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AN ALGEBRA OF INTEGRAL OPERATORS*

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Abstract. We introduce an algebra of integral operators related to a model of the q-harmonic oscillator and investigate some of its properties.

Key words. integral operators, divided difference operators, the continuous q-Hermite polynomials, generating functions, Poisson kernel, bilinear generating functions, q-harmonic oscillators

AMS subject classifications. 33D45, 42C10, 45E10

1. Introduction. In this report a unification of the basic analog of Fourier transform and inverses of the Askey–Wilson divided difference operators will be given in a form of certain algebraic structure related to a model of the q-harmonic oscillator. We present here only the summary of results; a paper with detailed proofs will appear elsewhere [53].

To be more specific, let us consider a q-quadratic lattice of the form $x = \left(q^s + q^{-s}\right)/2$ with $q^s = e^{i\theta}$ and let us introduce the symmetric difference operator as

$$\delta f\left(x\left(s\right)\right) = f\left(x\left(s+1/2\right)\right) - f\left(x\left(s-1/2\right)\right).$$

The first order *Askey–Wilson divided difference operator* is given by

$$\mathcal{D}_q f\left(x\right) := \frac{\delta f\left(x\right)}{\delta x} = \frac{f\left(x\left(s+1/2\right)\right) - f\left(x\left(s-1/2\right)\right)}{x\left(s+1/2\right) - x\left(s-1/2\right)}. \tag{1.1}$$

Several "right" inverses $\mathcal{D}_q^{-1}$ of the Askey–Wilson divided difference operator, such that $\mathcal{D}_q^{-1}$ $\mathcal{D}_q = I$ and I is the identity operator, were constructed in [20, 31, 33],

It was Dick Askey who realized that Wiener's treatment of the Fourier integrals [59] contains the key to q-extensions [8, 11, 41]. Generalizing Wiener's method to the level of the Askey–Wilson polynomials one can introduce a set of one-parameter integral operators which resemble raising and lowering operators. These operators obey an interesting algebraic structure which allows to obtain one-sided inverses of the divided difference operators of the first order [20, 31, 33], and to find the resolvents of the second order Askey–Wilson operators in different spaces of functions. The aim of the present note and its extended version [53] is to consider the simplest case related to the continuous q-Hermite polynomials; a more general case including the Askey–Wilson polynomials will be discussed later; see also [49] and [50] for an extension of the Askey–Wilson polynomials orthogonality to a certain class of ${}_8\varphi_7$ functions.

The paper is organized as follows. In §1 to §4 we remind the reader basic facts about the continuous q-Hermite polynomials and consider a model of q-harmonic oscillator in terms of these polynomials. In §5 to §7 we introduce a family of one parameter integral operators, which extend rasing and lowering operators, and investigate some properties of these operators, their adjoints and inverses in a framework of a single algebraic structure. An analog

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of the q-Fourier transform is briefly discussed in §8. An explicit realization of the number operator in this model of the q-oscillator terms of Hadamard's principal values integral is outlined in §9. Inverses of the first order Askey–Wilson operators are constructed in §10. In conclusion, the resolvent and Green's function of the corresponding q-Hamiltonian are found in §11. More details can be found in the forthcoming paper [53].

2. Continuous q-Hermite Polynomials. Although the continuous q-Hermite polynomials were originally introduced by Rogers [42], [43], [44], their orthogonality relation and asymptotic properties had been established only recently by Allaway [2], Al-Salam and Chihara [4], and Askey and Ismail [9], [10]. These polynomials are given by

$$H_n(x|q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} e^{i(n-2k)\theta}, \qquad x = \cos\theta \tag{2.1}$$

and the continuous orthogonality relation is [2], [9], [10]

$$\begin{aligned} \int_0^\pi H_m(\cos\theta|q)\, H_n(\cos\theta|q) \left(e^{2i\theta}, e^{-2i\theta}; q\right)_\infty d\theta \\ = 2\pi \frac{(q;q)_n}{(q;q)_\infty}\ \delta_{mn}, \end{aligned} \tag{2.2}$$

or

$$\int_{-1}^{1} H_n(x|q)\, H_m(x|q)\ \rho(x)\ dx = d_n^2\, \delta_{mn}, \tag{2.3}$$

where the weight function is

$$\rho(x) = 4\left(1-x^2\right)^{1/2} \prod_{k=0}^{\infty} \left(1 + 2\left(1-2x^2\right) q^k + q^{2k}\right) \tag{2.4}$$

and the $\mathcal{L}^2$-norm is given by

$$d_n^2 = 2\pi \frac{(q;q)_n}{(q;q)_\infty}. \tag{2.5}$$

The continuous q-Hermite polynomials (2.1) obey a very important property, namely, the action of the Askey–Wilson divided difference operator (1.1) on $H_n(x|q)$ results in the same polynomial of the lower degree,

$$\frac{\delta}{\delta x} H_n(x|q) = 2q^{(1-n)/2} \frac{1-q^n}{1-q} H_{n-1}(x|q),$$

which is a q-analog of the familiar formula $H_n'(x) = 2nH_{n-1}(x)$. It is worth also noting that the continuous q-Hermite polynomials are the simplest special case of the fundamental Askey-Wilson polynomials $p_n(x;a,b,c,d)$ [14] corresponding to the zero-valued parameters, $H_n(x|q) = p_n(x;0,0,0,0)$. They satisfy a second-order difference equation and have the Rodrigues-type formula among other properties; see, for example, [5], [14], [16], [18], [30], [32], [40], and [47] for more details.

3. Bilinear Generating Functions. There are several important generating functions for the continuous q-Hermite polynomials; see, for example, [6], [30], [48], and [51]. The Poisson kernel of Rogers, or the q-Mehler formula, is one of them

$$T_t\left(x,y\right) := \sum_{n=0}^{\infty} \frac{t^n}{d_n^2}\, H_n\left(x|q\right) H_n\left(y|q\right) = \frac{\left(q,t^2;q\right)_\infty}{2\pi\left(te^{i\theta+i\varphi},te^{i\theta-i\varphi},te^{i\varphi-i\theta},te^{-i\theta-i\varphi};q\right)_\infty} \tag{3.1}$$

with $x=\cos\theta$, $y=\cos\varphi$, $|t|<1$ and d_n^2 defined by (2.5). A related kernel is

$$L_t(x,y) := \sum_{n=0}^{\infty} \frac{t^n}{d_n^2}\, H_n\left(x|q\right) H_{n+1}\left(y|q\right) = \frac{\left(y-tx\right)\left(q,qt^2;q\right)_\infty}{\pi\left(te^{i\theta+i\varphi},te^{i\theta-i\varphi},te^{i\varphi-i\theta},te^{-i\theta-i\varphi};q\right)_\infty}. \tag{3.2}$$

Both kernels (3.1) and (3.2) are special cases $k=0$ and $k=1$, respectively, of a more general Carlitz's formula,

$$\sum_{n=0}^{\infty} \frac{t^n}{\left(q;q\right)_n}\, H_n\left(x|q\right) H_{n+k}\left(y|q\right) = \frac{\left(t^2q^k;q\right)_\infty \left(q;q\right)_k\, p_k\left(\cos\varphi;te^{i\theta},te^{-i\theta}\right)}{\left(te^{i\theta+i\varphi},te^{i\theta-i\varphi},te^{i\varphi-i\theta},te^{-i\theta-i\varphi};q\right)_\infty}, \tag{3.3}$$

where

$$p_k\left(\cos\varphi;a,b\right) = \frac{\left(ab;q\right)_k}{a^k\left(q;q\right)_k}\, {}_3\varphi_2\left(\begin{matrix} q^{-k},\ ae^{i\varphi},\ ae^{-i\varphi} \\ ab,\quad 0 \end{matrix}\ ;\ q,q\right)$$

are the Al-Salam and Chihara polynomials; see, for example, [14] and [34]. Carlitz [22] derived (3.3) using series manipulations; Al-Salam and Ismail [5] gave another proof using the fact that the continuous q-Hermite polynomials are the moments of the distribution function of the Al-Salam and Carlitz polynomials [3].

Let us also consider another related kernel

$$M_t(x,y) := \sum_{m=0}^{\infty} H_m(x|q)\, H_{m+1}(y|q)\, \frac{t^m}{d_{m+1}^2} = \sum_{n=0}^{\infty} q^n L_{tq^n}(x,y) \tag{3.4}$$

and introduce the generalizations of the T, L and M kernels as follows:

$$L_t^{(k)}(x,y) := \sum_{n=0}^{\infty} \frac{t^n}{d_n^2}\, H_n(x|q)\, H_{n+k}(y|q) = \frac{\left(q,t^2q^k;q\right)_\infty (q;q)_k\, p_k\left(y;te^{i\theta},te^{-i\theta}\right)}{2\pi\left(te^{i\theta+i\varphi},te^{i\theta-i\varphi},te^{i\varphi-i\theta},te^{-i\theta-i\varphi};q\right)_\infty} \tag{3.5}$$

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and

$$M_t^{(k)}(x,y) := \sum_{m=0}^{\infty} H_m(x|q)\, H_{m+k}(y|q)\, \frac{t^m}{d_{m+k}^2} = \sum_{n=0}^{\infty} q^n\, \frac{(q^k;q)_n}{(q;q)_n}\, L_{tq^n}^{(k)}(x,y). \tag{3.6}$$

With the help of these kernels (3.1), (3.2), (3.4)–(3.6) we shall introduce in this paper a family of integral operators related to the so-called q-Heisenberg algebra.

4. The q-Heisenberg Algebra. Recent advances in quantum groups has led to a study of the so-called q-harmonic oscillators, originally introduced by Arik and Coon [7] and then rediscovered by Biedenharn [19] and Macfarline [38]; see, for example, [12], [13], [17], [27], [28], [29], [25], [26], [57], [61], and references therein. The q-oscillator is a simple quantum mechanical system described by an annihilation operator and a creation operator parameterized by a parameter q. The basic problem is to find realizations of these operators as differential, difference or integral operators acting on appropriate functional spaces. The first model of q-oscillator in a Hilbert space of analytic functions was discussed in [7]. Later introduced models of q-oscillators are closely related to the q-orthogonal polynomials. The q-analogs of boson operators have been studied by various authors and the corresponding wave functions were constructed in terms of the continuous q-Hermite polynomials of Rogers [42]–[44] by Atakishiyev and Suslov [15] and by Floreanini and Vinet [29]; in terms of the Stieltjes–Wigert polynomials [46], [60] by Atakishiyev and Suslov [17]; and in terms of q-Charlier polynomials of Al-Salam and Carlitz [3] by Askey and Suslov [12], [13] and by Zhedanov [61]. The model related to the Rogers–Szegő polynomials [54] was investigated by Macfarline [38] and by Floreanini and Vinet [27]. In this note we shall restrict ourselves only to the model related to the continuous q-Hermite polynomials where the weight function $\rho(x)$ given by (2.4) is continuous and positive on $(-1,1)$ and the corresponding wave functions form a complete system.

Just as the Hermite polynomials $H_n(x)$ are associated with the wave functions for the harmonic oscillator [36], the continuous q-Hermite polynomials $H_n(x|q)$ are associated with the normalized q-wave function for the q-harmonic oscillator,

$$\Psi_n(x|q) = \left[\frac{(q^{n+1};q)_\infty}{2\pi}\right]^{1/2} \sqrt{\rho(x)}\, H_n(x|q),$$

so that the orthogonality relation (2.2)–(2.5) now reads

$$\int_{-1}^{1} \Psi_n(x|q)\, \Psi_n(x|q)\; dx = \delta_{mn}.$$

The q-annihilation operator $a_q(x)$ and the q-creation operator $a_q^+(x)$ that satisfy the commutation rule

$$a_q(x)a_q^+(x) - q^{-1}a_q^+(x)a_q(x) = 1$$

were introduced explicitly in [15]. In this paper we shall consider another form of the q-boson operators which is equivalent to those given in [15], but more convenient for our purposes; see [28].

The q-annihilation operator $a = a_q(x)$ and the q-creation operator $b = a_q^+(x)$ satisfy the commutation rule

$$ab - q^{-1}ba = 1 \tag{4.1}$$

and act on the corresponding q-wave functions as follows

$$a\,|n\rangle = \left(\frac{1-q^{-n}}{1-q^{-1}}\right)^{1/2} |n-1\rangle\,, \tag{4.2}$$

$$b\,|n\rangle = \left(\frac{1-q^{-n-1}}{1-q^{-1}}\right)^{1/2} |n+1\rangle\,. \tag{4.3}$$

Let $\mathcal{S}$ be a space of analytic functions spanned by $\left\{H_n(x|q)\right\}|_{n=0}^{\infty}$ and let the weighted inner product in $\mathcal{S}$ be

$$(\psi,\ \chi)_\rho := \int_{-1}^{1} \psi^*\left(x\right)\chi\left(x\right)\ \rho\left(x\right)dx, \tag{4.4}$$

where $*$ denotes the complex conjugate; we need also impose certain analyticity condition on the functions ψ and χ [53]; see also [52], [45] for the maximum domain of analyticity of the series in the continuous q-Hermite polynomials.

In this paper we consider the following explicit realization of the q-annihilation a and q-creation b operators. One can easily verify that the divided difference operators

$$a = -\frac{q^{1/2}}{(1-q)^{1/2}}\left(q^s - q^{-s}\right)^{-1}\left(e^{1/2\ \partial} - e^{-1/2\ \partial}\right), \tag{4.5}$$

$$b = -\frac{1}{(1-q)^{1/2}}\left(q^s - q^{-s}\right)^{-1}\left(q^{-2s}e^{1/2\ \partial} - q^{2s}e^{-1/2\ \partial}\right), \tag{4.6}$$

acting on analytic functions of the form

$$\psi(s) = \Psi(x(s)), \qquad x(s) = \left(q^s + q^{-s}\right)/2,$$

where $\exp\left(\alpha\ \partial\right)$ is the shift operator,

$$\exp\left(\alpha\ \partial\right)\psi(s) = \psi(s+\alpha),$$

indeed, satisfy the q-commutation rule (4.1). Moreover, it is easy to see that these operators are adjoint to each other,

$$\left(b\psi,\chi\right)_\rho = \left(\psi, a\chi\right)_\rho,$$

with respect to the inner product (4.4) in the space of analytic functions under consideration.

5. Introducing Integral Operators. Using the T, L and M kernels given by (3.1)–(3.2), (3.4) for $|t| < 1$, let us consider the following *integral operators*

$$\mathbf{T}(t)\,\psi(x) = \int_{-1}^{1} T_t(x,y)\,\psi(y)\,\rho(y)\,dy, \tag{5.1}$$

$$\mathbf{A}(t)\,\psi(x) = \int_{-1}^{1} L_t(x,y)\,\psi(y)\,\rho(y)\,dy, \tag{5.2}$$

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$$\mathbf{C}(t)\,\psi(x)=\int_{-1}^{1}M_t(y,x)\,\psi(y)\,\rho(y)\,dy. \tag{5.3}$$

and the corresponding *adjoint operators*

$$\mathbf{B}(t)\,\psi(x)=\int_{-1}^{1}L_t(y,x)\,\psi(y)\,\rho(y)\,dy, \tag{5.4}$$

$$\mathbf{D}(t)\,\psi(x)=\int_{-1}^{1}M_t(x,y)\,\psi(y)\,\rho(y)\,dy. \tag{5.5}$$

with respect to the inner product (4.4). Indeed,

$$(\mathbf{A}\psi,\chi)_\rho=(\psi,\mathbf{B}\chi)_\rho\,,\quad(\mathbf{C}\psi,\chi)_\rho=(\psi,\mathbf{D}\chi)_\rho$$

by the Fubini theorem when $|t|<1$; see, for example, [1], [23], [35], [37], [56] for an extensive theory of the integral operators.

In a more general setting, let us introduce also

$$\mathbf{A}^{(k)}(t)\,\psi(x)=\int_{-1}^{1}L_t^{(k)}(x,y)\,\psi(y)\,\rho(y)\,dy,$$
$$\mathbf{B}^{(k)}(t)\,\psi(x)=\int_{-1}^{1}L_t^{(k)}(y,x)\,\psi(y)\,\rho(y)\,dy,$$
$$\mathbf{C}^{(k)}(t)\,\psi(x)=\int_{-1}^{1}M_t^{(k)}(y,x)\,\psi(y)\,\rho(y)\,dy,$$
$$\mathbf{D}^{(k)}(t)\,\psi(x)=\int_{-1}^{1}M_t^{(k)}(x,y)\,\psi(y)\,\rho(y)\,dy$$

and with the help of (3.6) verify that

$$\mathbf{C}^{(k)}(t)=\sum_{n=0}^{\infty}q^n\,\frac{\left(q^k;q\right)_n}{(q;q)_n}\,\mathbf{B}^{(k)}\left(tq^n\right),$$
$$\mathbf{D}^{(k)}(t)=\sum_{n=0}^{\infty}q^n\,\frac{\left(q^k;q\right)_n}{(q;q)_n}\,\mathbf{A}^{(k)}\left(tq^n\right).$$

Once again,

$$\left(\mathbf{A}^{(k)}\psi,\chi\right)_\rho=\left(\psi,\mathbf{B}^{(k)}\chi\right)_\rho,$$
$$\left(\mathbf{C}^{(k)}\psi,\chi\right)_\rho=\left(\psi,\mathbf{D}^{(k)}\chi\right)_\rho,$$

in the space of analytic functions, if $|t|<1$.

It is easy to show that

$$\mathbf{A}^{(k)}(t)H_m(x|q)=t^{m-k}\,\frac{(q;q)_m}{(q;q)_{m-k}}\,H_{m-k}(x|q), \tag{5.6}$$

$$\mathbf{B}^{(k)}(t)H_m(x|q)=t^m\,H_{m+k}(x|q), \tag{5.7}$$

$$\mathbf{C}^{(k)}(t)H_m(x|q)=t^m\,\frac{(q;q)_m}{(q;q)_{m+k}}\,H_{m+k}(x|q), \tag{5.8}$$

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$$\mathbf{D}^{(k)}(t)H_m(x|q) = t^{m-k}\, H_{m-k}(x|q), \quad m \geq k. \tag{5.9}$$

For $k=1$ these relations (5.6)–(5.9) define a set of integral operators that correspond to the so-called *raising and lowering operators* for the continuous q-Hermite polynomials:

$$\mathbf{A}(t)H_m(x|q) = t^{m-1}\,(1-q^m)\, H_{m-1}(x|q), \tag{5.10}$$

$$\mathbf{B}(t)H_m(x|q) = t^m\, H_{m+1}(x|q), \tag{5.11}$$

$$\mathbf{C}(t)H_m(x|q) = \frac{t^m}{1-q^{m+1}}\, H_{m+1}(x|q), \tag{5.12}$$

$$\mathbf{D}(t)H_m(x|q) = t^{m-1}H_{m-1}(x|q), \quad m \neq 0. \tag{5.13}$$

The continuous q-Hermite polynomials are eigenfunctions of the $\mathbf{T}$-operator:

$$\mathbf{T}(t)H_m(x|q) = t^m\, H_m(x|q). \tag{5.14}$$

Combining (5.10) and (5.11) we find

$$\mathbf{B}\,(t_1)\,\mathbf{A}\,(t_2)\,H_m(x|q) = (t_1t_2)^{m-1}\,(1-q^m)\,H_m(x|q). \tag{5.15}$$

This *integral equation* with two free parameters t_1 and t_2 extends the corresponding second order difference equation for the continuous q-Hermite polynomials; see [14], [18] and [40] for more details on this equation. Another integral equation follows from (5.12)–(5.13).

6. "Algebra" of Integral Operators. The integral operators $\mathbf{T}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ obey the following *multiplication rules*:

	$\mathbf{T}\,(t_2)$	$\mathbf{A}\,(t_2)$	$\mathbf{B}\,(t_2)$	$\mathbf{C}\,(t_2)$	$\mathbf{D}\,(t_2)$
$\mathbf{T}\,(t_1)$	Eq. (6.1)	Eq. (6.3)	Eq. (6.5)	Eq. (6.7)	Eq. (6.9)
$\mathbf{A}\,(t_1)$	Eq. (6.2)	Eq. (6.16)	Eq. (6.12)	Eq. (6.10)	Eq. (6.18)
$\mathbf{B}\,(t_1)$	Eq. (6.4)	Eq. (6.13)	Eq. (6.17)	Eq. (6.20)	Eq. (6.11)
$\mathbf{C}\,(t_1)$	Eq. (6.6)	Eq. (6.11)	Eq. (6.21)	Eq. (6.22)	Eq. (6.14)
$\mathbf{D}\,(t_1)$	Eq. (6.8)	Eq. (6.19)	Eq. (6.10)	Eq. (6.15)	Eq. (6.23)

All the products in this table can be evaluated directly from the definitions of the integral operators and corresponding kernels in the following manner:

$$\mathbf{T}\,(t_1)\,\mathbf{T}\,(t_2) = \mathbf{T}\,(t_1t_2)\,, \tag{6.1}$$

$$\mathbf{A}\,(t_1)\,\mathbf{T}\,(t_2) = t_2\;\mathbf{A}\,(t_1t_2)\,, \tag{6.2}$$

$$\mathbf{T}\,(t_1)\,\mathbf{A}\,(t_2) = \mathbf{A}\,(t_1t_2)\,, \tag{6.3}$$

$$\mathbf{B}\,(t_1)\,\mathbf{T}\,(t_2) = \mathbf{B}\,(t_1t_2)\,, \tag{6.4}$$

$$\mathbf{T}\,(t_1)\,\mathbf{B}\,(t_2) = t_1\;\mathbf{B}\,(t_1t_2)\,, \tag{6.5}$$

$$\mathbf{C}\,(t_1)\,\mathbf{T}\,(t_2) = \mathbf{C}\,(t_1t_2)\,, \tag{6.6}$$

$$\mathbf{T}\,(t_1)\,\mathbf{C}\,(t_2) = t_1\;\mathbf{C}\,(t_1t_2)\,, \tag{6.7}$$

$$\mathbf{D}\,(t_1)\,\mathbf{T}\,(t_2) = t_2\;\mathbf{D}\,(t_1t_2)\,, \tag{6.8}$$

$$\mathbf{T}\,(t_1)\,\mathbf{D}\,(t_2) = \mathbf{D}\,(t_1t_2)\,, \tag{6.9}$$

$$\mathbf{A}\,(t_1)\,\mathbf{C}\,(t_2) = \mathbf{D}\,(t_1)\,\mathbf{B}\,(t_2) = \mathbf{T}\,(t_1t_2)\,, \tag{6.10}$$

$$\mathbf{C}\,(t_1)\,\mathbf{A}\,(t_2) = \mathbf{B}\,(t_1)\,\mathbf{D}\,(t_2) = (t_1t_2)^{-1}\,(\mathbf{T}\,(t_1t_2) - \mathbf{T}(0))\,, \tag{6.11}$$

$$\mathbf{A}\,(t_1)\,\mathbf{B}\,(t_2) = \mathbf{T}\,(t_1t_2) - q\mathbf{T}\,(qt_1t_2)\,, \tag{6.12}$$

$$\mathbf{B}\,(t_1)\,\mathbf{A}\,(t_2) = (t_1t_2)^{-1}\,(\mathbf{T}\,(t_1t_2) - \mathbf{T}\,(qt_1t_2))\,, \tag{6.13}$$

$$\mathbf{C}\left(t_1\right)\mathbf{D}\left(t_2\right)=\left(t_1t_2\right)^{-1}\left(\sum_{k=0}^{\infty}\mathbf{T}\left(t_1t_2q^k\right)-\mathbf{T}(0)\right), \tag{6.14}$$

$$\mathbf{D}\left(t_1\right)\mathbf{C}\left(t_2\right)=\sum_{k=0}^{\infty}\mathbf{T}\left(t_1t_2q^k\right)q^k, \tag{6.15}$$

$$\mathbf{A}\left(t_1\right)\mathbf{A}\left(t_2\right)=t_2\ \mathbf{A}^{(2)}\left(t_1t_2\right), \tag{6.16}$$

$$\mathbf{B}\left(t_1\right)\mathbf{B}\left(t_2\right)=t_1\ \mathbf{B}^{(2)}\left(t_1t_2\right), \tag{6.17}$$

$$\mathbf{A}\left(t_1\right)\mathbf{D}\left(t_2\right)=t_2\sum_{k=0}^{\infty}q^{2k}\mathbf{A}^{(2)}\left(t_1t_2q^k\right), \tag{6.18}$$

$$\mathbf{D}\left(t_1\right)\mathbf{A}\left(t_2\right)=t_2\sum_{k=0}^{\infty}q^{k}\mathbf{A}\left(t_1t_2q^k\right), \tag{6.19}$$

$$\mathbf{B}\left(t_1\right)\mathbf{C}\left(t_2\right)=t_1\sum_{k=0}^{\infty}q^{k}\mathbf{B}^{(2)}\left(t_1t_2q^k\right), \tag{6.20}$$

$$\mathbf{C}\left(t_1\right)\mathbf{B}\left(t_2\right)=t_1\sum_{k=0}^{\infty}q^{2k}\mathbf{B}^{(2)}\left(t_1t_2q^k\right), \tag{6.21}$$

$$\mathbf{C}\left(t_1\right)\mathbf{C}\left(t_2\right)=t_1\sum_{k=0}^{\infty}q^{k}\frac{1-q^{k+1}}{1-q}\mathbf{B}^{(2)}\left(t_1t_2q^k\right), \tag{6.22}$$

$$\mathbf{D}\left(t_1\right)\mathbf{D}\left(t_2\right)=t_1\sum_{k=0}^{\infty}q^{k}\frac{1-q^{k+1}}{1-q}\mathbf{A}^{(2)}\left(t_1t_2q^k\right) \tag{6.23}$$

and so on. Here $\max\left(\left|t_1\right|,\left|t_2\right|\right)<1$, when all integral operators are bounded.

Although this "algebra" of integral operators is not closed, it unifies many important properties of these operators in a single algebraic structure and deserves detailed study. For instance, it contains the inverses of the Askey–Wilson divided difference operators [20] and the q-Fourier transform [8] as special cases after certain analytic continuation with respect to the free parameter. We shall consider several important examples.

7. Some Degenerate Cases of Integral Operators.

So far we have considered the integral operators (5.1)–(5.5) with $|t|<1$, when they are bounded. In this section we shall consider analytic continuation of these integral operators outside the interval $0\le t<1$. This leads to several important (unbounded) operators, when $t=1,\ q^{-1/2},\ q^{-1}$, etc.

7.1. Operator $\mathbf{T}\left(1\right)$.

It can be shown that

$$\lim_{t\to1^-}\int_{-1}^{1}T_t\left(x,y\right)\psi\left(y\right)\rho\left(y\right)\ dy=\psi\left(x\right),$$

or $\mathbf{T}\left(1\right)$ is the identity operator

$$\mathbf{T}\left(1\right)=I$$

in the space of analytic functions under consideration; see [53] for more details.

7.2. Operator $\mathbf{T}\left(q^{-1/2}\right)$.

In a similar fashion

$$\mathbf{T}\left(q^{-1/2}\right)=\frac{q^se^{-1/2\ \partial}-q^{-s}e^{1/2\ \partial}}{q^s-q^{-s}}. \tag{7.1}$$

This can be shows as a result of "collision" of the poles in the complex plane [53].

Now operators $\mathbf{T}\left(q^{-k/2}\right)$ can be found as products of $\mathbf{T}\left(q^{-1/2}\right)$ from (7.1):

$$\left(\mathbf{T}\left(q^{-1/2}\right)\right)^{k}=\mathbf{T}\left(q^{-k/2}\right).$$

For example,

$$\begin{aligned}\mathbf{T}\left(q^{-1}\right)&=\mathbf{T}\left(q^{-1/2}\right)\mathbf{T}\left(q^{-1/2}\right)\\&=\frac{q^{-2s-1/2}\,e^{\partial}}{\left(q^{s}-q^{-s}\right)\left(q^{s+1/2}-q^{-s-1/2}\right)}\\&+\frac{q^{2s-1/2}\,e^{-\partial}}{\left(q^{s}-q^{-s}\right)\left(q^{s-1/2}-q^{-s+1/2}\right)}\\&-\frac{q^{2s-1/2}\,I}{\left(q^{s-1/2}-q^{-s+1/2}\right)\left(q^{s+1/2}-q^{-s-1/2}\right)},\end{aligned}$$

where I is the identity operator.

The operator $\mathbf{T}\left(q^{-1}\right)$ is closely related to the *Hamiltonian* of the model of the q-harmonic oscillator under consideration, namely,

$$\begin{aligned}H=ba&=\frac{1-\mathbf{T}\left(q^{-1}\right)}{1-q^{-1}}\\&=\frac{\left(1-\mathbf{T}\left(q^{-1/2}\right)\right)\left(1+\mathbf{T}\left(q^{-1/2}\right)\right)}{\left(1-q^{-1/2}\right)\left(1+q^{-1/2}\right)}.\end{aligned}$$

Here a and b are the q-annihilation and q-creation operators, given by (4.5) and (4.6), respectively.

7.3. Operators $\mathbf{A}\left(q^{-1/2}\right)$ and $\mathbf{B}\left(q^{-1/2}\right)$. "Colliding" the poles in the complex plane [53], one can show that

$$\mathbf{A}\left(q^{-1/2}\right)=(1-q)^{1/2}\,a,$$

which is up to a factor just the first order Askey–Wilson divided difference operator; cf. (1.1) and (4.5).

In a similar manner,

$$\mathbf{B}\left(q^{-1/2}\right)=(1-q)^{1/2}\,b.$$

From the multiplication table of the integral operators we obtain the following (q)-*commutators*:

$$\mathbf{A}\left(t_1\right)\mathbf{B}\left(t_2\right)-t_1t_2\mathbf{B}\left(t_2\right)\mathbf{A}\left(t_1\right)=(1-q)\,\mathbf{T}\left(qt_1t_2\right),$$

$$\mathbf{A}\left(t_1\right)\mathbf{B}\left(t_2\right)-qt_1t_2\mathbf{B}\left(t_2\right)\mathbf{A}\left(t_1\right)=(1-q)\,\mathbf{T}\left(t_1t_2\right).$$

The special cases $t_1=t_2=q^{-1/2}$ are well-known in the theory of q-oscillators [19], [38]:

$$ab-q^{-1}ba=I,\quad ab-ba=\mathbf{T}\left(q^{-1}\right).$$

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Also, from the multiplication table of the integral operators,

$$\mathbf{A}\left(t_1\right)\mathbf{T}\left(t_2\right)=t_2\mathbf{T}\left(t_2\right)\mathbf{A}\left(t_1\right),$$

$$\mathbf{B}\left(t_1\right)\mathbf{T}\left(t_2\right)=t_2^{-1}\mathbf{T}\left(t_2\right)\mathbf{B}\left(t_1\right),$$

and when $t_1=q^{-1/2}$, $t_2=t$ one gets

$$a\mathbf{T}\left(t\right)=t\mathbf{T}\left(t\right)a,$$

$$b\mathbf{T}\left(t\right)=t^{-1}\mathbf{T}\left(t\right)b.$$

In the case $t=q^{-1/2}$ we can use these relations in order to determine the spectrum of the q-Hamiltonian H in a pure algebraic form.

The normalized *q-wave functions* in the model of q-oscillator under consideration are

$$\psi_n\left(x\right)=\left[\left(q^{n+1};q\right)_\infty/2\pi\right]^{1/2}H_n\left(x|q\right)$$

with the *orthogonality relation*

$$\int_{-1}^{1}\psi_n\left(x\right)\psi_m\left(x\right)\ \rho\left(x\right)dx=\delta_{mn}$$

and the explicit action of the q-annihilation and q-creation operators on these wave functions is

$$a\psi_n=\left(\frac{1-q^{-n}}{1-q^{-1}}\right)^{1/2}\psi_{n-1},$$

$$b\psi_n=\left(\frac{1-q^{-n-1}}{1-q^{-1}}\right)^{1/2}\psi_{n+1}$$

according to the general rule (4.2)–(4.3).

8. Generalized q-Fourier Transform and its Inverse. The *semi-group property* from the multiplication table is

$$\mathbf{T}\left(t_1\right)\mathbf{T}\left(t_2\right)=\mathbf{T}\left(t_1t_2\right),$$

when $\max\left(\left|t_1\right|,\left|t_2\right|\right)<1$. "Analytic continuation" of the integral operators $\mathbf{T}\left(t_{1,2}\right)$ on the unit circle $\left|t_{1,2}\right|=1$ results in the *q-Fourier transform* [8], [11], [41], [53] (usually, in the classical case, $\tau=\pi/2$ [55], [59], but we discuss the general case with $0<\tau<\pi$). Then, formally,

$$\mathbf{T}\left(e^{i\tau}\right)\mathbf{T}\left(e^{-i\tau}\right)=I,$$

where I is the identity operator. The explicit transformation formulas in the spaces of analytic functions can be given in terms of *Cauchy's principal value integral*. The q-Fourier transform and its inverse are certain singular integral equations, somewhat similar to the case of the classical Hilbert transform; see [24], [39] for an account of the theory of singular integral equations.

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9. The "Number" Operator. The concept of number operator is well-known in quantum mechanics [36]. Similar operators were formally introduced in the theory of q-harmonic operators [19], [38], but explicit realizations of these "number" operators were not constructed. In the model of the q-oscillator under consideration it is natural to introduce this operator as the *generator* of the semi-group of the integral operators $\mathbf{T}(t)$ [53]. Denote

$$\mathbf{T}_{\alpha} = \mathbf{T}(t)|_{t=e^{\alpha}},$$

or in the form of a contour integral,

$$\mathbf{T}_{\alpha}\psi(x) = \int_{\Gamma} T_{e^{\alpha}}(x,y)\ \psi(y)\ \rho(y)\,dy,$$

where Γ is a contour corresponding to analytic continuation of the operator $\mathbf{T}(t)$ to the values $|t| > 1$; see [53] for the details. Then the semi-group properties can be written as usual

$$\mathbf{T}_0 = I, \qquad \mathbf{T}_{\alpha}\mathbf{T}_{\beta} = \mathbf{T}_{\alpha+\beta}$$

and formally

$$\mathbf{T}_{\alpha} = \exp(\alpha\mathcal{I}) = \sum_{n=0}^{\infty} \frac{(\alpha\mathcal{I})^n}{n!},$$

where by the definition the "infinitesimal" operator is

$$\mathcal{I} := \left(\frac{d\mathbf{T}_{\alpha}}{d\alpha}\right)\Bigg|_{\alpha=0}.$$

Explicit realization of the *number operator* $\mathcal{I}$ can be given in terms of *Hadamard's principal value integral* in the space of analytic functions; see [53] for the details.

10. Inversions of Operators $\mathbf{A}(t)$ and $\mathbf{B}(t)$. The operators $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are bounded integral operators for $|t| < 1$; they admit an analytic continuation in the larger domain $|t| > 1$ and become unbounded divided difference operators when $t = q^{-1/2}$. The problem of finding inverses of these operators is similar to the familiar classical results

$$\frac{d}{dx}\int f(x)\ dx = f(x),$$
$$\int \frac{d}{dx} f(x)\ dx = f(x) + \text{constant}.$$

In the model of the q-harmonic oscillator under consideration we can extend these relations to q-derivatives, or even to q-"fractional" derivatives, namely, our integral operators $\mathbf{A}(t)$ and $\mathbf{B}(t)$, with the help of the integral operators $\mathbf{C}(t)$ and $\mathbf{D}(t)$. Indeed, for $|t| < 1$ one can write from the table of multiplication of the operators that

$$\mathbf{A}\left(t^{-1}\right)\mathbf{C}(t) = \mathbf{T}(1) = I,$$
$$\mathbf{D}(t)\,\mathbf{B}\left(t^{-1}\right) = \mathbf{T}(1) = I,$$

where I is the identity operator. Thus the bounded integral operator $\mathbf{C}(t)$ ($\mathbf{D}(t)$) with $|t| < 1$ gives the right (left) inverse of the unbounded operator $\mathbf{A}\left(t^{-1}\right)$ ($\mathbf{B}\left(t^{-1}\right)$), provided that this operator is properly analytically continued to the domain $\left|t^{-1}\right| > 1$. When $t \to q^{-1/2}$ one

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gets, as a special case, the right inverse of the Askey–Wilson first order divided difference operator $\delta/\delta x$ originally found by Brown and Ismail [20] in this model of q-oscillator.

In a similar manner,

$$\mathbf{C}(t)\,\mathbf{A}\left(t^{-1}\right) = \mathbf{T}(1) - \mathbf{T}(0),$$
$$\mathbf{B}\left(t^{-1}\right)\mathbf{D}(t) = \mathbf{T}(1) - \mathbf{T}(0),$$

when $|t| < 1$ and

$$\mathbf{A}(t_1)\,\mathbf{C}(t_2) - t_1 t_2 \mathbf{C}(t_2)\,\mathbf{A}(t_1) = \mathbf{T}(0),$$
$$\mathbf{D}(t_2)\,\mathbf{B}(t_1) - t_1 t_2 \mathbf{B}(t_1)\,\mathbf{D}(t_2) = \mathbf{T}(0),$$

i.e., these "commutators" act on a vector ψ as the projection operator to the "vacuum" vector ψ_0. Indeed,

$$\begin{aligned}\mathbf{T}(0)\,\psi(x) &= \int_{-1}^{1} T_0(x,y)\,\psi(y)\;\rho(y)\,dy \\ &= (\psi_0,\psi)_\rho\;\psi_0,\end{aligned}$$

where $\psi_0 = d_0^{-1} H_0(x|q)$ is the "vacuum" vector.

11. Resolvents and Green's Functions. The continuous q-Hermite polynomials, or the wave functions in the model of the q-harmonic oscillator under consideration, satisfy two difference equations. We derive corresponding resolvents and Green's function.

11.1. First difference operator. Let us start from the following difference equation for the continuous q-Hermite polynomials

$$\mathbf{T}\left(q^{-1/2}\right) H_n(x|q) = q^{-n/2} H_n(x|q),$$

which is the special case $t = q^{-1/2}$ of (5.14) due to (7.1), and consider

$$\left(\mathbf{T}\left(q^{-1/2}\right) - \lambda I\right)\psi = \chi \tag{11.1}$$

with the *resolvent*

$$\mathbf{R}_\lambda = \left(\mathbf{T}\left(q^{-1/2}\right) - \lambda I\right)^{-1}, \quad \psi = \mathbf{R}_\lambda\,\chi.$$

Multiplying (11.1) by the corresponding bounded integral operator $\mathbf{T}\left(q^{1/2}\right)$, one gets

$$\left(I - \lambda \mathbf{T}\left(q^{1/2}\right)\right)\psi = \mathbf{T}\left(q^{1/2}\right)\chi,$$

where I is the identity operator. Thus,

$$\begin{aligned}\left(I - \lambda \mathbf{T}\left(q^{1/2}\right)\right)^{-1} &= \sum_{k=0}^{\infty} \lambda^k \left(\mathbf{T}\left(q^{1/2}\right)\right)^k \\ &= \sum_{k=0}^{\infty} \lambda^k \mathbf{T}\left(q^{k/2}\right)\end{aligned}$$

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by (6.1), and

$$\mathbf{R}_{\lambda} = \left(\mathbf{T}\left(q^{-1/2}\right) - \lambda I\right)^{-1} = \sum_{k=0}^{\infty} \lambda^k \mathbf{T}\left(q^{(k+1)/2}\right).$$

So, the resolvent is an integral operator

$$\mathbf{R}_{\lambda}\, \chi(x) = \int_{-1}^{1} R_{\lambda}(x,y)\, \chi(y)\ \rho(y)\, dy$$

with the kernel

$$\begin{aligned} R_{\lambda}(x,y) &= \sum_{k=0}^{\infty} \lambda^k T_{q^{(k+1)/2}}(x,y) \\ &= \sum_{n=0}^{\infty} \frac{H_n(x|q)\, H_n(y|q)}{d_n^2\left(1-\lambda q^{n/2}\right)}\, q^{n/2}. \end{aligned}$$

Introducing the eigenvalues $\lambda_n = q^{-n/2}$ and the orthonormal eigenfunctions $\psi_n = d_n^{-1} H_n(x|q)$, one can finally write

$$R_{\lambda}(x,y) = \sum_{n=0}^{\infty} \frac{\psi_n(x)\, \psi_n(y)}{\lambda_n - \lambda}.$$

The resolvent identity holds

$$(\mu - \lambda)\, \mathbf{R}_{\lambda}\mathbf{R}_{\mu} = \mathbf{R}_{\lambda} - \mathbf{R}_{\mu}, \tag{11.2}$$

see [1], [23], [35], [37] for more properties of the resolvent.

11.2. Second difference operator. Let us factor, first of all, the corresponding Askey–Wilson difference equation of the second order (or the q-Hamiltonian) in the following manner. At the level of the integral operators in Eq. (5.15) we have

$$\mathbf{B}(t_1)\, \mathbf{A}(t_2) = (t_1 t_2)^{-1}\left(\mathbf{T}(t_1 t_2) - \mathbf{T}(q t_1 t_2)\right)$$

and, hence, for $t_1 = t_2 = q^{-1/2}$,

$$q^{-1}\mathbf{B}\left(q^{-1/2}\right) \mathbf{A}\left(q^{-1/2}\right) = \mathbf{T}\left(q^{-1}\right) - I.$$

Therefore, instead of solving

$$\left(q^{-1}\mathbf{B}\left(q^{-1/2}\right) \mathbf{A}\left(q^{-1/2}\right) + \lambda\right) \psi = \chi,$$

one can solve a simpler equation

$$\left(\mathbf{T}\left(q^{-1}\right) - \mu I\right) \psi = \chi, \quad \mu = 1 - \lambda \tag{11.3}$$

with the help of the *resolvent*

$$\mathbf{R}_{\mu} = \left(\mathbf{T}\left(q^{-1}\right) - \mu I\right)^{-1}, \quad \psi = \mathbf{R}_{\mu}\, \chi.$$

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Multiplying (11.3) by $\mathbf{T}(q)$,

$$(I - \mu \mathbf{T}(q))\, \psi = \mathbf{T}(q)\, \chi,$$

where I is the identity operator, and once again

$$\begin{aligned}(I - \mu \mathbf{T}(q))^{-1} &= \sum_{k=0}^{\infty} \mu^k \left(\mathbf{T}(q)\right)^k \\ &= \sum_{k=0}^{\infty} \mu^k \mathbf{T}\left(q^k\right),\end{aligned}$$

or

$$\mathbf{R}_\mu = \left(\mathbf{T}\left(q^{-1}\right) - \mu I\right)^{-1} = \sum_{k=0}^{\infty} \mu^k \mathbf{T}\left(q^{k+1}\right).$$

This consideration gives an *explicit representation for the resolvent* in terms of the integral operator

$$\mathbf{R}_\mu\, \chi(x) = \int_{-1}^{1} R_\mu(x, y)\, \chi(y)\ \rho(y)\, dy$$

with the kernel

$$\begin{aligned}R_\mu(x, y) &= \sum_{k=0}^{\infty} \mu^k T_{q^{k+1}}(x, y) \\ &= \sum_{n=0}^{\infty} \frac{H_n(x|q)\, H_n(y|q)}{d_n^2\left(1 - \mu q^n\right)}\, q^n.\end{aligned}$$

The resolvent identity (11.2) holds.

11.3. Green's function. Let

$$\chi = \delta(x - x'),$$

where $\delta(y)$ is the Dirac delta function. Then

$$G_\mu(x, x') = \mathbf{R}_\mu \delta(x - x') = R_\mu(x, x')\, \rho(x'),$$

or

$$\left(\mathbf{T}\left(q^{-1}\right) - \mu I\right) G_\mu(x, y) = \delta(x - y)$$

with

$$\begin{aligned}G_\mu(x, y) &= R_\mu(x, y)\, \rho(y) \\ &= \sum_{n=0}^{\infty} \frac{H_n(x|q)\, H_n(y|q)}{d_n^2\left(1 - \mu q^n\right)}\, q^n \rho(y).\end{aligned}$$

See [53] for more details.

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