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ON THE CALCULATION OF APPROXIMATE FEKETE POINTS: THE UNIVARIATE CASE*

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Abstract. We discuss some theoretical aspects of the univariate case of the method recently introduced by Sommariva and Vianello [Comput. Math. Appl., to appear] for the calculation of approximate Fekete points for polynomial interpolation.

Key words. polynomial interpolation, Fekete points

AMS subject classifications. 41A05, 42A15, 65D05

1. Introduction. Fekete points are a set of points which are good for polynomial interpolation that may be defined for any compact set in any dimension. They are therefore a natural and important set of points from the point of view of polynomial interpolation. However, their computation involves an expensive multivariate optimization which is moreover numerically challenging for higher degrees; cf. the webpage of Womersley [13]. Recently Sommariva and Vianello [12] proposed a method for calculating *approximate* Fekete points that is highly efficient and may easily be used on different compact sets.

Polynomial interpolation, at least in one variable, is a classical subject. However, to make the notions we consider here precise, we briefly outline the main features of (multivariate) polynomial interpolation. Consider $K \subset \mathbb{R}^d$ a compact set. The polynomials of degree at most n in d real variables, when restricted to K, form a certain vector space which we will denote by $\mathcal{P}_n(K)$. The space $\mathcal{P}_n(K)$ has a dimension $N_n := \dim(\mathcal{P}_n(K))$. The polynomial interpolation problem for K is then, given a set of N_n distinct points $A_n \subset K$ and a function $f : K \to \mathbb{R}$, to find a polynomial $p \in \mathcal{P}_n(K)$ such that

$$p(a) = f(a), \quad \forall a \in A_n. \tag{1.1}$$

If we choose a basis,

$$B_n = \{P_1, P_2, \cdots, P_{N_n}\},$$

of $\mathcal{P}_n(K)$, then any polynomial $p \in \mathcal{P}_n(K)$ may be written in the form

$$p = \sum_{j=1}^{N_n} c_j P_j$$

for some constants $c_j \in \mathbb{R}$. Hence the conditions (1.1) may be expressed as

$$p(a) = \sum_{j=1}^{N_n} c_j P_j(a) = f(a), \quad a \in A_n, \tag{1.2}$$

which are exactly N_n linear equations in N_n unknowns c_j. In matrix form this becomes

$$[P(a)]_{a \in A_n, P \in B_n} c = F,$$

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where $c \in \mathbb{R}^{N_n}$ is the vector formed of the c_j and F is the vector of function values $f(a)$, $a \in A_n$. This linear system has a unique solution precisely when the so-called Vandermonde determinant

$$\mathrm{vdm}(A_n; B_n) := \det([P(a)]_{P\in B_n, a\in A_n}) \neq 0. \tag{1.3}$$

If this is the case, then the interpolation problem (1.1) is said to be *correct*.

Supposing then that the interpolation problem (1.1) is correct, we may write the interpolating polynomial in so-called Lagrange form as follows. For $a \in A_n$ set

$$\ell_a(x) := \frac{\mathrm{vdm}(A_n\backslash\{a\} \cup \{x\}; B_n)}{\mathrm{vdm}(A_n; B_n)}. \tag{1.4}$$

Note that the numerator is just the Vandermonde determinant with the interpolation point $a \in A_n$ replaced by the variable $x \in \mathbb{R}^d$.

It is easy to see that $\ell_a \in \mathcal{P}_n(K)$. Moreover $\ell_a(b) = \delta_{ab}$, the Kronecker delta, for $b \in A_n$. Using these so-called Fundamental Lagrange Interpolating Polynomials we may write the interpolant of (1.1) as

$$p(x) = \sum_{a\in A_n} f(a)\ell_a(x). \tag{1.5}$$

The mapping $f \to p$ is a projection and hence we write $p = L_{A_n}(f)$. If we regard both $f, p \in C(K)$ then the operator L_{A_n} has operator norm (as is not difficult to see)

$$\|L_{A_n}\| = \max_{x\in K} \sum_{a\in A_n} |\ell_a(x)|.$$

This operator norm, called the Lebesgue constant, gives a bound on how far the interpolant is from the best uniform polynomial approximant to f. It follows that the quality of approximation to f provided by the interpolant p is indicated by the size of the Lebsegue constant, the smaller it is the better.

Now, suppose that $F_n \subset K$ is a subset of N_n distinct points for which $A_n = F_n$ maximizes $|\mathrm{vdm}(A_n; B_n)|$. Then by (1.4), each supremum norm

$$\max_{x\in K} |\ell_a(x)| \le 1, \quad a \in F_n \tag{1.6}$$

and hence the corresponding Lebesgue constants are such that

$$\|L_{A_n}\| \le N_n,$$

i.e., the Lebesgue constants grow polynomially in n, which is the best that is known in general. Such a set F_n (it may not be unique) is called a set of (true) Fekete points of degree n for K and provide, for any K, a good (typically excellent) set of interpolation points. For more on Fekete points (and polynomial interpolation) we refer the reader to [4] (and its references). Note that the Fekete point sets F_n and also the Lebesgue constants $||L_{A_n}||$ are independent of the basis B_n. We also remark that for each degree n, the Fekete points F_n, form a *set*, i.e., they do not provide an ordering of the points. In applications, especially for high degrees, the ordering of the points can be important. One such ordering is provided by the so-called Leja points; see, for example, [10, 1]. Note, however, that the Leja points provide an ordering of all the points *up to* those of degree n, whereas the Fekete points are for *only* degree n. Once a set of Fekete points has been calculated they can be ordered by the Leja method, but we do not pursue that here.

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Now for the Sommariva-Vianello method. Here is some MATLAB code that implements the algorithm to find 21 (degree 20) approximate Fekete interpolation points for the interval $[-1,1]$:

```
n=21;  % number of interpolation points
m=1000; x=linspace(-1,1,m); % discrete model of [-1,1]
A = gallery('chebvand',n,x);
% A is the n by m Vandermonde matrix
% in the Chebyshev polynomial basis
b=rand(n,1);  % a random rhs
y=A\b;  % y is the MATLAB solution of Ay=b
pp=y~=0; %  vector of indices of the non-zero elements of y
pts=x(pp) % selects the points from x according to pp
```

Before explaining what this code does, we give some example results. In Figure 1.1 we show the approximate Fekete points computed by the above code (marked by +) as well as the true Fekete points, the extended Chebyshev points and also the so-called Leja points. Note how remarkably close the approximate Fekete points are to the true Fekete points. In comparison, the Vandermonde determinant (using the Chebyshev basis) for the true Fekete points is $1.532 \cdot 10^{11}$ (the maximum possible), whereas, for the approximate Fekete points, it is $1.503 \cdot 10^{11}$. For the extended Chebyshev points (a well-known set of *excellent* interpolation points for $[-1,1]$), this determinant is $1.265 \cdot 10^{11}$. As we shall see in Theorem 4.1 below, the values of the Vandermonde determinants for the approximate Fekete points are sufficiently close to the values of the Vandermonde determinants for the true Fekete points to allow us to conclude that these two sets of points are asymptotically the same.

Further, the Lebesgue constant for the true Fekete points is approximately 2.6, while for the approximate Fekete points it is approximately 2.8 and for the extended Chebyshev points approximately 2.9. The reader may be interested to note that for 21 equally spaced points on the interval $[-1,1]$ the Lebesgue constant is approximately 10986.5, i.e., significantly greater. We note the classical results that for $n+1$ Chebyshev points in the interval and $2n+1$ equally spaced points on a circle, the Lebesgue constants grow like $c\ln(n)$ (and this order of growth is optimal). This is further discussed in Section 3.2.

We hope that these results convince the reader that the Sommariva-Vianello algorithm is indeed promising. The purpose of this paper is to discuss some of the theoretical aspects of the algorithm; to hopefully explain why this algorithm gives such good results. Numerical implementation is discussed in [12]. In the next section we show how the procedure is related to a natural greedy algorithm for constructing submatrices of maximal determinant. We give concrete examples of the algorithm in Section 3. Finally, in Section 4 we prove that the algorithm produces points which asymptotically exhibit the same behavior as that of the true Fekete points for a finite union of nondegenerate compact, connected sets in the complex plane.

2. The relation to a greedy algorithm for maximum volume submatrices. The key to understanding how the code segment works is the command `y=A\b`. First observe that the command `A=chebvand(n,x)` produces the Vandermonde matrix of the first n Chebyshev polynomials evaluated at the points of the vector x. Specifically, the (i,j) entry of A is $T_{i-1}(x_j)$ so that the ith row of A corresponds to the Chebyshev basis polynomial T_{i-1} and the jth column of A corresponds to the jth point in x, x_j. Hence selecting a subset of columns of A corresponds to selecting a subset of the points of the vector x.

Now, note that the matrix $A \in \mathbb{R}^{n\times m}$ with $n = 21$ and $m = 1000$, in this case, so that the linear system $Ay = b$ is severely underdetermined. MATLAB resolves this problem

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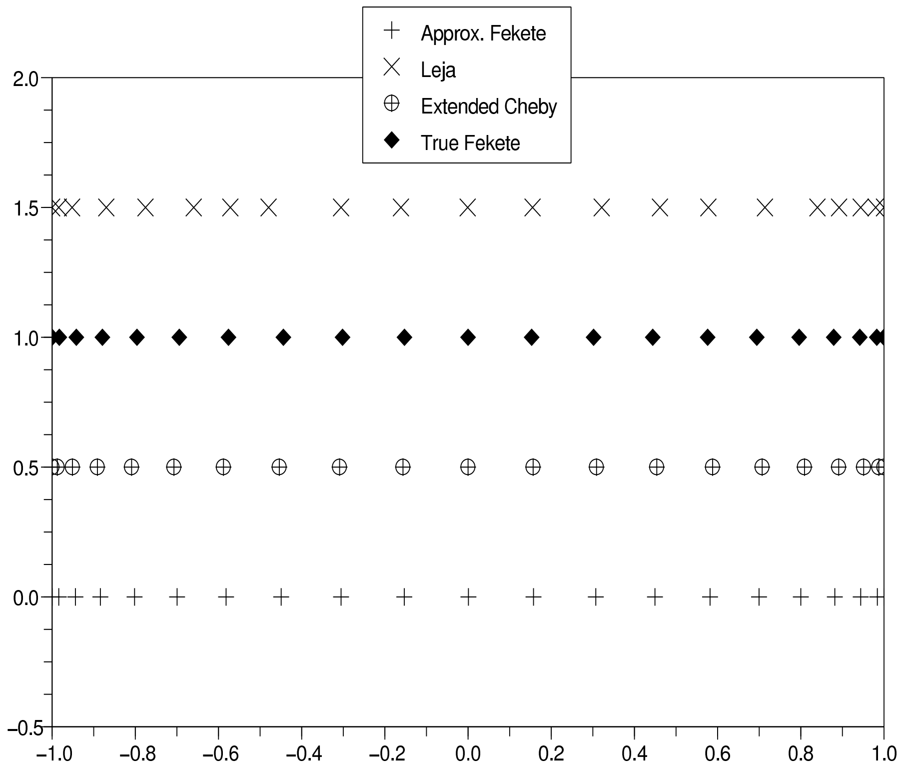

FIG. 1.1. *Plot of various point sets for* $n = 21$

by first computing the QR factorization of A ($Q \in \mathbb{R}^{n\times n}$, orthogonal and $R \in \mathbb{R}^{n\times m}$, upper triangular) with column pivoting; cf. [7, §5.4]. The basics of this algorithm are easy to describe. In fact, the pivoting is a procedure to select the n "most significant" among the $m >> n$ columns of A.

The first column selected is the one of largest euclidean length – call this column $a_1 \in \mathbb{R}^n$. Then, an orthogonal matrix $Q_1 \in \mathbb{R}^{n\times n}$ is chosen that maps a_1 to the first column of an upper triangular matrix, i.e., $\pm\|a_1\|_2 e_1$. Here we use the notation $\|\cdot\|_2$ to denote the euclidean (ℓ_2) norm of a vector in $\mathbb{R}^m$ or $\mathbb{C}^m$ (m may vary). Also, we use a dot "$\cdot$" to denote the euclidean inner product. We then compute

$$Q_1 A = \begin{bmatrix} \pm\|a_1\|_2 & * & \cdots & * \\ 0 & & & \\ 0 & & A_1 & \\ 0 & & & \\ 0 & & & \end{bmatrix}, \tag{2.1}$$

where $A_1 \in \mathbb{R}^{(n-1)\times(m-1)}$. Then these two operations are repeated to the matrix A_1, $A_2 \in \mathbb{R}^{(n-2)\times(m-2)}$ and so on. After n steps, we arrive at $Q = Q_1 \cdots Q_{n-2} Q_{n-1}$ and $R = A_{n-1}$. Once these have been calculated, MATLAB solves the system

$$\widehat{A}\hat{y} = b,$$

where $\widehat{A} \in \mathbb{R}^{n\times n}$ consists of the n columns so selected and $\hat{y} \in \mathbb{R}^n$. The other entries of

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$y \in \mathbb{R}^m$ are set to 0. Hence the command `pp=y~=0;` in the code segment above gives the indices of the selected columns.

Let us look now at the second column of A that the algorithm selects (the first is the one of longest euclidean length). This would correspond to the column of A_1 (in $\mathbb{R}^{n-1}$) of longest length. Let us call this longest column of A_1, $b_1 \in \mathbb{R}^{n-1}$, and the corresponding column of A, a_2. Then by (2.1) we have

$$Q_1 a_1 = \begin{bmatrix} \alpha \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \qquad Q_1 a_2 = \begin{bmatrix} \beta \\ \\ b_1 \\ \\ \end{bmatrix},$$

for $\alpha = \pm\|a_1\|_2$ and some $\beta \in \mathbb{R}$. We have then

$$\begin{bmatrix} 0 \\ \\ b_1 \\ \\ \end{bmatrix} = Q_1 a_2 - \left\{ (Q_1 a_2) \cdot \frac{Q_1 a_1}{\|Q_1 a_1\|_2} \right\} \frac{Q_1 a_1}{\|Q_1 a_1\|_2},$$

from which we see that

$$\begin{bmatrix} 0 \\ \\ b_1 \\ \\ \end{bmatrix}$$

is the component of $Q_1 a_2$ orthogonal to $Q_1 a_1$ and hence $\|b_1\|_2 \times \|Q_1 a_1\|_2$ is the area of the parallelogram generated by $Q_1 a_1$ and $Q_1 a_2$. But Q_1 is an orthogonal matrix and so $\|b_1\|_2 \times \|Q_1 a_1\|_2$ is also the area of the parallelogram generated by a_1 and a_2. It follows that the second column is chosen so that the area it generates with fixed a_1 is maximal. Similarly, the third column is chosen so that the volume it generates with fixed a_1 and a_2 is maximal, and so on.

In summary, the pivoting procedure for the QR factorization selects columns as follows:

(1) a_1 is the column of A of maximum euclidean length.

(2) Given $a_1, \cdots, a_k$, the $(k+1)$st column a_{k+1} is chosen so that the volume of the "box" generated by $a_1, \cdots, a_k, a_{k+1}$ is as large as possible.

This is precisely the standard Greedy Algorithm for constructing a maximal volume box from a collection of vectors. Note that, in principle, the algorithm selects the columns in a certain order. However, in the final result of the MATLAB command `A\b` this information is lost. As mentioned in the Introduction, in numerical applications the ordering of the points is important and a version of the command that saves the order information would be very useful. In this paper, however, we concentrate on the theoretical aspects of the *set* of points that the algorithm selects.

3. The continuous version of the algorithm. Suppose that $K \subset \mathbb{R}^d$ is compact. Given a basis $B_n = \{P_1, P_2, \cdots, P_N\}$ for $\mathcal{P}_n(K)$ the columns of the associated Vandermonde matrix are of the form

$$\vec{V}(x) := \begin{bmatrix} P_1(x) \\ P_2(x) \\ \cdot \\ \cdot \\ P_N(x) \end{bmatrix}$$

for some $x \in K$. Recalling that, for a Vandemonde matrix, selecting a subset of columns is equivalent to selecting a subset of points, we describe a continuous version of the Sommariva-Vianello Algorithm as follows:

(1) The first point $x_1 \in K$ is chosen to maximize $\|\vec{V}(x)\|_2$.

(2) Given $x_1, x_2, \cdots, x_k$ the $(k+1)$st point $x_{k+1} \in K$ is chosen so that the volume generated by the columns $\vec{V}(x_{k+1})$ and $\vec{V}(x_1), \vec{V}(x_2), \cdots, \vec{V}(x_k)$ is as large as possible.

3.1. First example: the unit circle. Here we take K to be the unit circle so that $\mathcal{P}_n(K)$ is the trigonometric polynomials of degree n with dimension $N = 2n+1$. We take the orthonormal basis,

$$B_n = \{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(\theta), \frac{1}{\sqrt{\pi}}\sin(\theta), \cdots, \frac{1}{\sqrt{\pi}}\cos(n\theta), \frac{1}{\sqrt{\pi}}\sin(n\theta)\},$$

with respect to the inner-product,

$$\langle P_1, P_2 \rangle := \int_0^{2\pi} P_1(\theta) P_2(\theta) d\theta, \tag{3.1}$$

so that

$$\vec{V}(\theta) = \frac{1}{\sqrt{\pi}} \begin{bmatrix} 1/\sqrt{2} \\ \cos(\theta) \\ \sin(\theta) \\ \cdot \\ \cdot \\ \sin(n\theta) \end{bmatrix}.$$

In particular, we have

$$\|\vec{V}(\theta)\|_2 = \sqrt{\frac{2n+1}{2\pi}}, \quad \forall \theta \in [0, 2\pi]. \tag{3.2}$$

More generally,

$$\begin{aligned} \vec{V}(\theta) \cdot \vec{V}(\phi) &= \frac{1}{\pi}\left\{\frac{1}{2} + \sum_{k=1}^{n} \cos(k\theta)\cos(k\phi) + \sin(k\theta)\sin(k\phi)\right\} \\ &= \frac{1}{\pi}\left\{\frac{1}{2} + \sum_{k=1}^{n} \cos(k(\theta-\phi))\right\} \\ &= \frac{1}{2\pi}\frac{\sin(\frac{2n+1}{2}(\theta-\phi))}{\sin(\frac{\theta-\phi}{2})}, \end{aligned}$$

which the reader will notice is the reproducing kernel for the space of trigonometric polynomials of degree at most n, equipped with the inner product (3.1).

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It follows that for *any* set of $2n+1$ equally spaced angles $\{\theta_k\}$, $k=0,1,\cdots,2n$, i.e., with $\theta_k=\theta_0+2k\pi/(2n+1)$,

$$\begin{aligned}\vec{V}(\theta_j)\cdot\vec{V}(\theta_k) &= \frac{1}{2\pi}\frac{\sin(\frac{2n+1}{2}(\theta_j-\theta_k))}{\sin(\frac{\theta_j-\theta_k}{2})}\\ &= \frac{1}{2\pi}\frac{\sin(\frac{2n+1}{2}\frac{2(j-k)\pi}{2n+1})}{\sin(\frac{(j-k)\pi}{2n+1})}\\ &= \frac{1}{2\pi}\frac{\sin((j-k)\pi)}{\sin(\frac{(j-k)\pi}{2n+1})}\\ &= 0 \quad \text{for } j\neq k.\end{aligned} \tag{3.3}$$

In other words, the column vectors $\vec{V}(\theta_j)$ and $\vec{V}(\theta_k)$ are orthogonal for $j\neq k$.

Conversely, for $\theta\in[0,2\pi]$ and fixed j,

$$\begin{aligned}\vec{V}(\theta_j)\cdot\vec{V}(\theta)=0 &\Longrightarrow \sin(\frac{2n+1}{2}(\theta_j-\theta))=0\\ &\Longrightarrow \theta_j-\theta=\frac{2k}{2n+1}\pi \quad \text{for some } k\\ &\Longrightarrow \theta=\theta_{j-k}.\end{aligned}$$

What is the result of the Algorithm in this case? Since by (3.2), the length of $\vec{V}(\theta)$ is constant for all $\theta\in[0,2\pi]$, the first point chosen will be *any* $\theta_0\in[0,2\pi]$. The second point θ will be so that the area generated by $\vec{V}(\theta_0)$ and $\vec{V}(\theta)$ is as large as possible. But as shown above, $\vec{V}(\theta)\perp\vec{V}(\theta_0)$ iff $\theta=\theta_j$ for some $j\neq 0$. Hence (noting that the lengths of $\vec{V}(\theta)$ are the same for all $\theta\in\mathbb{R}$) this area will be maximized by *any* θ_j, $j\neq 0$. Continuing, we see that the output of the Algorithm is a set of equally spaced angles $\{\theta_j\}$, generated in a random order, for some $\theta_0\in[0,2\pi]$. This is also a set of true Fekete points and we see that, in this case, the approximate Fekete points of the Algorithm are even true Fekete points.

3.2. Second example: the interval $[-1,1]\subset\mathbb{R}^1$. This example is a bit indirect. We construct a good set of interpolation points on $[-1,1]$ by first calculating approximate Fekete points on the unit circle $S^1\subset\mathbb{R}^2$ and projecting down, i.e., $(x,y)\to x\in[-1,1]$. Such indirect procedures are likley to be very useful also in several variables.

Since $(x,\pm y)$ project to the same point x, in order to obtain $n+1$ points in $[-1,1]$ it is natural to look for an even number, $2n$, points, symmetric under the mapping $y\to -y$, on the unit circle.

In this case the number of points (on the unit circle), $2n$, is not the dimension of trigonometric polynomials of some total degree and hence choosing the "best" basis is a bit more subtle. Here we use the basis

$$B_n:=\{\frac{1}{\sqrt{2}},\cos(\theta),\sin(\theta),\cdots,\cos((n-1)\theta),\sin((n-1)\theta),\frac{1}{\sqrt{2}}\cos(n\theta)\}. \tag{3.4}$$

Some words of explanation are in order. First of all, B_n spans the trigonometric polynomials of degree at most n with $\sin(n\theta)$ removed. This choice is made in anticipation that the $2n$ equally spaced angles $\theta_k=k\pi/n$, $k=0,1,\cdots,2n-1$ are the best possible for the purposes of interpolation. Note that $\sin(n\theta)$ is equal to 0 precisely at these points, and hence must be removed from any candidate basis.

The normalization constants $1/\sqrt{2}$ for the first and last elements of B_n are in order that B_n is orthonormal with respect to the equally weighted discrete inner-product based on the equally spaced points θ_k. Specifically, we have:

LEMMA 3.1. *Suppose that $\theta_k := k\pi/n$, $0 \le k \le 2n-1$. Then the elements of B_n are orthonormal with respect to the inner-product*

$$\langle p, q\rangle := \frac{1}{n}\sum_{k=0}^{2n-1} p(\theta_k)q(\theta_k) \tag{3.5}$$

(and corresponding norm $||p|| = \sqrt{\langle p, p\rangle}$).

Proof. We first calculate

$$\begin{aligned}\langle\cos(j\theta), \cos(m\theta)\rangle &= \frac{1}{n}\sum_{k=0}^{2n-1}\cos(j\theta_k)\cos(m\theta_k)\\ &= \frac{1}{2n}\sum_{k=0}^{2n-1}\{\cos((j+m)\theta_k) + \cos((j-m)\theta_k)\}. \end{aligned}\tag{3.6}$$

Now, for integer t, $0 < t < 2n$,

$$\begin{aligned}\sum_{k=0}^{2n-1}\cos(t\theta_k) &= \Re\left(\sum_{k=0}^{2n-1}\exp(it\theta_k)\right)\\ &= \Re\left(\sum_{k=0}^{2n-1}\exp(it\pi/n)^k\right)\\ &= \Re\left(\frac{\exp(2it\pi)-1}{\exp(it\pi/n)-1}\right)\\ &= 0.\end{aligned}$$

Hence, by (3.6), for $0 \le j < m \le n$ (and consequently $0 < j+m < 2n$ and also $0 < |j-m| < 2n$), we have

$$\langle\cos(j\theta), \cos(m\theta)\rangle = 0.$$

Also, for $0 < j = m < n$, we have

$$\begin{aligned}\|\cos(j\theta)\|^2 &= \langle\cos(j\theta), \cos(j\theta)\rangle\\ &= \frac{1}{2n}\sum_{k=0}^{2n-1}(\cos(2j\theta_k)+1)\\ &= \frac{1}{2n}\left\{0 + \sum_{k=0}^{2n-1}1\right\} = 1.\end{aligned}$$

For $j = m$, we have

$$\begin{aligned}\|\frac{1}{\sqrt{2}}\cos(n\theta)\|^2 &= \frac{1}{4n}\sum_{k=0}^{2n-1}(\cos(2n\theta_k)+1)\\ &= \frac{1}{4n}\sum_{k=0}^{2n-1}(\cos(2k\pi)+1) = 1.\end{aligned}$$

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For $j = m = 0$, we have

$$\|\frac{1}{\sqrt{2}}\|^2 = \frac{1}{2n}\sum_{k=0}^{2n-1} 1 = 1.$$

The calculations with the sines are similar and so we omit them. □

With the basis B_n the columns of the Vandermonde matrix are

$$\vec{V}(\theta) := \begin{bmatrix} 1/\sqrt{2} \\ \cos(\theta) \\ \sin(\theta) \\ \cdot \\ \cdot \\ \sin((n-1)\theta) \\ \cos(n\theta)/\sqrt{2} \end{bmatrix} \in \mathbb{R}^{2n}.$$

Note that by Lemma 3.1, the *rows* of the matrix

$$V := [\vec{V}(\theta_0), \vec{V}(\theta_1), \cdots, \vec{V}(\theta_{2n-1})]$$

are orthogonal and have the same euclidean length. It follows that V is a constant multiple of an orthognal matrix so that the *column* vectors are also orthogonal. In other words,

$$\vec{V}(\theta_j) \perp \vec{V}(\theta_k) \qquad j \neq k.$$

Note also that

$$\begin{aligned} \|\vec{V}(\theta)\|_2^2 &= \frac{1}{2} + (n-1) + \frac{1}{2}\cos^2(n\theta) \\ &= \frac{2n-1}{2} + \frac{1}{2}\cos^2(n\theta), \end{aligned}$$

which is maximized precisely at $\theta = \theta_k$ for some k. Moreover,

$$\begin{aligned} \|\vec{V}(\theta_k)\|_2^2 &= \frac{2n-1}{2} + \frac{1}{2}\cos^2(k\pi) \\ &= 2n, \qquad k = 0, 1, \cdots, 2n-1. \end{aligned}$$

From these considerations it follows that the Algorithm, applied to the basis B_n and $\theta \in [0, 2\pi]$ will select the angles θ_k, $k = 0, 1, \cdots, 2n-1$, in *some* random order.

The projected points are then $\cos(k\pi/n)$, $0 \le k \le n$, which are precisely the so-called extended Chebyshev points. Hence the Algorithm again produces an excellent set of interpolation points.

Actually, from this setup it is easy to see why the Chebyshev points have small Lebesgue constant. Although this is somewhat tangential to our main purpose it is useful to record this here as it may also shed some light on the multivariate case.

The basic fact to note is that the trigonometric (span of B_n) fundamental Lagrange polynomial for θ_k is just

$$\ell_{\theta_k}(\theta) = \frac{1}{n} K_n(\theta, \theta_k), \tag{3.7}$$

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where $K_n(\theta,\phi)$ is the reproducing kernel for the span of B_n and inner-product (3.5). That this holds is an example of a more general phenomenon. Indeed, we have for any $p \in \mathrm{span}(B_n)$,

$$\begin{aligned} p(\theta_j) &= \langle p(\theta), K_n(\theta_j,\theta)\rangle \quad \text{(reproducing property)} \\ &= \frac{1}{n}\sum_{k=0}^{2n-1} p(\theta_k)K_n(\theta_j,\theta_k) \quad \text{(definition of (3.5))} \\ &= \sum_{k=0}^{2n-1} p(\theta_k)\left(\frac{1}{n}K_n(\theta_j,\theta_k)\right). \end{aligned}$$

Then since $p \in \mathrm{span}(B_n)$ was arbitrary, it follows that

$$\frac{1}{n}K_n(\theta_j,\theta_k) = \delta_{jk},$$

and hence $(1/n)K_n(\theta,\theta_k) \in \mathrm{span}(B_n)$ indeed coincides with the fundamental Lagrange polynomial $\ell_{\theta_k}(\theta)$.

We may easily compute K_n. Indeed, by Lemma 3.1, B_n is orthonormal with respect to the inner-product (3.5) and we have

$$\begin{aligned} K_n(\theta,\phi) &= \frac{1}{2} + \sum_{k=1}^{n-1}(\cos(k\theta)\cos(k\phi) + \sin(k\theta)\sin(k\phi)) + \frac{1}{2}\cos(n\theta)\cos(n\phi) \\ &= \frac{1}{2} + \sum_{k=1}^{n-1}\cos(k(\theta-\phi)) + \frac{1}{2}\cos(n\theta)\cos(n\phi). \end{aligned} \tag{3.8}$$

Now note that from this formula for K_n, together with (3.7) and the fact that $\sin(n\theta_k) = 0,\ \forall k$, it follows that

$$\ell_{\theta_k}(\theta) = \ell_{\theta_0}(\theta - \theta_k).$$

We may simplify

$$\begin{aligned} \ell_{\theta_0}(\theta) &= \frac{1}{n}K_n(\theta,0) \\ &= \frac{1}{n}\left(\frac{1}{2} + \sum_{k=1}^{2n-1}\cos(k\theta) + \frac{1}{2}\cos(n\theta)\right) \\ &= \frac{1}{2n}\frac{\sin(n\theta)}{\sin(\theta/2)}\cos(\theta/2) \end{aligned} \tag{3.9}$$

by a standard calculation.

From this it is easy to calculate

$$\sum_{k=0}^{2n-1} \ell_{\theta_k}^2(\theta) = \frac{4n-1+\cos(2n\theta)}{4n} \le 1$$

from which it follows that the Lebesgue function

$$\sum_{k=0}^{2n-1} |\ell_{\theta_k}(\theta)| \le \sqrt{2n}.$$

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A more refined, but standard, calculation shows that actually

$$\sum_{k=0}^{2n-1} |\ell_{\theta_k}(\theta)| = \sum_{k=0}^{2n-1} |\ell_{\theta_0}(\theta - \theta_k)| \le c\ln(n) \tag{3.10}$$

for some constant $c > 0$, whose exact value is not important here; see, e.g., [11, §1.3] for similar calculations.

Further, since

$$\begin{aligned}\theta_{2n-k} &= (2n-k)\pi/n \\ &= 2\pi - k\pi/n \\ &= 2\pi - \theta_k \\ &= -\theta_k \qquad \text{mod } 2\pi,\end{aligned}$$

we have

$$\begin{aligned}\ell_{\theta_{2n-k}}(\theta) &= \ell_{\theta_0}(\theta - \theta_{2n-k}) \\ &= \ell_{\theta_0}(\theta + \theta_k) \\ &= \ell_{\theta_0}(-\theta - \theta_k) \quad \text{(since } \ell_{\theta_0} \text{ is even)} \\ &= \ell_{\theta_k}(-\theta).\end{aligned}$$

We may use these symmetry properties to obtain a relation between the *trigonometric* fundamental Lagrange polynomials for the angles θ_k and the *algebraic* fundamental Lagrange polynomials for the points $x_k = \cos(\theta_k) \in [-1, 1]$. (We emphasize that there are exactly $n+1$ different x_k as $x_{2n-k} = x_k$, as is easily seen.)

Specifically, note that $\ell_{\theta_0}(\theta)$ is a combination of cosines only and hence even in θ. Therefore the substitution $x = \cos(\theta)$ results in an algebraic polynomial of degree n,

$$\mathcal{L}_{x_0}(x) := \ell_{\theta_0}(\theta), \quad x = \cos(\theta).$$

It is easy to check that

$$\mathcal{L}_{x_0}(x_k) = \ell_{\theta_0}(\theta_k) = \delta_{k0},$$

and hence $\mathcal{L}_{x_0}(x)$ is the algebraic Lagrange polynomial for x_0 among the $n+1$ points x_k.

Similarly, $\ell_{\theta_n}(\theta)$ is also even in θ and so

$$\mathcal{L}_{x_n}(x) := \ell_{\theta_n}(\theta), \quad x = \cos(\theta)$$

is the algebraic fundamental Lagrange polynomial corresponding to x_n.

The construction for $\mathcal{L}_{x_k}(x)$, $0 < k < n$ is slightly different. Although in this case $\ell_{\theta_k}(\theta)$ is *not* even, the combination

$$\ell_{\theta_k}(\theta) + \ell_{\theta_{2n-k}}(\theta) = \ell_{\theta_k}(\theta) + \ell_{\theta_k}(-\theta)$$

is even. It is easy then to check that

$$\mathcal{L}_{x_k}(x) = \ell_{\theta_k}(\theta) + \ell_{\theta_{2n-k}}(\theta), \quad x = \cos(\theta),\ 0 \le \theta \le \pi,$$

for $0 < k < n$.

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It follows that the algebraic Lebesgue function (for the extended Chebyshev points)

$$\begin{aligned}\sum_{k=0}^{n}|\mathcal{L}_{x_k}(x)| &= |\ell_{\theta_0}(\theta)| + \sum_{k=1}^{n-1}|\ell_{\theta_k}(\theta)+\ell_{\theta_{2n-k}}(\theta)| + |\ell_{\theta_n}(\theta)| \\ &\le \sum_{k=0}^{2n-1}|\ell_{\theta_k}(\theta)| \\ &\le c\ln(n) \quad \text{(by (3.10))}.\end{aligned}$$

3.3. Using a general orthonormal basis. Suppose that μ is a (regular) measure on $[-1,1]$ with associated inner-product

$$\langle f,g\rangle := \int_{-1}^{1} f(x)g(x)d\mu(x). \tag{3.11}$$

We consider an orthonormal basis

$$B_n := \{P_0, P_1, \cdots, P_n\} \subset \mathcal{P}_n([-1,1])$$

for the polynomials of degree at most n, $\mathcal{P}_n([-1,1])$.

The columns of the Vandermonde matrix are then

$$\vec{V}(x) = \begin{bmatrix} P_0(x) \\ P_1(x) \\ \cdot \\ \cdot \\ P_n(x)\end{bmatrix}$$

with norm (squared)

$$\|\vec{V}(x)\|_2^2 = \sum_{k=0}^{n} P_k^2(x).$$

Note that this just the diagonal of the reproducing kernel for the space $\mathcal{P}_n([-1,1])$ with inner-product (3.11), sometimes also referred to as the reciprocal of the associated Christoffel function. To emphasize this relation we set

$$K_n(x) := \sum_{k=0}^{n} P_k^2(x),$$

so that

$$\|\vec{V}(x)\|_2 = \sqrt{K_n(x)}.$$

To the measure μ is associated the corresponding Gauss-Christoffel quadrature formula. Specifically, if $a_0, a_1 \cdots, a_n \in (-1,1)$ are the zeros of $P_{n+1}(x)$, then, for all polynomials $Q(x)$ of degree at most $2n+1$,

$$\int_{-1}^{1} Q(x)d\mu(x) = \sum_{k=0}^{n} w_k Q(a_k), \tag{3.12}$$

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where the weights are given by

$$w_k = \frac{1}{K_n(a_k)}.$$

Now consider the normalized columns

$$\vec{U}(x) := \frac{1}{\|\vec{V}\|_2}\vec{V}(x) = \frac{1}{\sqrt{K_n(x)}}\begin{bmatrix} P_0(x) \\ P_1(x) \\ \cdot \\ \cdot \\ P_n(x) \end{bmatrix}.$$

Specializing to $x = a_k$, we have

$$\vec{U}(a_k) = \frac{1}{\sqrt{K_n(a_k)}}\begin{bmatrix} P_0(a_k) \\ P_1(a_k) \\ \cdot \\ \cdot \\ P_n(a_k) \end{bmatrix} = \sqrt{w_k}\begin{bmatrix} P_0(a_k) \\ P_1(a_k) \\ \cdot \\ \cdot \\ P_n(a_k) \end{bmatrix}.$$

Define the matrix

$$U = [\vec{U}(a_0), \vec{U}(a_1), \cdots, \vec{U}(a_n)] \in \mathbb{R}^{(n+1)\times(n+1)}.$$

The ith row of U is the vector

$$U_i := [\sqrt{w_0}P_i(a_0), \sqrt{w_1}P_i(a_1), \cdots, \sqrt{w_n}P_i(a_n)],$$

so that the euclidean inner-product of U_i and U_j, by Gauss-Christoffel quadrature (3.12), is

$$\begin{aligned} U_i \cdot U_j &= \sum_{k=0}^{n} w_k P_i(a_k)P_j(a_k) \\ &= \int_{-1}^{1} P_i(x)P_j(x)d\mu(x) \quad \text{(since deg}(P_iP_j) \le 2n) \\ &= \delta_{ij}, \quad \text{(the Kronecker delta)} \end{aligned}$$

since the polynomials P_i are orthonormal.

In other words, the rows of the matrix U are orthonormal vectors. It follows that U is an orthogonal matrix and that also the columns of U are orthonormal vectors, i.e.,

$$\vec{U}(a_i) \cdot \vec{U}(a_j) = \delta_{ij}.$$

Now, if we apply the Sommariva-Vianello algorithm to the normalized colums, $\vec{U}(x)$, all of length 1, the first point chosen will be random, as there is no way for the Algorithm to distinguish first points. However, if by some means, we set the first point to $x_0 = a_j$ for some j, $0 \le j \le n$, then since the vectors $\vec{U}(a_i)$, $i \ne j$, are orthogonal to $\vec{U}(a_j)$ (and to each other), the remaining n points selected by the algorithm will be just the a_i, $i \ne j$, in some random order.

To summarize, if the Sommariva-Vianello algorithm, using an orthonormal basis, is applied to normalized columns *and* the first point is (artificially) set to one of the Gauss-Christoffel quadrature points, then the Algorithm selects precisely the Gauus-Christoffel quadrature points associated to the measure of orthogonality. We remark that if, on the other hand, the first point is *not* so set, the first point would be randomly chosen in the interval and would in general be a poor choice (although the algorithm would eventually self-correct).

3.3.1. An illustrative example. When we normalize the columns to have length one, it is also possible that the Algorithm selects good interpolation points, other than the Gauss-Christoffel points, depending on how the first point is set. For example, consider the measure

$$d\mu = \frac{1}{\sqrt{1-x^2}}dx.$$

The orthonormal polynomials with respect to this measure are the Chebyshev polynomials of the first kind. Specifically,

$$B_n = \sqrt{\frac{2}{\pi}}\{\frac{1}{\sqrt{2}}, T_1(x), T_2(x), \cdots, T_n(x)\},$$

so that

$$\vec{V}(x) = \sqrt{\frac{2}{\pi}}\begin{bmatrix} 1/\sqrt{2} \\ T_1(x) \\ T_2(x) \\ \cdot \\ \cdot \\ T_n(x) \end{bmatrix}.$$

If we set $x = \cos(\theta)$ and $y = \cos(\phi)$, then we may compute

$$\begin{aligned} \vec{V}(x)\cdot\vec{V}(y) &= \frac{2}{\pi}\left\{\frac{1}{2} + \sum_{k=0}^{n} T_k(x)T_k(y)\right\} \\ &= \frac{2}{\pi}\left\{\frac{1}{2} + \sum_{k=0}^{n} \cos(k\theta)\cos(k\phi)\right\} \qquad (3.13) \\ &= \frac{1}{2\pi}\left\{\frac{\sin((n+1/2)(\theta-\phi))}{\sin(\frac{\theta-\phi}{2})} + \frac{\sin((n+1/2)(\theta+\phi))}{\sin(\frac{\theta+\phi}{2})}\right\}, \end{aligned}$$

by a standard calculation.

We note that the Gauss-Christoffel quadrature points are the zeros of $T_{n+1}(x)$, i.e.,

$$a_k = \cos(\frac{2k+1}{2(n+1)}\pi), \qquad k = 0, 1, \cdots, n.$$

By the general theory of the previous section (or else by direct calculation), it follows that $\{\vec{V}(a_k)\}$ is a set of orthogonal vectors.

But notice that by (3.13), if we set

$$b_k = \cos(\frac{2k}{2n+1}\pi), \qquad k = 0, 1, \cdots, n,$$

then the set of vectors

$$\left\{\vec{V}(b_k) \,:\, k = 0, 1, \cdots, n\right\}$$

is also orthogonal. It follows that if we (artificially) set the first point to $x_0 = b_0 = \cos(0) = 1$, a not unnatural choice, then the Algorithm applied to normalized colums will select the points b_k in some random order.

Note, however, that these selected points are *not* symmetric, specificially, $x = 1$ is included, but $x = -1$ is not. We record, for comparison's sake, that for 21 points (degree 20), the Vandermonde determinant is approximately $5.796 \cdot 10^{10}$ and the Lebesgue constant approximately 3.3.

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3.4. Gram determinants and other bases. The volume of the parallelotope generated by the vectors $\vec{V}_1, \vec{V}_2, \cdots, \vec{V}_k \in \mathbb{R}^n$ is given by the associated Gram determinant. Specifically,

$$\begin{aligned}\mathrm{Vol}^2(\vec{V}_1, \vec{V}_2, \cdots, \vec{V}_k) &= \det\left(\left[\vec{V}_i \cdot \vec{V}_j\right]_{1\le i,j\le k}\right) \\ &=: g(\vec{V}_1, \vec{V}_2, \cdots, \vec{V}_k).\end{aligned}$$

For more information on Gram determinants, see, for example, [6, §8.7].

The continuous version of the Sommariva-Vianello algorithm can then be re-phrased as

(1) The first point $x_1 \in K$ is chosen to maximize $\|\vec{V}(x)\|_2$.

(2) Given $x_1, x_2, \cdots, x_k$ the $(k+1)$st point $x_{k+1} \in K$ is chosen so that the Gram determinant,

$$g(\vec{V}(x_1), \vec{V}(x_2), \cdots, \vec{V}(x_k), \vec{V}(x_{k+1})),$$

is as large as possible.

3.4.1. The standard basis. In this section we briefly consider the use of the standard polynomial basis,

$$B_n := \{1, x, x^2, \cdots, x^{n-1}\}.$$

In this case the columns of the Vandermonde matrix are

$$\vec{V}(x) = \begin{bmatrix} 1 \\ x \\ x^2 \\ \cdot \\ \cdot \\ x^{n-1} \end{bmatrix}.$$

Then we compute

$$\vec{V}(x) \cdot \vec{V}(y) = \sum_{k=0}^{n-1} x^k y^k = \begin{cases} \frac{(xy)^n - 1}{xy - 1} & : \quad xy \neq 1 \\ n & : \quad xy = 1 \end{cases}.$$

In particular, for $x = y$,

$$\|\vec{V}(x)\|^2 = \sum_{k=0}^{n-1} x^{2k} = \begin{cases} \frac{x^{2n} - 1}{x^2 - 1} & : \quad x \neq \pm 1 \\ n & : \quad x = \pm 1 \end{cases}.$$

Clearly, $\|\vec{V}(x)\|$ is maximized for $x = \pm 1$ and the algorithm will choose one of these at random. Let us suppose without loss of generality that the first point chosen is $x_1 = +1$. In general, it is difficult to calculate the precise points that the algorithm chooses. For n even it is not too difficult to show that the second point is $x_2 = -1$. Specifically, the second point will be the one for which $g(\vec{V}(1), \vec{V}(x))$ is maximal. But we calculate

$$\begin{aligned} g(\vec{V}(1), \vec{V}(x)) &= \begin{vmatrix} \vec{V}(1)\cdot\vec{V}(1) & \vec{V}(1)\cdot\vec{V}(x) \\ \vec{V}(x)\cdot\vec{V}(1) & \vec{V}(x)\cdot\vec{V}(x) \end{vmatrix} \\ &= \begin{vmatrix} n & \vec{V}(1)\cdot\vec{V}(x) \\ \vec{V}(x)\cdot\vec{V}(1) & \vec{V}(x)\cdot\vec{V}(x) \end{vmatrix} \\ &= n\|\vec{V}(x)\|^2 - (\vec{V}(x)\cdot\vec{V}(1))^2. \end{aligned}$$

We claim that, for n even, this is maximized for $x = -1$. To see this, first note that then

$$\vec{V}(-1)\cdot\vec{V}(1) = \frac{(-1)^n - 1}{-1-1} = 0$$

and hence, for all $x \in [-1,1]$,

$$\begin{aligned} g(\vec{V}(1),\vec{V}(x)) &= n\|\vec{V}(x)\|^2 - (\vec{V}(x)\cdot\vec{V}(1))^2 \\ &\le n\|\vec{V}(x)\|^2 \\ &\le n^2 \qquad (\text{since } \|\vec{V}(x)\|^2 \le n) \\ &= n\|\vec{V}(-1)\|^2 - (\vec{V}(-1)\cdot\vec{V}(1))^2 \\ &= g(\vec{V}(1),\vec{V}(-1)). \end{aligned}$$

In other words, $g(\vec{V}(1),\vec{V}(x))$ is indeed maximized for $x = -1$ and $x_2 = -1$ will be the second point chosen.

If n is odd, then

$$\vec{V}(-1)\cdot\vec{V}(1) = \frac{(-1)^n - 1}{-1-1} = 1 \neq 0,$$

and already proving that $x_2 = -1$ (although we conjecture that is the case) seems to be a bit difficult.

To see the effect of the basis, we will compute the four points chosen for $n = 4$ using both the standard and Chebyshev basis.

Continuing with the standard basis, since $n = 4$ is even we already know that $x_1 = +1$ (by choice) and $x_2 = -1$. Let us compute x_3. This will be the point for which $g(\vec{V}(1),\vec{V}(-1),\vec{V}(x))$ is maximized. But

$$\begin{aligned} g(\vec{V}(1),\vec{V}(-1),\vec{V}(x)) &= \begin{vmatrix} \vec{V}(1)\cdot\vec{V}(1) & \vec{V}(1)\cdot\vec{V}(-1) & \vec{V}(1)\cdot\vec{V}(x) \\ \vec{V}(-1)\cdot\vec{V}(1) & \vec{V}(-1)\cdot\vec{V}(-1) & \vec{V}(-1)\cdot\vec{V}(x) \\ \vec{V}(x)\cdot\vec{V}(1) & \vec{V}(x)\cdot\vec{V}(-1) & \vec{V}(x)\cdot\vec{V}(x) \end{vmatrix} \\ &= \begin{vmatrix} 4 & 0 & \frac{1-x^4}{1-x} \\ 0 & 4 & \frac{1-x^4}{1+x} \\ \frac{1-x^4}{1-x} & \frac{1-x^4}{1+x} & \frac{1-x^8}{1-x^2} \end{vmatrix} \\ &= 8(1-x^4)(1-x^2). \end{aligned}$$

The derivative of this expression is

$$\frac{d}{dx}8(1-x^4)(1-x^2) = 16x(x^2-1)(3x^2+1),$$

and hence the Gram determinant is maximized at $x = 0$ and $x_3 = 0$.

To find the fourth point, we must maximize

$$\begin{aligned} &g(\vec{V}(1),\vec{V}(-1),\vec{V}(\;0\;),\vec{V}(x)) \\ &\quad = \begin{vmatrix} \vec{V}(1)\cdot\vec{V}(1) & \vec{V}(1)\cdot\vec{V}(-1) & \vec{V}(1)\cdot\vec{V}(0) & \vec{V}(1)\cdot\vec{V}(x) \\ \vec{V}(-1)\cdot\vec{V}(1) & \vec{V}(-1)\cdot\vec{V}(-1) & \vec{V}(-1)\cdot\vec{V}(0) & \vec{V}(-1)\cdot\vec{V}(x) \\ \vec{V}(0)\cdot\vec{V}(1) & \vec{V}(0)\cdot\vec{V}(-1) & \vec{V}(0)\cdot\vec{V}(0) & \vec{V}(0)\cdot\vec{V}(x) \\ \vec{V}(x)\cdot\vec{V}(1) & \vec{V}(x)\cdot\vec{V}(-1) & \vec{V}(x)\cdot\vec{V}(0) & \vec{V}(x)\cdot\vec{V}(x) \end{vmatrix} \end{aligned}$$

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$$= \begin{vmatrix} 4 & 0 & 1 & \frac{1-x^4}{1-x} \\ 0 & 4 & 1 & \frac{1-x^4}{1+x} \\ 1 & 1 & 1 & 1 \\ \frac{1-x^4}{1-x} & \frac{1-x^4}{1+x} & 1 & \frac{1-x^8}{1-x^2} \end{vmatrix}$$
$$= 4x^2(x^2-1)^2.$$

The derivative of this determinant is

$$\frac{d}{dx}4x^2(x^2-1)^2 = 8x(x^2-1)(3x^2-1).$$

Hence the fourth point is $x_4 = \pm 1/\sqrt{3}$. In summary, the four points for $n = 4$ and the standard basis are

$$\{1, -1, 0, \pm 1/\sqrt{3}\}$$

which are not particularly good for interpolation.

Now let us compute the points for $n = 4$ using the Chebyshev basis

$$B_4 = \{T_0(x), T_1(x), T_2(x), T_3(x)\},$$

so that

$$\vec{V}(x) = \begin{bmatrix} T_0(x) \\ T_1(x) \\ T_2(x) \\ T_3(x) \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ 2x^2-1 \\ 4x^3-3x \end{bmatrix}.$$

It is easy to check that again $x_1 = 1$ (by choice) and $x_2 = -1$. To find the third point x_3, we must maximize

$$g(\vec{V}(1), \vec{V}(-1), \vec{V}(x)) = \begin{vmatrix} \vec{V}(1)\cdot\vec{V}(1) & \vec{V}(1)\cdot\vec{V}(-1) & \vec{V}(1)\cdot\vec{V}(x) \\ \vec{V}(-1)\cdot\vec{V}(1) & \vec{V}(-1)\cdot\vec{V}(-1) & \vec{V}(-1)\cdot\vec{V}(x) \\ \vec{V}(x)\cdot\vec{V}(1) & \vec{V}(x)\cdot\vec{V}(-1) & \vec{V}(x)\cdot\vec{V}(x) \end{vmatrix}$$
$$= \begin{vmatrix} 4 & 0 & 4x^3+2x^2-2x \\ 0 & 4 & -4x^3+2x^2+2x \\ 4x^3+2x^2-2x & -4x^3+2x^2+2x & 16x^6-20x^4+6x^2+2 \end{vmatrix}$$
$$= 32(x^2-1)^2(4x^2+1).$$

The derivative of this determinant is

$$\frac{d}{dx}32(x^2-1)^2(4x^2+1) = 128x(x^2-1)(6x^2-1).$$

Hence the third point will be $x_3 = \pm 1/\sqrt{6} = .4082482...$

If we take $x_3 = +1/\sqrt{6}$, then, after some tedious calculations, we find that the fourth point $x_4 = (\sqrt{6} - \sqrt{114})/18 = -.457088...$

For comparison's sake, the Fekete points are

$$\{-1, -\frac{1}{\sqrt{5}}, +\frac{1}{\sqrt{5}}, +1\} = \{-1, -.4472135954, +.4472135954, +1\},$$

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and we see that using the Chebyshev basis has produced a better result than for the standard basis.

We mention in passing that for the interval there is always an artificial basis for which the Algorithm selects precisely the true Fekete points. This is, of course, not very useful as one must know the true Fekete points a priori to construct the basis.

PROPOSITION 3.2. *Suppose that $K=[-1,1]$ and that B_n is the Lagrange basis based at the true Fekete points for K. Then the Sommariva-Vianello algorithm will select the true Fekete points in some order.*

Proof. Let us denote the set of true Fekete points by

$$F_n=\{a_0,a_1,\cdots,a_n\}.$$

Then,

$$\vec{V}(x)=\begin{bmatrix}\ell_0(x)\\ \ell_1(x)\\ \cdot\\ \cdot\\ \ell_n(x)\end{bmatrix},$$

so that $\vec{V}(a_j)=\vec{e}_j$, the standard basis vector. Hence the vectors $\vec{V}(a_j)$ are mutually orthogonal. Moreover, by Féjer's Theorem (cf. [3]),

$$\|\vec{V}(x)\|_2^2=\sum_{k=0}^{n}\ell_k^2(x)\le 1,\quad x\in[-1,1],$$

and $\|\vec{V}(x)\|_2=1$ only for $x\in F_n$. The result folows. □

We remark that the same result holds in several variables for any compact set K and degree n with the property that the sum of the squares of the fundamental Lagrange polynomials based at the Fekete points is at most one, and attains 1 only at the Fekete points (an unfortunately rare occurence; see, e.g., [3] for a discussion of this problem).

Of course, in applications, it will be important to select the "right" basis. In general, we would suggest the use of a basis of polynomials orthonormal with respect to the so-called equilibrium measure for K of potential theory. (For $K=[-1,1]$ this would be $\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx$ and the orthonormal polynomials are (multiples) of the Chebyshev polynomials; for K the unit circle this measure is just $\frac{1}{2\pi}d\theta$ and the orthonormal polynomials are (multiples of) the trigonometric polynomials.) However, for general K this measure (and the associated polynomials) are difficult to calculate. Sommariva and Vianello [12] discuss a form of "iterative improvement" that adjusts the basis dynamically.

4. A convergence theorem for $K\subset\mathbb{C}$. In this section, we work in the complex plane $\mathbb{C}$. Although the basis used does have a strong influence on the points the Algorithm selects, it turns out that *asymptotically* the points will have the same distribution, that of the true Fekete points.

For $x_1,x_2,\cdots,x_{n+1}\in\mathbb{C}$, we let

$$\mathrm{vdm}(x_1,x_2,\cdots,x_{n+1}):=\prod_{i<j}(x_j-x_i)$$

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denote the Vandermonde determinant for these points with respect to the *standard* basis. For $K \subset \mathbb{C}$ compact, we call

$$\tau(K) := \lim_{n\to\infty} \max_{z_1,z_2,\cdots,z_{n+1}\in K} |\text{vdm}(z_1, z_2, \cdots, z_{n+1})|^{1/\binom{n+1}{2}}$$

the transfinite diameter of K. Thus if $\{f_1^{(n)}, f_2^{(n)}, \cdots, f_{n+1}^{(n)}\}$ is a set of true Fekete points of degree n for K,

$$\lim_{n\to\infty} |\text{vdm}(f_1^{(n)}, f_2^{(n)}, \cdots, f_{n+1}^{(n)})|^{1/\binom{n+1}{2}} = \tau(K).$$

We formulate a general convergence theorem.

THEOREM 4.1. *Suppose that $K = \cup_i K_i \subset \mathbb{C}$ is a finite union of continua (i.e., each K_i is compact and connected, not a single point). Suppose further that $\phi : \mathbb{Z}_+ \to \mathbb{R}$ has the property that*

$$\lim_{n\to\infty} n^2\phi(n) = 0 \tag{4.1}$$

and that $A_n \subset K$, $n = 1, 2, \cdots$, are discrete subsets of K such that for all $x \in K$,

$$\min_{a\in A_n} |x - a| \le \phi(n).$$

Then for any basis the points $b_1, ..., b_{n+1} \in A_n \subset K$ generated by the Sommariva-Vianello Algorithm based on points in A_n satisfy

$$\lim_{n\to\infty} |vdm(b_1, ..., b_{n+1})|^{\frac{1}{\binom{n+1}{2}}} = \tau(K)$$

and the discrete probability measures $\mu_n := \frac{1}{n+1}\sum_{j=1}^{n+1} \delta_{b_j}$ converge weak- to the potential-theoretic equilibrium measure $d\mu_K$ of K.*

REMARK 4.2. For $K = [-1, 1]$, $d\mu_{[-1,1]} = \frac{1}{\sqrt{1-x^2}}dx$; for K the unit circle S^1, $d\mu_{S^1} = \frac{1}{2\pi}d\theta$. We refer the reader to [9] for more on complex potential theory. As an example of the condition (4.1), if $K = [-1, 1]$, A_n consisting of order $n^{2+\epsilon}$, $\epsilon > 0$, equally spaced points would suffice.

Proof. Suppose that $\{f_1^{(n)}, f_2^{(n)}, \cdots, f_{n+1}^{(n)}\}$ is a set of true Fekete points of degree n for K. Let then $\{a_1, a_2, \cdots, a_{n+1}\} \subset A_n$ be such that

$$|a_i - f_i^{(n)}| \le \phi(n), \qquad i = 1, \cdots, n+1,$$

the existence of which is guaranteed by our hypotheses.

By a result of Kövari and Pommerenke [8] it follows that there is a constant $c_1 > 0$ such that

$$|f_i^{(n)} - f_j^{(n)}| \ge \frac{c_1}{n^2}, \quad i \ne j. \tag{4.2}$$

Consequently, for $i \ne j$,

$$\begin{aligned} |a_i - a_j| &= |(a_i - f_i^{(n)}) + (f_i^{(n)} - f_j^{(n)}) + (f_j^{(n)} - a_j)| \\ &\ge |f_i^{(n)} - f_j^{(n)}| - |f_i^{(n)} - a_i| - |f_j^{(n)} - a_j| \\ &\ge \frac{c_1}{n^2} - 2\phi(n) \\ &\ge \frac{c_2}{n^2} \quad \text{for some } c_2 > 0, \end{aligned} \tag{4.3}$$

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since $\phi(n) = o(1/n^2)$ by (4.1).

We will first show that

$$\lim_{n\to\infty} |\mathrm{vdm}(a_1, a_2, \cdots, a_{n+1})|^{1/\binom{n+1}{2}} = \tau(K). \tag{4.4}$$

To see this, first note that

$$|\mathrm{vdm}(a_1, a_2, \cdots, a_{n+1})| \le |\mathrm{vdm}(f_1^{(n)}, f_2^{(n)}, \cdots, f_{n+1}^{(n)})|$$

by the definition of Fekete points.

Also,

$$\begin{aligned}
|\mathrm{vdm}(f_1^{(n)}, f_2^{(n)}, \cdots, f_{n+1}^{(n)})| &= \prod_{i<j} |f_j^{(n)} - f_i^{(n)}| \\
&= \prod_{i<j} |(f_j^{(n)} - a_j) - (f_i^{(n)} - a_i) + (a_j - a_i)| \\
&\le \prod_{i<j} \left\{ |f_j^{(n)} - a_j| + |f_i^{(n)} - a_i| + |a_j - a_i)| \right\} \\
&\le \prod_{i<j} |a_j - a_i| \prod_{i<j} \left\{ 1 + \frac{|f_j^{(n)} - a_j| + |f_i^{(n)} - a_i|}{|a_j - a_i|} \right\} \\
&\le \left(\prod_{i<j} |a_j - a_i| \right) \prod_{i<j} \left\{ 1 + \frac{2\phi(n)}{c_2/n^2} \right\} \\
&= \left(\prod_{i<j} |a_j - a_i| \right) \prod_{i<j} \left\{ 1 + \frac{2}{c_2} n^2 \phi(n) \right\} \\
&= |\mathrm{vdm}(a_1, a_2, \cdots, a_{n+1})| (1 + c_3 n^2 \phi(n))^{\binom{n+1}{2}},
\end{aligned}$$

where we have set $c_3 := 2/c_2 > 0$.

Hence, we have

$$\begin{aligned}
(1 + c_3 n^2 \phi(n))^{-\binom{n+1}{2}} |\mathrm{vdm}(f_1^{(n)}, \cdots, f_{n+1}^{(n)})| &\le |\mathrm{vdm}(a_1, \cdots, a_{n+1})| \\
&\le |\mathrm{vdm}(f_1^{(n)}, \cdots, f_{n+1}^{(n)})|
\end{aligned}$$

and then (4.4) follows by taking $1/\binom{n+1}{2}$ roots, and using (4.1).

Continuing, suppose that the Sommariva-Vianello Algorithm selects the points $b_1, b_2 \cdots, b_{n+1} \in A_n \subset K$. Then Çivril and Magdon-Ismail [5] have shown that

$$|\mathrm{vdm}(b_1, b_2, \cdots, b_{n+1})| \ge \frac{1}{(n+1)!} |\mathrm{vdm}(a_1^*, a_2^*, \cdots, a_{n+1}^*)|,$$

where $a_1^*, \cdots, a_{n+1}^*$ are the points among A_n which maximize the Vandermonde determinant. Note that such an inequality is independent of the basis used in calculating the determinant. Hence,

$$\begin{aligned}
|\mathrm{vdm}(f_1^{(n)}, f_2^{(n)}, \cdots, f_{n+1}^{(n)})| &\ge |\mathrm{vdm}(b_1, b_2, \cdots, b_{n+1})| \\
&\ge \frac{1}{(n+1)!} |\mathrm{vdm}(a_1^*, a_2^*, \cdots, a_{n+1}^*)| \\
&\ge \frac{1}{(n+1)!} |\mathrm{vdm}(a_1, a_2, \cdots, a_{n+1})|,
\end{aligned}$$

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by the definition of the a_j^*. Thus,

$$\lim_{n\to\infty} |\mathrm{vdm}(b_1, b_2, \cdots, b_{n+1})|^{1/\binom{n+1}{2}} = \tau(K)$$

as

$$\lim_{n\to\infty} (n+1)!^{1/\binom{n+1}{2}} = 1.$$

The final statement, that μ_n converges weak-* to $d\mu_K$, then follows by [2, Theorem 1.5]. ◻

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