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NEW QUADRILATERAL MIXED FINITE ELEMENTS*

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Abstract. In this paper, we introduce a new family of mixed finite element spaces of higher order ($k \geq 1$) on general quadrilateral grids. A typical element has two fewer degrees of freedom than the well-known Raviart-Thomas finite element $RT_{[k]}$, yet enjoys an optimal-order approximation for the velocity in L^2-norm. The order of approximation in the divergence norm is one less than the velocity, as is common to all other known elements, except for a recent element introduced by Arnold et al. [SIAM J. Numer. Anal., 42 (2005), pp. 2429–2451]. However, we introduce a local post-processing technique to obtain an optimal order in L^2-norm of divergence. This technique can be used to enhance the result of $RT_{[k]}$ element as well, and hence, can be easily incorporated into existing codes.

Our element has one lower order of approximation in pressure than the $RT_{[k]}$ element. However, the pressure also can be locally post-processed to produce an optimal-order approximation. The greatest advantage of our finite element lies in the fact that it has the fewest degrees of freedom among all the known quadrilateral mixed finite elements and thus, together with the post-processing techniques, provides a very efficient way of computing flow variables in mixed formulation. Numerical examples are in quite good agreement with the theory even for the case of almost degenerate quadrilateral grids.

Key words. Mixed finite element method, quadrilateral grid, optimal velocity, post-processing.

AMS subject classifications. 65N15, 65N30

1. Introduction. The mixed finite element method has been widely used as a tool to obtain a direct approximation of physical quantities such as fluxes and velocities for flow problems. In this method, one introduces a new variable $\mathbf{u} = -\kappa\nabla p$ and designs a finite element method which approximates $\mathbf{u}$ and p simultaneously. For this purpose, one needs to define finite-dimensional subspaces $\mathbf{V}_h$ of $\mathbf{H}(\mathrm{div}\,;\Omega)$ and W_h of $L^2(\Omega)$ which satisfy some stability condition. A variety of optimal-order methods, such as $RT_{[k]}$, $BDM_{[k+1]}$, or $BDFM_{[k]}$, have been developed for triangular and rectangular grids [4, 5, 6, 10] since its introduction by Raviart and Thomas [15]. Among these, $BDM_{[k]}$ has the fewest degrees of freedom and has the same order of accuracy for velocity. For some other aspects of mixed finite elements, we refer to [1, 2, 7, 8, 9, 11].

However, for quadrilateral grids, $BDM_{[k]}$ or $BDFM_{[k+1]}$ suffers from a loss of accuracy, unless the grids are almost parallel, which arise as a result of repeated refinements of a coarse grid, are assumed [3, 9]. So far, the only mixed finite element for general quadrilaterals having optimal order for velocity is $RT_{[k]}$, as shown recently by Arnold, Boffi, and Falk [3]. In fact, they showed that a necessary and sufficient condition for any finite element space $\mathbf{V}_h$ of $\mathbf{H}(\mathrm{div}\,;\Omega)$ to have an optimal order in velocity is for it to contain (in the reference space) a subspace $\mathcal{S}_k$ of the Raviart-Thomas element space of order k, where the two elements $(\hat{x}^{k+1}\hat{y}^k, 0)^T$, $(0, \hat{x}^k\hat{y}^{k+1})^T$ are replaced by the single element $(\hat{x}^{k+1}\hat{y}^k, -\hat{x}^k\hat{y}^{k+1})^T$. This is a proper subspace of $RT_{[k]}$ but properly contains $BDM_{[k]}$ and $BDFM_{[k+1]}$. Their idea of obtaining such a condition is this: in order to have an optimal order in the mapped space, the reference space must contain the inverse image of $\mathbf{P}_k$, the space of polynomials up to degree k, under the Piola map. The resulting condition is the one mentioned earlier.

On the other hand, $RT_{[k]}$ does not have enough polynomials to have optimal order in divergence norm. A necessary and sufficient condition is that the divergence of the local

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velocity space contains (again in the reference space) the space $\mathcal{R}_k$ defined as the space of all polynomials in each variable up to degree $k+1$ except constant multiples of the term $\hat{x}^{k+1}\hat{y}^{k+1}$. However, the pressure space of $RT_{[k]}$ is a proper subspace of $\mathcal{R}_k$. Thus, one has to enrich the pressure space and this, in turn, necessitates the enrichment of the velocity space to satisfy the stability condition: $\operatorname{div}\hat{\mathbf{V}}(\hat{K}) = \hat{W}(\hat{K})$. Hence, Arnold, Boffi, and Falk introduced a new element, called $ABF_{[k]}$, which has significantly more degrees of freedom than $RT_{[k]}$: $ABF_{[0]}$ has six for velocity and three for pressure, and $ABF_{[1]}$ has sixteen and eight, respectively, for each element.

The purpose of this paper is to propose a new mixed finite element space which lies between $BDFM_{[k+1]}$ and $RT_{[k]}$ ($k \geq 1$), yet has an optimal order for velocity on general quadrilateral grids. Obviously, this element has the smallest number of degrees of freedom among all possible mixed finite elements having optimal order for velocity on general quadrilateral grids. Our element for $k = 1$ has eleven degrees of freedom for velocity and three for pressure, and a total of $4k+6$ fewer degrees of freedom than $ABF_{[k]}$ on each element.

Next we introduce a local post-processing of pressure variable to have optimal order, after which we show how this post-processed pressure solution can be used to find optimal divergence.

The organization of this paper is as follows. In the next section, we introduce some basic material for mixed methods, focused on quadrilateral grids. Our new element is introduced and analyzed in Section 3. In Section 4, post-processing techniques to obtain an optimal order in pressure and divergence are presented. Finally, numerical results for our new elements together with the post-processing of pressure and divergence are presented in Section 5.

2. Mixed finite element for quadrilateral grids.

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$. We consider the following second-order elliptic boundary value problem:

$$\begin{aligned} -\operatorname{div}(\kappa\,\nabla p) + cp &= f, \quad \text{in } \Omega,\\ p &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\kappa = \kappa(\mathbf{x})$ is a symmetric and uniformly positive definite matrix, and c and f are any reasonable functions that guarantee the existence of a unique solution. Let us introduce a vector variable $\mathbf{u} = -\kappa\nabla p$ and rewrite the problem (2.1) in the mixed form

$$\begin{aligned} \mathbf{u} + \kappa\nabla p &= 0, \quad \text{in } \Omega,\\ \operatorname{div}\mathbf{u} + cp &= f, \quad \text{in } \Omega,\\ p &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

We need to describe some function spaces. For any domain Ω, we let $L^2(\Omega)$ be the space of all square integrable functions on Ω equipped with the usual inner product $(\cdot,\cdot)_\Omega$. Let $H^i(\Omega) = W^{i,2}(\Omega)$ be the Sobolev spaces of order $i = 0, 1, \cdots$, with obvious norms. Now, let $\mathbf{H}^i(\Omega)$ be the space of vectors $\mathbf{u} = (u, v)$ whose components lie in $H^i(\Omega), i = 0, 1, \cdots$. For both of the spaces $H^i(\Omega)$ and $\mathbf{H}^i(\Omega), i = 0, 1, 2, \cdots$, we shall denote their norms(seminorms) by $\|\cdot\|_{i,\Omega}(|\cdot|_{i,\Omega})$, and the subscript Ω will be dropped when it is clear from the context. Also, let $\mathbf{V} = H(\operatorname{div};\Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 : \operatorname{div}\mathbf{v} \in L^2(\Omega)\}$ with norm $\|\mathbf{v}\|^2_{H(\operatorname{div};\Omega)} = \|\mathbf{v}\|_0^2 + \|\operatorname{div}\mathbf{v}\|_0^2$, and let $W = L^2(\Omega)$. Then we have the following variational form for (2.2):

$$\begin{aligned} (\kappa^{-1}\mathbf{u}, \mathbf{v}) - (\operatorname{div}\mathbf{v}, p) &= 0, \quad \forall \mathbf{v} \in \mathbf{V},\\ (\operatorname{div}\mathbf{u}, q) + (cp, q) &= (f, q), \quad \forall q \in W. \end{aligned} \tag{2.3}$$

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This problem is well-posed by the theory of Brezzi [16], since the form $(\kappa^{-1}\cdot,\cdot)$ is coercive and the form $(\mathrm{div}\,\mathbf{u},q)$ satisfies the inf-sup condition. Let $(\mathbf{u},p)\in\mathbf{V}\times W$ be the unique solution pair for which we would like to find an approximation using finite element spaces. For each $h>0$, let $\mathcal{T}_h=\{K\}$ be a triangulation of the domain $\bar{\Omega}$ into closed triangles, rectangles, or convex quadrilaterals whose diameters are bounded by h. Assume that we have some approximating spaces $\mathbf{V}_h\subset\mathbf{V}$ and $W_h\subset W$ based on these grids. Then the corresponding finite dimensional problem becomes: Find $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times W_h$ such that

$$
\begin{aligned}
(\kappa^{-1}\mathbf{u}_h,\mathbf{v}_h)-(\mathrm{div}\,\mathbf{v}_h,p_h)&=0,\quad \forall\mathbf{v}_h\in\mathbf{V}_h,\\
(\mathrm{div}\,\mathbf{u}_h,q_h)+(cp_h,q_h)&=(f,q_h),\quad \forall q_h\in W_h.
\end{aligned}
\tag{2.4}
$$

First, we assume a triangular or rectangular grid. If the spaces $\mathbf{V}_h$ and W_h are chosen to satisfy a certain compatibility condition known as discrete inf-sup condition together with a certain approximation property, then it is well-known [15, 16], under a certain shape-regularity of $\mathcal{T}_h$, that

$$
\|\mathbf{u}-\mathbf{u}_h\|_0+\|\mathrm{div}\,(\mathbf{u}-\mathbf{u}_h)\|_0+\|p-p_h\|_0\le Ch^{k+1}(\|\mathbf{u}\|_{k+1}+|\mathrm{div}\,\mathbf{u}|_{k+1}),
\tag{2.5}
$$

where k is the order of approximation of the space $\mathbf{V}_h$ and W_h.

Next, we consider quadrilateral grids. Let K be a quadrilateral with diameter h_K whose vertices are $\mathbf{a}_i=(x_i,y_i)$, $i=1,2,3,4$. Also, let $\hat{K}$ be the unit square reference element with vertices $\hat{\mathbf{a}}_i=(\hat{x}_i,\hat{y}_i), i=1,2,3,4$. Then there exists a unique bilinear map F_K from $\hat{K}$ onto K satisfying

$$
F_K(\hat{\mathbf{a}}_i)=\mathbf{a}_i,\quad i=1,\dots,4.
$$

We let DF_K denote its derivative and let J_K be the Jacobian determinant. Let S_i be the subtriangle of K with vertices $\mathbf{a}_{i-1}$, $\mathbf{a}_i$, and $\mathbf{a}_{i+1}$, $i=1,\dots,4$, where $\mathbf{a}_{i+4}=\mathbf{a}_i$.

We assume the usual shape regularity of $\mathcal{T}_h$ in the sense of [12]: There exists a positive constant σ such that

$$
h_K\le\sigma\rho_K,\quad \forall K\in\mathcal{T}_h,
$$

where ρ_K is the minimum of the diameters of the circles inscribed in $S_i, i=1,\cdots,4$.

Now, we need to define the spaces $\mathbf{V}_h$ and W_h. Assuming that $\hat{\mathbf{V}}(\hat{K})$ and $\hat{W}(\hat{K})$ are given, we let

$$
\mathbf{V}_h(K)=\{\mathbf{v}=\mathcal{P}_K\hat{\mathbf{v}}:\ \hat{\mathbf{v}}\in\hat{\mathbf{V}}(\hat{K})\},
$$

and define

$$
\mathbf{V}_h=\{\mathbf{v}\in\mathbf{V}:\mathbf{v}|_K\in\mathbf{V}_h(K)\},
\tag{2.6}
$$

where $\mathcal{P}_K:\mathbf{H}(\mathrm{div}\,;\hat{K})\to\mathbf{H}(\mathrm{div}\,;K)$ is the Piola transform defined by

$$
\mathbf{v}=\mathcal{P}_K\hat{\mathbf{v}}=\frac{DF_K}{J_K}\hat{\mathbf{v}}\circ F_K^{-1}.
$$

This transformation preserves the divergence and flux in the following sense (cf. [6]): Let $q=\hat{q}\circ F_K^{-1}$, where $\hat{q}$ is any scalar function on $\hat{K}$. Then

$$
\begin{aligned}
\mathrm{div}\,\mathbf{v}&=\frac{1}{J_K}\hat{\mathrm{div}}\,\hat{\mathbf{v}},\\
\int_K\nabla q\cdot\mathbf{v}d\mathbf{x}&=\int_{\hat{K}}\hat{\nabla}\hat{q}\cdot\hat{\mathbf{v}}d\hat{\mathbf{x}}\quad \text{for } q\in H^1(K),\\
\int_{\partial K}\mathbf{v}\cdot\mathbf{n}\,q ds&=\int_{\partial\hat{K}}\hat{\mathbf{v}}\cdot\hat{\mathbf{n}}\,\hat{q}d\hat{s}\quad \text{for } q\in H^{1/2}(\partial K).
\end{aligned}
$$

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Finally, we define the finite element space W_h. First, let

$$W_h(K) = \{q = \hat{q} \circ F_K^{-1} : \hat{q} \in \hat{W}(\hat{K})\}$$

and then define

$$W_h = \{q \in L^2(\Omega) : q|_K \in W_h(K)\}. \tag{2.7}$$

A most common example of a mixed element is $RT_{[k]}$, which is defined as

$$\hat{\mathbf{V}}(\hat{K}) = Q_{k+1,k}(\hat{K}) \times Q_{k,k+1}(\hat{K}), \quad \hat{W}(\hat{K}) = Q_{k,k}(\hat{K}).$$

Here, $Q_{i,j}(\Omega)$ for any domain Ω is the space of polynomials of total degree i and j in each variable. For later use, we shall denote by $P_k(\Omega)$ the space of polynomials of total degree k on Ω. The $RT_{[k]}$ element, as mentioned earlier, does not have optimal order in divergence: one has an estimate similar to (2.5), but one order lower in divergence [15]. Here, we present a slightly improved form given by Arnold et al. [3]:

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_0 &\le Ch^{k+1}\|\mathbf{u}\|_{k+1}, \\ \|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h)\|_0 &\le Ch^k\|\mathrm{div}\,\mathbf{u}\|_k, \\ \|p - p_h\|_0 &\le \begin{cases} Ch^{k+1}\|p\|_{k+1} & k \ge 1, \\ Ch\|p\|_2 & k = 0. \end{cases} \end{aligned}$$

One of the reason why one does not have optimal order in divergence is that $\mathrm{div}\,\mathbf{V}_h$ does not contain enough polynomials. So, in order to improve this situation, one has to add more terms in the definition of $\hat{\mathbf{V}}(\hat{K})$. As a result, Arnold et al. introduced a new space, called $ABF_{[k]}(k \ge 0)$, where

$$\hat{\mathbf{V}}(\hat{K}) = Q_{k+2,k}(\hat{K}) \times Q_{k,k+2}(\hat{K}), \quad \hat{W}(\hat{K}) = \mathcal{R}_k,$$

where $\mathcal{R}_k$ is the subspace of $Q_{k+1,k+1}(\hat{K})$ which is spanned by all the polynomials $x^i y^j$, $1 \le i, j \le k+1$, except for $\hat{x}^{k+1}\hat{y}^{k+1}$.

The degrees of freedoms are $2(k+3)(k+1)$ and $(k+2)^2 - 1$, respectively. In this case, it is shown that $\mathbf{V}_h(K) \supset \mathbf{P}_k(K)$, $W(K) \supset P_k(K)$ and $\mathrm{div}\,\hat{\mathbf{V}}(\hat{K}) \supset \hat{W}(\hat{K})$, and therefore,

$$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h)\|_0 \le Ch^{k+1}\|\mathrm{div}\,\mathbf{u}\|_{k+1}. \tag{2.8}$$

3. A new mixed finite element.

In this section, we introduce a new mixed finite element, inspired by the study of Arnold et al. [3]. They introduced necessary and sufficient conditions for the optimal *velocity* and *divergence* approximations, hence designed a new element-ABF to incorporate those conditions fully. But we have found the condition for optimal *velocity* approximation is good enough to determine a new space. Based on this observation, we shall introduce a new space.

For this purpose, let us present necessary and sufficient conditions for optimal velocity and divergence approximations. Let $\mathcal{S}_k$ $(k \ge 1)$ be the subspace of $Q_{k+1,k}(\hat{K}) \times Q_{k,k+1}(\hat{K})$, where $(\hat{x}^{k+1}\hat{y}^k, 0)$ and $(0, \hat{x}^k\hat{y}^{k+1})$ are replaced by the single element $(\hat{x}^{k+1}\hat{y}^k, -\hat{x}^k\hat{y}^{k+1})$. Then we have [3]:

THEOREM 3.1. *Suppose that $\hat{\mathbf{V}}(\hat{K})$ contains $\mathcal{S}_k$. Then there exists a constant C independent of $\mathbf{u}$ such that*

$$\inf_{\mathbf{v} \in \mathbf{V}_h(K)} \|\mathbf{u} - \mathbf{v}\|_0 \le Ch^{k+1}|\mathbf{u}|_{k+1}, \text{ for all } \mathbf{u} \in \mathbf{H}^{k+1}(K).$$

THEOREM 3.2. *Suppose that* $\hat{W}(\hat{K})$ *contains* $\mathcal{R}_k (k \geq 0)$. *Then there exists a constant* C *independent of* $\mathbf{u}$ *such that*

$$\inf_{\mathbf{v}\in\mathbf{V}_h} \|\mathrm{div}\,\mathbf{u} - \mathrm{div}\,\mathbf{v}\|_0 \leq Ch^{k+1}|\mathrm{div}\,\mathbf{u}|_{k+1}, \text{ for all } \mathbf{u} \in \mathbf{H}^{k+1}(K) \text{ with } \mathrm{div}\,\mathbf{u} \in \mathbf{H}^{k+1}(K).$$

Our new element is based on the pair $(\mathcal{S}_k, \mathcal{R}_{k-1})$ for $k \geq 1$. Define

$$\hat{\mathbf{V}}(\hat{K}) = \mathcal{S}_k,\ \hat{W}(\hat{K}) = \mathcal{R}_{k-1}$$

as reference spaces for our new element, and define $\mathbf{V}_h$ and W_h through (2.6) and (2.7). Then we see that the stability condition $\hat{\mathrm{div}}\,\hat{\mathbf{V}}(\hat{K}) = \hat{W}(\hat{K})$ holds. Note that our pair has the degrees of freedoms $2(k+2)(k+1) - 1$ and $(k+1)^2 - 1$, respectively, hence a total of two fewer than $RT_{[k]}$ and $4k + 6$ fewer than $ABF_{[k]}$.

We will now show the unisolvence of this element. Let $\mathbf{\Psi}_k(\hat{K})$ be a subspace of $Q_{k-1,k}(K) \times Q_{k,k-1}(\hat{K})$ where $(\hat{x}^{k-1}\hat{y}^k, 0)$ and $(0, \hat{x}^k\hat{y}^{k-1})$ are replaced by the single element $(\hat{x}^{k-1}\hat{y}^k, -\hat{x}^k\hat{y}^{k-1})$.

LEMMA 3.3 (Unisolvence). *For any* $\hat{\mathbf{u}} = (\hat{u}, \hat{v}) \in \mathcal{S}_k$, *the conditions*

$$\int_{\hat{e}} \hat{\mathbf{u}} \cdot \hat{\mathbf{n}}\, \hat{q}\, d\hat{s}, \quad \hat{q} \in P_k(\hat{e}), \quad \text{for each edge } \hat{e} \text{ of } \hat{K}, \tag{3.1}$$

$$\int_{\hat{K}} \hat{\mathbf{u}} \cdot \hat{\mathbf{v}}\, d\hat{\mathbf{x}}, \quad \hat{\mathbf{v}} \in \mathbf{\Psi}_k(\hat{K}) \tag{3.2}$$

uniquely determine $\hat{\mathbf{u}}$.

Proof. Since the number of conditions, $4(k+1)+(k+1)^2-2+k^2 = 2(k+2)(k+1)-1$ equals the dimension of $\mathcal{S}_k$, it suffices to show that if the degrees of freedom (3.1) – (3.2) are all zero then $\hat{\mathbf{u}} = 0$. The first degree of freedom (3.1) implies that $\hat{\mathbf{u}} \equiv 0$ for each edge $\hat{e}$ of the reference element $\hat{K}$, that is, $\hat{\mathbf{u}} = (u, v)$ satisfies $u = x(1-x)u_1$, $v = y(1-y)v_1$ where $(u_1, v_1) \in \mathbf{\Psi}_k(\hat{K})$. Immediately, the degree of freedom (3.2) gives the desired result. $\square$

For the error estimate we need to define a projection operator $\hat{\Pi}_{\hat{K}} : \mathbf{H}^{k+1}(\hat{K}) \to \hat{\mathbf{V}}(\hat{K})$ satisfying

$$\int_{\hat{e}} (\hat{\mathbf{u}} - \hat{\Pi}_{\hat{K}}\hat{\mathbf{u}}) \cdot \hat{\mathbf{n}}\, \hat{q}\, d\hat{s} = 0, \quad \hat{q} \in P_k(\hat{e}), \quad \text{for each edge } \hat{e} \text{ of } \hat{K}, \tag{3.3}$$

$$\int_{\hat{K}} (\hat{\mathbf{u}} - \hat{\Pi}_{\hat{K}}\hat{\mathbf{u}}) \cdot \hat{\mathbf{v}}\, d\hat{\mathbf{x}} = 0, \quad \hat{\mathbf{v}} \in \mathbf{\Psi}_k(\hat{K}). \tag{3.4}$$

This operator has the following property:

LEMMA 3.4.

$$(\hat{\mathrm{div}}\,(\hat{\mathbf{u}} - \hat{\Pi}_{\hat{K}}\hat{\mathbf{u}}), \hat{q}) = 0, \quad \forall \hat{\mathbf{u}} \in \hat{\mathbf{V}}(\hat{K}),\ \forall \hat{q} \in \hat{W}(\hat{K}). \tag{3.5}$$

Proof. First, note that $\hat{q}|_{\hat{e}} \in P_k(\hat{e})$ for $\hat{q} \in \mathcal{R}_{k-1}$, $\hat{\nabla}\mathcal{R}_{k-1} \subset \mathbf{\Psi}_k(\hat{K})$. Hence we see by (3.3) and (3.4)

$$\begin{aligned}(\hat{\mathrm{div}}\,\hat{\Pi}_{\hat{K}}\hat{\mathbf{u}}, \hat{q}) &= \int_{\partial\hat{K}} \hat{\Pi}_{\hat{K}}\hat{\mathbf{u}} \cdot \hat{\mathbf{n}}\, \hat{q}\, d\hat{s} - \int_{\hat{K}} \hat{\Pi}_{\hat{K}}\hat{\mathbf{u}} \cdot \hat{\nabla}\hat{q}\ d\hat{\mathbf{x}} \\ &= \int_{\partial\hat{K}} \hat{\mathbf{u}} \cdot \hat{\mathbf{n}}\, \hat{q}\, d\hat{s} - \int_{\hat{K}} \hat{\mathbf{u}} \cdot \hat{\nabla}\hat{q}\ d\hat{\mathbf{x}} = (\hat{\mathrm{div}}\,\hat{\mathbf{u}}, \hat{q}). \qquad \square\end{aligned}$$

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Define the projection operators $\Pi_K : \mathbf{H}^{k+1}(K) \to \mathbf{V}_h(K)$ and $\Pi_h : \mathbf{H}^{k+1}(\Omega) \to \mathbf{V}_h$ by

$$\Pi_K(\mathbf{u}|_K) = \mathcal{P}_K(\hat{\Pi}_{\hat{K}}(\hat{\mathbf{u}}|_{\hat{K}}))$$

and

$$(\Pi_h \mathbf{u})|_K = \Pi_K(\mathbf{u}|_K).$$

We also need an operator $\Phi_h : L^2(\Omega) \to W_h$. First, we let $\hat{\Phi}_{\hat{K}}$ be the local L^2-projection onto $\hat{W}(\hat{K}) = \mathcal{R}_{k-1}$. Then define $\Phi_K p = (\hat{\Phi}_{\hat{K}}\hat{p}) \circ F_K^{-1}$. Finally, we let $(\Phi_h p)|_K = \Phi_K(p|_K)$. Now since $\hat{\mathbf{V}}(\hat{K}) \supset \mathcal{S}_k$, the approximation property of Π_K follows from [3, Theorem 4.1]:

$$\|\mathbf{u} - \Pi_K \mathbf{u}\|_{0,K} \leq Ch^{k+1}|\mathbf{u}|_{k+1,K}, \quad \forall \mathbf{u} \in \mathbf{H}^{k+1}(K). \tag{3.6}$$

LEMMA 3.5. *We have the following approximation property of the projection operator Π_h:*

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_0 \leq Ch^{k+1}|\mathbf{u}|_{k+1}, \quad \forall \mathbf{u} \in \mathbf{H}^{k+1}(\Omega). \tag{3.7}$$

Also, the following is valid:

$$(\mathrm{div}\,(\mathbf{u} - \Pi_h \mathbf{u}), q) = 0, \quad \forall \mathbf{u} \in \mathbf{V}, q \in W_h, \tag{3.8}$$

$$\|\mathrm{div}\,(\mathbf{u} - \Pi_h \mathbf{u})\|_0 \leq Ch^k|\mathrm{div}\,\mathbf{u}|_k, \ \forall \mathbf{u} \in \mathbf{H}^k(\Omega) \text{ with } \mathrm{div}\,\mathbf{u} \in H^k(\Omega). \tag{3.9}$$

Proof. The estimate (3.7) is a result of (3.6). For (3.8), we see from (3.5) that for each K,

$$\begin{aligned} (\mathrm{div}\,\Pi_K \mathbf{u}, q)_K &= (\hat{\mathrm{div}}\,\hat{\Pi}_{\hat{K}}\hat{\mathbf{u}}, \hat{q})_{\hat{K}} \\ &= (\hat{\mathrm{div}}\,\hat{\mathbf{u}}, \hat{q})_{\hat{K}} \\ &= (\mathrm{div}\,\mathbf{u}, q)_K, \ q \in W_h. \end{aligned} \tag{3.10}$$

The estimate (3.9) now follows along the lines of [3, Theorem 4.2]. ∎

REMARK 3.6.

1. The operator Φ_h defined above is different from the L^2-projection P_h onto W_h, which is defined as $((P_h p)|_K, q)_K = (p, q)_K = (\hat{p}, \hat{q}J_K)_{\hat{K}}$, $q \in W_h(K)$. In fact, $((\Phi_h p)|_K, q)_K = (\hat{\Phi}_{\hat{K}}\hat{p}, \hat{q}J_K)_{\hat{K}}$.
2. Note that since the divergence of $\mathbf{V}_h$ is not equal to W_h, the relation (3.10) does not imply the relation $\mathrm{div}\,\Pi_K = \Phi_K \mathrm{div}$, even though $\hat{\mathrm{div}}\,\hat{\Pi}_{\hat{K}} = \hat{\Phi}_{\hat{K}}\hat{\mathrm{div}}$ holds. However, one can verify that $\mathrm{div}\,\Pi_K = P_h \mathrm{div}$ holds.

Now we have the following error estimates.

THEOREM 3.7. *Let $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$ and $p \in H^{k+1}(\Omega)$ be the solution of (2.3) and $\mathbf{u}_h$ and p_h be the solution of (2.4). Then*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_0 &\leq Ch^{k+1}\|\mathbf{u}\|_{k+1}, \\ \|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h)\|_0 &\leq Ch^k\|\mathrm{div}\,\mathbf{u}\|_k, \\ \|\Phi_h p - p_h\|_0 &\leq Ch^{k+1}\|\mathbf{u}\|_{k+1}, \end{aligned} \tag{3.11}$$

$$\|p - p_h\|_0 \leq Ch^k\|p\|_{k+1}. \tag{3.12}$$

Proof. These estimates essentially follow along the same lines as [3, Theorem 6.1, 6.2] using Theorems 3.1, 3.2 and Lemma 3.5. However, the estimates (3.11), (3.12) are not optimal. Note that the loss of order results from the fact that $\mathcal{R}_{k-1} \subsetneq Q_{k,k}$. ∎

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4. Post-processing. In this section, we present some post-processing techniques that produce an optimal-order error for pressure and divergence.

4.1. Local post-processing for the pressure. In order to enhance the convergence order in pressure, we apply a simple local postprocessing scheme using the pressure space of $RT_{[k]}$. We first define the pressure space, W_h^{RT}, related to $RT_{[k]}$,

$$W_h^{RT}(K) = \{w = \hat{w} \circ F_K^{-1},\ \hat{w} \in Q_{k,k}\}.$$

Given the solution $(\mathbf{u}_h, p_h)$ of (2.4), we define a new pressure solution $p_h^\# \in W_h^{RT}$ locally on each element $K \in \mathcal{T}_h$ as follows:

$$\int_K \kappa \nabla p_h^\# \cdot \nabla q \, d\mathbf{x} = -\int_K \mathbf{u}_h \cdot \nabla q \, d\mathbf{x}, \quad \forall q \in W_h^{RT}(K), \tag{4.1}$$

$$\int_K p_h^\# \, d\mathbf{x} = \int_K p_h \, d\mathbf{x}. \tag{4.2}$$

This technique has been suggested by Stenberg in the case of BDM on affine elements [17]. Since the space W_h contains non-polynomials on quadrilateral element, the proof needs a modification. Here we present a modified proof which also handles the general coefficients.

THEOREM 4.1. *If* $p \in H^{k+1}(\Omega)$ *and* $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$ *are the solutions of* (2.2) *(*$k \geq 1$*) and* $p_h^\#$ *is given by* (4.1) *and* (4.2)*, then we have*

$$\|p - p_h^\#\|_0 \leq C h^{k+1} |\mathbf{u}|_{k+1}. \tag{4.3}$$

Proof. Let $\Phi_K^{RT} : L^2(K) \to W_h^{RT}(K)$ be the local projection operator defined by $\Phi_K^{RT} p = (\hat{\Phi}_{\hat{K}}^{RT} \hat{p}) \circ F_K^{-1}$, where $\hat{\Phi}_{\hat{K}}^{RT}$ is the L^2-projection onto $\hat{W}^{RT}(\hat{K})$, and put $q = \Phi_K^{RT} p - p_h^\# \in W_h^{RT}$. Then using the weighted norm $\|\cdot\|_{0,\kappa,K} := (\kappa \cdot, \cdot)_K^{1/2}$ and weighted semi-norm $|\cdot|_{1,\kappa,K} := (\kappa \nabla \cdot, \nabla \cdot)_K^{1/2}$, we have

$$\begin{aligned}
|q|_{1,\kappa,K}^2 &= \int_K \kappa \nabla ((\Phi_K^{RT} p - p_h^\#) \cdot \nabla q \, dx \\
&= \int_K \kappa \nabla (\Phi_K^{RT} p - p) \cdot \nabla q \, dx + \int_K \kappa \nabla (p - p_h^\#) \cdot \nabla q \, dx \\
&= \int_K \kappa \nabla (\Phi_K^{RT} p - p) \cdot \nabla q \, dx + \int_K (-\mathbf{u} + \mathbf{u}_h) \cdot \nabla q \, dx,
\end{aligned}$$

where (4.1) was used. Now by the Cauchy-Schwarz inequality, we have

$$|q|_{1,\kappa,K} \leq C_1 |p - \Phi_K^{RT} p\,|_{1,\kappa,K} + C_2 \|\mathbf{u} - \mathbf{u}_h\|_{0,\kappa,K},$$

where C_i, $i = 1, 2$, are constants. By the norm equivalence, we have

$$|q|_{1,K} \leq C |p - \Phi_K^{RT} p\,|_{1,K} + C \|\mathbf{u} - \mathbf{u}_h\|_{0,K}.$$

Next, we let $\dot{q} = q - \bar{q}$, where $\bar{q} = \frac{1}{\text{Area}(K)} \int_K q \, d\mathbf{x}$ is the average of q over K. Then we have by the Poincaré inequality

$$\begin{aligned}
\|\dot{q}\|_{0,K} &\leq C h |\dot{q}|_{1,K} = C h |q|_{1,K} \\
&\leq C h (|p - \Phi_K^{RT} p\,|_{1,K} + \|\mathbf{u} - \mathbf{u}_h\|_{0,K}).
\end{aligned} \tag{4.4}$$

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Also, by (4.2) and the fact that $(\Phi_K^{RT}p,1)_K = (\Phi_K p,1)_K$, we have

$$\begin{aligned}\|\bar{q}\|_\infty &= \left|\frac{1}{\text{Area}(K)}\int_K (\Phi_K^{RT}p - p_h^{\#})d\mathbf{x}\right| = \left|\frac{1}{\text{Area}(K)}\int_K (\Phi_K p - p_h)d\mathbf{x}\right| \\ &\le Ch^{-1}\|\Phi_K p - p_h\|_{0,K}.\end{aligned}$$

Hence,

$$\|\bar{q}\|_{0,K} \le Ch\|\bar{q}\|_\infty \le C\|\Phi_K p - p_h\|_{0,K}. \tag{4.5}$$

Finally, using (4.4) and (4.5), we have

$$\begin{aligned}&\|p - p_h^{\#}\|_{0,K} \\ &= \|p - \Phi_K^{RT}p + \Phi_K^{RT}p - p_h^{\#}\|_{0,K} = \|p - \Phi_K^{RT}p + \dot{q} + \bar{q}\|_{0,K} \\ &\le \|p - \Phi_K^{RT}p\|_{0,K} + \|\dot{q}\|_{0,K} + \|\bar{q}\|_{0,K} \\ &\le \|p - \Phi_K^{RT}p\|_{0,K} + Ch(|p - \Phi_K^{RT}p|_{1,K} + \|\mathbf{u} - \mathbf{u}_h\|_{0,K}) + C\|\Phi_K p - p_h\|_{0,K}.\end{aligned}$$

Now the estimate follows from the approximation property of Φ_K^{RT}, the estimates (3.11), (3.12), and summation over all $K \in \mathcal{T}_h$. ◻

4.2. Local post-processing for divergence. According to the discussion in the previous sections, one has to enrich $\hat{\mathbf{V}}(\hat{K})$ in order to obtain optimal order in divergence norm, so that $\hat{\text{div}}\,\hat{\mathbf{V}}(\hat{K}) \supset \mathcal{R}_k$. The result is $ABF_{[k]}$ mentioned earlier, for which an improved estimate (3.9) holds, with $k+1$ in place of k, at the cost of extra degrees of freedom. However, if one wants an optimal divergence, there is a simple way as we show below. First, we introduce some notations:

$$\begin{aligned}\mathbf{V}_h^{ABF}(K) &= \{\mathbf{v} = \mathcal{P}_K\hat{\mathbf{v}},\ \hat{\mathbf{v}} \in ABF_{[k]}\}, \\ W_h^{ABF}(K) &= \{w = \hat{w}\circ F_K^{-1},\ \hat{w} \in \mathcal{R}_k\}.\end{aligned}$$

The corresponding global spaces $\mathbf{V}_h^{ABF}$ and W_h^{ABF} are defined in an obvious manner. The notation $\mathbf{V}_h^{RT}$ is used for the velocity space of $RT_{[k]}$. We consider the following problem: Find $\text{div}\,\mathbf{e}_h^* \in \mathbf{V}_h^{ABF}$ such that

$$(\text{div}\,\mathbf{e}_h^*, q_h) = (f, q_h) - (cp_h^{\#}, q_h) - (\text{div}\,\mathbf{u}_h, q_h), \quad \forall q_h \in W_h^{ABF}. \tag{4.6}$$

Let $\text{div}\,\mathbf{u}_h^{\#} = \text{div}\,\mathbf{u}_h + \text{div}\,\mathbf{e}_h^*$. Then, we see that (4.6) is equivalent to solving

$$(\text{div}\,\mathbf{u}_h^{\#}, q_h) = (f - cp_h^{\#}, q_h), \quad \forall q_h \in W_h^{ABF}.$$

In other words,

$$\text{div}\,\mathbf{u}_h^{\#} = P_K^{ABF}(f - cp_h^{\#}), \tag{4.7}$$

where $P_K^{ABF} : L^2(K) \to W_h^{ABF}(K)$ is the local L^2 projection operator defined by

$$(P_K^{ABF}p, q)_K = (p,q)_K = (\hat{p}, \hat{q}J_K)_{\hat{K}}, \text{ for all } \hat{q} \in \hat{W}^{ABF} = \mathcal{R}_k.$$

For the analysis, subtract (4.7) from the second equation of (2.4) to see

$$\begin{aligned}\text{div}\,(\mathbf{u} - \mathbf{u}_h^{\#}) &= f - cp - P_h^{ABF}(f - cp_h^{\#}) \\ &= (I - P_K^{ABF})f - (cp - P_K^{ABF}(cp_h^{\#})) \\ &= (I - P_K^{ABF})(f - cp) - P_K^{ABF}(cp - cp_h^{\#}).\end{aligned}$$

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Hence,

$$
\begin{aligned}
\|\mathrm{div}\,(\mathbf{u}-\mathbf{u}_h^{\#})\|_{0,K} &\le \|(I-P_K^{ABF})(f-cp)\|_{0,K} \\
&\quad + \|P_K^{ABF}\| \cdot \|c\|_{0,K}\|p-p_h^{\#}\|_{0,K} \\
&\le Ch^{k+1}|\mathrm{div}\,\mathbf{u}|_{k+1,K} + Ch^{k+1}\|P_K^{ABF}\||\mathbf{u}|_{k+1},
\end{aligned}
\tag{4.8}
$$

where $\|P_K^{ABF}\|$ is the operator norm. Let us divert briefly and show that $\|P_K^{ABF}\| \le 1$.

For any $\phi \in L^2(K)$, we have

$$
\begin{aligned}
\|P_K^{ABF}\phi\|_{0,K}^2 &= (P_K^{ABF}\phi, P_K^{ABF}\phi)_K = (\hat{\phi}, \widehat{P_K^{ABF}}\phi J_K)_K \\
&\le (\hat{\phi}, \hat{\phi}J_K)_K^{1/2} \cdot (\widehat{P_K^{ABF}}\phi, \widehat{P_K^{ABF}}\phi J_K)_K^{1/2} = \|\phi\|_{0,K}\|P_K^{ABF}\phi\|_{0,K}.
\end{aligned}
$$

Thus, $\|P_K^{ABF}\| \le 1$.

Now the following result follows from (4.3), (4.8), the approximation property, and the boundedness of P_K^{ABF}.

PROPOSITION 4.2. *Let* $\mathbf{u}$ *be the solution of problem* (2.1) *such that* $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$ *and* $\mathrm{div}\,\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, *and* $\mathrm{div}\,\mathbf{u}_h^{\#}$ *be defined as in* (4.7). *Then we have:*

$$
\|\mathrm{div}\,(\mathbf{u}-\mathbf{u}_h^{\#})\|_0 \le Ch^{k+1}|\mathrm{div}\,\mathbf{u}|_{k+1}.
$$

REMARK 4.3.

1. To compute $\mathrm{div}\,\mathbf{u}_h^{\#}$, we do not solve (4.6). Instead, we obtain it as a projection of $f - cp_h^{\#}$ as in (4.7). In particular, when $c = 0$, $\mathrm{div}\,\mathbf{u}_h^{\#}$ can be obtained without computing $\mathbf{u}_h$.
2. This procedure can be easily incorporated into the existing codes written using RT-element.
3. It would be interesting to consider the three-dimensional case, but the 3D Raviart-Thomas-Nedelec element space [14] does not achieve optimal L^2 approximation as numerical experiments show [13]. In fact, one can verify with tedious calculation that the Raviart-Thomas-Nedelec element does not contain $\mathbf{P}_k$ under the Piola map even for $k = 0$. Further investigations are needed for three-dimensional problems.

5. Numerical results. In this section, we report some numerical simulations to confirm our theoretical results. We solve problem (2.4) with $\kappa = I$ and $c = 1$ on the unit square $\bar{\Omega} = [0,1] \times [0,1]$. The function $p(x,y) = \log(x^3 + y^2 + 4)\sin(\pi x)(y^2 - y)$ is chosen as the exact solution. When $k = 1$, there are eleven degrees of freedom for the velocity space $\mathcal{S}_1$, and three for the pressure space $\mathcal{R}_0$, on each element. Grids are distorted as in Figure 5.1, where α $(0 \le \alpha < 1)$ is the measure of distortion. The results for $\alpha = 0, 0.2, 0.6$ and $\alpha = 0.99$ are reported. In all cases, the discrete L^2-norm is measured at nine Gauss points.

Our new element has second order accuracy for all variables in case of a rectangular element ($\alpha = 0$). As the element becomes distorted, only the velocity shows second order accuracy, while the post-processing shows second order for the other variable. The odd numbered tables show the results without post-processing (p_h, $\mathbf{u}_h$, and $\mathrm{div}\,\mathbf{u}_h$), while the even numbered tables show those with post-processing ($p_h^{\#}$ and $\mathrm{div}\,\mathbf{u}_h^{\#}$). Note that our scheme works even when the element almost degenerates into a triangle and the shape regularity does not hold ($\alpha = 0.99$). As a comparison, we test the $RT_{[1]}$ element. The results are listed in Tables 5.9 (for $\alpha = 0$) and 5.10 (for $\alpha = 0.99$). The orders of convergence are exactly as predicted by the theory, and post-processing increases pressure and divergence orders by one.

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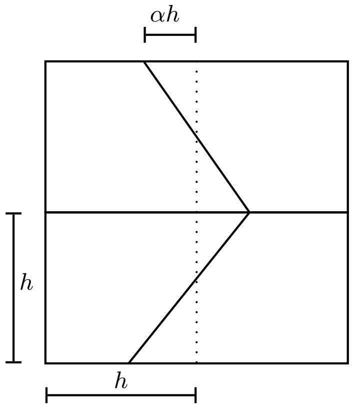

FIGURE 5.1. *An example : trapzoidal grid with α factor.*

Alternatively, the divergence can be obtained from the L^2-projection of $f - cp_h^{\#}$ as indicated by the remark above.

It would be interesting to compare the total cost of the new element and $RT_{[1]}$. The new element has $11 + 3 = 14$ unknowns per element, while the $RT_{[1]}$ has $12 + 4 = 16$. Thus, the total number of unknowns are roughly $N_1 = 14/h^2$ versus $N_2 = 16/h^2$ where h is the grid size. The exact comparison is not possible, but the cost to solve the saddle point system in N unknown is at least $O(N^2)$. So we can save relatively about $(N_2^2 - N_1^2)/N_2^2 = (1 - 7^2/8^2) \approx 12.5\%$. Instead, the added cost for local post-processing is at most proportional to N_1, which is negligible.

TABLE 5.1
Results with $\alpha = 0.0$.

	$\|p - p_h\|_0$		$\|\mathbf{u} - \mathbf{u}_h\|_0$		$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h)\|_0$	
n	error	order	error	order	error	order
4	1.1668e-02		2.1495e-02		1.1806e-01	
8	2.9998e-03	1.95	5.3625e-03	2.00	2.9998e-02	1.97
16	7.5518e-04	1.98	1.3401e-03	2.00	7.5309e-03	1.96
32	1.8912e-04	1.99	3.3501e-04	2.00	1.8846e-03	1.99
64	4.7301e-05	1.99	8.3752e-05	2.00	4.7129e-04	1.99

TABLE 5.2
Post-processed results with $\alpha = 0.0$.

	$\|p - p_h^{\#}\|_0$		$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h^{\#})\|_0$	
n	error	order	error	order
4	6.7860e-03		6.9514e-03	
8	1.6974e-03	2.00	1.7048e-03	2.02
16	4.2445e-04	2.00	4.2487e-04	2.00
32	1.0612e-04	2.00	1.0614e-04	2.00
64	2.6532e-05	2.00	2.6532e-05	2.00

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TABLE 5.3
Results with $\alpha = 0.2$.

	$\|p - p_h\|_0$		$\|\mathbf{u} - \mathbf{u}_h\|_0$		$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h)\|_0$	
n	error	order	error	order	error	order
4	1.2990e-02		2.4174e-02		2.2704e-01	
8	3.9874e-03	1.70	6.1410e-03	1.97	1.0397e-01	1.12
16	1.4835e-03	1.42	1.5494e-03	1.98	5.0588e-02	1.03
32	6.6133e-04	1.16	3.8912e-04	1.99	2.5111e-02	1.01
64	3.1977e-04	1.04	9.7495e-05	1.99	1.2533e-02	1.00

TABLE 5.4
Post-processed results with $\alpha = 0.2$.

	$\|p - p_h^{\#}\|_0$		$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h^{\#})\|_0$	
n	error	order	error	order
4	7.7270e-03		8.0530e-03	
8	1.9349e-03	1.99	1.9653e-03	2.03
16	4.8417e-04	1.99	4.8922e-04	2.00
32	1.2108e-04	1.99	1.2219e-04	2.00
64	3.0274e-05	1.99	3.0542e-05	2.00

TABLE 5.5
Results with $\alpha = 0.6$.

	$\|p - p_h\|_0$		$\|\mathbf{u} - \mathbf{u}_h\|_0$		$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h)\|_0$	
n	error	order	error	order	error	order
4	2.1076e-02		4.4702e-02		6.4683e-01	
8	8.4839e-03	1.31	1.1927e-02	1.90	3.3039e-01	0.96
16	3.9115e-03	1.11	3.0764e-03	1.95	1.6580e-01	0.99
32	1.9113e-03	1.03	7.8044e-04	1.97	8.2979e-02	0.99
64	9.5004e-04	1.00	1.9647e-04	1.98	4.1500e-02	0.99

TABLE 5.6
Post-processed results with $\alpha = 0.6$.

	$\|p - p_h^{\#}\|_0$		$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h^{\#})\|_0$	
n	error	order	error	order
4	1.3710e-02		1.7683e-02	
8	3.4603e-03	1.98	4.1843e-03	2.07
16	8.6822e-04	1.99	1.0325e-03	2.01
32	2.1730e-04	1.99	2.5733e-04	2.00
64	5.4342e-05	1.99	6.4283e-05	2.00

TABLE 5.7
Results with $\alpha = 0.99$.

n	$\|p - p_h\|_0$ error	order	$\|\mathbf{u} - \mathbf{u}_h\|_0$ error	order	$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h)\|_0$ error	order
4	3.2052e-02		8.4730e-02		1.3579e-00	
8	1.3586e-02	1.23	2.4275e-02	1.80	7.1167e-01	0.93
16	6.3981e-03	1.08	6.4658e-03	1.90	3.5943e-01	0.98
32	3.1466e-03	1.02	1.6648e-03	1.95	1.8017e-01	0.99
64	1.5666e-03	1.00	4.2207e-04	1.97	9.0150e-02	0.99

TABLE 5.8
Post-processed results with $\alpha = 0.99$.

n	$\|p - p_h^{\#}\|_0$ error	order	$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h^{\#})\|_0$ error	order
4	2.1044e-02		3.4112e-02	
8	5.3703e-03	1.97	7.6588e-03	2.15
16	1.3507e-03	1.99	1.8551e-03	2.04
32	3.3822e-04	1.99	4.5992e-04	2.01
64	8.4590e-05	1.99	1.1473e-04	2.00

TABLE 5.9
Results with $\alpha = 0$ for $RT_{[1]}$.

n	$\|p - p_h\|_0$ error	order	$\|\mathbf{u} - \mathbf{u}_h\|_0$ error	order	$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h)\|_0$ error	order
4	6.7709e-03		2.1413e-02		9.7288e-02	
8	1.6966e-03	1.99	5.3582e-03	1.99	2.4330e-02	2.00
16	4.2441e-04	1.99	1.3399e-03	2.00	6.0836e-03	2.00
32	1.0611e-04	1.99	3.3500e-04	2.00	1.5209e-03	2.00
64	2.6530e-05	1.99	8.3751e-05	2.00	3.8025e-04	2.00

TABLE 5.10
Results with $\alpha = 0.99$ for $RT_{[1]}$.

n	$\|p - p_h\|_0$ error	order	$\|\mathbf{u} - \mathbf{u}_h\|_0$ error	order	$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h)\|_0$ error	order
4	1.8171e-02		7.1774e-02		6.7905e-01	
8	4.6367e-03	1.97	1.9508e-02	1.87	3.3798e-01	1.00
16	1.1652e-03	1.99	5.0727e-03	1.94	1.6870e-01	1.00
32	2.9163e-04	1.99	1.2929e-03	1.97	8.4357e-02	1.00
64	7.2927e-05	1.99	3.2636e-04	1.98	4.2191e-02	1.00

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