Electronic Transactions on Numerical Analysis.
Volume 36, pp. 195-223, 2010.

ISSN 1068-9613.

ETNA
Kent State University
http://etna.math.kent.edu

SLIT MAPS AND SCHWARZ-CHRISTOFFEL MAPS FOR MULTIPLY CONNECTED DOMAINS*

THOMAS K. DELILLO† AND EVERETT H. KROPF†

Dedicated to Richard Varga on his 80th birthday
and to the memory of Dieter Gaier and their Oberwolfach meetings
on Constructive Methods in Complex Analysis.

Abstract. We review recent derivations of formulas for conformal maps from finitely connected domains with circular holes to canonical radial or circular slit domains. The formulas are infinite products based on simple reflection arguments. An earlier similar derivation of the Schwarz-Christoffel formula for the bounded multiply connected case and recent progress in its numerical implementation are also reviewed. We give some sample calculations with a reflection method and an estimate of its accuracy. We also discuss the relation of our approach to that of D. Crowdy and J. Marshall. In addition, a slit map calculation using Laurent series computed by the least squares method in place of the reflection method is given as an example of a possible direction for future improvements in the numerics.

Key words. conformal mapping, Schwarz-Christoffel transformation, multiply connected domains, canonical slit domains, Schottky-Klein prime function

AMS subject classifications. 30C30, 65E05

1. Introduction.

Conformal mapping has been a topic of theoretical interest and a useful tool for solving boundary value problems of classical potential theory in the plane for over 100 years. With the development of modern computers, many numerical methods have been proposed for approximating conformal maps. The books by Gaier [27] and Henrici [30] provide introductions to this field. The survey paper by Wegmann [38] and the book on Schwarz-Christoffel mapping by Driscoll and Trefethen [26] review more recent work especially relevant to computations. In spite of the ability of today's computers to solve many fully three dimensional problems, there is a continuing interest in these inherently two dimensional methods of function theory due the power of the techniques and the clarity of the understanding that they bring to many important applications.

In the last several years there have been a number of advances in methods for multiply connected domains; see, e.g., [3, 38]. In particular, the Schwarz-Christoffel transformation for domains with polygonal boundaries has been extended to to multiply connected domains in [17, 20, 21] using reflection arguments and in [8, 9] using the closely related Schottky-Klein prime function; see also [10]. These results were the topic of a recent article in SIAM Review [6]. The methods use multiply connected domains with circular boundaries as their computational domains and involve infinite products. Explicit formulas for conformal maps from the circular domains to the canonical slit domains [34, 35] for the multiply case case can be derived using the same techniques [13, 19]. Canonical slit maps can be used to represent Green's functions for the Dirichlet, Neumann, and mixed boundary value problems for the Laplace equation in multiply connected domains; see [14]. One advantage of using circle domains is the possibility of using fast computational methods based on Fourier/Laurent expansions centered at the circles.

In this paper, we review these results for multiply connected polygonal and slit domains and attempt to clarify some of the relations among the alternative approaches. We will discuss

*Received March 15, 2009. Accepted for publication December 15, 2010. Published online on December 21, 2010. Recommended by R. Freund.

†Department of Mathematics & Statistics, Wichita State University, Wichita, KS 67260-0033 ({delillo,kropf}@math.wichita.edu).

ETNA
Kent State University
http://etna.math.kent.edu

mainly the case of bounded multiply connected domains. The results for unbounded domains are similar and most of them have already been treated in the references. In Section 2, we recall some useful preliminary facts about conformal maps and reflections in circles. We include a listing of a simplified version of our MATLAB code for calculating these reflections. All of the computed examples in this paper, except for the Laurent series example in the last section, are performed using variations of this reflection algorithm. Section 3 discusses maps to canonical circular and radial slit domains. As an example of our techniques, we derive the formula for the map from a disk with circular holes to a half plane with radial slits. The derivation is based on extension of analytic functions by Schwarz reflection through circular arcs and radial slits leading to infinite product formulas. A listing of a short MATLAB code for computing this map is given. We prove that the products converge and satisfy the required boundary conditions, namely, that the arguments of the map on the circles are constant. The convergence is based on an estimate of the rate at which the reflected circles shrink. We also relate the slit maps to the Schottky-Klein prime function. In addition, we give a brief discussion of Green's functions, which, for circular domains, can be given explicitly in terms of our product formulas for maps to circular slit disks and rings. In Section 4, we discuss the Schwarz-Christoffel map to multiply connected polygonal domains. The derivative of the map is represented as an infinite product based on reflections. We suggest some alternative representations of this transformation which may allow us to replace the infinite products with finite products yielding a completely general formula. These alternatives are based on the maps to radial slit half planes derived in Section 3. In Section 5, we review recent progress on the numerical implementation of the Schwarz-Christoffel transformation [23]. (We only discuss the cases of connectivity greater than 2, since the simply and doubly connected cases have been thoroughly treated elsewhere by somewhat more specialized techniques.) We give a practical error estimate in terms of the radii of the reflected circles. We also discuss some potential difficulties; for instance, in cases where slits or polygonal boundaries form narrow channels, the corresponding circles in the computational domain are close-to-touching. This may be thought of as a form of the *crowding phenomenon* [16, 24] for multiply connected domains. In the final subsection, we discuss a method for computing maps to radial slit half-planes using least squares to find a Laurent series approximation to the map satisfying the boundary conditions. We expect that such techniques will lead to improvements in our numerical solutions.

2. Preliminaries.

In the cases below, we are seeking a conformal map f from D, the interior of the unit disk, D_0, minus m closed nonintersecting disks, D_k, in the interior of D_0, onto a region Ω with exterior boundary, Γ_0, and m nonintersecting interior boundary curves, $\Gamma_k, 1 \le k \le m$. Therefore, the *connectivity* of D and Ω is $m+1$. For the slit maps in Section 3, Ω will be a half-plane (or disk), Γ_0 will be a straight line through the origin (or the unit circle), and the Γ_k's, $k \neq 0$ will be radial or circular slits. For the Schwarz-Christoffel maps in Section 4, Γ_0 will be the outer polygonal boundary and the Γ_k's, $k \neq 0$, will be the inner polygonal boundaries. The boundaries of the circular disks, D_k, are the circles, C_k, with centers, $c_k (= s_k)$, and radii, r_k, and are parametrized by $C_k : c_k + r_k e^{i\theta}$. The boundary of D is thus $C = C_0 + C_1 + \cdots + C_m$. The boundary of Ω is $\Gamma = \Gamma_0 + \Gamma_1 + \cdots + \Gamma_m$. f extends to the boundary, $f(C_k) = \Gamma_k$. If Ω is given, then fixing the value of $w = f(z)$ at three boundary points on the unit circle C_0 or at an interior point and one boundary point uniquely determines the map f and the other circles $C_k, k \neq 0$ [30, 34]. (For the unbounded case, the outer boundaries D_0 and Γ_0 are not included, the connectivity is m, and $w = f(z) = O(z), z \approx \infty$. In this case, fixing $w = f(z) = z + O(1/z), z \approx \infty$ uniquely determines the map and the circles. In either case, the domains are conformally equivalent to an annulus with circular slits (or holes) [34]. For connectivity $m = 2$, there is one conformal modulus, the ratio of the outer

ETNA
Kent State University
http://etna.math.kent.edu

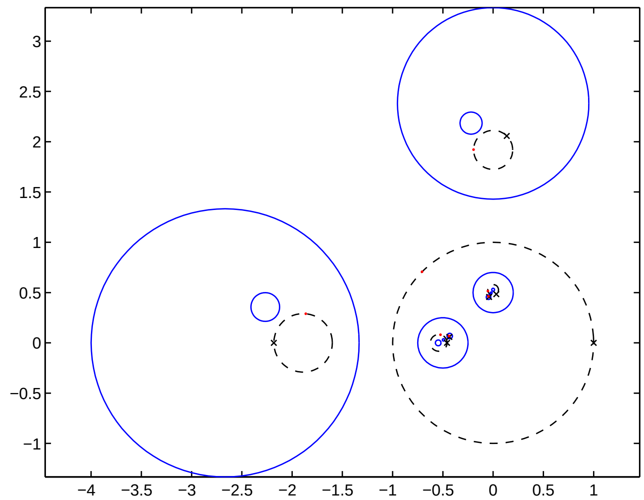

FIG. 2.1. *$N = 2$ levels of reflected circles and zeros (·) and poles (x) on the outer boundary for the map to the radial slit half-plane in Figure 3.1. The outer unit circle and its reflections are plotted with dashed boundaries.*

to inner radii. For $m = 3$, two more moduli are needed to determine the length and radius of the circular slit (or center and radius of the circular hole), since the annulus can be rotated to place the tip of the slit (or center of the hole) on the positive real axis. For connectivity $m > 3$, each additional slit (or hole) is determined by three real parameters: its length, radius, and tip location (or center and radius). Therefore, for connectivity $m \geq 3$ the number of conformal moduli needed to uniquely determine the class of conformally equivalent domains is $3m - 6$. We will mainly discuss the cases of connectivity $m \geq 3$ here, since the simply and doubly connected cases are thoroughly treated in [26, 30].)

Next, we introduce notation and recall basic facts about reflections in circles from [17, 18, 21]. The *reflection* of z through a circle C_k with center c_k and radius r_k is given by

$$\rho_k(z) = \rho_{C_k}(z) := c_k + \frac{r_k^2}{\overline{z} - \overline{c}_k}.$$

The set of multi-indices of length n will be denoted $\sigma_n := \{\nu_1\nu_2 \cdots \nu_n : 0 \leq \nu_k \leq m, \nu_k \neq \nu_{k+1}, k = 1, ..., n-1\}, \quad n > 0$, and $\sigma_0 = \phi$, in which case $\nu i = i$. Note that consecutive indices are not equal, since two consecutive reflections through the same circle is just the identity, $\rho_k(\rho_k(z)) = z$. In addition, $\sigma_n\,(i) = \{\nu \in \sigma_n : \nu_n \neq i\}$ denotes sequences in σ_n whose last factor never equals i, e.g., for $m+1 = 3$, $\sigma_3 = \{010, 012, 020, 012, 101, 102, \ldots\}$, $\sigma_3(0) = \{101, 121, 012, \ldots\}$. The following lemma [21, Lemma 1] says that ν just indexes successive reflections through the C_k's.

LEMMA 2.1. *$a_\nu = \rho_{\nu_1}(\rho_{\nu_2}(\cdots(\rho_{\nu_{n-1}}(a_{\nu_n}))\cdots))$ for $\nu = \nu_1\nu_2\cdots\nu_n \in \sigma_n$.*

Similarly, reflections of a circle C_k will be also be circles denoted by $C_{\nu k} = \rho_\nu(C_k)$ with centers and radii denoted $c_{\nu k}$ and $r_{\nu k}$, respectively. Our figures are produced with a MATLAB code which performs all reflections to level $n = N$. The reflections to two levels $N = 2$ of $m + 1 = 3$ circles and two points on the boundary of the unit circle are shown in Figure 2.1. Note that the number of new reflections of the a_ν's at a given level is m times that at the previous level.

Here is a simplified MATLAB code illustrating the reflection procedure and used to produce Figures 2.1 and 3.1.

ETNA
Kent State University
http://etna.math.kent.edu

ALGORITHM 2.2.
```
function [anu,cnu,rnu,jla,jlr] = reflect_circ(a,c,r,N)
% This code reflects circles through each other N times
% cnu(nu,j) = center of reflection nu of circle j
% rnu(nu,j) = radius of reflection nu of circle j
% anu(k,nu,j) = reflections of a(k)
% jlr(nu,j) = leading index = index of circle of last reflection
m = length(r); cnu(1,1:m)=c; snu=cnu; rnu(1,1:m)=r; ma = length(a);
anu(1,1:ma) = a;  % place vector a in first row of anu
jla(1)=1;
for j=1:m
  jlr(1,j)=j;
end
 num = 0;
for level=1:N
  nul = num+1;
  if m ~= 2
     num = ((m-1)^level - 1)/(m-2);
   elseif m == 2
     num = nul;
  end
    nuja=num;  nujc(1:m)=num*ones(1,m);
  for nu = nul:num
   for jl=1:m
    if jl ~= jla(nu) % do not reflect over same circle twice in a row
     nuja=nuja+1;
     jla(nuja)=jl;
   % reflect a_nu thru C_j1
     anu(nuja,1:ma)= c(jl) + r(jl)^2./conj(anu(nu,1:ma) - c(jl));
    end
    for j=1:m
     if jl ~= jlr(nu,j) % do not reflect over same circle twice in a row
      nujc(j)=nujc(j)+1;
      jlr(nujc(j),j)=jl;  % save index of current reflection
     % compute centers and radii of reflected circles:
      cnu(nujc(j),j) = c(jl) + r(jl)^2*(cnu(nu,j) - c(jl)) ...
                        /(abs(cnu(nu,j) - c(jl))^2 - rnu(nu,j)^2);
      rnu(nujc(j),j) = ...
        r(jl)^2*rnu(nu,j)/abs(abs(cnu(nu,j) - c(jl))^2 - rnu(nu,j)^2);
     end
    end
   end
 end
end
```

In order to state our convergence results, we need the following definition and lemma. The *separation parameter* of the region is

$$\Delta := \max_{i,j;i\neq j} \frac{r_i + r_j}{|c_i - c_j|} < 1, \;\; 0 \leq i,j \leq m,$$

ETNA
Kent State University
http://etna.math.kent.edu

for the assembly of $m+1$ mutually exterior circles that form the boundary of Ω; see [30, p. 501]. (Our Δ is actually defined by C_0 and the first reflections of the interior C_k's, $k \neq 0$, through C_0.) Let $\widetilde{C}_j$ denote the circle with center c_j and radius r_j/Δ. Then geometrically, $1/\Delta$ is the smallest magnification of the m radii such that at least two $\widetilde{C}_j$'s just touch. Our proof of convergence of the infinite products is based on estimating how fast the successively reflected circles shrink. For this estimate, we use the following inequality from [30, p. 505].

LEMMA 2.3.

$$\sum_{\nu \in \sigma_{n+1}} r_\nu^2 \leq \Delta^{4n} \sum_{i=0}^{m} r_i^2.$$

3. Slit maps. This section uses simple reflection arguments to derive infinite product formulas for the maps from circular domains to canonical circular and radial slit domains; see [34]. These techniques were used in [19] to derive the maps for unbounded domains. Convergence of the infinite products is proven if the circles are sufficiently well-separated. We will present the details only for the map from a bounded circle domain to a radially slit half plane where selected points on a circle are mapped to 0 and ∞. This case has not been treated in detail before. However, the methods we use are quite similar to our previous results for the slit maps [19] and Schwarz-Christoffel maps [17, 21] and will serve to illustrate our proofs for this overview paper. We also derive an expression for the radial slit map in terms of the Schottky-Klein prime function. This expression allows us to relate our formulas to those of [13, 14], where the canonical maps and the related Green's functions are given in terms of Schottky-Klein prime functions [2]. The formulas for other canonical maps are stated without proofs.

3.1. Radial slit map-bounded case. In this section, we discuss the map $w = f(z)$ from interior of a disk with circular holes to the a half plane with the origin on boundary and with slits radial with respect to the origin; see Figure 3.1. We will show that, for circle domains satisfying our separation criterion, the map can be represented by an infinite product formula. This map will be useful as a basic factor in our derivation of an alternative representation of the Schwarz-Christoffel transformation for multiply connected domains in Section 4, following in the framework of [26].

The idea for the product formula for the map is based on a simple reflection argument. Let $w = f(z)$ map a bounded circle domain of connectivity m to an unbounded radial slit domain. Let a and b be the two distinct points on one of the circles such that $f(a) = \infty$ and $f(b) = 0$. By the Reflection Principle we can extend f to the $z-$plane. Since reflections across the radial slits in the $w-$plane will just leave 0 and ∞ fixed, reflections $b_\nu = \rho_\nu(b)$ of b will be all of the (simple) zeros and reflections $a_\nu = \rho_\nu(a)$ of a will be all of the (simple) poles of f. The function therefore has the form

$$f(z) = C \prod_\nu \frac{z - \rho_\nu(b)}{z - \rho_\nu(a)}.$$

A MATLAB code implementing this formula is given in Algorithm 3.1, which uses Algorithm 2.2.

ETNA
Kent State University
http://etna.math.kent.edu

ALGORITHM 3.1.

```
% reflect_circ_driver.m % brief code for map to slit half-plane
% centers c and radii r of m mutually exterior circles
m=3; c(1)=0; r(1)=1; c(2)=.5*i; r(2)=0.2; c(3)=-.5; r(3)=.25;
theta_0=3*pi/4; theta_inf=0; % arg of pts mapped to 0 and \infty
a = [exp(i*theta_inf) exp(i*theta_0)];
N = 4; % compute N levels of reflections:
[anu,cnu,rnu,jla,jlr] = reflect_circ(a,c,r,N);
 z(1,:) = cnu(1,1)+rnu(1,1)*exp(i*(theta_inf+2*pi*[1:101]/102));
for j=2:m
 z(j,:) = cnu(1,j)+rnu(1,j)*exp(i*2*pi*[0:100]/100);
end
% evaluate product formula for map on circles:
 zprod = ones(size(z));
for nu = 1:length(anu(:,1))
 zprod = zprod.*(z-anu(nu,2))./(z-anu(nu,1));
end
for j=1:m
  plot(real(zprod(j,:)),imag(zprod(j,:))); % plot map
  hold on; axis equal;
end
```

We will now prove these statements. Our proof is similar to the proof of the Schwarz-Christoffel formula in [21], but easier. Note that, if a radial slit in the w-plane is at angle θ, then w reflects to $e^{i2\theta}\overline{w}$. Therefore, an even number successive reflections through radial slits will take $w = f(z)$ to $Aw = Af(z)$, for some A with $|A| = 1$. As a result, the extended function $f'(z)/f(z) = Af'(z)/Af(z)$ is invariant under even numbers of reflections and is single-valued. (For the case of the multiply connected Schwarz-Christoffel map, below the preSchwarzian $f''(z)/f'(z)$ is invariant under reflections, and we used this same "method of images" to construct a singularity function, $S(z) = f''(z)/f'(z)$, as an infinite sum satisfying appropriate boundary conditions.) Here, our singularity function is

$$S(z) = f'(z)/f(z) = \frac{d}{dz}\log f(z) = \sum_{\nu}\left(\frac{1}{z-\rho_\nu(b)} - \frac{1}{z-\rho_\nu(a)}\right)$$
$$= \sum_{\nu}\left(\frac{\rho_\nu(b)-\rho_\nu(a)}{(z-\rho_\nu(a))(z-\rho_\nu(b))}\right).$$

Since $f(z)$ maps to radial slits, $\arg f(z) =$constant, for $z \in C_k$. This boundary condition is given in the following lemma.

LEMMA 3.2. $\mathrm{Re}\{(z-c_k)f'(z)/f(z)\} = 0, z \in C_k$.

Proof. For $z \in C_k$, we have $z = c_k + r_k e^{i\theta}$ and since $f(z)$ maps to radial slits, we have $\arg f(z) =$const. Therefore,

$$0 = \frac{\partial}{\partial\theta}\arg f(z) = \frac{\partial}{\partial\theta}\mathrm{Im}\log f(c_k + r_k e^{i\theta}) = \mathrm{Im}\, ir_k e^{i\theta}\frac{f'}{f} = \mathrm{Re}\, r_k e^{i\theta}\frac{f'}{f}(c_k + r_k e^{i\theta}). \qquad \square$$

We show below that $S(z)$ satisfies this condition and that, indeed, $f'(z)/f(z) = S(z)$.

We now state our main theorem for radial slit maps.

ETNA
Kent State University
http://etna.math.kent.edu

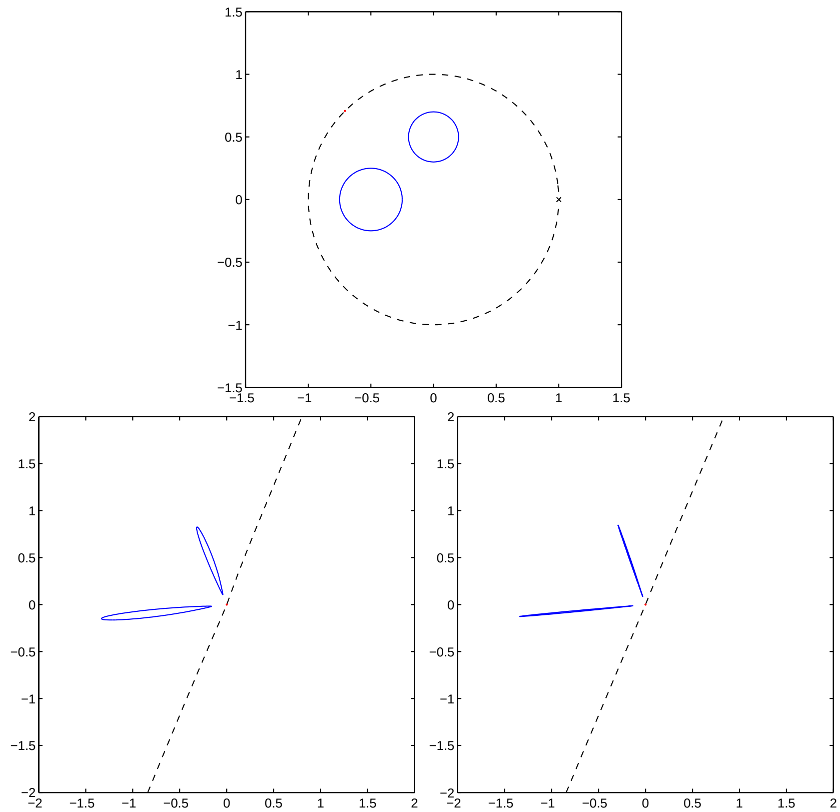

FIG. 3.1. *Conformal map $w = f(z)$ from the unit disk with $m = 2$ circular holes (top) to a radial slit half-plane with $N = 2$ (lower left) and $N = 4$ (lower right) reflections using Algorithm 3.1. Note that increasing N causes the slits to close.*

THEOREM 3.3. *Let Ω be an unbounded $m+1$-connected radial slit upper half-plane and D a conformally equivalent bounded circular domain, $a, b \in C_0$. Further, suppose Ω satisfies the separation property $\Delta < m^{-1/4}$, for $m \geq 1$. Then D is mapped conformally onto Ω by f with $f(b) = 0$ and $f(a) = \infty$ if and only if*

$$f(z) = C \prod_{\substack{j=0 \\ \nu \in \sigma_j(0)}}^{\infty} \frac{z - \rho_\nu(b)}{z - \rho_\nu(a)}$$

for some constant C.

Proof. Once we establish that $S_N(z)$ converges to $S(z)$ and satisfies the boundary condition, we can show that $f(z) = C \exp(\int S(z) dz)$. The proof follows closely the proof in [19]. In fact, by mapping the circle to the upper half-plane and extending the map and the image slit half-plane to a full plane with $2m$ radial slits by reflection across the real axis, we may just use the proof in [19]. We omit the details. □

The proof of convergence of the $S_N(z)$ also closely follows [19, Theorem 3.3]. We will show that the sums truncated to N levels of reflection,

$$S_N(z) = \sum_{\substack{j=0 \\ \nu \in \sigma_j(0)}}^{N} \left(\frac{1}{z - \rho_\nu(b)} - \frac{1}{z - \rho_\nu(a)} \right),$$

ETNA
Kent State University
http://etna.math.kent.edu

converge uniformly to $S(z)$ for $z \in \overline{\Omega}$ as $N \to \infty$, provided the circles satisfy our separation condition. (In the special case when $m+1=2$, there is no restrictive separation hypothesis, since then $\Delta < m^{-1/4} = 1$ is equivalent to the fact that the two boundary components are disjoint.)

We now prove the convergence of $S_N(z)$ to $S(z)$ for sufficiently well-separated circles. For $j = 0, 1, 2, \ldots,$ we write

$$A_j\left(z\right) = \sum_{\nu \in \sigma_j(0)} \left(\frac{1}{z-b_\nu} - \frac{1}{z-a_\nu} \right) = \sum_{\nu \in \sigma_j(0)} \frac{b_\nu - a_\nu}{(z-a_\nu)(z-b_\nu)}$$

and hence, in brief notation,

$$S_N(z) = \sum_{j=0}^{N} A_j(z), \quad S(z) := \lim_{N\to\infty} S_N(z).$$

Let

$$\delta = \delta_\Omega = \inf_{z\in\Omega} \left\{ |z-a_\nu|, |z-b_\nu| : |\nu| \geq 0 \right\}.$$

Then, clearly $\delta > 0$ holds since the a_ν's and the b_ν's lie inside the circles for $|\nu| \neq 0$.

We have the following result.

THEOREM 3.4. *For connectivity $m \geq 1$, $S_N(z)$ converges to $S(z)$ uniformly on $\overline{\Omega}$ satisfying the estimate,*

$$|S(z) - S_N(z)| = O((\Delta^2\sqrt{m})^{N+1}),$$

for regions satisfying the separation condition,

$$\Delta < \frac{1}{m^{1/4}}.$$

Proof. Note that the number of terms in the $A_j(z)$ sum is $O(m^j)$. This exponential increase in the number of terms is the principal difficulty in establishing convergence. Recall that r_ν is the radius of circle C_ν. We bound $A_j(z)$ for $z \in \overline{\Omega}$ by using the inequality $|a_\nu - b_\nu| < 2r_\nu$. First, note that

$$|A_j(z)| \leq \sum_{\nu\in\sigma_j(0)} \frac{|a_\nu - b_\nu|}{|z-a_\nu||z-b_\nu|} \leq \frac{2}{\delta^2} \sum_{\nu\in\sigma_j(0)} r_\nu, \tag{3.1}$$

where $\delta = \delta_\Omega$. (In practice, the sum of the r_ν's above at the $j = N$th level gives a good estimate of the truncation error. We will give an example of this below for a Schwarz-Christoffel map.) In order to prove convergence, we estimate the rate of decrease of the r_ν's using Lemma 2.3 and the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{\nu\in\sigma_j(0)} r_\nu &\leq \left(\sum_{\nu\in\sigma_j(0)} r_\nu^2\right)^{1/2} \left(\sum_{\nu\in\sigma_j(0)} 1\right)^{1/2} = \left(\sum_{\nu\in\sigma_j(0)} r_\nu^2\right)^{1/2} m^{j/2} \\ &\leq \Delta^{2j} \left(\sum_{i=0}^{m} r_i^2\right)^{1/2} m^{j/2} \leq C\Delta^{2j} m^{j/2}.\end{aligned}$$

ETNA
Kent State University
http://etna.math.kent.edu

Therefore, the series converges if $\Delta^2\sqrt{m} < 1$. ☐

The proof that $f(z)$ defined by the (convergent) infinite product formula satisfies the boundary conditions in Lemma 3.2 is nearly identical to [19, Theorem 3.4]. Again, we will use the formula

$$\operatorname{Re}\left\{\frac{w}{w-1} + \frac{w^*}{w^*-1}\right\} = 1, \tag{3.2}$$

where w and $w^* = 1/\overline{w}$ are symmetric points with respect to the unit circle. Then the following theorem gives the result. We prove the theorem for $a, b \in C_i$ for arbitrary i, but we can assume $i = 0$ and $C_i = C_0$ is the unit circle, without loss of generality.

THEOREM 3.5. *If $\Delta < m^{-1/4}$, then for $z \in C_k$,*

$$\operatorname{Re}\{(z - c_k)S_N(z)\} = O((\Delta^2\sqrt{m})^N)$$

and

$$\operatorname{Re}\{(z - c_k)S(z)\} = 0.$$

Proof. The idea of the proof is, for $z \in C_p, p \neq i$, to use properties of the reflections, $b_{p\nu} = \rho_p(b_\nu)$, to group terms in $S_N(z)$ related by reflection ρ_p through C_p with $z \in C_p$ as follows:

$$\begin{aligned} S_N(z) = &\left(\frac{1}{z-b} + \frac{1}{z-b_p}\right) - \left(\frac{1}{z-a} + \frac{1}{z-a_p}\right) + \cdots \\ &+ \left(\frac{1}{z-b_\nu} + \frac{1}{z-b_{p\nu}}\right) - \left(\frac{1}{z-a_\nu} + \frac{1}{z-a_{p\nu}}\right) + \cdots . \end{aligned}$$

Then, multiplying by $z - c_p$ and denoting $a_i := a, b_i := b$, we have in more detail,

$$\begin{aligned} (z-c_p)S_N(z) = &\frac{(z-c_p)/(b_i-c_p)}{(z-c_p)/(b_i-c_p)-1} + \frac{(z-c_p)/(b_{pi}-c_p)}{(z-c_p)/(b_{pi}-c_p)-1} \\ &- \frac{(z-c_p)/(a_i-c_p)}{(z-c_p)/(a_i-c_p)-1} + \frac{(z-c_p)/(a_{pi}-c_p)}{(z-c_p)/(a_{pi}-c_p)-1} \\ &+ \sum_{j=2}^{N-1} \sum_{\substack{\nu\in\sigma_j(i),\\ \nu i,\nu_1\neq p}} \left(\frac{(z-c_p)/(b_{\nu i}-c_p)}{(z-c_p)/(b_{\nu i}-c_p)-1} + \frac{(z-c_p)/(\rho_p(b_{\nu i})-c_p)}{(z-c_p)/(\rho_p(b_{\nu i})-c_p)-1}\right) \\ &- \sum_{j=2}^{N-1} \sum_{\substack{\nu\in\sigma_j(i)\\ \nu i,\nu_1\neq p}} \left(\frac{(z-c_p)/(a_{\nu i}-c_p)}{(z-c_p)/(a_{\nu i}-c_p)-1} + \frac{(z-c_p)/(\rho_p(a_{\nu i})-c_p)}{(z-c_p)/(\rho_p(a_{\nu i})-c_p)-1}\right) \\ &+ (z-c_p)\sum_{\substack{j=1,\\ j\neq p}}^{m} \sum_{j\nu\in\sigma_N(i)} \left(\frac{b_{j\nu}-a_{j\nu}}{(z-a_{j\nu})(z-b_{j\nu})}\right). \end{aligned} \tag{3.3}$$

We take the real part of the above expression and, using, for instance, $w = (z - c_p)/(a_{\nu i} - c_p)$ and noting that $w^* = (z - c_p)/(\rho_p(a_{\nu i}) - c_p)$, (3.2) gives

$$\operatorname{Re}\left\{\frac{(z-c_p)/(a_{\nu i}-c_p)}{(z-c_p)/(a_{\nu i}-c_p)-1} + \frac{(z-c_p)/(\rho_p(a_{\nu i})-c_p)}{(z-c_p)/(\rho_p(a_{\nu i})-c_p)-1}\right\} = \operatorname{Re}\left\{\frac{w}{w-1} + \frac{w^*}{w^*-1}\right\} = 1.$$

ETNA
Kent State University
http://etna.math.kent.edu

Taking the real part of (3.3), we see that the first four lines sum to 0. The final m terms, all lying inside circles C_j, $j \neq p$, approximate the truncation error and are bounded by

$$\sum_{\nu \in \sigma_{n+1}} r_\nu^2 \le \Delta^{4N} \sum_{i=0}^{m} r_i^2.$$

This gives

$$\mathrm{Re}\,\{(z-c_p)S_N(z)\} = O(\sqrt{m}(\Delta^{2N} m^{N/2}).$$

Next we prove the boundary condition for $z = c_i + r_i e^{i\theta} \in C_i$. (In our case $i = 0$ and $C_i = C_0$ is the unit circle, $r_i = 1, c_i = 0$. However, this is not necessary in general.) Using $z = c_i + r_i e^{i\theta}$, $a = a_i = c_i + r_i e^{i\theta_a} \in C_i$, and $b = b_i = c_i + r_i e^{i\theta_b} \in C_i$, we have

$$\begin{aligned}
&(z-c_i)S_N(z) = \\
&(z-c_i)\left[\frac{1}{z-b_i} - \frac{1}{z-a_i} + \cdots + \left(\frac{1}{z-b_{\nu i}} + \frac{1}{z-b_{i\nu i}}\right) - \left(\frac{1}{z-a_{\nu i}} + \frac{1}{z-a_{i\nu i}}\right) + \cdots\right] \\
&= \frac{z-c_i}{z-b_i} - \frac{z-c_i}{z-a_i} + \cdots + \left(\frac{(z-c_i)/(b_{\nu i}-c_i)}{(z-c_i)/(b_{\nu i}-c_i)-1} + \frac{(z-c_i)/(b_{i\nu i}-c_i)}{(z-c_i)/(b_{i\nu i}-c_i)-1}\right) \\
&\quad - \left(\frac{(z-c_i)/(a_{\nu i}-c_i)}{(z-a_i)/(a_{\nu i}-c_i)-1} + \frac{(z-c_i)/(a_{i\nu i}-c_i)}{(z-c_i)/(a_{i\nu i}-c_i)-1}\right) + \cdots .
\end{aligned}$$

Taking the real parts, we get our boundary condition,

$$\begin{aligned}
\mathrm{Re}\,\{(z-c_i)S_N(z)\} &= \mathrm{Re}\left\{\frac{e^{i\theta}}{e^{i\theta}-e^{i\theta_b}}\right\} - \mathrm{Re}\left\{\frac{e^{i\theta}}{e^{i\theta}-e^{i\theta_a}}\right\} + (1-1) + (1-1) + \cdots \\
&= \mathrm{Re}\left\{\frac{e^{i(\theta-\theta_b)/2}}{e^{i(\theta-\theta_b)/2} - e^{-i(\theta-\theta_b)/2}}\right\} - \mathrm{Re}\left\{\frac{e^{i(\theta-\theta_a)/2}}{e^{i(\theta-\theta_a)/2} - e^{-i(\theta-\theta_a)/2}}\right\} \\
&= \mathrm{Re}\left\{\frac{1}{2} - \frac{i}{2}\cot\frac{\theta-\theta_b}{2}\right\} - \mathrm{Re}\left\{\frac{1}{2} - \frac{i}{2}\cot\frac{\theta-\theta_a}{2}\right\} \\
&= \frac{1}{2} - \frac{1}{2} = 0. \quad \square
\end{aligned}$$

REMARK 3.6. The case of the map from the unbounded circle domain containing ∞ to the *unbounded radial slit domain*, with b in the domain and $f(b) = 0$, was treated in [19]; see Figure 5.5 (left). The formula is nearly identical to the bounded case above, except that $f(a) = \infty$ is replaced by $f(\infty) = \infty$, and hence, for any reflection of a center $\rho_{\nu k}(\infty) = \rho_\nu(c_k) = s_{\nu k}$, we have $f(s_{\nu k}) = \infty$. The infinite product formula is then

$$f(z) = (z-b) \prod_{k=1}^{m} \prod_{\substack{j=0 \\ \nu \in \sigma_j(k)}}^{\infty} \frac{z - \rho_\nu(b_k)}{z - \rho_\nu(c_k)}.$$

3.2. The Schottky-Klein prime function. Crowdy [8, 9] expresses his formula in terms of Moebius maps $\theta_j(z)$ which generate the *Schottky group* associated with the bounded, circular domains. Here, we relate his maps to our reflections, as in [17]. Crowdy defines the maps,

$$\phi_j(z) := \overline{c}_j + \frac{r_j^2}{z - c_j} \text{ and } \theta_j(z) := \overline{\phi}_j(1/z) = \overline{\phi_j(1/\overline{z})} = c_j + \frac{r_j^2}{1/z - \overline{c_j}}.$$

ETNA
Kent State University
http://etna.math.kent.edu

In terms of our reflections $\rho_j, j \neq 0$, it's easy to see that $\phi_j(z) = \overline{\rho_j(z)} = \rho_{\overline{C_j}}(\overline{z})$ and $\theta_j(z) = \overline{\phi_j(1/\overline{z})} = \rho_j(\rho_0(z))$, where $\rho_0(z) = 1/\overline{z}$ = reflection through the unit circle C_0. Note that $\theta_j^{-1} = \rho_0\rho_j$.

The full *Schottky group* Θ consists of all products of the θ_i's and θ_i^{-1}'s where

$$\theta_i(z) = \frac{a_i z + b_i}{c_i z + d_i} \text{ and } \theta_i^{-1}(z) = \frac{d_i z - b_i}{-c_i z + a_i} \text{ with } a_i d_i - b_i c_i = 1.$$

Therefore, Θ is the Moebius group generated by all compositions of the basic $\theta_j = \rho_j\rho_0$ and their inverses. The $\theta_i(z_{k,j})$ generate exactly the reflections, $z_{k,\nu j}$, of the prevertices $z_{k,j}$ used in the formulas in Section 4. For instance, using our reflection notation and the fact that $\rho_j^2 = id$ (the group of reflections is a "free" group), we have

$$z_{k,102320} = \rho_1\rho_0\rho_2\rho_3\rho_2(z_{k,0}) = \rho_1\rho_0\rho_2\rho_0\rho_0\rho_3\rho_2\rho_0(z_{k,0}) = \theta_1\theta_2\theta_3^{-1}\theta_2(z_{k,0}).$$

The *Schottky-Klein (SK) prime functions* used by Crowdy are

$$\omega(z,\gamma) := (z-\gamma)\omega'(z,\gamma) = (z-\gamma)\prod_{\theta_i\in\Theta''}\frac{(\theta_i(z)-\gamma)(\theta_i(\gamma)-z)}{(\theta_i(z)-z)(\theta_i(\gamma)-\gamma)}, \tag{3.4}$$

where $\theta_i \in \Theta''$ involve all compositions of the "forward" maps $\theta_j = \rho_j\rho_0$ giving "half" of the Schottky group Θ, and Θ'' does not include any θ_i^{-1} or the identity map, id; see [2, Chapter 12]. The relation between the slit maps and the Schottky-Klein prime functions from [17] is given by the following theorem, which explicitly states the relation of the ratios of the SK prime functions to radial slit maps. The theorem gives an alternate representation of ratios of Schottky-Klein prime functions using the full Schottky group Θ.

THEOREM 3.7. *] If $\Delta < m^{-1/4}$, then the infinite products converge and*

$$\frac{\omega(z,a)}{\omega(z,b)} = C(a,b)\prod_{\theta_i\in\Theta}\frac{z-\theta_i(a)}{z-\theta_i(b)},$$

where $C(a,b)$ is a ratio of integration constants. Therefore, for $a, b \in C_i$,

$$\frac{\omega(z,a)}{\omega(z,b)} = C\prod_{\nu}\frac{z-\rho_\nu(a)}{z-\rho_\nu(b)}$$

is a slit map *to a half-plane with radial slits, and so*

$$\arg\frac{\omega(z,a)}{\omega(z,b)} = \text{constant}$$

for $z \in C_j, j = 0,\ldots,m$.

Proof. A proof of this of is given in [17], based on a calculation in [4, 5]. Here, we give a shorter, alternate proof suggested by a referee of [17, Remark 2]. The idea is to shift the Moebius transformations, θ_i, from z to γ in (3.4), so that the infinite product can be taken over the entire Schottky group, Θ. This is accomplished using the calculations,

$$\begin{aligned}
\gamma - \theta_i(z) &= \gamma - \frac{a_i z + b_i}{c_i z + d_i}\\
&= \frac{c_i\gamma z + d_i\gamma - a_i z - b_i}{c_i z + d_i}\\
&= \left(\frac{c_i\gamma - a_i}{c_i z + d_i}\right)\left(z - \frac{d_i\gamma - b_i}{-c_i\gamma + a_i}\right)\\
&= \left(\frac{c_i\gamma - a_i}{c_i z + d_i}\right)(z - \theta_i^{-1}(\gamma))
\end{aligned}$$

ETNA
Kent State University
http://etna.math.kent.edu

and

$$\begin{aligned} z - \theta_i(z) &= \left(\frac{c_i z - a_i}{c_i z + d_i} \right) (z - \theta_i^{-1}(z)) \\ &= \left(\frac{c_i z - a_i}{c_i z + d_i} \right) \left(z + \frac{d_i z - b_i}{c_i z - a_i} \right) \\ &= \left(\frac{c_i}{c_i z + d_i} \right) (z - A_i)(z - B_i), \end{aligned}$$

where A_i, B_i are distinct fixed points of $\theta_i, A_i = \theta_i(A_i), B_i = \theta_i(B_i)$. Substituting these results into ω in (3.4), gives

$$\begin{aligned} \omega(z,\gamma) &= (z-\gamma) \prod_{\theta_j \in \Theta''} \frac{(z-\theta_j(\gamma))(\gamma - \theta_j(z))}{(z-\theta_j(z)(\gamma-\theta_j(\gamma))} \\ &= (z-\gamma) \prod_{\theta_j \in \Theta''} \frac{(c_j\gamma - a_j)(z-\theta_j(\gamma))(z-\theta_j^{-1}(\gamma))}{c_j(z-A_j)(z-B_j)(\gamma-\theta_j(\gamma))} \\ &= K(\gamma) \prod_{\theta_j \in \Theta''} \frac{1}{(z-A_j)(z-B_j)} \prod_{\theta_j \in \Theta} (z-\theta_j(\gamma)), \end{aligned}$$

where

$$K(\gamma) := \prod_{\theta_j \in \Theta''} \frac{(c_j\gamma - a_j)}{c_j(\gamma - \theta_j(\gamma))},$$

giving finally

$$\frac{\omega(z,a)}{\omega(z,b)} = \frac{K(a)}{K(b)} \prod_{\theta_j \in \Theta} \frac{z-\theta_j(a)}{z-\theta_j(b)}.$$

If $a \in C_i$ and $b \in C_i$, by the observations at the beginning of this subsection, we can replace the $\theta_j(a)$'s by the corresponding reflections, ρ_ν, i.e., $\theta_j(a) = \rho_\nu(a)$ and $\theta_j(b) = \rho_\nu(b)$. This just yields our formula for the map to a radially slit half plane. ☐

REMARK 3.8. Crowdy and Marshall [15] give a Laurent series method for evaluating the prime function for general circle domains where the convergence condition above need not hold. In Section 5.3, we discuss a similar method for the map to a radial slit half plane.

REMARK 3.9. In [26, pp. 65–68], the annulus map is derived by taking successive products of maps that gradually "straighten" the circles. Here, we "unwrap" that derivation and show that, in the general multiply connected case, it just leads to our radial slit map, above. We will illustrate this process on an annulus where the outer boundary is the the unit circle C_0 for the factors that take prevertex $z_{k,0}$ to 0 and 1 to ∞. We will denote the succesive straightening factors by $g_{k,\nu 0}$. The first factor is

$$g_{k,0}(z) = \frac{1 - z/z_{k,0}}{1-z} = \frac{z - z_{k,0}}{z_{k,0}(z-1)}.$$

We can ignore constant factors like the $z_{k,0}$ in the denominator above, since they all be absorbed in a multiplicative constant in the end. $g_{k,0}(z)$ straightens out the 0 circle, but distorts

ETNA
Kent State University
http://etna.math.kent.edu

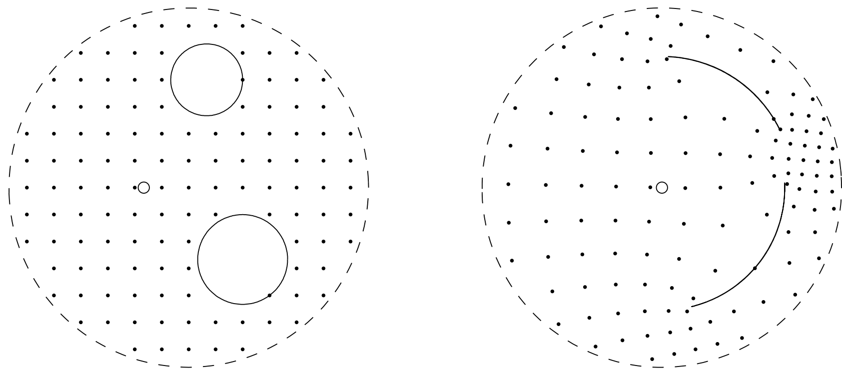

FIG. 3.2. *The map $w = f(z)$ with $m+1 = 3$ and $f(a) = 0$ from the interior circle domain to the interior circular slit (unit) disk using product formula with $N = 4$ levels of reflection. The modified (hydrodynamic) Green's function is given by $\log|f(z)/f(z_0)|$ for some $z_0 \in C_0$. Here $f(C_0)$ =outer circle. In Nehari's notation [34], $R_0(z;a) = f(z)/f(z_0)$.*

the other circles in the process. To cancel out this effect and straighten the image of circle 1 into a list, we multiply by

$$
\begin{aligned}
g_{k,10}(z) &= \overline{g_{k,0}(\rho_1(z))} = \overline{g_{k,0}\left(c_1 + \frac{r_1^2}{\overline{z} - \overline{c}_1}\right)} \\
&= \overline{\left(\frac{c_1 + \frac{r_1^2}{\overline{z}-\overline{c}_1} - z_{k,0}}{z_{k,0}\left(c_1 + \frac{r_1^2}{\overline{z}-\overline{c}_1} - 1\right)}\right)} = \frac{\frac{r_1^2}{z-c_1} - (\overline{z}_{k,0} - \overline{c}_1)}{\overline{z}_{k,0}\left(\overline{c}_1 + \frac{r_1^2}{z-c_1} - 1\right)} \\
&= \left(\frac{\overline{z}_{k,0} - \overline{c}_1}{\overline{z}_{k,0}(1-\overline{c}_1)}\right)\left(\frac{z - \left(c_1 + \frac{r_1^2}{\overline{z}_{k,0}}\right)}{z - \left(c_1 + \frac{r_1^2}{1-\overline{c}_1}\right)}\right) = \left(\frac{\overline{z}_{k,0} - \overline{c}_1}{\overline{z}_{k,0}(1-\overline{c}_1)}\right)\left(\frac{z - \rho_1(z_{k,0})}{z - \rho_1(1)}\right).
\end{aligned}
$$

Continuing this process, we get a constant multiple of our slit map,

$$
f(z) = C' g_{k,0}(z) g_{k,10}(z) \cdots = C \frac{(z - z_{k,0})(z - \rho_1(z_{k,0}))\cdots}{(z-1)(z-\rho_1(1))\cdots} = C \prod_{\nu} \frac{z - \rho_\nu(z_{k,0})}{z - \rho_\nu(1)},
$$

which is just the map in Theorem 3.3 with $a = 1, b = z_{k,0}$, and the reflections taken over the two concentric circular boundaries of the annulus. This example illustrates the fact that our formulas are, in effect, just the "method of image" with successive reflections of zeros and singularities applied to impose desired boundary behavior.

3.3. Circular slit map. These maps were derived in [19] for the unbounded case; see Figure 3.2 for the bounded case and Figure 5.5 (right) for the unbounded case. The formulas are identical. To get the bounded map, one just evaluates the formula for z in the bounded circle domain interior to one of the circles. To get the unbounded map, one evaluates the formula for z in the unbounded circle domain. The map $w = f(z)$ from the (un)bounded circle domain to the conformally equivalent, (un)bounded circular slit domain with the slits centered at the origin can be derived in a similar fashion to the radial slit map. Once again $f(a) = 0$ and $f(\infty) = \infty$ with $f(z) \sim z, z \approx \infty$. Again, $a_k = \rho_k(a)$ is the reflection of a across circle C_k and $c_k = s_k = \rho_k(\infty)$, the center of circle C_k, is the reflection of ∞ across C_k. In the w-plane 0 and ∞ just reflect back and forth to each other. Therefore, when we

ETNA
Kent State University
http://etna.math.kent.edu

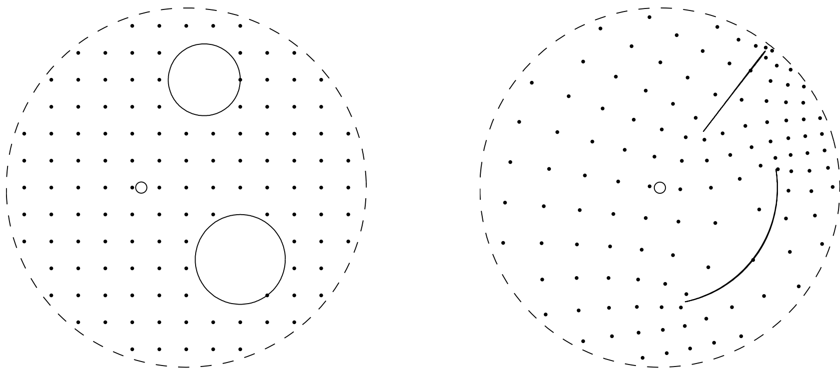

FIG. 3.3. *Map to combined radial and circular slit domain.*

extend f, we will have $f(a_k) = \infty$ and $f(c_k) = 0$. In this way, we see that all odd numbers of reflections, $a_{\nu_o k}, |\nu_o| = 2l+1$, of a_k and all even numbers of reflections, $s_{\nu_e k}, |\nu_e| = 2l$, of c_k will be simple zeros, $f(a_{\nu_o k}) = f(s_{\nu_e k}) = 0$. Likewise, all odd numbers of reflections, $s_{\nu_o k}, |\nu_o| = 2l+1$, of c_k and all even numbers of reflections, $a_{\nu_e k}, |\nu_e| = 2l$, of a_k will be simple poles, $f(a_{\nu_e k}) = f(s_{\nu_o k}) = \infty$. The infinite product for $w = f(z)$ therefore has the form,

$$f(z) = (z-a)\prod_{k=1}^{m}\prod_{\substack{j=0 \\ \nu_e,\nu_o\in\sigma_j(k)}}^{\infty}\frac{(z-\rho_{\nu_o}(a_k))(z-\rho_{\nu_e}(c_k))}{(z-\rho_{\nu_e}(a_k))(z-\rho_{\nu_o}(c_k))},$$

(where reflections back to a or ∞ are excluded from the product) with $f(a) = 0$, provided the m circles with centers c_k satisfy our standard separation criterion.

Following [19], we note that, if a circular slit in the w-plane is at radius r_1, then w reflects to $r_1^2/\overline{w}$. Reflection through another circular slit with radius r_2 will then take w to $(r_2/r_1)^2 w$, and so on. Therefore, an even number successive reflections through circular slits will take $w = f(z)$ to $Aw = Af(z)$, for some A real. As a result, the extended function $f'(z)/f(z) = Af'(z)/Af(z)$ is invariant under even numbers of reflections and hence is single-valued. Here, our singularity function is

$$S(z) = f'(z)/f(z) = \frac{d}{dz}\log f(z) =$$
$$\frac{1}{z-a} + \sum_{k=1}^{m}\sum_{\substack{j=0 \\ \nu_e,\nu_o\in\sigma_j(k)}}^{\infty}\left(\frac{1}{z-\rho_{\nu_o}(a_k)} - \frac{1}{z-\rho_{\nu_o}(c_k)}\right) + \left(\frac{1}{z-\rho_{\nu_e}(c_k)} - \frac{1}{z-\rho_{\nu_e}(a_k)}\right),$$

For $z \in C_k$, since $f(z)$ maps to circular slits, we have $\log|f(z)| = \mathrm{Re}\log f(z) =$const. Our boundary conditions are given by the following lemma.

LEMMA 3.10. $\mathrm{Im}\{(z-c_k)f'(z)/f(z)\} = 0, z \in C_k$.

3.4. Combined circular and radial slit map. Here we consider the map $w = f(z)$ from the bounded circle domain to the interior a disk bounded by a mixture of radial and circular slits with $f(a) = 0$. This map was also given in [19] for the unbounded case and is identical to that case, except, again one evaluates the map in the interior of C_0 instead of in the exterior; see Figure 3.3. This map will lead to the Robin function, i.e., the Green's function for the mixed boundary value problem. The log of the function maps to a domain exterior to horizontal and vertical slits. Reflections through radial slits will keep 0 and ∞ fixed, whereas, reflections through circular slits will swap 0 and ∞ as in the circular slit map above. Let ρ_{ν_e} denote a sequence of reflections with an even number of reflections through circular slits and let ρ_{ν_o} denote a sequence with an odd number of reflections through circular slits. Then $\rho_{\nu_e}(a)$ and $\rho_{\nu_o}(\infty)$ are simple zeros of $f(z)$ and $\rho_{\nu_e}(\infty)$ and $\rho_{\nu_o}(a)$ are simple poles. Therefore, we have

$$f(z) = (z-a) \prod_{\nu_e,\nu_o} \frac{(z-\rho_{\nu_e}(a))(z-\rho_{\nu_o}(\infty))}{(z-\rho_{\nu_e}(\infty))(z-\rho_{\nu_o}(a))}.$$

3.5. Green's functions. It is interesting to note that Green's functions for circle domains for the Dirichlet, Neumann, and mixed cases [31, 32, 34, 35, 39] can be written explicitly in terms of the slit maps. For instance, the Dirichlet Green's function would have the form [34, p. 357]

$$g(z,a) = -G(z,a) + \sum_{i=1}^{m} \gamma_i \omega_i(z)$$

where $G(z,a) = \mathrm{Re}\{\log f(z)\} = \log|f(z)|$ and $f(z)$ is the map of the circle domain onto the circular slit unit disk with the C_0 mapped to the unit circle and $f(a) = 0$, so that $G(z,a)$ has exactly one logarithmic singularity at a and $G(z,a) = 0, z \in C_0$. Since $f(z)$ maps the circles, $C_i, i = 1,\ldots,m$, to concentric arcs, $G(z,a) = \log|f(z)| = \gamma_i$, constant, for $z \in C_i$. ($G(z,a)$ is the so-called modified or hydrodynamic Green's function.) The $\omega_i(z)$'s are the *harmonic measures* of the C_i's. That is, $\omega_i(z)$ is harmonic in the circle domain with $\omega_i(z) = 1, z \in C_i$ and $\omega_i(z) = 0$ for $z \in C_j, j \neq i, j = 0,1,\ldots,m$. Therefore, $g(z,a)+\log|z-a|$ is harmonic in the circle domain and $g(z,a) = 0, z \in C_i, i = 0,1,\ldots,m$, i.e., $g(z,a)$ is the (Dirichlet) Green's function for the circle domain.

Nehari [34, Chap. VII, Sec. 3] shows how to construct the harmonic measures using maps to canonical slit domains. We will outline this briefly here in order to show how these functions can be explicitly constructed for circle domains. In Nehari's notation [34], the map to the circular slit unit disk taking the jth boundary to the unit circle is denoted by $R_j(z;a)$ with $R(a;a) = 0$ and the normalization $R'(a;a) > 1$. In [34, Chap. VII, Sec. 1], it is shown that the map to a circular slit annulus, taking the jth boundary to the outer circle and the kth boundary to the inner circle and the other boundaries to the slits, can be written as a ratio of maps to circular slit disks, $S_{jk}(z) := R_j(z;a)/R_k(z;a)$. For circle domains, $S_{jk}(z)$ can, therefore, be given explicitly using our infinite product formulas. A computed example is shown in Figure 3.4. Note that $S_{jk}(z) \neq 0$. Now let $\sigma_j(z) := \log|S_{j0}(z)|$. Nehari shows that constants a_{ji} can be found, such that

$$\omega_i(z) = \sum_{j=1}^{m} a_{ji}\sigma_j(z), \quad i = 1,\ldots,m.$$

The a_{ji}'s can be found as solutions to linear systems, but we will not discuss this here.

Similar expressions using the Schottky-Klein prime functions are given in [12, 13, 14]. In cases where reflections are not feasible, all of these maps can be computed efficiently using the least squares/Laurent series approach; see [15, 19, 36] and Section 5.3. These functions could potentially be combined with conformal maps of circle domains [3, 23] and Section 5.1, below, to provide Green's functions for general multiply connected domains.

ETNA
Kent State University
http://etna.math.kent.edu

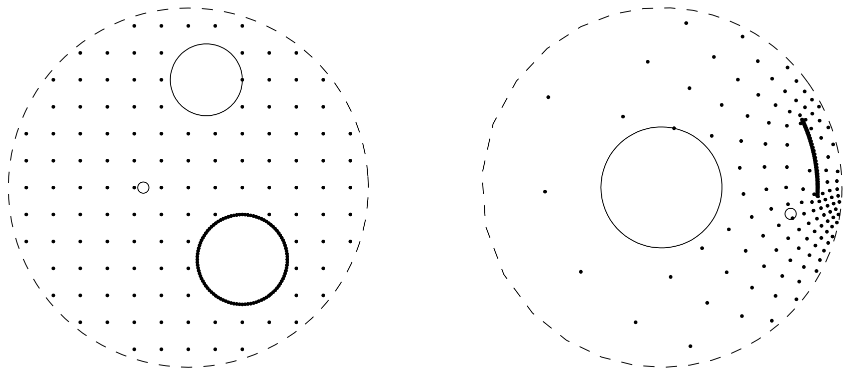

FIG. 3.4. *The map* $w = f(z), f(z) \neq 0$, *from interior circle domain to interior circular slit ring domain using product formulas with* $N = 5$ *levels of reflection. Here* $f(C_0) =$ *outer circle and* $f(C_1) =$ *inner circle. In Nehari's notation* $f(z) = S_{01}(z) = R_0(z; a)/R_1(z; a)$.

4. Schwarz-Christoffel maps for multiply connected domains. In this section, we review results for the Schwarz-Christoffel maps from circular domains to multiply connected polygonal domains [17, 18, 21] and further clarify the relation of the formulation of Crowdy [8, 9] to ours. We will concentrate on the bounded case [17] here, since the unbounded case [21] is similar. Our treatment attempts to unify three forms of these formulas: (i) the original formulas in terms of infinite products involving reflections of the mapping parameters as first derived by [20] for the annulus, [21] for the unbounded case, and [17] for the bounded case, using the invariance of the preSchwarzian, $S(z) = f''(z)/f'(z)$, under extension by Schwarz reflection, (ii) the formulas of [8, 9] for expressing the bounded and unbounded cases in terms of (finite) products of Schottky-Klein prime functions, $\omega(z, a)$, and (iii) a new form of the formulas for the bounded and unbounded cases expressed in terms of (finite) products of maps from the circle domains to radial slit domains. The last form fits into the framework of [26] wherein the derivative of the mapping function, $f'(z)$, is expressed as a product,

$$f'(z) = A \prod_k f_k(z), \tag{4.1}$$

of factors $f_k(z)$ that guarantee that f' has piecewise constant argument for the given geometry. For instance, for the case of simply connected maps from the disk, $f_k(z) := (z - z_k)^{\beta_k}$, $-\beta_k\pi =$ the turning angle at prevertex z_k, $\beta_k = \alpha_k - 1$, and $\sum_k \beta_k = -2$. In this case, the mapping function is

$$f(z) = A \int^z \prod_k (\zeta - z_k)^{\beta_k} d\zeta + B,$$

where a normalization condition, such as fixing an interior point and one boundary point, gives a unique map. (The numerical problem [25] in this case is to find A, B, z_k's by matching side lengths of the polygon.) There are several variations in which other domains are used, e.g., a rectangle or an infinite strip, [26, Chapter 4].

(i) The multiply connected Schwarz-Christoffel (MCSC) formulas can be written in terms of reflections. We will only discuss the bounded case [17] here. The outer circle C_0 is the unit circle. Here, $\alpha_{k,i}\pi$ are the interior angles of the polygons at the corners, $w_{k,i}$, $\beta_{k,i}\pi, k = 1, \ldots, K_i, i = 0, \ldots, m$ are the turning angles of f', with $\beta_{k,i} = \alpha_{k,i} - 1$ and

$\sum_{k=1}^{K_0} \beta_{k,0} = -2, \sum_{k=1}^{K_i} \beta_{k,i} = 2, i = 1, \ldots, m, z_{k,i} = c_i + r_i e^{i\theta_{k,i}}$ are the prevertices and $w_{k,i}$ are the corners, with $w_{k,i} = f(z_{k,i})$. Also, $z_{k,\nu i} = \rho_\nu(z_{k,i})$ denotes reflections of the kth prevertex $z_{k,i}$ on the ith circle. The MCSC formula for in this case (i) is

$$f'(z) = A \prod_{k=1}^{K_0} \prod_{\substack{j=0 \\ \nu \in \sigma_j(0)}}^{\infty} (z - z_{k,\nu 0})^{\beta_{k,0}} \prod_{i=1}^{m} \prod_{k=1}^{K_i} \prod_{\substack{j=0 \\ \nu \in \sigma_j(i)}}^{\infty} (z - z_{k,\nu i})^{\beta_{k,i}} . \tag{4.2}$$

The derivation [17] of the mapping formula for the bounded case is similar to the derivation for the unbounded case [21]. We repeat some of the details and main theorems here. As in the slit map cases, we analytically continue f from D by reflection across an arc $\gamma_{k,i}$ between prevertices $z_{k,i}, z_{k+1,i}$ on C_i. This extension, $\tilde{f}_{k,i}$, has the form

$$\tilde{f}_{k,i}(z) = a_{k,i} \overline{f(c_i + r_i^2/(\overline{z} - \overline{c_i})} + b_{k,i}$$

for z in the reflected domain with $a_{k,i}, b_{k,i}$ determined by the line containing the edge $f(\gamma_{k,i})$ joining $w_{k,i}$ and $w_{k+1,i}$ in the boundary of the polygon, Γ_i. This extended f maps the reflected circle domain conformally onto the reflected polygonal domain. By repeated application of the reflection process one obtains from the initial function in D a global (many-valued) analytic function $\widehat{f}$ defined on $\mathbb{C}_\infty \backslash \overline{\{z_{k,\nu}\}}$. Any two values, $\widehat{f}_r(z)$ and $\widehat{f}_s(z)$ of $\widehat{f}$ at a point $z \in \mathbb{C} \setminus \overline{\{z_{k,\nu}\}}$ are related by an even number of reflections in lines and hence $\widehat{f}_s(z) = c\widehat{f}_r(z) + d$ for some $c, d \in \mathbb{C}$. Therefore, the *preSchwarzian* of f, $f''(z)/f'(z)$, is invariant under affine maps $w \longmapsto aw + b$; that is,

$$\frac{(af(z) + b)''}{(af(z) + b)'} = \frac{f''(z)}{f'(z)}$$

is defined and *single-valued* on $\mathbb{C} \backslash \overline{\{z_{k,\nu}\}}$.

The preSchwarzian is determined by its singularities, $z_{k,\nu}$. By the usual argument

$$(f(z) - f(z_{k,i}))^{1/\alpha_{k,i}} = (z - z_{k,i}) h_{k,i}(z),$$

where $h_{k,i}(z)$ is analytic and nonvanishing near $z_{k,i}$. This gives the local expansion,

$$\frac{f''(z)}{f'(z)} = \frac{\beta_{k,i}}{(z - z_{k,i})} + H_{k,i}(z), \ \beta_{k,i} = \alpha_{k,i} - 1,$$

where $H_{k,i}(z)$ is analytic in a neighborhood of $z_{k,i}$. The *singularity function*, $S(z)$, of the global preSchwarzian is, in nonconvergent form,

$$S(z) = \sum_{j=0}^{\infty} \sum_{i=0}^{m} \sum_{\nu \in \sigma_j(i)} \sum_{k=1}^{K_i} \frac{\beta_{k,i}}{z - z_{k,\nu i}}.$$

To give the correct form, we truncate $S(z)$ and regroup terms as

$$S_N(z) = \sum_{k=1}^{K_0} \frac{\beta_{k,0}}{z - z_{k,0}} + \sum_{j=0}^{N} \sum_{i=1}^{m} \sum_{\nu \in \sigma_j(i)} \left[\sum_{k=1}^{K_i} \frac{\beta_{k,i}}{z - z_{k,\nu i}} + \sum_{k=1}^{K_0} \frac{\beta_{k,0}}{z - z_{k,\nu i 0}} \right].$$

We then show that $S_N(z)$ converges and we define $S(z) := \lim_{N \to \infty} S_N(z)$. We also show that $S(z)$ obeys the same boundary conditions as the preSchwarzian, as given by the following lemma from [21].

ETNA
Kent State University
http://etna.math.kent.edu

LEMMA 4.1. $\mathrm{Re}\left\{(z-c_i)\, f''(z)\,/ f'(z)\right\}_{|z-c_i|=r_i} = -1$, $i = 0, 1, ..., m$.

Proof. We repeat the proof for the reader's convenience. The tangent angle, $\psi(\theta) = \arg\{ir_i e^{i\theta} f'(c_i + r_i e^{i\theta})\} = \mathrm{Im}\{\log(ir_i e^{i\theta} f'(c_i + r_i e^{i\theta}))\}$, of the boundary C_i is constant on each of the arcs between prevertices. Hence, for $|z - c_i| = r_i, z \neq z_{k,i}$, we have $\psi'(\theta) = \mathrm{Re}\left\{(z - c_i)f''(z)/f'(z) + 1\right\} = 0$. $\square$

Our main theorem for the bounded polygonal case is as follows.

THEOREM 4.2. *Let Ω be an bounded m+1-connected polygonal region, and D a conformally equivalent circle domain. Further, suppose D satisfies the separation property $\Delta < m^{-1/4}$ for $m \geq 1$. Then D is mapped conformally onto Ω by a function of the form $Af(z) + B$, where*

$$f(z) = \int^z \prod_{i=0}^{m} \prod_{k=1}^{K_i} \left[\prod_{\substack{j=0 \\ \nu\in\sigma_j(i)}}^{\infty} (\zeta - z_{k,\nu i}) \right]^{\beta_{k,i}} d\zeta.$$

The turning parameters satisfy $-1 < \beta_{k,i} \leq 1$ and $\sum_{k=1}^{m} \beta_{k,i} = 2, \sum_{k=1}^{m} \beta_{k,0} = -2$. The separation parameter, Δ, is given explicitly in terms of the radii and centers of the (exterior) circular boundary components of $C_0, C_1, \ldots, C_m$.

The proof of convergence is given in the following theorem.

THEOREM 4.3. *For connectivity $m + 1 \geq 2$, $S_N(z)$ converges to $S(z)$ uniformly on closed sets $G \subset H = \overline{\Omega} \setminus \{z_{k,i}\}$ by the following estimate*

$$|S(z) - S_N(z)| = O((\Delta^2 \sqrt{m})^{N+1})$$

for regions satisfying the separation condition

$$\Delta < \frac{1}{m^{1/4}}.$$

The next theorem, like the corresponding one for the unbounded case in [21], shows, for general m, that $S(z)$ satisfies the boundary condition, Lemma 4.1, for $f''(z)/f'(z)$ for well-separated domains.

THEOREM 4.4. *If $\Delta < m^{-1/4}$ then for $z \in C_i, z \neq z_{k,i}$*

$$\mathrm{Re}\left\{(z - c_i)S_N(z)\right\} = -1 + O((\Delta^2\sqrt{m})^N)$$

and

$$\mathrm{Re}\left\{(z - c_i)S(z)\right\} = -1.$$

(ii) Crowdy's formula for the bounded case [8] is

$$f'(z) = \tilde{A} S_c(z) \prod_{k=1}^{K_0} \left[\omega(z, z_{k,0})\right]^{\beta_{k,0}} \prod_{i=1}^{m} \prod_{k=1}^{K_i} \left[\omega(z, z_{k,i})\right]^{\beta_{k,i}}, \tag{4.3}$$

where $\omega(z, a)$ are the SK prime functions (above) and

$$S_c(z) := \frac{\omega_z(z, \alpha)\omega(z, \overline{\alpha}^{-1}) - \omega_z(z, \overline{\alpha}^{-1})\omega(z, \alpha)}{\prod_{j=1}^{n} \omega(z, \gamma_1^j)\omega(z, \gamma_2^j)}.$$

In [17], formula (4.3) is reduced to the infinite product formula (4.2) above.

(iii) The text [26, pp. 65–68] gives a "geometric" derivation of $f'(z)$ for the annulus [21]. We now generalize this for arbitrary connectivity m: Pick one point a_i, not a prevertex, on each circle C_i. Then we define the radial slit map to a half-plane as in Section 3,

$$f_{k,i}(z) := \prod_{\nu} \frac{z - \rho_\nu(z_{k,i})}{z - \rho_\nu(a_i)},$$

with $f_{k,i}(z_{k,i}) = 0$ and $f_{k,i}(a_i) = \infty$. Note, that the radial slit maps do not directly satisfy the condition of Lemma 4.1 (their derivatives do), since they have constant arguments on the circles, C_i. The *DT framework* (4.1) for the bounded case is, thus,

$$f'(z) = A f_a(z) \prod_{i=0}^{m} \prod_{k=1}^{K_i} f_{k,i}(z)^{\beta_{k,i}},$$

where

$$f_a(z) = (z - a_0)^{-2} \prod_{i=1}^{m} \prod_{\substack{j=0 \\ \nu \in \sigma_j(0)}}^{\infty} (z - a_{\nu i0})^{-2} (z - a_{\nu i})^2$$

gives the correct boundary behavior. We have used $\sum_{k=1}^{K_i} \beta_{k,i} = 2, i = 1, \ldots, m$ and $\sum_{k=1}^{K_0} \beta_{k,0} = -2$ to collect the a_i terms in $f_a(z)$. Properties of $f_a(z)$ defined independently of the infinite product representation, above, would be needed to prove the theorem below, in the general case, without the separation condition. A calculation shows that $f_a(z)$ has the correct boundary behavior. With these observations, we can link the three approaches based on reflections, slit maps, and SK prime functions (≈ Crowdy) roughly as follows.

THEOREM 4.5. *(*DEP=DT≈Cr *for bounded case) If the infinite products converge, then*

$$\begin{aligned} f'(z) &= A \prod_{k=1}^{K_0} \prod_{\substack{j=0 \\ \nu \in \sigma_j(0)}}^{\infty} (z - z_{k,\nu 0})^{\beta_{k,0}} \prod_{i=1}^{m} \prod_{k=1}^{K_i} \prod_{\substack{j=0 \\ \nu \in \sigma_j(i)}}^{\infty} (z - z_{k,\nu i})^{\beta_{k,i}} \\ &= A f_a(z) \prod_{i=0}^{m} \prod_{k=1}^{K_i} f_{k,i}(z)^{\beta_{k,i}} \\ &= \tilde{A} f_a(z) \prod_{i=0}^{m} \prod_{k=1}^{K_i} \left(\frac{\omega(z, z_{k,i})}{\omega(z, a_i)} \right)^{\beta_{k,i}}. \end{aligned}$$

Proof. This is an obvious consequence of the previous results. □

Similar expressions can be given for the unbounded case. The slit maps exist in all cases independently of their infinite product expansion and can be evaluated efficiently with Laurent series [19]. Crowdy and Marshall [15] have developed an efficient Laurent series expansion of the SK functions for general domains which do not necessarily satisfy the separation criterion. The last form in the theorem above is close to [8, 9] in the sense that the SK prime functions are used. Note that additional parameters, the a_i's, must be selected. In effect, they take the place of the zeros $\gamma_j^{1,2}$ of the derivatives of the circular slit maps in the factor $S_c(z)$ in Crowdy's original formulation [8]. It is hoped that better understanding of these various formulations will lead to efficient numerical methods for the general mapping problem in cases when connectivity is too high or the circles are too close-to-touching to make use of reflections feasible.

ETNA
Kent State University
http://etna.math.kent.edu

5. Numerical and theoretical issues. In this section, we will discuss numerical aspects of the Schwarz-Christoffel map, including the solution to the parameter problem, ill-conditioning due to crowding of the circles, and possible improvements to the reflection procedure using Laurent series and least squares techniques.

5.1. Numerics for Schwarz-Christoffel maps. We will briefly review here some recent progress on the numerical computation of Schwarz-Christoffel maps for multiply connected domains reported in [23]. We will give an example of the computations for a bounded domain; see Figures 5.1 and 5.2. Several examples of both bounded and unbounded domains are computed in [23]. Preliminary work on this problem was done in [18] for the unbounded case. A highly symmetric example for the bounded case was computed in [17].

In order to compute of the MCSC maps for given polygonal boundaries, one must solve the so-called *parameter problem* of finding the prevertices and the centers and radii of the circles. This is done by solving a nonlinear set of equations that guarantee that the side lengths of the polygons and their locations are correct. The examples computed in [17, 18] were particularly sensitive to initial guesses and small changes in the polygonal domains. This was due to the use of constrained variables, the $\theta_{k,i}$'s which must remain in order on the circles. In [23], a transformation to unconstrained coordinates similar to [26, p. 25] is used. The method proves to be extremely robust and rarely fails to converge. The selection of integration paths between circles (required to position the circles so that the locations of the polygons are correct) rarely causes problems, unless the paths pass between or very close to singularities in the inside of the circles. This situation can generally be avoided by an expedient choice of integration intervals. As in [18], a homotopy search method [1, Program 3] is used in [23] and found to be very effective compared to other solvers. Convergence is almost always achieved even with a deliberately poor initial guess. Some additional code used here, such as Gauss-Jacobi integration routines to handle the singularities in the Schwarz-Christoffel integrals, is taken from SC Toolbox [25, 26], an existing MATLAB package for computing Schwarz-Christoffel maps for various simply and doubly connected geometries.

We will summarize some details for the bounded maps from [23]. The prevertices on C_i are parametrized by $\theta_{k,i}$, where $z_{k,i} = c_i + r_i e^{i\theta_{k,i}}$ for $k = 1, \ldots, K_i$, and constrained to lie in order,

$$\theta_{1,i} < \theta_{2,i} < \ldots < \theta_{K_i,i}. \tag{5.1}$$

The unknown c_i's, r_i's, and $\theta_{k,i}$'s amount to a total of $K_0 + K_1 + \cdots + K_m + 3m + 3$ real parameters. Since the circles determine the reflections, these are precisely the parameters needed to determine the infinite product for $f'(z)$. We approximate the infinite product by a finite product truncated after N levels of reflection ($N + 1$ for the outer unit circle),

$$p_b(z) = \prod_{k=1}^{K_0} (z - z_{k,0})^{\beta_{k,0}} \prod_{i=1}^{m} \prod_{\substack{j=0 \\ \nu \in \sigma_j(i)}}^{N} \left(\prod_{k=1}^{K_0} (z - z_{k,\nu i 0})^{\beta_{k,0}} \prod_{k=1}^{K_i} (z - z_{k,\nu i})^{\beta_{k,i}} \right).$$

In the bounded case, the map can be normalized by fixing one boundary point, $f(1) = w_{1,0}$, and one interior point, $f(z_0) = w_0$. Letting

$$C = \frac{w_{2,0} - w_{1,0}}{\int_{z_{1,0}}^{z_{2,0}} p_b(z)\, dz},$$

we have

$$f(z) = C \int_{z_{1,0}}^{z} p_b(\zeta)\, d\zeta + D,$$

ETNA
Kent State University
http://etna.math.kent.edu

with $f(z_{1,0}) = D = w_{1,0}$. We require that $c_0 = 0$ and $r_0 = 1$, and fixing $f(1) = w_{1,0}$ is equivalent to setting $\theta_{1,0} = 0$. This amounts to fixing four of the real parameters, so we have

$$K_0 + \cdots + K_m + 3m - 1$$

unknowns parameters to determine.

The remaining parameters are determined from the geometry of the polygonal domain. First, we have the *side-length conditions*,

$$|f(z_{k+1,i}) - f(z_{k,i})| = |w_{k+1,i} - w_{k,i}|,$$

for $i = 0, \dots, m$ and $k = 1, \dots, K_i$, where here and below

$$f(z_{k+1,i}) - f(z_{k,i}) = C \int_{z_{k,i}}^{z_{k+1,i}} p_b(\zeta)\, d\zeta$$

is calculated by numerical integration from $z_{k,i}$ to $z_{k+1,i}$ using the Gauss-Jacobi weights. These side-length conditions give $K_0 + \cdots + K_m$ real equations, but the calculation of C removes one from this count. In addition, we leave off the calculation of the last two side-lengths of the outer boundary polygon ($i = 0$), which can be done since the known turning angles of the polygon allow the last vertex to be uniquely determined by the intersection of lines drawn from the adjacent vertices; see [26, Fig. 3.1, p. 24]. The final side-length conditions then add up to $K_0 + \cdots + K_m - 3$ real equations. The *positions* of Γ_0 through Γ_m with respect to w_0 are fixed by requiring that

$$f(z_{1,i}) - f(z_0) = w_{1,i} - w_0$$

for $i = 0, \dots, m$. These conditions give $2m + 2$ real equations. The integration paths for our example below with $m = 3$ are shown in Figure 5.3 (left) as the three straight line segments connecting z_0 and $z_{1,0} = 1$, $z_{1,1}$, and $z_{1,2}$. The circles and paths move during the iterations of the solver and can be monitored. Finally, the *orientations* of Γ_1 through Γ_m (the orientation of Γ_0 is determined by the calculation of C) are given by the m real equations,

$$\arg(f(z_{2,i}) - f(z_{1,i})) = \arg(w_{2,i} - w_{1,i})$$

for $i = 0, \dots, m$. Therefore, the side-length, position, and orientation conditions give

$$K_0 + \cdots + K_m + 3m - 1$$

real equations. This is exactly want is needed. Other selections of conditions are possible and useful; for instance, the polygons and vertices can be numbered differently or different integration paths between circles can be chosen. However, it is important that these equations give a complete and independent set of conditions.

The constraints (5.1) on the $\theta_{k,i}$'s are difficult to enforce. We therefore use a transformation to unconstrained variables similar to [26, p. 25]. Let $\phi_{k,i} := \theta_{k+1,i} - \theta_{k,i}$, $k = 1, \dots, K_i$. Then the unconstrained variables are

$$\psi_{k,i} := \ln \frac{\phi_{k+1,i}}{\phi_{1,i}} \quad \text{for } k = 1, \dots, K_i - 1. \tag{5.2}$$

Given $\theta_{1,j}$, the transformation (5.2) can be inverted by

$$\theta_{k,i} = \theta_{1,i} + 2\pi \frac{1 + \sum_{l=1}^{k-2} e^{\psi_{l,j}}}{1 + \sum_{l=1}^{K_i-1} e^{\psi_{l,j}}} \tag{5.3}$$

ETNA
Kent State University
http://etna.math.kent.edu

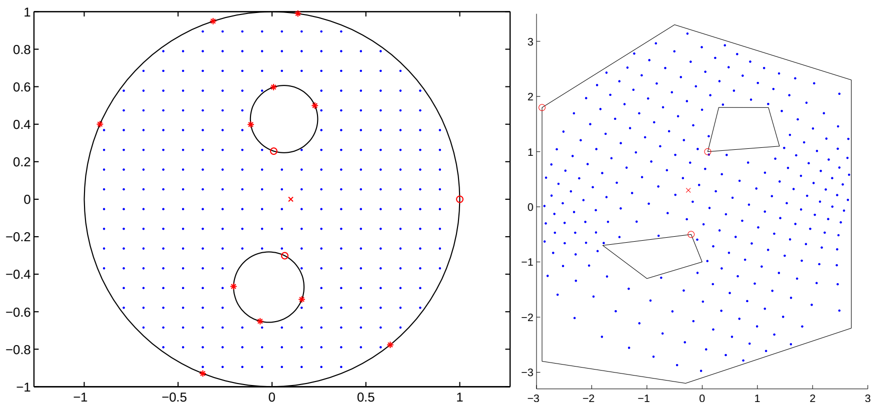

FIG. 5.1. *Schwarz-Christoffel map to a triply connected bounded polygonal domain with a Cartesian grid. The map is normalized by fixing one boundary point and one interior point. Note that the upper inner circle maps to the lower inner polygon.*

for $k = 2, \ldots, K_i$. Our unconstrained parameters are, therefore,

$$\theta_{1,i}, \psi_{1,i}, \psi_{2,i}, \ldots, \psi_{K_i-1,i},$$

for $i = 0, \ldots, m$. (Recall that $\theta_{1,0} = 0$). The parameters are placed in a real vector X of length $n := K_0 + \cdots + K_m + 3m - 1$ and the nonlinear equations form an $n \times n$ system $F(X) = 0$, where $F(X)$ is the objective function for our nonlinear solver.

The solution for a polygon with $m = 3$ is shown in the figures. Figure 5.1 shows the map from the circle domain to the polygonal domain evaluated on a Cartesian grid. Cartesian grids are not necessarily the most natural grids for these maps. In Figure 5.2, we plot a boundary-fitted orthogonal grid mapped from a polar grid for the conformally equivalent radial slit disk in Figure 5.3 (right). Clearly images of polar grids for mixed radial and circular slit disks and annuli could also be used and might be appropriate for the associated boundary value problems. An indication of the accuracy of the mapping function is given in Figure 5.4, where the error $\|X_N - X_{N-1}\|_\infty$ between solutions X_N for successive levels of reflections N is plotted against N. The proof of convergence of the infinite product formulas for Schwarz-Christoffel maps [17, 21] is similar to the proof above of convergence for the slit map formulas. An estimate similiar to (3.1) is used and we have found in [23] that $C\sum_{|\nu|=N} r_\nu$ gives a good estimate of the errors. We illustrate this in Figure 5.4. Also, we find in [23] that the sufficient condition for convergence, $\Delta < (m-1)^{-1/4}$ for connectivity m, is not necessary in practice. In such cases, (3.1) usually gives a good estimate of convergence, even though convergence may be very slow.

Although reasonable levels of accuracy can be achieved for domains of low connectivity or small Δ by evaluation of truncated infinite products, for the general case the reflections will need to be replaced by much more efficient methods. One option is to approximate the factors in one of the alternative formulations in Theorem 4.5 using least squares based on Laurent series, discussed below in Section 5.3. Since series approximations of functions on domains with very close-to-touching circles may be inaccurate, ultimately, some combination of reflection and series methods, similar to that suggested in [7], will most likely be needed. Some initial efforts are under development in [11].

ETNA
Kent State University
http://etna.math.kent.edu

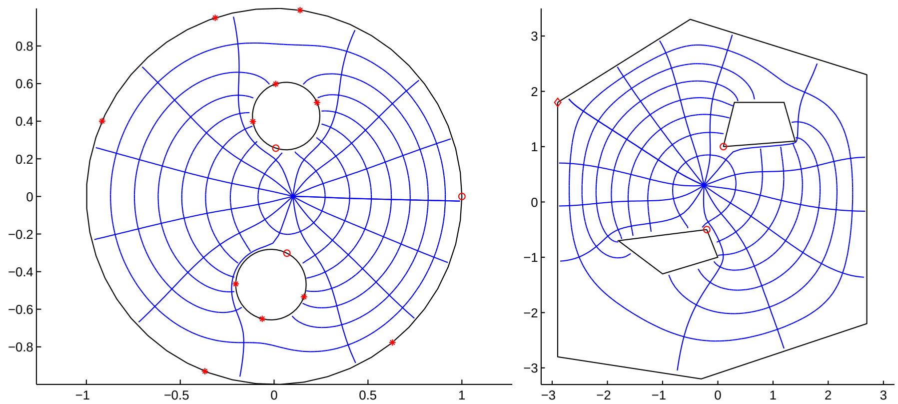

FIG. 5.2. *Schwarz-Christoffel map to a triply connected bounded polygonal domain with a boundary-fitted orthogonal grid from the radial slit disk map. The map is normalized by fixing one boundary point and one interior point. Note that the upper inner circle maps to the lower inner polygon.*

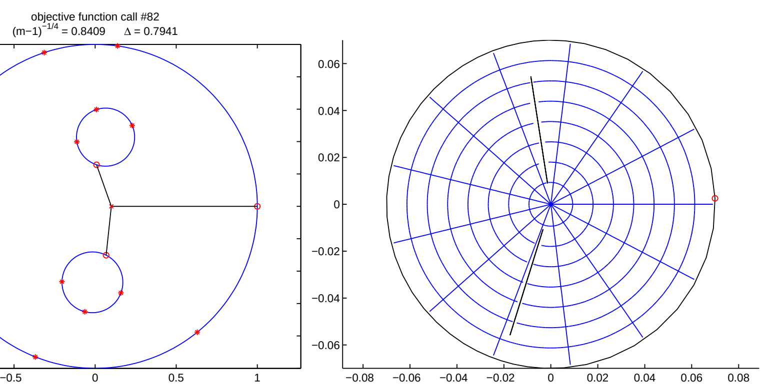

FIG. 5.3. *The integration paths (left) for the Schwarz-Chrstoffel map in Figure 5.2. The orthogonal grid for Figure 5.2 is the inverse image of the polar grid under the infinite product map from the circle domain to the radial slit disk (right). The connectivity is* 3 *here and the convergence criteria is satisfied since* $\Delta = 0.7941 < 2^{-1/4} = 0.8409$.

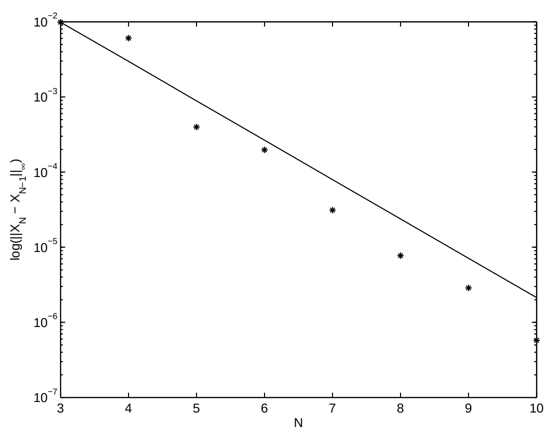

FIG. 5.4. *The log of the error in the parameters vs. the level of reflections* N *for the example in Figure 5.2 fit with* $\log \sum_{|\nu|=N} r_\nu - c$*; see (3.1).*

ETNA
Kent State University
http://etna.math.kent.edu

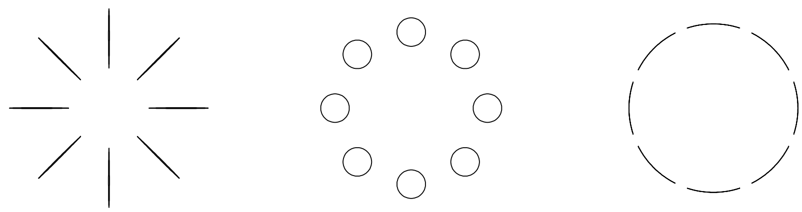

FIG. 5.5. *Maps from unbounded circle domain to radial and circular slit domains exhibiting crowding*

5.2. Crowding. In the words of Rudolf Wegmann [38],

> The behavior of conformal mapping depends on the local property of smoothness—and the global property of shape.
> On small scales a conformal mapping maps disks to disks, but on large scales a disk can be mapped to any simply-connected bounded region, however elongated and distorted it may be. But it takes some effort for a mapping which has such a strong tendency to map disks to disks, to map a disk to an elongated region. The mapping suffers, lying on a Procrustean bed, and the numerical conformal mapper must share the pains.

Difficult cases for computing the maps occur (i) when arcs of a circle map to the ends of boundaries of elongated regions and (ii) when the circles are close to touching. Some of the first general estimates for (i), the so-called *crowding phenomemon* for simply connected domains, were given in [16, 24] based on Pflüger's Theorem. In this section, we remark on (ii). How close the circles are to touching, though predetermined by the geometry, is not known in advance. Already in Halsey [29, Figure 5], an example was given of (ii) for the map from the exterior of three circles to the exterior of three thin, closely-spaced, parallel ellipses. Numerical examples are also given in [22]. As in the simply connected case, the crowding of the circles is due to elongated or narrow sections in the region to be mapped. In [22] we use an estimate [28, 37] of harmonic measure $u(0)$ with respect to 0 of, e.g., a boundary arc at the outer edge of a wedge shaped region with $\delta < \rho < R, \theta = \theta(\rho)$,

$$u(0) \leq \frac{8}{\pi} \exp\left(-\pi \int_{\delta}^{R} \frac{d\rho}{\rho\theta(\rho)}\right),$$

and similar results to derive asymptotic estimates of the distance r between the conformally equivalent circles as the distance d or angle θ between the boundaries go to 0. Our estimates, though somewhat heuristic, are close to the exact asymptotic behavior for explicit maps (see [3]) to parallel slits, $r \sim \pi^2 d^2/2$, and to collinear slits, $d \sim 8\exp(-\pi^2/\sqrt{8r})$. Such estimates may be interpreted as the probability that a particle in Brownian motion [28, 33] starting at 0 will exit the region through the given boundary arc. For instance, the probability that the particle will exit through the m symmetric radial slits $1 < \rho < R$ in Figure 5.5 (left) is bounded by the harmonic measure $(8m/\pi)\exp(-m\ln(R)/2)$ of the gaps in the conformally equivalent circular arc domain, Figure 5.5 (right).

5.3. Laurent series/least squares methods. The use of Laurent series with coefficients determined by a least squares fit to the boundary conditions was discussed in [19]. It was demonstrated there that this method was an efficient alternative to the infinite product formulas for the map from the exterior of a circle domain to the exterior of a radial slit domain. We will present this method here for the map $w = f(z)$ from the bounded circle domain to the radial slit half plane discussed above in Section 3.1 and used in one of the factorizations

ETNA
Kent State University
http://etna.math.kent.edu

in Theorem 4.5. The formulation here is a slight variation of the approach in [19] in order to allow a zero and a singularity on one of the boundaries, C_i. The conditions on $f(z)$ from Lemma 3.2 and Theorem 3.3 are, given $a, b \in C_i$, $a \neq b$, that

1. $f(b) = 0$ and $f(a) = \infty$ and
2. $f(z)$ satisfies the boundary conditions, $\arg f(z) \equiv \text{const.}$ for $z \in C_p, p = 0, 1, \ldots, m$.

For computation of the map, we write

$$f(z) = \frac{z-b}{z-a} e^{g(z)}$$

for some $g(z)$ analytic in the (bounded) circle domain. We represent g as the sum of the series expansions,

$$g(z) = \sum_{\ell=1}^{\infty} \alpha_{0,\ell} z^{\ell} + \sum_{p=1}^{m} \sum_{\ell=1}^{\infty} \frac{\alpha_{p,\ell} r_p^{\ell}}{(z-c_p)^{\ell}}.$$

Now consider

$$\log(f(z)) = \log(z-b) - \log(z-a) + g(z).$$

Since $\arg f(z) = \operatorname{Im}(\log f(z))$, the boundary conditions for $p \neq i$ are

$$\operatorname{Im}(\log f(z)) = \arg \frac{z-b}{z-a} + \operatorname{Im}(g(z)) \equiv \text{const.}$$

or

$$\operatorname{Im} g(z) = \text{const.} - \arg \frac{z-b}{z-a}, \quad \text{for all } z \in C_p.$$

For $p = i$, we recall Lemma 3.2 and express the boundary condition as

$$\operatorname{Re}\left((z-c_i)\frac{f'(z)}{f(z)}\right) = 0,$$

for $z = c_i + r_i e^{i\theta}$, where

$$\frac{f'(z)}{f(z)} = \frac{d}{dz}\log f(z) = \frac{1}{z-b} - \frac{1}{z-a} + g'(z).$$

Therefore, following the proof of Theorem 3.5, we have

$$\operatorname{Re}\left((z-c_i)\frac{f'(z)}{f(z)}\right) = \operatorname{Re}\left(\frac{z-c_i}{z-b} - \frac{z-c_i}{z-a} + (z-c_i)g'(z)\right) = 0,$$

and the boundary condition, using $z = c_i + r_i e^{i\theta}$, $a = c_i + r_i e^{i\theta_a}$, and $b = c_i + r_i e^{i\theta_b}$, is

$$\operatorname{Re}\{(z-c_i)g'(z)\} = \operatorname{Re}\left\{\frac{e^{i\theta}}{e^{i\theta} - e^{i\theta_a}}\right\} - \operatorname{Re}\left\{\frac{e^{i\theta}}{e^{i\theta} - e^{i\theta_b}}\right\} = \frac{1}{2} - \frac{1}{2} = 0.$$

Summarizing, the boundary conditions may stated as

$$\operatorname{Im}(g(z)) = \text{const.} - \arg \frac{z-b}{z-a}, \quad \text{for all } z \in C_p, \quad p \neq i. \tag{5.4}$$

and

$$\mathrm{Re}\,((z-c_i)g'(z)) = 0, \quad \text{for all } z \in C_i. \tag{5.5}$$

Our aim is to calculate the series $g(z)$ truncated to $(m+1)L$ terms for some L by a matrix-vector multiplication Ax where $x = [\alpha_{p,\ell}]$ is a $((m+1)L \times 1)$ column vector of the coefficients of g and A is given by

$$A = \begin{bmatrix} z^\ell & r_p^\ell/(z-c_p)^\ell \end{bmatrix}_{M\times(m+1)L}.$$

The factors of r_p^ℓ prevent the production of large entries in A when $|z-c_p| < 1$ and ℓ is large, which leads to severe ill-conditioning of A. The coefficients will be found using a linear system of equations given by the boundary conditions (5.4) and (5.5).

Discretize by taking M equally spaced points z around each circle C_p, $p = 0, 1, \ldots, m$. Define the matrices

$$F_p = \begin{bmatrix} z^\ell & r_p^\ell/(z-c_p)^\ell \end{bmatrix}_{M\times(m+1)L} \quad \text{for} \quad z \in C_p,\ p \neq i,$$

and using

$$g'(z) = \sum_{\ell=1}^{L} \ell\alpha_{0,\ell} z^{\ell-1} + \sum_{p=1}^{m}\sum_{\ell=1}^{L} \frac{-\ell\alpha_{p,\ell} r_p^\ell}{(z-c_p)^{\ell+1}},$$

define

$$G = \begin{bmatrix} \ell(z-c_i)z^{\ell-1} & -\ell(z-c_i)r_p^\ell/(z-c_p)^{\ell-1} \end{bmatrix}_{M\times(m+1)L} \quad \text{for} \quad z \in C_i.$$

With $F_p = F_{R_p} + iF_{I_p}, G = G_R + iG_I$ and $x = x_R + ix_I$, a simple calculation shows that

$$\mathrm{Im}(g(z)) \approx F_{I_p}x_R + F_{R_p}x_I \quad \text{on any} \quad C_p$$

and

$$\mathrm{Re}\,((z-c_i)g'(z)) \approx G_R x_R - G_I x_I \quad \text{on} \quad C_i.$$

The values of $\mathrm{Im}(f(z))$ may not be known, but the difference of $\mathrm{Im}(f(z))$ for any pair of points on a circle C_p, $p \neq i$, is zero. Therefore, defining

$$P = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}_{(M-1)\times M},$$

for $z \in C_p$, we have

$$P\begin{bmatrix} F_{I_p} & F_{R_p} \end{bmatrix}\begin{bmatrix} x_R \\ x_I \end{bmatrix} = -P\begin{bmatrix} \arg \frac{z-b}{z-a} \end{bmatrix}_{(M-1)\times 1}$$

by the boundary condition (5.4). By the boundary condition (5.5), we also have

$$\begin{bmatrix} G_R & -G_I \end{bmatrix}\begin{bmatrix} x_R \\ x_I \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}.$$

For the sake of exposition suppose $i \notin \{0, m\}$. Define the coefficient matrices,

$$B_1 = \begin{bmatrix} F_{I_1} \\ \vdots \\ F_{I_{i-1}} \\ G_R \\ F_{I_{i+1}} \\ \vdots \\ F_{I_m} \end{bmatrix}_{(m+1)M\times(m+1)L} \quad \text{and} \quad B_2 = \begin{bmatrix} F_{R_1} \\ \vdots \\ F_{R_{i-1}} \\ -G_I \\ F_{R_{i+1}} \\ \vdots \\ F_{R_m} \end{bmatrix}_{(m+1)M\times(m+1)L},$$

and the difference matrix,

$$E = \begin{bmatrix} P & & & & \\ & \ddots & & & \\ & & I & & \\ & & & \ddots & \\ & & & & P \end{bmatrix}_{(m+1)(M-1)+1\times(m+1)M},$$

where the identity matrix occupies the ith block-row. This gives the system

$$E\begin{bmatrix} B_1 & B_2 \end{bmatrix}\begin{bmatrix} x_R \\ x_I \end{bmatrix} = -E\begin{bmatrix} \arg\frac{z-b}{z-a} \\ \vdots \\ 0 \\ \vdots \\ \arg\frac{z-b}{z-a} \end{bmatrix},$$

which can be solved efficiently with the MATLAB backslash operator. A computed example with $m = 2$ radial slits and $L = 16$ terms in the series (similar to Figure 3.1 using the product formula) is shown in Figure 5.6. A listing of the least squares code for the exterior radial slits was included in [19]. A comparison of the efficiency and accuracy of the series approach and the infinite product approach and the use of the series approximations for the Schwarz-Christoffel computations will be undertaken in future work.

Acknowledgment. We thank one of the referees for requesting changes which strengthened the paper significantly.

REFERENCES

[1] E. L. ALLGOWER AND K. GEORG, *Numerical Continuation Methods: An Introduction*, Springer, New York, 1990.

[2] H. F. BAKER, *Abelian Functions: Abel's Theorem and the Allied Theory including the Theory of the Theta Functions*, Cambridge University Press, Cambridge, 1897, reissued 1995.

[3] N. BENCHAMA, T. DELILLO, T. HRYCAK, AND L. WANG, *A simplified Fornberg-like method for conformal mapping of multiply connected regions—Comparisons and crowding*, J. Comput. Appl. Math., 209 (2007), pp. 1–21.

[4] W. BURNSIDE, *On functions determined from their discontinuities and a certain form of boundary condition*, Proc. London Math. Soc., 22 (1891), pp. 346–358.

[5] W. BURNSIDE, *On a class of automorphic functions*, Proc. London Math. Soc., 23 (1891), pp. 49–88.

[6] J. CASE, *Breakthrough in conformal mapping*, SIAM News, 41, 1 (Jan./Feb. 2008).

[7] H. CHENG AND L. GREENGARD, *A method of images for the evaluation of electrostatic fields in systems of closely spaced conducting cylinders*, SIAM J. Appl. Math., 58 (1998), pp. 122–141.

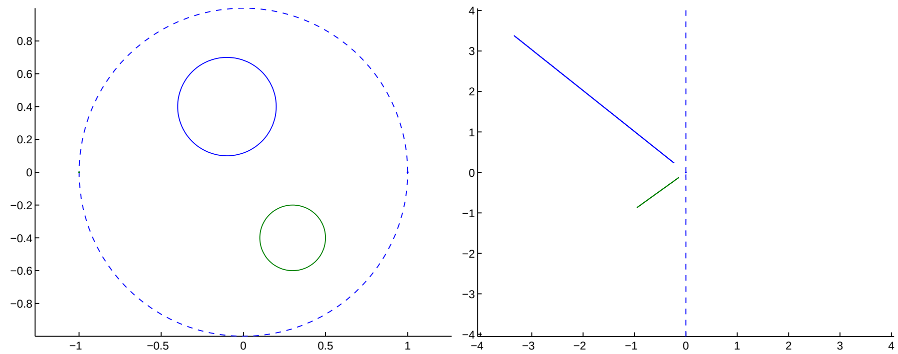

FIG. 5.6. *Conformal map $w = f(z)$ from the unit disk with $m = 2$ circular holes to a radial slit half-plane calculated with the least squares approach using Taylor/Laurent series of length $L = 16$ on each circle. This is an alternative approach to the product formula used for the similar domain in Figure 3.1.*

[8] D. CROWDY, *The Schwarz-Christoffel mapping to bounded multiply connected polygonal domains*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 461 (2005), pp. 2653–2678.
[9] D. CROWDY, *Schwarz-Christoffel mapping to unbounded multiply connected polygonal regions*, Math. Proc. Camb. Phil. Soc., 142 (2007), pp. 319–339.
[10] D. CROWDY, *Geometric function theory: a modern view of a classical subject*, Nonlinearity 21 (2008), pp. T205–T219.
[11] D. CROWDY, T. DELILLO, AND J. MARSHALL, work in progress on computing electrostatic fields exterior to close-to-touching disks.
[12] D. CROWDY AND J. MARSHALL, *Analytical formulae for the Kirchhoff-Routh path function in multiply connected domains*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 461 (2005), pp. 2477–2501.
[13] D. CROWDY AND J. MARSHALL, *Conformal mapping between canonical multiply connected domains*, Comput. Methods Funct. Theory, 6 (2006), pp. 59–76.
[14] D. CROWDY AND J. MARSHALL, *Green's functions for Laplace's equation in multiply connected domains*, IMA J. Appl. Math., 72 (2007), pp. 278–301.
[15] D. CROWDY AND J. MARSHALL, *Computing the Schottky-Klein prime function on the Schottky double of planar domains*, Comput. Methods Funct. Theory, 7 (2007), pp. 293–308.
[16] T. K. DELILLO, *The accuracy of numerical conformal mapping methods: a survey of examples and results*, SIAM J. Numer. Anal., 31 (1994), pp. 788–812.
[17] T. K. DELILLO, *Schwarz-Christoffel mapping of bounded, multiply connected domains*, Comput. Methods Funct. Theory, 6, (2006), pp. 275–300.
[18] T. K. DELILLO, T. A. DRISCOLL, A. R. ELCRAT, AND J. A. PFALTZGRAFF, *Computation of multiply connected Schwarz-Christoffel maps for exterior domains*, Comput. Methods Funct. Theory, 6 (2006), pp. 301–315.
[19] T. K. DELILLO, T. A. DRISCOLL, A. R. ELCRAT, AND J. A. PFALTZGRAFF, *Radial and circular slit maps of unbounded multiply connected circle domains*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 464 (2008), pp. 1719–1737.
[20] T. K. DELILLO, A. R. ELCRAT, AND J. A. PFALTZGRAFF, *Schwarz-Christoffel mapping of the annulus*, SIAM Rev., 43 (2001), pp. 469–477.
[21] T. K. DELILLO, A. R. ELCRAT, AND J. A. PFALTZGRAFF, *Schwarz-Christoffel mapping of multiply connected domains*, J. d'Analyse Math., 94 (2004), pp. 17–47.
[22] T. K. DELILLO, T. HRYCAK, AND L. WANG, *Examples of the crowding phenomenon for conformal maps of doubly connected regions*, preprint.
[23] T. K. DELILLO AND E. H. KROPF, *Numerical computation of the Schwarz-Christoffel transformation for multiply connected domains*, submitted for publication.
[24] T. K. DELILLO AND J. A. PFALTZGRAFF, *Extremal distance, harmonic measure and numerical conformal mapping*, J. Comput. Appl. Math., 46 (1993), pp. 103–113.
[25] T. A. DRISCOLL, *A MATLAB Toolbox for Schwarz-Christoffel mapping*, ACM Trans. Math. Software, 22 (1996), pp. 168–186.
http://www.math.udel.edu/~driscoll/software/SC/index.html
[26] T. A. DRISCOLL AND L. N. TREFETHEN, *Schwarz-Christoffel Mapping*, Cambridge University Press, Cambridge, 2002.
[27] D. GAIER, *Konstruktive Methoden der konformen Abbildung*, Springer, Berlin, 1964.

[28] J. GARNETT AND D. MARSHALL, *Harmonic Measure*, Cambridge University Press, Cambridge, 2005.
[29] N. D. HALSEY, *Potential flow analysis of multielement airfoils using conformal mapping*, AIAA J., 17 (1979), pp. 1281–1288.
[30] P. HENRICI, *Applied and Computational Complex Analysis*, Vol. III, John Wiley, New York, 1986.
[31] G. JULIA, *Lecons sur la Representation Conforme des Aires Multiplement Connexes*, Gauthier-Villars, Paris, 1934.
[32] P. KOEBE, *Abhandlungen zur Theorie der konformen Abbildung. IV. Abbildung mehrfach zusammenhängender schlichter Bereiche auf Schlitzbereiche*, Acta Math., 41 (1916), pp. 305–344.
[33] G. F. LAWLER, *Conformally Invariant Processes in the Plane*, American Mathematical Society, Providence, RI, 2005.
[34] Z. NEHARI, *Conformal Mapping*, McGraw-Hill, New York, 1952.
[35] M. SCHIFFER, *Some recent developments in the theory of conformal mapping*, appendix to R. Courant, *Dirichlet's Principal, Conformal Mapping, and Minimal Surfaces*, Interscience, New York, 1950.
[36] L. N. TREFETHEN, *Ten digit algorithms: Ten digits, five seconds, just one page*, Report No. 05/13, Oxford University Computing Laboratory, 2005.
[37] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.
[38] R. WEGMANN, *Methods for numerical conformal mapping*, in *Handbook of Complex Analysis, Geometric Function Theory, Vol. 2*, R. Kühnau, ed., Elsevier, 2005, pp. 351–477.
[39] G. WEN, *Conformal Mapping and Boundary Value Problems*, American Mathematical Society, Providence, RI, 1992.