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ANALYSIS OF THE DGFEM FOR NONLINEAR CONVECTION-DIFFUSION PROBLEMS*

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Abstract. The paper is concerned with the analysis of error estimates of the discontinuous Galerkin finite element method (DGFEM) for the numerical solution of nonstationary nonlinear convection-diffusion problems equipped with Dirichlet boundary conditions. First, the case of nonlinear diffusion as well as nonlinear convection is considered. Then, the optimal $L^\infty(L^2)$-error estimate is discussed in the case of nonlinear convection and linear diffusion.

Key words. nonlinear nonstationary convection-diffusion problems, nonlinear convection, nonlinear diffusion, discontinuous Galerkin finite element method, NIPG, SIPG and IIPG versions, optimal error estimates

AMS subject classifications. 65M60, 76M10

1. Introduction. In a number of complex problems from science and technology, it is necessary to approximate nonlinear singularly perturbed systems in domains with a complex geometry, whose solutions contain internal or boundary layers. A possible numerical method for an efficient solution of such problems is the discontinuous Galerkin finite element method (DGFEM). This technique uses piecewise polynomial approximations of the sought solution on a finite element mesh without any requirement on the continuity between neighbouring elements. It allows the construction of higher order schemes in a natural way and it is suitable for the approximation of discontinuous solutions of conservation laws, or solutions of singularly perturbed convection-diffusion problems having steep gradients. This method uses advantages of the finite element method and finite volume schemes with an approximate Riemann solver and can be applied to unstructured grids, of the kind which are generated for most complex geometries.

The original DGFEM was introduced in [26] for the solution of a neutron transport linear equation and analyzed theoretically in [24] and later in [23]. Nearly simultaneously the DGFE techniques were developed for the numerical solution of elliptic problems in [33] and for space semidiscretization of parabolic problems in [1] and [16] , using the interior penalty Galerkin methods. In these papers the symmetric approximation of the diffusion terms is used, and therefore the method is called SIPG (symmetric interior penalty Galerkin). Quite popular is the NIPG (nonsymmetric interior penalty Galerkin) method, which was first introduced in [28]. Theoretical analyses of various types of the DGFE method applied to elliptic problems can be found, e.g., in [1], [2], [3], [21], and [29].

The DGFEM has been used in a number of applications. Let us mention conservation laws in [7], [14], [22], compressible flow in [4], [5], [8], [10], [12], [18], [20], [32], and convection-diffusion problems in time-dependent domains in [31]. A survey of DGFE methods, techniques, and some applications can be found in [6].

In the discretization of nonstationary problems, one often uses the *space semidiscretization,* also called the *method of lines*. In [9], [11], and [15] the problem with a nonlinear convection and linear diffusion was analyzed. The work in [27] is concerned with the DGFEM applied to the solution of a parabolic problem with a nonlinear diffusion, equipped with the Neumann boundary condition prescribed on the whole boundary.

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In this paper, we shall analyze the DGFEM applied to nonstationary convection-diffusion problems with nonlinear convection as well as diffusion and Dirichlet conditions on nonconforming meshes. Further, we discuss here the optimal $L^\infty(L^2)$-error estimates.

Let us note that we consider a problem in two space dimensions for simplicity only. All results presented here can be extended to more space dimensions.

2. Formulations of the problem. First we shall introduce the formulations of the continuous and discrete problems.

2.1. Continuous problem. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with a Lipschitz-continuous boundary $\partial\Omega$ and $T > 0$. We shall deal with the following *initial-boundary value problem*: find $u : Q_T = \Omega \times (0,T) \to \mathbb{R}$, such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} = \operatorname{div}\big(\beta(u)\nabla u\big) + g, \quad \text{in } Q_T, \tag{2.1}$$

$$u|_{\partial\Omega\times(0,T)} = u_D, \tag{2.2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega. \tag{2.3}$$

Let $g : Q_T \to \mathbb{R}$, $u_D : \partial\Omega \times (0,T) \to \mathbb{R}$ and $u^0 : \Omega \to \mathbb{R}$ be given functions, and let $f_s \in C^1(\mathbb{R})$, $s = 1, 2$, be prescribed Lipschitz-continuous fluxes. Without the loss of generality we suppose that $f_s(0) = 0, s = 1, 2$. We assume that the function β satisfies the conditions

$$\beta : \mathbb{R} \to [\beta_0, \beta_1], \quad 0 < \beta_0 < \beta_1 < \infty, \tag{2.4}$$

$$|\beta(u_1) - \beta(u_2)| \le L|u_1 - u_2|, \quad \forall u_1, u_2 \in \mathbb{R}. \tag{2.5}$$

This problem represents a simplified model of heat and mass transfer problems, where nonlinear convection as well as diffusion appear. As an example we can mention the system of compressible Navier-Stokes equations; see, e.g., [17].

The application of techniques from [30] allows to prove the existence and uniqueness of a weak solution of the above problem.

2.2. Discretization. Let $\mathcal{T}_h$ be a partition of the closure $\overline{\Omega}$ into a finite number of closed triangles, whose interiors are mutually disjoint.

For any $K \in \mathcal{T}_h$, we set $|K| = \text{meas}_2(K)$ (two dimensional Lebesgue measure), $h_K = \text{diam}(K)$, the diameter of K, and $h = \max_{K\in\mathcal{T}_h} h_K$. We consider an index set $I \subset \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, such that all elements of $\mathcal{T}_h$ are numbered by indices from I, i.e., $\mathcal{T}_h = \{K_i\}_{i\in I}$. If two elements $K_i, K_j \in \mathcal{T}_h$ share a common face, which by definition has to be a linear segment, we call them *neighbours* and set $\Gamma_{ij} = \partial K_i \cap \partial K_j$ and $d(\Gamma_{ij}) = \text{meas}_1\Gamma_{ij} =$ length of Γ_{ij}. For $i \in I$ we define $s(i) = \{j \in I; K_j \text{ is a neighbour of } K_i\}$. The boundary $\partial\Omega$ is formed by a finite number of faces of elements K_i adjacent to $\partial\Omega$. We denote all these boundary faces by S_j, where $j \in I_b \subset \mathbb{Z}^- = \{-1, -2, \ldots\}$ and set $\gamma(i) = \{j \in I_b; S_j \text{ is a face of } K_i\}, \Gamma_{ij} = S_j$ for $K_i \in \mathcal{T}_h$, such that $S_j \subset \partial K_i, j \in I_b$. If K_i is not adjacent to $\partial\Omega$, we set $\gamma(i) = \emptyset$. Furthermore we set $S(i) = s(i) \cup \gamma(i)$. We can see that

$$s(i) \cap \gamma(i) = \emptyset, \quad \partial K_i = \bigcup_{j\in S(i)} \Gamma_{ij}, \quad \partial K_i \cap \partial\Omega = \bigcup_{j\in\gamma(i)} \Gamma_{ij}.$$

By $\boldsymbol{n}_{ij}$ we denote the unit outer normal to ∂K_i on the face Γ_{ij}. In our case, $\boldsymbol{n}_{ij}$ is constant along Γ_{ij}.

Over $\mathcal{T}_h$ we define the *broken Sobolev space*

$$H^k(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^k(K)\ \forall K \in \mathcal{T}_h\},$$

equipped with the seminorm

$$|v|_{H^k(\Omega,\mathcal{T}_h)} = \left(\sum_{i\in I} |v|^2_{H^k(K_i)} \right)^{1/2}.$$

For $v \in H^1(\Omega, \mathcal{T}_h)$ we set

$$\begin{aligned} v|_{\Gamma_{ij}} &= \text{ \textit{trace} of } v|_{K_i} \text{ on } \Gamma_{ij}, \\ \langle v \rangle_{\Gamma_{ij}} &= \frac{1}{2}(v|_{\Gamma_{ij}} + v|_{\Gamma_{ji}}), \text{ \textit{average} of traces of } v \text{ on } \Gamma_{ij}, \\ [v]_{\Gamma_{ij}} &= v|_{\Gamma_{ij}} - v|_{\Gamma_{ji}}, \text{ \textit{jump} of traces of } v \text{ on } \Gamma_{ij}. \end{aligned}$$

Finally, we define the space of discontinuous piecewise polynomial functions

$$S_h = S^{p,-1}(\Omega, \mathcal{T}_h) = \{v; v|_K \in P_p(K)\ \forall K \in \mathcal{T}_h\},$$

where $P_p(K)$ is the space of all polynomials on K of degree $\leq p$.

In order to approximate the diffusion terms, we introduce the following forms defined for $u, \varphi \in H^2(\Omega, \mathcal{T}_h)$:

$$\begin{aligned} a_h(u,\varphi) = &\sum_{i\in I} \int_{K_i} \beta(u) \nabla u \cdot \nabla \varphi \, dx \\ &- \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \langle \beta(u)\nabla u\rangle \cdot \boldsymbol{n}_{ij} [\varphi] \, dS - \Theta \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \langle \beta(u)\nabla \varphi\rangle \cdot \boldsymbol{n}_{ij} [u] \, dS \\ &- \sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \beta(u)\nabla u \cdot \boldsymbol{n}_{ij} \varphi \, dS - \Theta \sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \nabla \beta(u) \varphi \cdot \boldsymbol{n}_{ij} u \, dS \end{aligned}$$

and

$$\begin{aligned} l_h(u,\varphi)(t) = &\int_\Omega g(t)\varphi \, dx - \Theta \sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \beta(u)\nabla\varphi \cdot \boldsymbol{n}_{ij} u_D(t) \, dS \\ &+ \sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \sigma u_D(t) \varphi \, dS. \end{aligned}$$

Further, we define the *interior and boundary penalty*

$$J_h(u,\varphi) = \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \sigma [u][\varphi] \, dS + \sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \sigma u \varphi \, dS.$$

The weight σ is defined by

$$\sigma|_{\Gamma_{ij}} = C_W / d(\Gamma_{ij}),$$

where $C_W > 0$ is a suitable constant.

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Taking $\Theta = 1$, 0 and -1, we obtain the symmetric (SIPG), incomplete (IIPG) and nonsymmetric (NIPG) variants of the approximation of the diffusion terms.

Finally, we define the *convective* form

$$b_h(u,\varphi) = -\sum_{i\in I}\int_{K_i}\sum_{s=1}^{2} f_s(u)\frac{\partial\varphi}{\partial x_s}\,dx + \sum_{i\in I}\sum_{j\in S(i)}\int_{\Gamma_{ij}} H(u|_{\Gamma_{ij}}, u|_{\Gamma_{ji}}, \boldsymbol{n}_{ij})\varphi|_{\Gamma_{ij}}\,dS.$$

Here H is the so-called *numerical flux* with properties specified later.

Now we can introduce the discrete problem (space semidiscretization with continuous time, also called the method of lines).

DEFINITION 2.1. *We say that u_h is a DGFE solution of the convection-diffusion problem* (2.1)–(2.3)*, if*

$$\begin{aligned} &a)\ \ u_h \in C^1([0,T]; S_h),\\ &b)\ \ \frac{d}{dt}(u_h(t),\varphi_h) + b_h(u_h(t),\varphi_h) + \beta_0 J_h(u_h(t),\varphi_h) + a_h(u_h(t),\varphi_h)\\ &\qquad = l_h(u_h,\varphi_h)(t), \quad \forall\varphi_h\in S_h,\ \forall t\in(0,T),\\ &c)\ \ u_h(0) = u_h^0, \end{aligned} \tag{2.6}$$

where u_h^0 is an S_h approximation of the initial condition u^0 and $\beta_0 > 0$ is a constant from assumption (2.4).

The discrete problem (2.6) is equivalent to a large system of ordinary differential equations. If we apply a suitable ODE solver, we get a fully discrete problem. However, this subject lies outside the framework of this paper. Here we shall be concerned with the analysis of the semidiscrete problem (2.6).

3. Error analysis.

3.1. Some assumptions. We assume that the numerical flux H has the following properties:

(H1) $H(u,v,\boldsymbol{n})$ is defined in $\mathbb{R}^2\times B_1$, where $B_1 = \{\boldsymbol{n}\in\mathbb{R}^d; |\boldsymbol{n}| = 1\}$, and is *Lipschitz-continuous* with respect to u, v:

$$|H(u,v,\boldsymbol{n}) - H(u^*,v^*,\boldsymbol{n})| \le C_L(|u-u^*| + |v-v^*|),$$
$$u,\ v,\ u^*,\ v^*\in\mathbb{R},\ \boldsymbol{n}\in B_1.$$

(H2) $H(u,v,\boldsymbol{n})$ is *consistent*:

$$H(u,u,\boldsymbol{n}) = \sum_{s=1}^{d} f_s(u)\,n_s, \quad u\in\mathbb{R},\ \boldsymbol{n} = (n_1,\dots,n_d)\in B_1.$$

(H3) $H(u,v,\boldsymbol{n})$ is *conservative*:

$$H(u,v,\boldsymbol{n}) = -H(v,u,-\boldsymbol{n}), \quad u,\ v\in\mathbb{R},\ \boldsymbol{n}\in B_1.$$

We shall assume that the weak solution u of problem (2.1)–(2.3) is regular, namely

$$\frac{\partial u}{\partial t} \in L^2\big((0,T); H^{p+1}(\Omega)\big), \tag{3.1}$$

where $p \geq 1$ denotes the given degree of approximation. Then $u \in C\big([0,T]; H^{p+1}(\Omega)\big)$ and u satisfies (2.1)–(2.3) pointwise. It is possible to show that the regular solution satisfies the identity

$$\begin{aligned}\frac{d}{dt}(u(t),\varphi_h) + b_h(u(t),\varphi_h) + \beta_0 J_h(u(t),\varphi_h) + a_h(u(t),\varphi_h)\\ = l_h(u,\varphi_h)(t), \quad \forall \varphi_h \in S_h, \text{ for a. a. } t \in (0,T).\end{aligned} \tag{3.2}$$

To treat the nonlinear diffusion terms, we need one more regularity assumption on the solution u of the continuous problem:

$$\|\nabla u(t)\|_{L^\infty(\Omega)} \leq C_R \quad \text{for a. a. } t \in (0,T). \tag{3.3}$$

Let us consider a system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$, of partitions of the domain Ω with the following properties:

(A1) The system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ is regular: there exists a constant $C_T > 0$, such that

$$\frac{h_K}{\rho_K} \leq C_T, \quad \forall K \in \mathcal{T}_h, \quad \forall h \in (0,h_0).$$

(A2) There exists a constant $C_D > 0$, such that

$$h_{K_i} \leq C_D\, d(\Gamma_{ij}), \quad \forall i \in I, \quad \forall j \in S(i), \quad \forall h \in (0,h_0).$$

3.2. Some auxiliary results. First we state some results necessary for our analysis; see, e.g., [9].

LEMMA 3.1 (Multiplicative trace inequality). *There exists a constant $C_M > 0$ independent of h, K, v, such that*

$$\begin{aligned}\|v\|^2_{L^2(\partial K)} \leq C_M\big(\|v\|_{L^2(K)}|v|_{H^1(K)} + h_K^{-1}\|v\|^2_{L^2(K)}\big),\\ \forall K \in \mathcal{T}_h,\ v \in H^1(K),\ h \in (0,h_0).\end{aligned} \tag{3.4}$$

LEMMA 3.2 (Inverse inequality). *There exists a constant $C_I > 0$ independent of h, K, v, such that*

$$|v|_{H^1(K)} \leq C_I h_K^{-1}\|v\|_{L^2(K)}, \quad \forall K \in \mathcal{T}_h,\ v \in P_p(K),\ h \in (0,h_0). \tag{3.5}$$

Now, for $v \in L^2(\Omega)$ we denote by $\Pi_h v$ the $L^2(\Omega)$-projection of v on S_h:

$$\Pi_h v \in S_h, \quad (\Pi_h v - v,\, \varphi_h) = 0 \qquad \forall\, \varphi_h \in S_h.$$

Obviously, if $K \in \mathcal{T}_h$, then the function $\Pi_h v|_K$ is the $L^2(K)$-projection of $v|_K$ on $P_p(K)$. Let $k \in [1,p]$ be an integer. It is possible to show, cf., e.g., [19, Lemma 4.1], that the operator Π_h has the following properties.

LEMMA 3.3. *There exists a constant $C_A > 0$ independent of h, K, v, such that*

$$\|\Pi_h v - v\|_{L^2(K)} \leq C_A h_K^{k+1}|v|_{H^{k+1}(K)}, \tag{3.6}$$

$$|\Pi_h v - v|_{H^1(K)} \leq C_A h_K^{k}|v|_{H^{k+1}(K)}, \tag{3.7}$$

$$|\Pi_h v - v|_{H^2(K)} \leq C_A h_K^{k-1}|v|_{H^{k+1}(K)}, \tag{3.8}$$

for all $v \in H^{k+1}(K)$, $K \in T_h$ and $h \in (0,h_0)$, where $k \in [1,p]$ is an integer.

In what follows, we shall denote by C a generic constant independent of h, K, and/or other quantities attaining different values in different places.

We set $\eta(t) = \Pi_h u(t) - u(t) \in H^{p+1}(\Omega, \mathcal{T}_h)$ for a. a. $t \in (0, T)$. Then (3.4)–(3.8) yield the following estimates.

LEMMA 3.4. *There exists a constant $C > 0$ independent of h, K, such that for all $h \in (0, h_0)$*

$$
\begin{aligned}
&\text{a) } \|\eta\|_{L^2(\Omega,\mathcal{T}_h)} \le Ch^{p+1}|u|_{H^{p+1}(\Omega)},\\
&\text{b) } |\eta|_{H^1(\Omega,\mathcal{T}_h)} \le Ch^{p}|u|_{H^{p+1}(\Omega)},\\
&\text{c) } |\eta|_{H^2(\Omega,\mathcal{T}_h)} \le Ch^{p-1}|v|_{H^{p+1}(\Omega)},\\
&\text{d) } \left\|\frac{\partial \eta}{\partial t}\right\| \le Ch^{p+1}\left|\frac{\partial u}{\partial t}\right|_{H^{p+1}(\Omega)},\\
&\text{e) } J_h(\eta,\eta) \le Ch^{2p}|u|^2_{H^{p+1}(\Omega)}.
\end{aligned}
$$

3.3. Properties of the convective form. We use the following notation:

$$
\|w\|_{DG} = \left(\frac{1}{2}\left(|w|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h(w,w)\right)\right)^{1/2}.
$$

It is possible to show that $\|\cdot\|_{DG}$ is a norm in $H^1(\Omega, \mathcal{T}_h)$.

Now, we shall be concerned with the properties of the form b_h. Under assumptions (H1)–(H3) and (A1)–(A2) the convective form b_h is Lipschitz-continuous in the following sense.

LEMMA 3.5. *Let $u, \bar{u}, v \in H^1(\Omega, \mathcal{T}_h)$ and $h \in (0, h_0)$. Then there exists a constant $C > 0$ independent of $u, \bar{u}, v, h$, such that*

$$
\begin{aligned}
|b_h(u,v) - b_h(\bar{u},v)| \le{}& C\left(J_h(v,v)^{1/2} + |v|_{H^1(\Omega,\mathcal{T}_h)}\right)\\
&\times\left(\|u-\bar{u}\|_{L^2(\Omega)} + \left(\sum_{i\in I} h_{K_i}\|u-\bar{u}\|^2_{L(\partial K_i)}\right)^{1/2}\right).
\end{aligned}
$$

From this and results from Section 3.2 we obtain

LEMMA 3.6. *Let u be the solution of the continuous problem, u_h the solution of the discrete problem and let $\xi = u_h - \Pi_h u \in S_h$. Then there exists a constant $C > 0$, independent of $h \in (0, h_0)$, such that*

$$
|b_h(u,\xi) - b_h(u_h,\xi)| \le C\|\xi\|_{DG}\left(h^{p+1}|u|_{H^{p+1}(\Omega)} + \|\xi\|_{L^2(\Omega)}\right).
$$

3.4. Coercivity. An important question is the coercivity of the discrete problem (2.6).

LEMMA 3.7 (Nonsymmetric case). *Let $C_W > 0$. Then for the nonsymmetric diffusion form a_h and nonsymmetric right hand side l_h (i.e., $\Theta = -1$), we have*

$$
a_h(u_h,\xi) - a_h(u,\xi) - l_h(u_h,\xi) + l_h(u,\xi) = A + B,
$$

where

$$
\begin{aligned}
&A \ge \beta_0|\xi|^2_{H^1(\Omega,\mathcal{T}_h)},\\
&|B| \le C\Big((2\beta_1-\beta_0)h^p|u|_{H^{p+1}(\Omega)} + h^{p+1}|u|_{H^{p+1}(\Omega)} + \|\xi\|_{L^2(\Omega)}\Big)\\
&\qquad\times\left(|\xi|_{H^1(\Omega,\mathcal{T}_h)} + J_h(\xi,\xi)^{1/2}\right).
\end{aligned}
\tag{3.9}
$$

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LEMMA 3.8 (Symmetric case). *Let*

$$C_W \geq 4\left(\frac{\beta_1}{\beta_0}\right)^2 C_M(1+C_I),$$

where C_M and C_I are the constants from Lemmas 3.1 and 3.2, respectively. Then for the symmetric diffusion form a_h and symmetric right hand side l_h (i.e., $\Theta = 1$), we have

$$a_h(u_h,\xi) - a_h(u,\xi) - l_h(u_h,\xi) + l_h(u,\xi) = A + B,$$

where

$$\begin{aligned} A &\geq \frac{\beta_0}{2}\big(|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} - J_h(\xi,\xi)\big),\\ |B| &\leq C\Big((2\beta_1-\beta_0)h^p|u|_{H^{p+1}(\Omega)} + h^{p+1}|u|_{H^{p+1}(\Omega)} + \|\xi\|_{L^2(\Omega)}\Big)\\ &\quad\times\big(|\xi|_{H^1(\Omega,\mathcal{T}_h)} + J_h(\xi,\xi)^{1/2}\big). \end{aligned}$$

LEMMA 3.9 (Incomplete case). *Let*

$$C_W \geq 2\left(\frac{\beta_1}{\beta_0}\right)^2 C_M(1+C_I),$$

where C_M and C_I are the constants from Lemmas 3.1 and 3.2, respectively. Then for the incomplete *diffusion form a_h and* incomplete *right hand side l_h, we have*

$$a_h(u_h,\xi) - a_h(u,\xi) - l_h(u_h,\xi) + l_h(u,\xi) = A + B,$$

where

$$\begin{aligned} A &\geq \frac{\beta_0}{2}\big(|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} - J_h(\xi,\xi)\big),\\ |B| &\leq C\Big((2\beta_1-\beta_0)h^p|u|_{H^{p+1}(\Omega)} + h^{p+1}|u|_{H^{p+1}(\Omega)} + \|\xi\|_{L^2(\Omega)}\Big)\\ &\quad\times\big(|\xi|_{H^1(\Omega,\mathcal{T}_h)} + J_h(\xi,\xi)^{1/2}\big). \end{aligned}$$

The proofs of Lemmas 3.7–3.9 are rather technical. Therefore, we shall give here only a sketch of the proof of Lemma 3.7.

Proof of Lemma 3.7. We break down $a_h(u,\xi) - l_h(u,\xi)$ into individual terms by setting

$$\sigma^1(u,\xi) = \sum_{i\in I} \int_{K_i} \beta(u)\nabla u \cdot \nabla\xi \, dx,$$

$$\sigma^2(u,\xi) = -\sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \langle \beta(u)\nabla u\rangle \cdot \boldsymbol{n}_{ij}[\xi]\, dS,$$

$$\sigma^3(u,\xi) = \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \langle \beta(u)\nabla \xi\rangle \cdot \boldsymbol{n}_{ij}[u]\, dS,$$

$$\sigma^4(u,\xi) = -\sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \beta(u)\nabla u \cdot \boldsymbol{n}_{ij}\xi \, dS,$$

$$\sigma^5(u,\xi) = \sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \beta(u)\nabla \xi \cdot \boldsymbol{n}_{ij} u \, dS - \\ -\sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \beta(u)\nabla \xi \cdot \boldsymbol{n}_{ij} u_D \, dS.$$

Therefore

$$a_h(u_h,\xi) - a_h(u,\xi) - l_h(u_h,\xi) + l_h(u,\xi) = \sum_{i=1}^{5} (\sigma^i(u_h,\xi) - \sigma^i(u,\xi))$$

and we shall treat these terms separately:

1) First term:

$$\sigma^1(u_h,\xi) - \sigma^1(u,\xi) = \sum_{i\in I}\int_{K_i} \big(\beta(u_h)\nabla u_h - \beta(u)\nabla u\big)\cdot\nabla\xi\, dx$$
$$= \sum_{i\in I}\int_{K_i} \Big(\big(\beta(u_h)\nabla u_h - \beta(u_h)\nabla\Pi_h u\big) + \big(\beta(u_h)\nabla\Pi_h u - \beta(u)\nabla\Pi_h u\big)$$
$$+\big(\beta(u)\nabla\Pi_h u - \beta(u)\nabla u\big)\Big)\cdot\nabla\xi\, dx = \sigma_1^1 + \sigma_2^1 + \sigma_3^1$$

and, using (2.4), we estimate

$$\sigma_1^1 = \sum_{i\in I}\int_{K_i} \beta(u_h)\nabla\xi\cdot\nabla\xi\, dx \ge \beta_0|\xi|^2_{H^1(\Omega,\mathcal{T}_h)}.$$

Further, by (2.4), (2.5), the Cauchy inequality, (3.3) and Lemma 3.4, we get

$$|\sigma_2^1| = \left|\sum_{i\in I}\int_{K_i}\Big(\big(\beta(u_h)-\beta(u)\big)\nabla\eta + \big(\beta(u_h)-\beta(u)\big)\nabla u\Big)\cdot\nabla\xi\, dx\right|$$
$$\le \Big(\big((\beta_1-\beta_0) + LC_R h\big)Ch^p|u|_{H^{p+1}(\Omega)} + LC_R\|\xi\|_{L^2(\Omega)}\Big)|\xi|_{H^1(\Omega,\mathcal{T}_h)}.$$

Finally, (2.4), the Cauchy inequality and Lemma 3.4 imply that

$$|\sigma_3^1| = \left|\sum_{i\in I}\int_{K_i}\beta(u)\big(\nabla\Pi_h u - \nabla u\big)\cdot\nabla\xi\, dx\right|$$
$$\le \sum_{i\in I}\int_{K_i}\beta_1|\nabla\eta||\nabla\xi|\, dx \le \beta_1 Ch^p|u|_{H^{p+1}(\Omega)}|\xi|_{H^1(\Omega,\mathcal{T}_h)}.$$

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2) Second term:

$$\begin{aligned}
&\sigma^2(u_h,\xi)-\sigma^2(u,\xi)\\
&\quad=-\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\big\langle\big(\beta(u_h)\nabla u_h-\beta(u_h)\nabla\Pi_h u\big)\\
&\qquad+\big(\beta(u_h)\nabla\Pi_h u-\beta(u)\nabla\Pi_h u\big)\\
&\qquad+\big(\beta(u)\nabla\Pi_h u-\beta(u)\nabla u\big)\big\rangle\cdot\boldsymbol{n}_{ij}[\xi]\,dS=\sigma_1^2+\sigma_2^2+\sigma_3^2,
\end{aligned}$$

where

$$\sigma_1^2=-\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\langle\beta(u_h)\nabla\xi\rangle\cdot\boldsymbol{n}_{ij}[\xi]\,dS.$$

We do not estimate σ_1^2, since it will cancel out a similar term in the following. After applying (2.4) and (2.5), we get

$$\begin{aligned}
|\sigma_2^2|&=\left|\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\Big\langle\big(\beta(u_h)-\beta(u)\big)\nabla\eta+\big(\beta(u_h)-\beta(u)\big)\nabla u\Big\rangle\cdot\boldsymbol{n}_{ij}[\xi]\,dS\right|\\
&\le\frac{(\beta_1-\beta_0)}{C_W}\left(\sum_{i\in I}h_{K_i}||\nabla\eta||^2_{L^2(\partial K_i)}\right)^{1/2}J_h(\xi,\xi)^{1/2}\\
&\quad+\frac{L||\nabla u||_{L^\infty(\Omega)}}{C_W}\left(\sum_{i\in I}h_{K_i}||u_h-u||^2_{L^2(\partial K_i)}\right)^{1/2}J_h(\xi,\xi)^{1/2}.
\end{aligned}$$

The multiplicative trace inequality implies that

$$|\sigma_2^2|\le C\Big(\big((\beta_1-\beta_0)+LC_Rh\big)h^p|u|_{H^{p+1}(\Omega)}+LC_R||\xi||_{L^2(\Omega)}\Big)J_h(\xi,\xi)^{1/2}$$

and

$$|\sigma_3^2|\le\beta_1Ch^p|u|_{H^{p+1}(\Omega)}J_h(\xi,\xi)^{1/2}.$$

3) Third term:

$$\begin{aligned}
&\sigma^3(u_h,\xi)-\sigma^3(u,\xi)\\
&=\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\langle\beta(u_h)\nabla\xi\rangle\cdot\boldsymbol{n}_{ij}[u_h]-\langle\beta(u)\nabla\xi\rangle\cdot\boldsymbol{n}_{ij}[u]\,dS\\
&=\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\big\langle\beta(u_h)\nabla\xi\big\rangle\cdot\boldsymbol{n}_{ij}[u_h-\Pi_h u]\\
&\quad+\big\langle\big(\beta(u_h)-\beta(u)\big)\nabla\xi\big\rangle\cdot\boldsymbol{n}_{ij}[\Pi_h u]+\big\langle\beta(u)\nabla\xi\big\rangle\cdot\boldsymbol{n}_{ij}[\Pi_h u-u]\,dS\\
&=\sigma_1^3+\sigma_2^3+\sigma_3^3.
\end{aligned}$$

By (3.4), we get

$$\sigma_1^3=\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\langle\beta(u_h)\nabla\xi\rangle\cdot\boldsymbol{n}_{ij}[\xi]\,dS=-\sigma_1^2.$$

Due to the regularity condition (3.3), the function u is continuous and, thus, $[u] = 0$ and $[\Pi_h u] = [\eta]$. We get the estimate:

$$
\begin{aligned}
|\sigma_2^3| &= \left| \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \Big\langle \big(\beta(u_h) - \beta(u)\big)\nabla\xi \Big\rangle \cdot \boldsymbol{n}_{ij}[\eta]\, dS \right| \\
&\le (\beta_1 - \beta_0) C |\xi|_{H^1(\Omega,\mathcal{T}_h)} h^p |u|_{H^{p+1}(\Omega)}.
\end{aligned}
$$

Finally, we have

$$
|\sigma_3^3| \le \beta_1 C |\xi|_{H^1(\Omega,\mathcal{T}_h)} h^p |u|_{H^{p+1}(\Omega)}.
$$

4) Fourth term:

$$
\begin{aligned}
&\sigma^4(u_h,\xi) - \sigma^4(u,\xi) = -\sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \big(\beta(u_h)\nabla u_h - \beta(u)\nabla u\big) \cdot \boldsymbol{n}_{ij}\xi\, dS \\
&= -\sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \big(\beta(u_h)\nabla\xi + \big(\beta(u_h) - \beta(u)\big)\nabla\Pi_h u + \beta(u)\nabla\eta\big) \cdot \boldsymbol{n}_{ij}\xi\, dS \\
&= \sigma_1^4 + \sigma_2^4 + \sigma_3^4
\end{aligned}
$$

and these terms can be treated similarly as σ_1^2, σ_2^2 and σ_3^2 to obtain

$$
\begin{aligned}
\sigma_1^4 \; &= -\sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \beta(u_h)\nabla\xi \cdot \boldsymbol{n}_{ij}\xi\, dS, \\
|\sigma_2^4| &\le C\Big(\big((\beta_1 - \beta_0) + LC_R h\big) h^p |u|_{H^{p+1}(\Omega)} + LC_R \|\xi\|_{L^2(\Omega)}\Big) J_h(\xi,\xi)^{1/2}, \\
|\sigma_3^4| &\le \beta_1 C h^p |u|_{H^{p+1}(\Omega)} J_h(\xi,\xi)^{1/2}.
\end{aligned}
$$

5) Fifth term:

$$
\begin{aligned}
&\sigma^5(u_h,\xi) - \sigma^5(u,\xi) = \sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \big(\beta(u_h)\nabla\xi \cdot \boldsymbol{n}_{ij} u_h \\
&\quad -\beta(u)\nabla\xi \cdot \boldsymbol{n}_{ij} u - \big(\beta(u_h) - \beta(u)\big)\nabla\xi \cdot \boldsymbol{n}_{ij} u_D\big)\; dS \\
&= \sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \big(\beta(u_h)\nabla\xi \cdot \boldsymbol{n}_{ij}\xi \\
&\quad +\big(\beta(u_h) - \beta(u)\big)\nabla\xi \cdot \boldsymbol{n}_{ij}\big(\Pi_h u - u_D\big) + \beta(u)\nabla\xi \cdot \boldsymbol{n}_{ij}\eta\big)\; dS \\
&= \sigma_1^5 + \sigma_2^5 + \sigma_3^5.
\end{aligned}
$$

We have $\sigma_1^5 = -\sigma_1^4$ and

$$
\sigma_2^5 = \sum_{i\in I} \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \big(\beta(u_h) - \beta(u)\big)\nabla\xi \cdot \boldsymbol{n}_{ij}\eta\, dS,
$$

since $u = u_D$ on $\partial\Omega$. This, the Cauchy inequality and the multiplicative trace inequality yield

$$
|\sigma_2^5| \le (\beta_1 - \beta_0) C |\xi|_{H^1(\Omega,\mathcal{T}_h)} h^p |u|_{H^{p+1}(\Omega)},
$$

and

$$
|\sigma_3^5| \le \beta_1 C |\xi|_{H^1(\Omega,\mathcal{T}_h)} h^p |u|_{H^{p+1}(\Omega)}.
$$

The use of the derived inequalities gives us the sought estimates (3.9). □

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3.5. Main result. On the basis of the above analysis, we get the apriori error estimate for the problem with the nonlinear convection and diffusion.

THEOREM 3.10. *Let assumptions* (H1)–(H3) *and* (A1)–(A2) *be satisfied and let the constant* C_W *satisfy the assumption from Lemmas 3.7, 3.8* and *3.9 corresponding to the NIPG, SIPG and IIPG version, respectively. Let* u *be the exact solution of problem* (2.1)–(2.3) *satisfying the regularity conditions* (3.1) *and* (3.3) *and let* u_h *be the approximate solution defined by* (2.6)*with* $u_h^0 = \Pi_h u^0$. *Then the error* $e_h = u - u_h$ *satisfies the estimate*

$$\begin{aligned}\max_{t\in[0,T]} \|e_h(t)\|^2_{L^2(\Omega)} &+ \frac{\beta_0}{2}\int_0^t \left(|e_h(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h(e_h(\vartheta), e_h(\vartheta))\right) d\vartheta \\ &\le C\, h^{2p},\end{aligned} \tag{3.10}$$

with a constant $C > 0$ *independent of* h.

Proof. From (2.6), (3.2) and estimates in Sections 3.2–3.4 it follows that

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\|\xi(t)\|^2_{L^2(\Omega)} + \frac{\beta_0}{2}\left(|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h(\xi,\xi)\right)\\ &\le C\Bigg\{\left(J_h(\xi,\xi)^{1/2} + |\xi|_{H^1(\Omega,\mathcal{T}_h)}\right)\left(h^{p+1}|u|_{H^{p+1}(\Omega)} + \|\xi\|_{L^2(\Omega)}\right)\\ &\quad + h^{p+1}|\partial u/\partial t|_{H^{p+1}(\Omega)}\|\xi\|_{L^2(\Omega)} + \beta_0 h^p |u|_{H^{p+1}(\Omega)} J_h(\xi,\xi)^{1/2}\\ &\quad + \Big((2\beta_1 - \beta_0)h^p|u|_{H^{p+1}(\Omega)}\\ &\quad + h^{p+1}|u|_{H^{p+1}(\Omega)} + \|\xi\|_{L^2(\Omega)}\Big)\left(|\xi|_{H^1(\Omega,\mathcal{T}_h)} + J_h(\xi,\xi)\right)\Bigg\}.\end{aligned}$$

The application of Young's inequality gives us

$$\begin{aligned}&\frac{\partial}{\partial t}\|\xi(t)\|^2_{L^2(\Omega)} + \beta_0\left(|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h(\xi,\xi)\right)\\ &\le \frac{\beta_0}{2}\left(J_h(\xi,\xi) + |\xi|^2_{H^1(\Omega,\mathcal{T}_h)}\right) + C\Bigg\{\left(1 + \frac{1}{\beta_0}\right)\|\xi\|^2_{L^2(\Omega)}\\ &\quad + \frac{1}{\beta_0}\left(h^{2p+2} + \beta_0^2 h^{2p} + (2\beta_1 - \beta_0)^2 h^{2p} + h^{2p+2}\right)|u|^2_{H^{p+1}(\Omega)}\\ &\quad + h^{2p+2}|\partial u/\partial t|^2_{H^{p+1}(\Omega)}\Bigg\}.\end{aligned}$$

After integrating from 0 to $t \in [0,T]$ and noticing that $\xi(0) = u_h^0 - \Pi_h u^0 = 0$, we obtain

$$\begin{aligned}&\|\xi(t)\|^2_{L^2(\Omega)} + \beta_0\int_0^t \left(|\xi(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h(\xi(\vartheta),\xi(\vartheta))\right) d\vartheta\\ &\le C\Bigg\{\left(1 + \frac{1}{\beta_0}\right)\int_0^t \|\xi(\vartheta)\|^2_{L^2(\Omega)}\, d\vartheta + \frac{1}{\beta_0}h^{2p}\int_0^t \big(h^2 + \beta_0^2\\ &\quad + (2\beta_1 - \beta_0)^2 + h^2\big)|u(\vartheta)|^2_{H^{p+1}(\Omega)}\, d\vartheta + h^{2p+2}\int_0^t |\partial u/\partial t|^2_{H^p(\Omega)}\, d\vartheta\Bigg\}.\end{aligned}$$

Now the application of Gronwall's lemma implies that

$$\begin{aligned}
&\|\xi(t)\|^2_{L^2(\Omega)} + \beta_0 \int_0^t \left(|\xi(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h(\xi(\vartheta),\xi(\vartheta)) \right) d\vartheta \\
&\le C h^{2p} \frac{\beta_0+1}{\beta_0} \bigg(\frac{1}{\beta_0} \left(h^2 + \beta_0^2 + (2\beta_1 - \beta_0)^2 + h^2 \right) \\
&\quad \times \|u\|^2_{L^2(0,T;H^{p+1}(\Omega))} + h^2 \|\partial u/\partial t\|^2_{L^2(0,T;H^{p+1}(\Omega))} \bigg) \\
&\quad \times \exp\left(C \frac{\beta_0+1}{\beta_0} t \right), \quad t \in [0,T].
\end{aligned}$$

Finally, since $e_h = \xi + \eta$, the above estimate and estimates from Lemma 3.4 yield the sought result. ◻

4. Optimal error estimates. The error estimate (3.10) is optimal in the $L^2(H^1)$-norm, but suboptimal in the $L^\infty(L^2)$-norm. Our goal is to derive an optimal error estimate in the $L^\infty(L^2)$-norm. It was carried out in [13] under the following assumptions.

Assumptions (B):

- diffusion is linear, i.e., $\beta(u) = \varepsilon > 0$,
- the discrete diffusion form a_h is symmetric (i.e., we consider the SIPG version of the discrete problem),
- the polygonal domain Ω is convex,
- the meshes $\mathcal{T}_h$, $h \in (0, h_0)$, are conforming with standard properties from the finite element method (i.e., without hanging nodes),
- the exact solution u of problem (2.1)–(2.3) satisfies the regularity condition (3.1),
- conditions (H1)–(H3) and (A1) are satisfied.

The derivation of the $L^\infty(L^2)$-optimal error estimate was carried out with the aid of the Aubin-Nitsche technique based on the use of the elliptic dual problem considered for each $z \in L^2(\Omega)$:

$$-\Delta\psi = z \quad \text{in } \Omega, \quad \psi|_{\partial\Omega} = 0. \tag{4.1}$$

Then the weak solution $\psi \in H^2(\Omega)$ and there exists a constant $C > 0$, independent of z, such that

$$\|\psi\|_{H^2(\Omega)} \le C \|z\|_{L^2(\Omega)}. \tag{4.2}$$

We shall give here a sketch of the proof of the optimal error estimate. Let us set

$$A_h(w,v) = a_h(w,v) + \varepsilon J_h(w,v), \quad w, v \in H^2(\Omega, \mathcal{T}_h).$$

Now, for each $h \in (0, h_0)$ and $t \in [0,T]$ we define the function $u^*(t)$ $(= u_h^*(t))$ as the "A_h-projection" of $u(t)$ on S_h, i. e. a function satifying the conditions

$$u^*(t) \in S_h, \qquad A_h(u^*(t), \varphi_h) = A_h(u(t), \varphi_h) \quad \forall\, \varphi_h \in S_h, \tag{4.3}$$

and set $\chi = u - u^*$. The use of the coercivity of the form A_h and estimates from sections 3.2–3.4 allow us to prove the estimates

$$\|\chi\|_{DG} \le C\, h^p |u|_{H^{p+1}(\Omega)}, \quad \|\chi_t\|_{DG} \le C\, h^p |u_t|_{H^{p+1}(\Omega)}, \tag{4.4}$$

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for all $h \in (0, h_0)$.

It is necessary to derive optimal $L^2(\Omega)$-estimates of χ and χ_t.

LEMMA 4.1. *There exists a constant $C > 0$ such that for all $h \in (0, h_0)$*

$$\|\chi\|_{L^2(\Omega)} \leq C h^{p+1} |u|_{H^{p+1}(\Omega)}, \quad \textit{and} \quad \|\chi_t\|_{L^2(\Omega)} \leq C h^{p+1} |u_t|_{H^{p+1}(\Omega)}. \tag{4.5}$$

Proof. We have

$$\|\chi\|_{L^2(\Omega)} = \sup_{z \in L^2(\Omega)} \frac{(\chi, z)}{\|z\|_{L^2(\Omega)}}.$$

Let $\psi \in H^2(\Omega)$ be the solution of problem (4.1) for $z \in L^2(\Omega)$ satisfying (4.2) and let ψ_h be the piecewise linear Lagrange interpolant of the function ψ. Obviously $\psi_h \in C(\overline{\Omega}) \cap S_h$ and $\psi_h|_{\partial\Omega} = 0$. Thus, we have

$$\|\psi - \psi_h\|_{DG}^2 = \frac{1}{2} |\psi - \psi_h|^2_{H^1(\Omega, \mathcal{T}_h)} \leq C\, h^2 |\psi|^2_{H^2(\Omega)}.$$

The assumption that $\psi \in H^2(\Omega)$ implies that

$$[\psi]_{\Gamma_{ij}} = 0 = [\nabla \psi]_{\Gamma_{ij}} \qquad \forall i \in I,\ j \in s(i).$$

Now, using (4.1) and Green's theorem, we find after some calculations that

$$(\chi, z) = \frac{1}{\varepsilon} A_h(\psi, \chi).$$

Further, the symmetry of A_h and (4.3) give

$$A_h(\psi_h, \chi) = A_h(\chi, \psi_h) = A_h(u - u^*,\, \psi_h) = 0,$$

and thus

$$(\chi, z) = \frac{1}{\varepsilon} A_h(\psi - \psi_h,\, \chi).$$

Moreover, due to the properties of ψ and ψ_h,

$$(\psi - \psi_h)|_{\Gamma_{ij}} = 0 \qquad \forall\, i \in I,\ j \in \gamma(i),$$

and

$$[\psi - \psi_h]_{\Gamma_{ij}} = 0 \qquad \forall\, i \in I,\ j \in s(i).$$

After some calculation, we get

$$\begin{aligned}
(\chi, z) &= \sum_{i \in I} \int_{K_i} \nabla(\psi - \psi_h)\, \nabla\chi \, \mathrm{d}x \\
&\quad - \sum_{i \in I} \sum_{\substack{j \in s(i) \\ j < i}} \int_{\Gamma_{ij}} \left(\frac{d(\Gamma_{ij})}{C_W} \right)^{1/2} \langle \nabla(\psi - \psi_h) \rangle \cdot \boldsymbol{n}_{ij} \left(\frac{C_W}{d(\Gamma_{ij})} \right)^{1/2} [\chi]\, \mathrm{d}S \\
&\quad - \sum_{i \in I} \sum_{j \in \gamma(i)} \int_{\Gamma_{ij}} \left(\frac{d(\Gamma_{ij})}{C_W} \right)^{1/2} \nabla(\psi - \psi_h) \cdot \boldsymbol{n}_{ij} \left(\frac{C_W}{d(\Gamma_{ij})} \right)^{1/2} \chi\, \mathrm{d}S \\
&\leq |\psi - \psi_h|_{H^1(\Omega)} |\chi|_{H^1(\Omega, \mathcal{T}_h)} \\
&\quad + \frac{1}{C_W^{1/2}} \left(\sum_{i \in I} h_{K_i} \|\nabla(\psi - \psi_h)\|^2_{L^2(\partial K_i)} \right)^{1/2} \left(J_h^\sigma(\chi, \chi) \right)^{1/2}.
\end{aligned}$$

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According to the multiplicative trace inequality, (4.2) and (4.4), we can write

$$\begin{aligned}
&\frac{1}{C_W^{1/2}}\left(\sum_{i\in I} h_{K_i}\|\nabla(\psi-\psi_h)\|^2_{L^2(\partial K_i)}\right)^{1/2}(J_h(\chi,\chi))^{1/2}\\
&\le C\|\chi\|_{DG}\left(\sum_{i\in I} h^2_{K_i}|\psi|^2_{H^2(K_i)}\right)^{1/2}\le C\,h^{p+1}|u|_{H^{p+1}(\Omega)}\,\|z\|_{L^2(\Omega)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|\psi-\psi_h|_{H^1(\Omega)}|\chi|_{H^1(\Omega,\mathcal{T}_h)} &\le C\,h|\psi|_{H^2(\Omega)}\,\|\chi\|_{DG}\\
&\le C\,h^{p+1}|u|_{H^{p+1}(\Omega)}\|z\|_{L^2(\Omega)}.
\end{aligned}$$

Combining the previous estimates, we find that

$$(\chi,z)\le C\,h^{p+1}|u|_{H^{p+1}(\Omega)}\|z\|_{L^2(\Omega)}.$$

Hence,

$$\|\chi\|_{L^2(\Omega)}=\sup_{z\in L^2(\Omega)}\frac{(\chi,z)}{\|z\|_{L^2(\Omega)}}\le C\,h^{p+1}|u|_{H^{p+1}(\Omega)},$$

which completes the proof of the first estimate in (4.5).

In the derivation of the estimate of the norm $\|\chi_t\|_{L^2(\Omega)}$ we proceed similarly as above, using the differentiation of identity (4.3) with respect to t. □

Finally, we come to the optimal $L^\infty(L^2)$-error estimate.

THEOREM 4.2. *Let assumptions* (B) *be fulfilled. Then the error $e_h = u - u_h$ satisfies the estimate*

$$\|e_h\|_{L^\infty(0,T;L^2(\Omega))}\le Ch^{p+1},$$

with a constant $C>0$ independent of h.

Proof. Let u^* be defined by (4.3) and $\chi = u - u^*$, $\vartheta = u^* - u_h$. Then $e_h = u - u_h = \chi + \vartheta$. Let us subtract (3.2) from (2.6, b), substitute $\vartheta \in S_h$ for φ_h and use the relation

$$\left(\frac{\partial\vartheta(t)}{\partial t},\,\vartheta(t)\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vartheta(t)\|^2_{L^2(\Omega)}.$$

Then, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vartheta(t)\|^2_{L^2(\Omega)}+A_h(\vartheta(t),\,\vartheta(t))\\
&\qquad=[b_h(u(t),\,\vartheta(t))-b_h(u_h(t),\,\vartheta(t))]-(\chi_t(t),\,\vartheta(t)),
\end{aligned}\tag{4.6}$$

because $A_h(u(t)-u^*(t),\,\vartheta(t)) = 0$. The first right-hand side term can be estimated by Lemma 3.6 and Young's inequality (we omit the argument t):

$$b_h(u,\vartheta)-b_h(u_h,\vartheta)\le\frac{\varepsilon}{2}\|\vartheta\|^2_{DG}+\frac{C}{\varepsilon}\left(h^{2(p+1)}|u|^2_{H^{p+1}(\Omega)}+\|\vartheta\|^2_{L^2(\Omega)}\right).$$

For the second term of the right-hand side of (4.6), by the Cauchy and Young's inequalities and Lemma 4.1, we have

$$|(\chi_t,\vartheta)|\le\frac{1}{2}\left(C\,h^{2(p+1)}|u_t|^2_{H^{p+1}(\Omega)}+\|\vartheta\|^2_{L^2(\Omega)}\right).$$

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Finally, the coercivity of A_h following from Lemma 3.8 gives the estimate of the left-hand side of (4.6).

Hence, we find that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,\|\vartheta\|^2_{L^2(\Omega)} + \varepsilon\|\vartheta\|^2_{DG} \\ &\le C\,h^{2(p+1)}\left(\frac{1}{\varepsilon}\,|u|^2_{H^{p+1}(\Omega)} + |u_t|^2_{H^{p+1}(\Omega)}\right) + C\left(1+\frac{1}{\varepsilon}\right)\|\vartheta\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.7}$$

Now the proof is concluded in a standard way by the integration of (4.7) from 0 to $t \in [0, T]$, the estimate of $\|\vartheta(0)\|_{L^2(\Omega)}$, and the application of Gronwall's lemma. ∎

5. Conclusion. In the paper we derived error estimates for the NIPG, SIPG and IIPG versions of the DGFEM applied to the numerical solution of nonstationary convection-diffusion problems with nonlinear convection as well as diffusion, equipped with Dirichlet boundary condition. The computational grids can be nonconforming with hanging nodes. Further, we discussed optimal $L^\infty(L^2)$-error estimates of the SIPG method in the case of a linear diffusion and nonlinear convection, with the Dirichlet boundary condition on the whole boundary, over conforming meshes.

There are still some open problems:
- optimal $L^\infty(L^2)$-error estimates in the case of a nonlinear diffusion,
- derivation of optimal $L^\infty(L^2)$-error estimates over nonconforming meshes,
- the analysis of optimal error estimates in the case of a nonconvex polygonal domain Ω and mixed Dirichlet-Neumann boundary conditions under a realistic regularity of the exact solutions of the nonstationary initial-boundary value problem and the elliptic dual problem.

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