q \le \min\{1, \eta_Q\}$ *and* $0 < \kappa_Q < 2^{-m/a_Q}$ *for each vertex* $Q \in \mathcal{V}$. *Then, under the hypotheses of Theorem* 5.12,

$$\|u - u_n\|_{H^1(\Omega)} \le C\|u - u_n\|_{\mathcal{K}^1_1(\Omega)} \le C \dim(S_n)^{-m/2} \|f\|_{H^{m-1}(\Omega)},$$

for a constant $C > 0$ *independent of* $f \in H^{m-1}(\Omega)$ *and* n.

Note that we do not claim that $u \in \mathcal{K}^1_1(\Omega)$ (which is in general not true).

TABLE 6.1
Convergence history for a crack domain.

$j \backslash \kappa$	$e : \kappa = 0.1$	$e : \kappa = 0.2$	$e : \kappa = 0.3$	$e : \kappa = 0.4$	$e : \kappa = 0.5$
3	0.76	0.79	0.79	0.83	0.77
4	0.88	0.90	0.89	0.82	0.76
5	0.94	0.95	0.91	0.79	0.70
6	0.97	0.97	0.92	0.76	0.63
7	0.99	0.98	0.91	0.73	0.57
8	0.99	0.98	0.91	0.71	0.54
9	1.00	0.99	0.90	0.69	0.52

6. Numerical tests. In this section, we present numerical examples which test for the quasi-optimal rates of convergence established *a priori* in the previous section. The convergence history of the Finite Element solution supports our results. Recall the Finite Element solution $u_n \in S_n$ is defined by

$$a(u_n, v_n) := \sum_{i,j=1}^{2} \int_{\Omega} A^{ij} \partial_i u_n \partial_j v_n dx = (f, v_n), \qquad \forall v_n \in S_n.$$

To verify the theoretical prediction, we focus on the more challenging problem where Neumann-Neumann vertices and interfaces are present. We start by testing different configurations of mixed Dirichlet/Neumann boundary conditions, but no interface, on several different domains for the simple model problem (6.1),

$$(6.1) \qquad -\Delta u = 1 \ \text{in} \ \Omega, \quad u = 0 \ \text{on} \ \partial_D \Omega, \quad \partial_\nu u = 0 \ \text{on} \ \partial_N \Omega.$$

In particular, we consider non-convex domains Ω with a crack. In this case, the optimal grading can be computed explicitly beforehand. We then perform a test for the model transmission problem

$$(6.2) \qquad -\operatorname{div}(a(x, y)\nabla u) = 1 \ \text{in} \ \Omega, \quad u = 0 \ \text{on} \ \partial \Omega,$$

where a is a piece-wise constant function. We have run also a few tests with $m = 2$, which also seem to confirm our theoretical results. However, more refinement steps seem to be necessary in this case to achieve results that are as convincing as in the case $m = 1$. Thus more powerful (i. e., faster) algorithms and codes will need to be used to test the case $m = 2$ completely.

6.1. Domains with cracks and Neumann-Neumann vertices. We discuss the results of two tests for the mixed boundary value problem (6.1). In the first test, we impose pure Dirichlet boundary conditions, i. e., we take $\partial_D \Omega = \partial$, but on a domain with a crack. Specifically, we let $\Omega = (0, 1) \times (0, 1) \smallsetminus \{(x, 0.5), 0 < x < 0.5\}$ with a crack at the point $(0.5, 0.5)$; see Figure 6.1. The presence of the crack forces a singularity for H^2 solutions at the tip of the crack. By the arguments in Section 4, any mesh grading $0 < a < \eta = \pi/2\pi = 1/2$ should yield quasi-optimal rates of convergence as long as the decay ratio κ of triangles in subsequent refinements satisfies $\kappa = 2^{-1/a} < 2^{-1/\eta} = 0.25$ near the crack tip. In fact, in this case the solution is H^2 away from the crack, but is only in H^s, $s < 1 + \eta = 1.5$, near the crack (following [46]). Recall that the mesh size h after j refinements is $O(2^j)$. Thus, quasi-uniform meshes should give a convergence rate no better than $h^{0.5}$ [78].
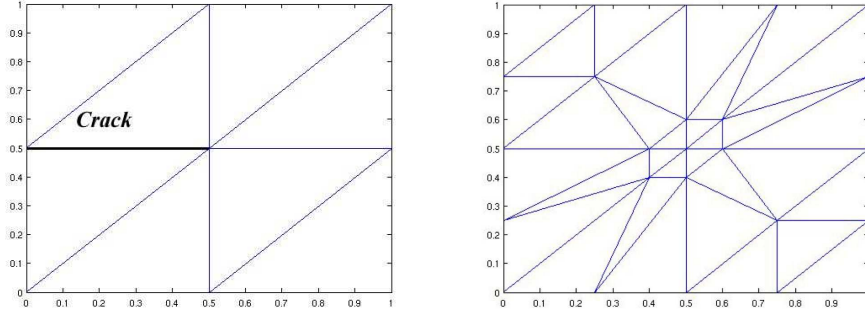
FIG. 6.1. *The domain with a crack: initial triangles (left); the triangulation after one refinement, $\kappa = 0.2$ (right).*
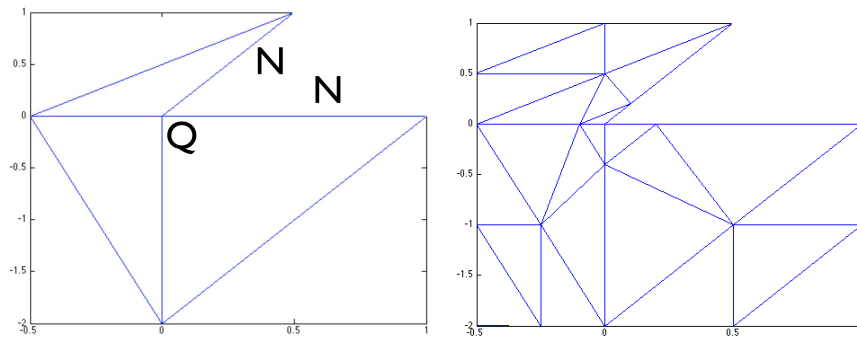

FIG. 6.2. *Initial triangles for a Neumann-Neumann vertex Q (left); the triangulation after one refinement, $\kappa = 0.2$ (right).*

In the second test, Ω is the non-convex domain of Figure 6.2 with a reentrant vertex Q. The interior angle at Q is 1.65π. We impose Neumann boundary conditions on *both* sides adjacent to the vertex Q, and Dirichlet boundary conditions on other edges. Again, an H^2 solution will have a singularity at the reentrant corner In this case, the arguments of Sections 4 and 5 imply that we can take $0 < a < \eta = \pi/1.65\pi \approx 0.61$ for the mesh grading, and consequently, the quasi-optimal rates of convergence should be recovered as long as the decay ratio κ of triangles in subsequent refinements satisfies $\kappa = 2^{-1/a} < 2^{-1/\eta} \approx 0.32$ near Q.

The convergence history for the FEM solutions in the two tests are given respectively in Table 6.1 and Table 6.2. Both tables confirm the predicted rates of convergence. The left-most column in each table of this section contains the number of refinements from the initial triangulation of the domain. In each of the other columns, we list the convergence rate of the numerical solution for the problem (6.1) computed by the formula

$$(6.3) \qquad e = \log_2 \left(\frac{|u_{j-1} - u_j|_{H^1}}{|u_j - u_{j+1}|_{H^1}} \right),$$

where u_j is the Finite Element solution after j mesh refinements. Therefore, since the dimension of the space S_n grows by the factor of 4 with every refinement for linear finite element approximations, e should be very close to 1 if the numerical solutions yield quasi-optimal rates of convergence, an argument convincingly verified in the two tables. In Table 6.2, for example, we achieve quasi-optimal convergence rate whenever the decay ratio $\kappa < 0.32$,
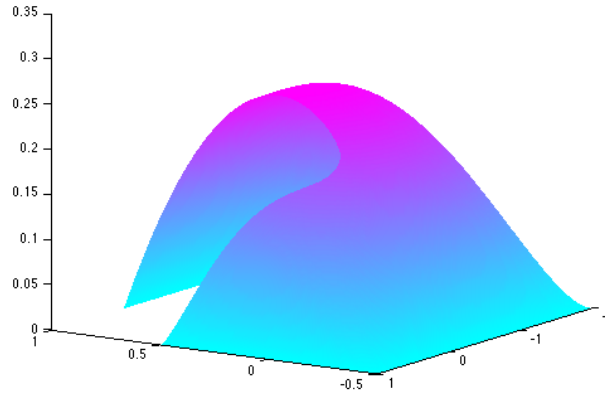
FIG. 6.3. *The numerical solution for the mixed problem with a Neumann-Neumann vertex.*

TABLE 6.2
Convergence history in the case of a Neumann-Neumann vertex.

$j \backslash \kappa$	$e : \kappa = 0.1$	$e : \kappa = 0.2$	$e : \kappa = 0.4$	$e : \kappa = 0.5$
3	0.91	0.93	0.95	0.94
4	0.96	0.97	0.97	0.96
5	0.98	0.99	0.98	0.95
6	0.99	1.00	0.98	0.93
7	1.00	1.00	0.97	0.89
8	1.00	1.00	0.96	0.84

since $e \to 1$ after a few refinements. On the other hand, if $\kappa > 0.32$, the convergence rates decrease with successive refinements due to the effect of the singularity at Q. In fact, for $\kappa = 0.5$ we expect the values of e to approach 0.61, which is the asymptotical convergence rate on quasi-uniform meshes for a function in $H^{1.61}$.

6.2. Domains with artificial vertices. We discuss again a test for the model mixed boundary value problem (6.1), but now we test convergence in the presence of an artificial vertex, where the boundary conditions change on a given side. We take the domain to be the unit square $\Omega = (0,1) \times (0,1)$ and we impose the the mixed boundary conditions $\partial_N \Omega = \{(x,0), 0 < x < 0.5\}$, $\partial_D \Omega = \Omega \smallsetminus \partial_N \Omega$ (see Figure 6.4). In this case, the solution is H^2 near all geometric vertices, as the interior angle is $\pi/2$, but it does possess a singularity at the artificial vertex $Q = (0.5, 0)$, where the boundary conditions change. Near such a vertex, the maximum mesh grading from Section 4 is $\eta_Q = 0.5\pi/\pi = 0.5$. Then, quasi-optimal rates of convergence can be obtained on graded meshes if the decay ratio κ of triangles in subsequent refinements satisfies $0 < \kappa = 2^{-1/a} < 2^{-1/\eta} = 0.25$ near the singular point $(0.5, 0)$. The optimal rate is again supported by the convergence history of the numerical solution in Table 6.3.

6.3. Transmission problems. We discuss finally a test for the model transmission problem (6.2), The singularities in the solution arise from jumps in the coefficient a across the interface. As discussed in Section 5, quasi-optimal rates of convergence can be achieved *a priori* by organizing triangles in the initial triangulation so that each side on the interface is a side of one the triangles as well. We verify *a posteriori* that this construction yields the predicted rates of convergence. We choose the domain again to be the square $\Omega = (-1, 1) \times (-1, 1)$
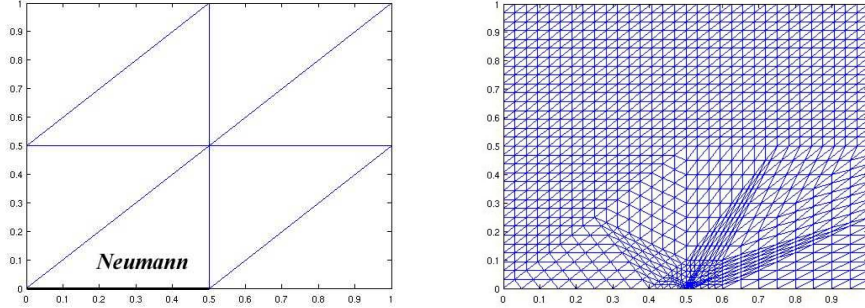
FIG. 6.4. *The domain with an artificial vertex: initial triangles (left); the triangulation after four refinements,*
$\kappa = 0.2$ *(right).*

TABLE 6.3
Convergence history in the case of an artificial vertex.

$j \backslash \kappa$	$e : \kappa = 0.1$	$e : \kappa = 0.2$	$e : \kappa = 0.3$	$e : \kappa = 0.4$	$e : \kappa = 0.5$
3	0.84	0.87	0.87	0.84	0.81
4	0.91	0.93	0.91	0.85	0.77
5	0.95	0.95	0.91	0.82	0.70
6	0.97	0.96	0.91	0.78	0.63
7	0.99	0.97	0.90	0.75	0.57
8	0.99	0.98	0.90	0.72	0.54
9	1.00	0.98	0.89	0.70	0.52

with a single, but nonsmooth, interface Γ as in Figure 6.5, which identifies two subdomains Ω_j, $j = 1, 2$. We also pick the coefficient $a(x, y)$ in (6.2) of the form

$$a(x, y) = \begin{cases} 1 & \text{on } \Omega_1, \\ 30 & \text{on } \Omega_2. \end{cases}$$

The large jump across the interface makes the numerical analysis more challenging. The solution of (6.2) may have singularities in H^2 at the points $Q_1 = (-1, 1)$, $Q_2 = (1, 0)$ where the interface joins the boundary, and at $Q_3 = (0, 0)$, which is a vertex for the interface (there are no singularities again in H^2 at the square geometric vertices).

Again based on the results of Sections 4 and 5, for each singular point Q_i, $i = 1, 2, 3$, there exists a positive number η_i, depending on the interior angle and the coefficients, such that, if the decay rate κ_i of triangles in successive refinements satisfies $0 < \kappa_i < 2^{-1/\eta_i}$ near each vertex Q_1, quasi-optimal rate of convergence can be obtained for the finite element solution. We observe that the solution belongs to H^2 in the neighborhood of a vertex, whenever $\eta_i \geq 1$, and therefore, a quasi-uniform mesh near that vertex is sufficient in this case.

Instead of computing η_i explicitly, as a formula is not readily available, we test different values of $\kappa_i < 0.5$ near each singular points until we obtain values of e approaching 1. This limit signals, as discussed above, that we have reached quasi-optimal rates of convergence for the numerical solution. The value of e is given in equation (6.3). Once again, the convergence history in Table 6.4 strongly supports the theoretical findings. In particular, no special mesh grading is needed near the points $(-1, 1)$ and $(1, 0)$. Near the internal vertex $(0, 0)$, however, we found the optimal grading ratio to be $\kappa_3 \in (0.3, 0.4)$, in agreement with the results of
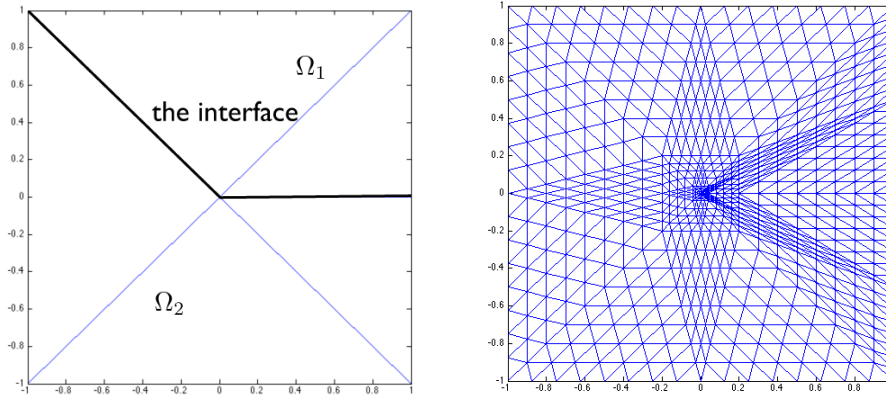
FIG. 6.5. *The transmission problem: initial triangles (left); the triangulation after four refinements, $\kappa = 0.2$ (right).*
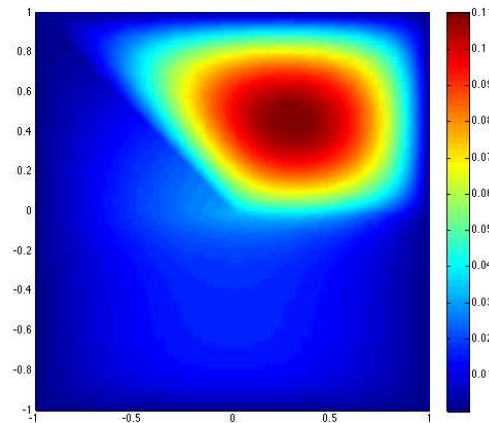

FIG. 6.6. *The numerical solution for the transmission problem.*

Theorem 4.7 and Theorem 5.12. Figure 6.5 shows the mesh refinement near $(0,0)$ when $\kappa = 0.2$.

We notice that our tests involve values of κ that are very small, yet the optimal convergence rate is preserved. We expect, however, that if κ becomes even smaller, it may take longer to observe the optimal rate of convergence. On the other hand, the angle condition is not an issue, by the results of Babuška and Aziz, who have shown that the problem arises when some angles of the triangles become large [9]. In fact, in our refinement, the maximum size of the resulting angles does not increase with each refinement. This maximum size can also be chosen not to be too large in the initial triangulation, and hence in all triangulations. However, as κ becomes smaller, our procedure leads to smaller and smaller angles, although the minimum size of these angles do not decrease with each refinement. Even the smallness of the angles can be dealt with by choosing a different method of dividing the triangles close to the singularities, leading to a slightly different graded mesh, as in [52]. The constant κ_Q associated to each singular point Q will be the same in the new family of graded meshes.

TABLE 6.4
Convergence history for the transmission problem.

$j\backslash\kappa$	$e:\kappa=0.1$	$e:\kappa=0.2$	$e:\kappa=0.3$	$e:\kappa=0.4$	$e:\kappa=0.5$
3	0.82	0.83	0.84	0.83	0.78
4	0.91	0.91	0.91	0.90	0.83
5	0.97	0.97	0.96	0.94	0.86
6	0.99	0.99	0.98	0.95	0.85
7	1.00	0.99	0.99	0.95	0.82
8	1.00	1.00	1.00	0.95	0.80

The methods used in this paper can be generalized to deal with polyhedral domains in three dimensions. See [19] and the references therein. However, the resulting algorithm is significantly more complicated and it leads to meshes that do not satisfy the minimum angle condition. On the other hand, for point singularities such as the ones arising in the study of Schrödinger operators, this procedure simplifies and is almost identical to the one in two dimensions presented here [15, 41].

6.4. Conclusion. It is well known that the singular solutions of elliptic equations in polyhedral domains can be conveniently studied using the weighted Sobolev spaces $\mathcal{K}_a^m$. However, the classical results on solvability (i. e., well-posedness) in weighted spaces $\mathcal{K}_a^m$ do not extend to the case of boundary-value problems where adjacent sides of a corner are endowed with Neumann boundary conditions, or to transmission problems. In this paper, we succeed to establish new a priori estimates (well-posedness, regularity and the Fredholm property) for the solution of the transmission problem (1.1) in augmented weighted Sobolev spaces (see Section 4) in the presence of non-smooth interfaces and Neumann–Neumann vertices. Using these theoretical results, we construct a class of graded meshes that recover the optimal rate of convergence of the Finite Element approximation. Our numerical tests for different problems give convincing evidence of the improvement in the convergence rate on these graded meshes. The use of augmented weighted Sobolev spaces in the analysis of other numerical methods for these transmission problems, for example in the study of the adaptive Finite Element Method, is a promising future direction of our research [59, 63].

REFERENCES

[1] B. AMMANN AND V. NISTOR, *Weighted sobolev spaces and regularity for polyhedral domains*, CMAM, 196 (2007), pp. 3650–3659.

[2] T. APEL, *Anisotropic finite elements: local estimates and applications*, Advances in Numerical Mathematics, B. G. Teubner, Stuttgart, 1999.

[3] T. APEL, A. SÄNDIG, AND J. R. WHITEMAN, *Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains*, Math. Methods Appl. Sci., 19 (1996), pp. 63–85.

[4] D. ARNOLD AND R. FALK, *Well-posedness of the fundamental boundary value problems for constrained anisotropic elastic materials*, Arch. Rational Mech. Anal., 98 (1987), pp. 143–165.

[5] D. ARNOLD, R. SCOTT, AND M. VOGELIUS, *Regular inversion of the divergence operator with Dirichlet boundary conditions on a polygon*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 15 (1988), pp. 169–192 (1989).

[6] A. K. AZIZ AND B. KELLOGG, *On homeomorphisms for an elliptic equation in domains with corners*, Differential Integral Equations, 8 (1995), pp. 333–352.

[7] I. BABUŠKA, *Finite element method for domains with corners*, Computing, 6 (1970), pp. 264–273.

[8] I. BABUŠKA AND A. K. AZIZ, *Survey Lectures on the Mathematical Foundations of the Finite Element Method*, Proc. Sympos., Univ. Maryland, Baltimore, 1972.

[9] ———, *On the angle condition in the finite element method*, SIAM J. Numer. Anal., 13 (1976), pp. 214–226.

[10] I. BABUŠKA, U. BANERJEE, AND J. OSBORN, *Survey of meshless and generalized finite element methods: a unified approach*, Acta Numer., 12 (2003), pp. 1–125.

[11] I. BABUŠKA AND B. Q. GUO, *The h-p version of the finite element method for domains with curved boundaries*, SIAM J. Numer. Anal., 25 (1988), pp. 837–861.

[12] I. BABUŠKA, R. B. KELLOGG, AND J. PITKÄRANTA, *Direct and inverse error estimates for finite elements with mesh refinements*, Numer. Math., 33 (1979), pp. 447–471.

[13] I. BABUŠKA, T. VON PETERSDORFF, AND B. ANDERSSON, *Numerical treatment of vertex singularities and intensity factors for mixed boundary value problems for the Laplace equation in $\mathbb{R}^3$*, SIAM J. Numer. Anal., 31 (1994), pp. 1265–1288.

[14] C. BACUTA, J. BRAMBLE, AND J. XU, *Regularity estimates for elliptic boundary value problems in Besov spaces*, Math. Comp., 72 (2003), pp. 1577–1595.

[15] C. BACUTA, H. LI, AND V. NISTOR, *Analysis of Schrödinger operators with more singluar potentials.* work in progress, preliminary title.

[16] C. BACUTA, A. MAZZUCATO, V. NISTOR, AND L. ZIKATANOV, *Interface and mixed boundary value problems on n-dimensional polyhedral domains.* IMA preprint, 2009.

[17] C. BACUTA, V. NISTOR, AND L. ZIKATANOV, *Improving the rate of convergence of 'high order finite elements' on polygons and domains with cusps*, Numer. Math., 100 (2005), pp. 165–184.

[18] ———, *Improving the rate of convergence of high-order finite elements on polyhedra. I. A priori estimates*, Numer. Funct. Anal. Optim., 26 (2005), pp. 613–639.

[19] ———, *Improving the rate of convergence of high-order finite elements on polyhedra. II. Mesh refinements and interpolation*, Numer. Funct. Anal. Optim., 28 (2007), pp. 775–824.

[20] U. BANERJEE AND J. OSBORN, *Estimation of the effect of numerical integration in finite element eigenvalue approximation*, Numer. Math., 56 (1990), pp. 735–762.

[21] M. BERNDT, T. A. MANTEUFFEL, AND S. MCCORMICK, *Analysis of first-order system least squares (FOSLS) for elliptic problems with discontinuous coefficients. II*, SIAM J. Numer. Anal., 43 (2005), pp. 409–436.

[22] M. BERNDT, T. A. MANTEUFFEL, S. MCCORMICK, AND G. STARKE, *Analysis of first-order system least squares (FOSLS) for elliptic problems with discontinuous coefficients. I*, SIAM J. Numer. Anal., 43 (2005), pp. 386–408.

[23] P. BINEV, W. DAHMEN, AND R. DEVORE, *Adaptive finite element methods with convergence rates*, Numer. Math., 97 (2004), pp. 219–268.

[24] J. BRANNICK, H. LI, AND L. ZIKATANOV, *Uniform convergence of the multigrid V-cycle on graded meshes for corner singularities*, Numer. Linear Algebra Appl., 15 (2008), pp. 291–306.

[25] S. C. BRENNER, *Multigrid methods for the computation of singular solutions and stress intensity factors. I. Corner singularities*, Math. Comp., 68 (1999), pp. 559–583.

[26] S. C. BRENNER, J. CUI, F. LI, AND L.-Y. SUNG, *A nonconforming finite element method for a two-dimensional curl-curl and grad-div problem*, Numer. Math., 109 (2008), pp. 509–533.

[27] S. C. BRENNER AND R. SCOTT, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer, New York, second ed., 2002.

[28] S. C. BRENNER AND L.-Y. SUNG, *Multigrid methods for the computation of singular solutions and stress intensity factors. II. Crack singularities*, BIT, 37 (1997), pp. 623–643.

[29] Z. CAI, T. A. MANTEUFFEL, S. F. MCCORMICK, AND J. RUGE, *First-order system LL^* (FOSLL*):*

scalar elliptic partial differential equations, SIAM J. Numer. Anal., 39 (2001), pp. 1418–1445.

[30] F. CAKONI, D. COLTON, AND P. MONK, *On the use of transmission eigenvalues to estimate the index of refraction from far field data*, Inverse Problems, 23 (2007), pp. 507–522.

[31] P. CIARLET, *The Finite Element Method for Elliptic Problems*, vol. 4 of Studies in Mathematics and Its Applications, North-Holland, Amsterdam, 1978.

[32] M. COSTABEL, *Boundary integral operators on curved polygons*, Ann. Mat. Pura Appl. (4), 133 (1983), pp. 305–326.

[33] M. COSTABEL AND M. DAUGE, *Singularities of electromagnetic fields in polyhedral domains*, Arch. Ration. Mech. Anal., 151 (2000), pp. 221–276.

[34] M. DAUGE, *Elliptic Boundary Value Problems on Corner Domains*, vol. 1341 of Lecture Notes in Mathematics, Springer, Berlin, 1988.

[35] C. DUARTE, O. HAMZEH, T. LISZKA, AND W. TWORZYDLO, *A generalized finite element method for the simulation of three dimensional dynamic crack propagation*, Comput. Methods Appl. Mech. Engrg., 190 (2001), pp. 2227–2262.

[36] P. GRISVARD, *Elliptic problems in nonsmooth domains*, vol. 24 of Monographs and Studies in Mathematics, Pitman, Boston, 1985.

[37] B. GUO AND I. BABUŠKA, *Regularity of the solutions for elliptic problems on nonsmooth domains in* $\mathbf{R}^3$. *I. Countably normed spaces on polyhedral domains*, Proc. Roy. Soc. Edinburgh Sect. A, 127 (1997), pp. 77–126.

[38] J. GUZMÁN, *Quadrature and Schatz's pointwise estimates for finite element methods*, BIT, 45 (2005), pp. 695–707.

[39] W. HACKBUSCH, *Multigrid methods and applications*, vol. 4 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1985.

[40] G. HARUTYUNYAN AND B.-W. SCHULZE, *Elliptic mixed, transmission and singular crack problems*, vol. 4 of EMS Tracts in Mathematics, European Mathematical Society, Zürich, 2008.

[41] E. HUNSICKER, V. NISTOR, AND J. SOFO, *Analysis of periodic Schrödinger operators: regularity and approximation of eigenfunctions*, J. Math. Phys., 49 (2008), pp. 083501, 21.

[42] R. B. KELLOGG, *Singularities in interface problems*, in Numerical Solution of Partial Differential Equations, II (SYNSPADE 1970) (Proc. Sympos., Univ. of Maryland, College Park, Md., 1970), Academic Press, New York, 1971, pp. 351–400.

[43] ———, *Higher order singularities for interface problems*, in The mathematical foundations of the finite element method with applications to partial differential equations (Proc. Sympos., Univ. Maryland, Baltimore, Md., 1972), Academic Press, New York, 1972, pp. 589–602.

[44] ———, *On the Poisson equation with intersecting interfaces*, Applicable Anal., 4 (1974/75), pp. 101–129.

[45] R. B. KELLOGG AND J. E. OSBORN, *A regularity result for the Stokes problem in a convex polygon*, J. Functional Analysis, 21 (1976), pp. 397–431.

[46] V. A. KONDRAT′EV, *Boundary value problems for elliptic equations in domains with conical or angular points*, Transl. Moscow Math. Soc., 16 (1967), pp. 227–313.

[47] V. KOZLOV, V. MAZ′YA, AND J. ROSSMANN, *Spectral problems associated with corner singularities of solutions to elliptic equations*, vol. 85 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2001.

[48] R. LAUTER AND V. NISTOR, *Analysis of geometric operators on open manifolds: a groupoid approach*, in Quantization of Singular Symplectic Quotients, N. Landsman, M. Pflaum, and M. Schlichenmaier, eds., vol. 198 of Progress in Mathematics, Birkhäuser, Basel - Boston - Berlin, 2001, pp. 181–229.

[49] E. LEE AND T. A. MANTEUFFEL, *FOSLL* method for the eddy current problem with three-dimensional edge singularities*, SIAM J. Numer. Anal., 45 (2007), pp. 787–809.

[50] E. LEE, T. A. MANTEUFFEL, AND C. WESTPHAL, *Weighted-norm first-order system least squares (FOSLS) for problems with corner singularities*, SIAM J. Numer. Anal., 44 (2006), pp. 1974–1996.

[51] R. LEVEQUE AND Z. LI, *The immersed interface method for elliptic equations with discontinuous coefficients and singular sources*, SIAM J. Numer. Anal., 31 (1994), pp. 1019–1044.

[52] H. LI, A. MAZZUCATO, AND V. NISTOR, *Numerical methods for exceptional domains*. in preparation.

[53] Y. LI AND L. NIRENBERG, *Estimates for elliptic systems from composite material*, Comm. Pure Appl. Math., 56 (2003), pp. 892–925.

[54] Y. LI AND M. VOGELIUS, *Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients*, Arch. Ration. Mech. Anal., 153 (2000), pp. 91–151.

[55] Z. LI AND S. LUBKIN, *Numerical analysis of interfacial two-dimensional Stokes flow with discontinuous viscosity and variable surface tension*, Internat. J. Numer. Methods Fluids, 37 (2001), pp. 525–540.

[56] J. LUBUMA AND S. NICAISE, *Dirichlet problems in polyhedral domains. II. Approximation by FEM and BEM*, J. Comput. Appl. Math., 61 (1995), pp. 13–27.

[57] V. MAZ′YA AND J. ROSSMANN, *Weighted L_p estimates of solutions to boundary value problems for second order elliptic systems in polyhedral domains*, Z. Angew. Math. Mech., 83 (2003), pp. 435–467.

[58] A. MAZZUCATO AND V. NISTOR, *Well posedness and regularity for the elasticity equation with mixed bound-*

ary conditions on polyhedral domains and domains with cracks, Arch. Ration. Mech. Anal., 195 (2010), pp. 25–73.

[59] K. MEKCHAY AND R. NOCHETTO, *Convergence of adaptive finite element methods for general second order linear elliptic PDEs*, SIAM J. Numer. Anal., 43 (2005), pp. 1803–1827.

[60] R. MELROSE AND J. WUNSCH, *Propagation of singularities for the wave equation on conic manifolds*, Invent. Math., 156 (2004), pp. 235–299.

[61] D. MITREA, M. MITREA, AND Q. SHI, *Variable coefficient transmission problems and singular integral operators on non-smooth manifolds*, J. Integral Equations Appl., 18 (2006), pp. 361–397.

[62] P. MONK, *Finite element methods for Maxwell's equations*, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2003.

[63] P. MORIN, R. NOCHETTO, AND K. SIEBERT, *Convergence of adaptive finite element methods*, SIAM Rev., 44 (2002), pp. 631–658.

[64] S. MOROIANU AND V. NISTOR, *Index and homology of pseudodifferential operators on manifolds with boundary*. in Perspectives in operator algebras and mathematical physics, F. P. Boca, ed., Theta Ser. Adv. Math., Theta, Bucharest, 8, 2008, pp. 123–148.

[65] S. A. NAZAROV AND B. A. PLAMENEVSKY, *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, vol. 13 of Expositions in Mathematics, de Gruyter, New York, 1994.

[66] S. NICAISE, *Polygonal Interface Problems*, Lang, Peter Publishing, 1993.

[67] S. NICAISE AND A. SÄNDIG, *General interface problems. I, II*, Math. Methods Appl. Sci., 17 (1994), pp. 395–429, 431–450.

[68] C. PESKIN, *Numerical analysis of blood flow in the heart*, J. Comput. Phys., 25 (1977), pp. 220–252.

[69] M. PETZOLDT, *Regularity results for laplace interface problems in two dimensions*, Z. Anal. Anwendungen, 20 (2001), pp. 431–455.

[70] G. RAUGEL, *Résolution numérique par une méthode d'éléments finis du problème de Dirichlet pour le laplacien dans un polygone*, C. R. Acad. Sci. Paris Sér. A-B, 286 (1978), pp. A791–A794.

[71] J. ROĬTBERG AND Z. ŠEFTEL′, *On equations of elliptic type with discontinuous coefficients*, Dokl. Akad. Nauk SSSR, 146 (1962), pp. 1275–1278.

[72] ———, *General boundary-value problems for elliptic equations with discontinuous coefficients*, Dokl. Akad. Nauk SSSR, 148 (1963), pp. 1034–1037.

[73] W. RUDIN, *Functional Analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York, second ed., 1991.

[74] A. SCHATZ, I. SLOAN, AND L. WAHLBIN, *Superconvergence in finite element methods and meshes that are locally symmetric with respect to a point*, SIAM J. Numer. Anal., 33 (1996), pp. 505–521.

[75] M. SCHECHTER, *A generalization of the problem of transmission*, Ann. Scuola Norm. Sup. Pisa (3), 14 (1960), pp. 207–236.

[76] E. SCHROHE, *Spectral invariance, ellipticity, and the Fredholm property for pseudodifferential operators on weighted Sobolev spaces.*, Ann. Global Anal. Geom., 10 (1992), pp. 237–254.

[77] C. SCHWAB, *p- and hp- Finite Element Methods: Theory and Applications in Solid and Fluid Mechanics*, Oxford University Press, New-York, 1999.

[78] L. B. WAHLBIN, *On the sharpness of certain local estimates for $\overset{\circ}{H}{}^1$ projections into finite element spaces: influence of a re-entrant corner*, Math. Comp., 42 (1984), pp. 1–8.

[79] L. YING, *Two dimensional interface problems for elliptic equations*, J. Partial Differential Equations, 16 (2003), pp. 37–48.

[80] S. YU, Y. ZHOU, AND G. WEI, *Matched interface and boundary (MIB) method for elliptic problems with sharp-edged interfaces*, J. Comput. Phys., 224 (2007), pp. 729–756.