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THE METHOD OF LOWER AND UPPER SOLUTIONS FOR PERIODIC AND ANTI–PERIODIC DIFFERENCE EQUATIONS*

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Abstract. In this work we do a survey on the method of lower and upper solutions for periodic and anti-periodic discrete problems. Some new existence results are also given.

Key words. lower and upper solutions, monotone iterative techniques, Green's functions

AMS subject classification. 39A10

1. Introduction. The method of lower and upper solutions for ordinary differential equations was introduced in 1931 by G. Scorza Dragoni [25] for a Dirichlet problem. This method allow us to enssure the existence of a solution of the considered problem lying between the lower and the upper solutions, i. e., we have information about the existence and location of the solutions. After this, there is a large number of works in which the method has been developed for different boundary value problems, thus first, second and higher order ordinary differential equations with different type of boundary conditions as, among others, the periodic, mixed, Dirichlet or Neumann conditions, and partial differential equations of first and second order, have been treated in the literature. In the classical books of S.R. Bernfeld and V. Lakshmikantham [7] and G.S. Ladde, V. Lakshmikantham and A.S. Vatsala [22] is exposed the classical theory of the method of lower and upper solutions and the monotone iterative technique, that give us the expression of the solution as the limit of a monotone sequence formed by functions that solve linear problems related with the nonlinear considered equations. We refer to the reader to the surveys in this field of C. De Coster and P. Habets [18, 19] in which one can found hystorical and biographycal references together with recent results and open problems.

The application of this kind of techniques to difference equations is recent. In this paper we present some of the results that have appeared in this field. Furthermore, some new results for first and second order difference equations are given. In §2, we consider first order equations. Second order equations are aborded in §3 and higher order in §4. As a consequence of the exposed results in that section, we present in §5 some existence and uniqueness results for first and second order anti-periodic difference equations, some of them (for order two) are new.

In all the paper, we denote by $\Delta u_k = u_{k+1} - u_k$ and $\Delta^j = \Delta \circ \Delta^{j-1}$. Moreover given $\alpha, \beta \in \mathbb{R}^p$, such that $\alpha \leq \beta$, we write

$$[\alpha, \beta] = \{u \in \mathbb{R}^p, \alpha_k \leq u_k \leq \beta_k,\ k = 0, \ldots, p-1\}.$$

2. First order difference equations. First order difference equations has been aborded by V. Otero-Espinar, R. L. Pouso and the author in [16]. There the following problem has been studied.

$$\Delta u_k = f(k, u_{k+1}), \quad k \in \{0, \ldots, N-1\}; \qquad B(u_0, u) = 0. \tag{2.1}$$

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Thus, $\alpha = \{\alpha_0, \ldots, \alpha_N\}$ is a lower solution if

$$\Delta\alpha_k \leq f(k, \alpha_{k+1}), \quad k \in \{0, \ldots, N-1\}; \qquad B(\alpha_0, \alpha) \leq 0$$

and $\beta = \{\beta_0, \ldots, \beta_N\}$ is an upper solution when

$$\Delta\beta_k \geq f(k, \beta_{k+1}), \quad k \in \{0, \ldots, N-1\}; \qquad B(\beta_0, \beta) \geq 0.$$

In that paper it is proved that if $\alpha \leq \beta$, $f : \{0, \ldots, N-1\} \times \mathbb{R} \to \mathbb{R}$ and $B : \{0, \ldots, N-1\} \times \mathbb{R}^{N+1} \to \mathbb{R}$ are continuous functions such that $B(\alpha_0, \cdot)$ and $B(\beta_0, \cdot)$ are nonincreasing, then this problem has at least one solution in the sector $[\alpha, \beta]$. Moreover, if $B(x, \cdot)$ is a nonincreasing function for each $x \in [\alpha_0, \beta_0]$, then problem (2.1) has extremal solutions lying between α and β. Where by extremal solutions we denote the biggest and the smallest solutions of the problem lying in the sector $[\alpha, \beta]$.

It is clear that this result is applicable to initial conditions $B(x, y) = x$ and periodic boundary value problems $B(x, y) = x - y_N$.

When we think about anti - periodic boundary value conditions ($u_0 = -u_N$), we cannot enssure the existence of extremal solutions, unless all of them start at $u_0 = 0$, but in this case we arrive at $u_N = 0$, and since final problem has a unique solution, we conclude that the anti-periodic problem has a unique solution too. As consequence, if the anti-periodic problem has more than one solution then there is no extremal solutions. Our investigation is directed to conclude the existence of at least one solution and, if we try to use monotone iterative techniques, to ensure the existence of only one solution.

With respect to the existence results, in [2] the definition of related lower and upper solutions for problem (2.1) is given as a pair $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_N\}$ and $\beta = \{\beta_0, \beta_1, \ldots, \beta_N\}$ of real sequences such that $\alpha \leq \beta$ in $\{0, \ldots, N\}$, such that

$$\Delta\alpha_k \leq f(k, \alpha_{k+1}), \quad k \in \{0, \ldots, N-1\}; \qquad B(\alpha_0, \beta) \leq 0$$

and

$$\Delta\beta_k \geq f(k, \beta_{k+1}), \quad k \in \{0, \ldots, N-1\}; \qquad B(\beta_0, \alpha) \geq 0,$$

and it is proved that if there exist α and β, a pair of related lower and upper solutions of (2.1), and $B(\alpha_0, \cdot)$ and $B(\beta_0, \cdot)$ are nondecreasing in $\mathbb{R}^{N+1}$, then, provided that f and B are continuous functions, problem (2.1) has at least one solution $u \in [\alpha, \beta]$.

Clearly, defining $B(x, y) = x + y_N$ we have an existence result for first order anti-periodic problems.

Both existence results have been generalized by D. Franco, D. O'Regan and J. Perán in [23]. There the authors define the concept of coupled lower and upper solutions for problem.

$$\Delta u_k + f(k, u_{k+1}) = 0, \quad k \in \{0, \ldots, N-1\}; \qquad B(u_0, u) = 0, \tag{2.2}$$

as a pair $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_N\}$ and $\beta = \{\beta_0, \beta_1, \ldots, \beta_N\}$ of real sequences such that $\alpha \leq \beta$ in $\{0, \ldots, N\}$, such that

$$\Delta\alpha_k + f(k, \alpha_{k+1}) \leq 0, \quad k \in \{0, \ldots, N-1\}; \qquad \max\{B(\alpha_0, \alpha), B(\alpha_0, \beta)\} \leq 0$$

and

$$\Delta\beta_k + f(k, \beta_{k+1}) \geq 0, \quad k \in \{0, \ldots, N-1\}; \qquad \min\{B(\beta_0, \alpha), B(\beta_0, \beta)\} \geq 0,$$

and prove that if there exist α and β, a pair of coupled lower and upper solutions of (2.2), and $B(\alpha_0, \cdot)$ and $B(\beta_0, \cdot)$ are monotone (nondecreasing or nonincreasing) in $\mathbb{R}^{N+1}$, and f and B are continuous functions, then problem (2.2) has at least one solution $u \in [\alpha, \beta]$.

In fact it is possible to give an existence result for problem (2.1) (and the equivalent expression (2.2)) replacing monotony properties if B by the weak one:

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(H) There exists a pair $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_N\}$ and $\beta = \{\beta_0, \beta_1, \ldots, \beta_N\}$ of real sequences such that $\alpha \leq \beta$ in $\{0, \ldots, N\}$, such that

$$\Delta \alpha_k \leq f(k, \alpha_{k+1}), \quad k \in \{0, \ldots, N-1\},$$

$$\Delta \beta_k \geq f(k, \beta_{k+1}), \quad k \in \{0, \ldots, N-1\}$$

and

$$B(\alpha_0, u) \leq 0 \leq B(\beta_0, u), \quad \text{for all } u \in [\alpha, \beta].$$

Under this assumption, we arrive at the following existence result. The proof follows the ideas exposed in [16], we present it here by the convenience of the reader.

THEOREM 2.1. *Suppose that function $f(k, \cdot)$ is continuous in $[\alpha_{k+1}, \beta_{k+1}]$ for all $k \in \{0, \ldots, N-1\}$ and that $B \in C(\mathbb{R} \times \mathbb{R}^{N+1}, \mathbb{R})$. If condition (H) holds then problem (2.1) has at least one solution $u \in [\alpha, \beta]$.*

Proof. Let us consider the following modified problem:

$$\Delta u_k = f(k, p(k+1, u_{k+1})), \quad k \in \{0, \ldots, N-1\}, \tag{2.3}$$

$$u_0 = p(0, u_0 - B(u_0, u)), \tag{2.4}$$

where $p(k, r) = \max\{\alpha_k, \min\{r, \beta_k\}\}$ for all $k \in \{0, \ldots, N\}$ and $r \in \mathbb{R}$.

Follow the proof of Theorem 2.1 in [16] one can see that (2.3) – (2.4) has at least one solution $u \in [\alpha, \beta]$ and that all of the possible solutions of that problem belong to the sector $[\alpha, \beta]$.

Now, we need to verify that all the solutions of (2.3) – (2.4) satisfy $B(u_0, u) = 0$. To this end, let u be a solution of problem (2.3) – (2.4), if $u_0 - B(u_0, u) < \alpha_0$, by definition of function p, we have that $u_0 = \alpha_0$, and then, since $u \in [\alpha, \beta]$, from condition (H), we arrive at

$$\alpha_0 > \alpha_0 - B(\alpha_0, u) \geq \alpha_0,$$

which is a contradiction.

The fact that $u_0 - B(u_0, u) \leq \beta_0$ holds similarly. □

If we are interested into approach the solutions of the anti-periodic problem via monotone iterative techniques, since, as we have exposed above, there is no extremal solutions in $[\alpha, \beta]$ and comparison results do not hold in this situation, the monotone method, valid for periodic equations, is not applicable in this situation. In [1] and [2] some criteria for existence and uniqueness results for n–th order anti-periodic difference equations have been developed. The particular case of first order equations has been also considered in [2] and will be exposed in §5 of this paper.

3. Second order difference equations. Second order boundary value problems have been studied for different authors under the assumption of the existence of a pair of lower and upper solutions. Thus, assuming that $f : \{1, \ldots, N\} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, monotone decreasing in the second and the third variables, and the existence of a pair of well ordered lower and upper solutions, that is, the lower solution is less than or equals to the upper one, P. W. Eloe, provided that the lower and the upper solution satisfy some additional conditions, proves in [21] the existence of at least one solution of the second order periodic boundary value problem

$$\Delta^2 u_k = f(k, u_k, u_{k+1}); \quad k \in \{0, \ldots, N\}, \quad u_0 = u_{N+1}, \ \Delta u_0 = \Delta u_{N+1},$$

in the sector $[\alpha, \beta]$.

F. Atici and the author prove in [4] the existence of a solution of the second order periodic boundary value problem

$$-\Delta^2 u_{k-1} + q_k\, u_k = f(k, u_k),\ k \in \{1, \dots, N\}, \quad u_0 = u_N,\ \Delta u_0 = \Delta u_N, \tag{3.1}$$

with $q \geq 0$ and $q \not\equiv 0$ in $\{1, \dots, N\}$.

Here we define a lower solution α as a vector in $\mathbb{R}^{N+2}$ that satisfies

$$-\Delta^2 \alpha_{k-1} + q_k \alpha_k \leq f(k, \alpha_k),\ k \in \{1, \dots, N\}, \quad \alpha_0 = \alpha_N,\ \Delta \alpha_0 \geq \Delta \alpha_N.$$

β will be an upper solution if the reversed inequalities hold.

These definitions are the natural adaptation to this case of the problem studied by P. W. Eloe in [21].

In this case no monotony conditions are imposed in the continuous function f. In the proof, some properties of the Green's function related with the linear operator

$$L_2\, u_k \equiv -\Delta^2 u_{k-1} + q_k\, u_k$$

in the space of periodic functions

$$S = \{u \equiv \{u_0, \dots, u_{N+1}\};\ u_0 = u_N,\ \Delta u_0 = \Delta u_N\},$$

given by F. Atici and S. Guseinov in [6], are used.

Due to the fact [6] that the Green's function related with operator L_2 has constant sign, it is also developed the monotone method, i. e., assuming a one - sided Lipschitz condition in funcion f, two monotone sequences that start at the lower solution α and the upper solution β and converge to a solutions φ and ϕ, are constructed; moreover every solution $u \in [\alpha, \beta]$ of problem (3.1) satisfies that $\varphi \leq u \leq \phi$ in $\{0, \dots N+1\}$.

These kind of results have been applied by F. Atici, A. Cabada and V. Otero-Espinar in [5], where, depending on the values of the real parameter λ, existence results for the following second order periodic boundary value problem are given

$$\begin{gathered} -\Delta[p_{k-1}\, \Delta u_{k-1}] + q_k\, u_k = \lambda\, f(k, u_k), \quad k \in \{0, \dots, N\}, \\ u_0 = u_N,\ p_0 \Delta u_0 = p_N \Delta u_N. \end{gathered}$$

Other type of boundary conditions have been also considered. In [26] Zhuang, Chen and Cheng study the Dirichlet boundary value problem

$$\Delta^2 u_k + f(k, u_k) = 0, \quad k \in \{1, \dots, N\}, \quad u_0 = u_{N+1} = 0,$$

and deduce the existence of a solution lying between a lower solution α and an upper solution β such that $\alpha \leq \beta$ of the Dirichlet problem. Here α is a lower solution when it satisfies (the reversed inequalities for an upper solution β)

$$\Delta^2 \alpha_{k-1} + f(k, \alpha_k) \geq 0, \quad k \in \{1, \dots, N\}, \quad \alpha_0 \leq 0,\ \alpha_{N+1} \leq 0.$$

Similar existence results are given by R. P. Agarwal and D. O'Regan in [3]. Moreover, assuming a one - sided Lipschitz condition in function f, the monotone method is also developed in [26].

In [8] is studied the second order difference equation with nonlinear functional boundary conditions

$$-\Delta[\phi(\Delta u_k)] = f(k, u_{k+1}), \quad k \in \{0, \dots, N-1\}, \tag{3.2}$$

$$B_1(u_0, u) = B_2(u, u_{N+1}) = 0, \tag{3.3}$$

with f a continuous function, $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ continuous, strictly increasing and $\phi(\mathbb{R}) = \mathbb{R}$, $B_1 : \mathbb{R} \times \mathbb{R}^{N+2} \longrightarrow \mathbb{R}$ continuous and nondecreasing in the second variable and $B_2 : \mathbb{R}^{N+2} \times \mathbb{R} \longrightarrow \mathbb{R}$ continuous and nonincreasing in the first variable.

This kind of problems is known as ϕ – Laplacian equation and arises in the theory of radial solutions for the p – Laplacian equation ($\phi(x) = \mid x \mid^{p-2} x$, $p > 1$) on an annular domain (see [20], and references therein), and has been exhaustively studied recently for differential equations (see, for instance, [9, 17]). This class of nonlinear boundary conditions allow functional depence of the solutions and include the Dirichlet,

$$B_1(x,y) = -x, \quad B_2(x,y) = y,$$

Neumann,

$$B_1(x,y) = y_1 - x, \quad B_2(x,y) = y - x_N$$

and periodic boundary conditions

$$B_1(x,y) = y_N - x, \quad B_2(x,y) = y - x_1,$$

as particular cases.

Obviously, defining ϕ as the identity, we have that every given result for problem (3.2) – (3.3) remains valid for second order difference equations. In that paper is proved an existence result of extremal solutions when $\alpha \leq \beta$ without assuming monotony properties in function f. In this case we say that α is a lower solution for problem (3.2) – (3.3) if it satisfies the following inequalities

$$-\Delta[\phi(\Delta \alpha_k)] \leq f(k, \alpha_{k+1}),\ k \in \{0, \ldots, N-1\},\ B_1(\alpha_0, \alpha) \geq 0 \geq B_2(\alpha, \alpha_{N+1}).$$

When f satisfies a one - sided Lipschitz condition, the monotone iterative technique is developed. These results generalize previous given results and some of them have been applied in [12] to deduce existence results in presence of lower and upper solutions for the functional ϕ – Laplacian problem with periodic boundary value conditions

$$\begin{aligned} -\Delta[\phi(\Delta u_k)] + q_{k+1} u_{k+1} &= g(k, u_{k+1}, u), \qquad k \in \{0, \ldots, N-1\}, \\ u_0 = u_N, \quad \Delta u_0 &= \Delta u_N, \end{aligned}$$

with $q > 0$ in $\{1, \ldots, N\}$ and $g : \{1, \ldots, N\} \times \mathbb{R} \times \mathbb{R}^{N+2} \to \mathbb{R}$ nondecreasing and discontinuous in the third variable. In this case the techniques developed by S. Heikkilä and V. Laksmikantham in the monograph [24] are the fundamental tools.

Nonlinear boundary value problems for second order difference equations have been also considered by D. Franco, D. O'Regan and J. Perán in [23], where is considered the problem

$$\begin{aligned} \Delta^2 u_{k-1} + f(k, u_k) &= 0, \qquad k \in \{1, \ldots, N\}, \\ g(u_0, u_{N+1}) = h(u_0, u_{N+1}) &= 0. \end{aligned}$$

The authors introduce the concept of coupled lower and upper solutions as a pair of vectors in $\mathbb{R}^{N+2}$, α and β, such that $\alpha \leq \beta$ and

$$\Delta^2 \alpha_{k-1} + f(k, \alpha_k) \geq 0 \geq \Delta^2 \beta_{k-1} + f(k, \beta_k), \quad k \in \{1, \ldots, N\},$$

with

$$\max\{g(\beta_0, \beta_{N+1}),\ g(\beta_0, \alpha_{N+1})\} \leq 0,$$

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$$\min\{g(\alpha_0,\alpha_{N+1}),\ g(\alpha_0,\beta_{N+1})\} \geq 0,$$

$$\max\{h(\beta_0,\beta_{N+1}),\ h(\alpha_0,\beta_{N+1})\} \leq 0,$$

$$\min\{h(\alpha_0,\alpha_{N+1}),\ h(\beta_0,\alpha_{N+1})\} \geq 0.$$

In this case it is proved that if functions $g(\alpha_0,\cdot)$, $g(\beta_0,\cdot)$, $h(\cdot,\alpha_{N+1})$ and $h(\cdot,\beta_{N+1})$ are monotone (nonincreasing or nondecreasing), then there exists at least one solution in $[\alpha,\beta]$ of the treated problem.

In this definition of coupled lower and upper solutions are included both functions, unless the four previous functions was nondecreasing.

The non usual case, in which the lower solution α is bigger than or equals to the upper solution β, has been aborded in different papers. In this case the existence results follow directly from the monotone iterative techniques. In [13], A. Cabada and V. Otero-Espinar study the ϕ-Laplacian problem

$$-\Delta[\phi(\Delta u_k)] = f(k,u_{k+1}),\ k \in \{0,\dots,N-1\}.$$

There, under suitable conditions in functions ϕ and f, some results of existence of extremal solutions are given for Neumann and peridic boundary value problems.

An exhaustive study of the second order Neumann boundary value problem

$$u_{k+2} = f(k,u_k,u_{k+1}),\ k \in \{0,\dots,N-1\};\quad \Delta u_0 = A,\ \Delta u_N = B,$$

has been done in [14], where comparison results for the second order linear operator

$$L[\gamma,\mu] = u_{k+2} - 2\,\gamma\,u_{k+1} + \mu\,u_k,$$

in the set

$$W_N = \{u \in \mathbb{R}^{N+2};\ \Delta u_0 \geq 0 \geq \Delta u_N\},$$

are given.

4. Higher order equations. As we have seen in the previous sections, the study of first and second order difference equations give us the existence of solutions lying between a pair of well ordered lower and upper solutions. The main arguments to enssure the location of the solutions are the oscillation properties. When α and β are given in the reversed order for first and second order problems or if we consider equations of order $n \geq 3$, this kind of techniques are insufficient to conclude the location of solutions. In this situation we must use iterative techniques to derive existence and approximation of solutions. This class of techniques has been applied to n-th order periodic equations in [14, 15] and for n-th order anti-periodic ones in [1, 2].

For the periodic case, the following existence and approximation result is proved in [14].

THEOREM 4.1. *Let $f : \{0,\dots,N-1\} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a continuous function and $\lambda_i \in \mathbb{R}$, $i = 0,\dots,n-1$, given. Suppose that there exist $\alpha = \{\alpha_0,\alpha_1,\dots,\alpha_{N+n-1}\}$ and $\beta = \{\beta_0,\beta_1,\dots,\beta_{N+n-1}\}$, satisfying $\alpha \leq \beta$ and*

$$\begin{aligned}
\alpha_{k+n} &\leq f(k,\alpha_k,\alpha_{k+1},\dots,\alpha_{k+n}), \quad k \in \{0,\dots,N-1\},\\
\alpha_i - \alpha_{N+i} &= \lambda_i, \quad i = 0,\dots,n-2,\\
\alpha_{n-1} - \alpha_{N+n-1} &\leq \lambda_{n-1}
\end{aligned}$$

and

$$\beta_{k+n} \geq f(k, \beta_k, \beta_{k+1}, \ldots, \beta_{k+n}), \quad k \in \{0, \ldots, N-1\},$$
$$\beta_i - \beta_{N+i} = \lambda_i, \quad i = 0, \ldots, n-2,$$
$$\beta_{n-1} - \beta_{N+n-1} \geq \lambda_{n-1}.$$

If function f satisfies the following condition
(H_1) *There exists* $\{M_0, \ldots, M_n\} \in \mathbb{R}^{n+1}$ *such that*

$$f(k, x_0, \ldots, x_n) + \sum_{i=0}^{n} M_i\, x_i \leq f(k, y_0, \ldots, y_n) + \sum_{i=0}^{n} M_i\, y_i,$$

for all $k \in \{0, \ldots, N-1\}$ *and* $\alpha_{k+i} \leq x_i \leq y_i \leq \beta_{k+i}, \quad i = 0, \ldots, n,$
for some $M_0, \ldots, M_n \in \mathbb{R}$ *for which it is satisfied that if*

$$u \in \{u \in \mathbb{R}^{N+n};\, u_i = u_{N+i},\, i = 0, \ldots, n-2,\; u_{n-1} \geq u_{N+n-1}\}$$

and

$$T_n[M_0, \ldots, M_n]u_k \equiv u_{k+n} + \sum_{i=0}^{n} M_i u_{k+i} \geq 0 \text{ in } \{0, \ldots, N-1\}$$

then $u \geq 0$ *in* $\{0, \ldots, N+n-1\}$.

Then there exist two monotone sequences in $\mathbb{R}^{N+n}$, $\{a_m\}$ *and* $\{b_m\}$ *with* $a_0 = \alpha$ *and* $b_0 = \beta$*, which converge pointwise to the extremal solutions in* $[\alpha, \beta]$ *of problem*

$$(P_n)\begin{cases} u_{k+n} &= f(k, u_k, u_{k+1}, \ldots, u_{k+n}), \quad k \in \{0, \ldots, N-1\}, \\ u_i - u_{N+i} &= \lambda_i, \quad i = 0, \ldots, n-1. \end{cases}$$

Proof. For each $\eta \in [\alpha, \beta]$, we consider the following linear problem:

$$(P_\eta)\begin{cases} T_n[M_0, \ldots, M_n]u_k &= f(k, \eta_k, \eta_{k+1}, \ldots, \eta_{k+n}) + \\ &\quad + \sum_{i=0}^{n} M_i\, \eta_{k+i}, \quad k \in \{0, \ldots, N-1\}, \\ u_i - u_{N+i} &= \lambda_i, \quad i = 0, \ldots, n-1. \end{cases}$$

It is not difficult to verify that problem (P_η) admits a unique solution u for each η given.

From condition (H_1) and the definition of α and β, we have that

$$T_n[M_0, \ldots, M_n](u - \alpha) \geq 0 \quad \text{on } \{0, \ldots, N-1\}$$

and we conclude that $u \geq \alpha$ on $\{0, \ldots, N+n-1\}$.

On the other hand, let u_i, $i = 1, 2$, be the unique solutions of problem (P_{η_i}), with $\eta_1 \leq \eta_2$ on $\{0, \ldots, N+n-1\}$. We know that

$$T_n[M_0, \ldots, M_n](u_2 - u_1) \geq 0 \quad \text{on } \{0, \ldots, N-1\}$$

and then $u_2 \geq u_1$ on $\{0,\dots,N+n-1\}$.

The sequences $\{a_m\}$ and $\{b_m\}$ are obtained by recurrence: $a_0=\alpha$, $b_0=\beta$ and a_m and b_m are given as the unique solutions of $(P_{a_{m-1}})$ and $(P_{b_{m-1}})$ respectively. $\square$

It is important to note that in the proof of the previous result is fundamental the study of the Green's function related with operator $T_n[M_0,\dots,M_n]$ and, more concisely, with the values of the parameters M_i for which such Green's function has constant sign. In [15], a formula to obtain such function is given. Using this expression, optimal estimates for first and second order equations are obtained. This study has been continued in [11], where, using those optimal estimates together with the expression of the n-th order linear operator

$$T_n[M]\,u_k \equiv \Delta^n\,u_k + M\,u_k$$

as a composition of suitable first and second order operators, the following result is proved for problem

$$(P)\left\{\begin{array}{lcl} \Delta^n u_k &=& f(k,u_k), \quad k\in\{0,\dots,N-1\},\\ \\ u_i &=& u_{N+i}, \quad i=0,\dots,n-1. \end{array}\right.$$

THEOREM 4.2. *Let $f:\{0,\dots,N-1\}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Suppose that there exist $\alpha\leq\beta$ satisfying*

$$\begin{aligned} \Delta^n\alpha_k &\leq f(k,\alpha_k), \quad k\in\{0,\dots,N-1\},\\ \alpha_i-\alpha_{N+i} &= 0, \quad i=0,\dots,n-2,\\ \alpha_{n-1}-\alpha_{N+n-1} &\leq 0 \end{aligned}$$

and

$$\begin{aligned} \Delta^n\beta_k &\geq f(k,\beta_k), \quad k\in\{0,\dots,N-1\},\\ \beta_i-\beta_{N+i} &= 0, \quad i=0,\dots,n-2,\\ \beta_{n-1}-\beta_{N+n-1} &\geq 0. \end{aligned}$$

If f satisfies condition (H_1) (with obvious notation) for some $M>0$ such that

$$M \leq \left[\frac{\tan\frac{\pi}{N}}{(1+\tan\frac{\pi}{N})\,\cos\frac{\pi}{n}}\right]^n, \textit{ when } n=4\,p,\ p\in\{1,2,\dots\},$$

$$M \leq \left[\frac{\tan\frac{\pi}{N}}{1+\tan\frac{\pi}{N}\,\cos\frac{\pi}{n}}\right]^n, \textit{ if } n=2+4\,p,\ p\in\{0,1,\dots\},$$

$$\textit{or } M \leq \left[\frac{\tan\frac{\pi}{N}}{\tan\frac{\pi}{N}\,\cos\frac{2\pi}{n}+\cos\frac{\pi}{2\,n}}\right]^n \textit{ if } n \textit{ odd.}$$

Then there exist two monotone sequences in $\mathbb{R}^{N+n}$, $\{a_m\}$ and $\{b_m\}$ with $a_0=\alpha$ and $b_0=\beta$, which converge pointwise to the extremal solutions in $[\alpha,\beta]$ of problem (P).

If we are interested into approach the solutions of the anti-periodic problem

$$(AP_n)\left\{\begin{array}{lcl} L_n\,u_k &=& f(k,u_k,u_{k+1},\dots,u_{k+n}), \quad k\in\{0,\dots,N-1\},\\ \\ u_i &=& -u_{N+i}, \quad i=0,\dots,n-1, \end{array}\right.$$

with L_n a n-th order linear operator, it is not possible, as we have noted before, to ensure existence of extremal solutions and, obviously, the Green's function associated to operator

$T_n[M_0,\ldots,M_n]$ in the set of anti-periodic functions changes sign. Is for this that the development given for periodic equations does not hold in this new situation. J. J. Nieto and the author present in [10] the concept of coupled lower and upper solutions for an abstract Hilbert-Schmith operator and deduce existence and uniqueness results for some ordinary differential equations. Under this point of view, existence and uniqueness results for anti-periodic difference equations have been obtained in [2] where, for any choice of $(K_0,\ldots,K_n)\in\mathbb{R}^{n+1}$, the following equivalent problem is considered

$$\begin{aligned} S_n[K_0,\ldots,K_n]\,u_k \equiv L_n\,u_k+\sum_{i=0}^{n}K_i\,u_{k+i} &= f(k,u_k,u_{k+1},\ldots,u_{k+n})+\sum_{i=0}^{n}K_i\,u_{k+i},\\ u_i &= -u_{N+i},\quad i=0,\ldots,n-1. \end{aligned}$$

If $S_n^{-1}[K_0,\ldots,K_n]$ exists in the set

$$D_n=\{u\in\mathbb{R}^{N+n}\,|\; u_i=-u_{N+i},\; i=0,\ldots,n-1\}, \tag{4.1}$$

defining for any $\eta\in\mathbb{R}^{P+n}$, the operators

$$A_n^+[K]\,\eta_k=\sum_{j=0}^{N-1}G_K^+(k,j)\left(P\,\eta_j+\sum_{i=0}^{n}K_i\,\eta_{j+i}\right)$$

and

$$A_n^-[K]\,\eta_k=\sum_{j=0}^{N-1}G_K^-(k,j)\left(P\,\eta_j+\sum_{i=0}^{n}K_i\,\eta_{j+i}\right).$$

Where $G_K(k,j)$ is the Green's function associated with the operator $S_n^{-1}[K_0,\ldots,K_n]$, P is the superposition operator induced by the nonlinear function

$$f:\{0,\ldots,N-1\}\times\mathbb{R}^{n+1}\to\mathbb{R},$$

$G_K^+(k,j)=\max\{G_K(k,j),0\}\geq 0$ and $G_K^-(k,j)=-\min\{G_K(k,j),0\}\geq 0$.

It is clear that the solutions of problem (AP_n) are the fixed points of the operator

$$A_n[K]=A_n^+[K]-A_n^-[K]\quad\text{ in }\mathbb{R}^{N+n}.$$

Now, given $\alpha,\,\beta\in\mathbb{R}^{N+n}$, such that $\alpha\leq\beta$ on $\{0,\ldots,N+n-1\}$, we say that α and β are coupled lower and upper solutions of (AP_n) if $S_n^{-1}[K_0,\ldots,K_n]$ exists in D_n and the inequalities

$$\alpha_k\leq A_n^+[K]\,\alpha_k-A_n^-[K]\,\beta_k,\qquad k\in\{0,\ldots,N+n-1\}$$

and

$$\beta_k\geq A_n^+[K]\,\beta_k-A_n^-[K]\,\alpha_k,\qquad k\in\{0,\ldots,N+n-1\},$$

hold for some $K=(K_0,\ldots,K_n)\in\mathbb{R}^{n+1}$.

Note that in this case we do not impose any additional conditions on α and β on the boundary of $\{0,\ldots,N+n-1\}$. One can verify, see [2], that this definition covers the definition of related lower and upper solutions for first order anti-periodic equations.

Thus, it is proved in [2] the following result.

THEOREM 4.3. *Suppose that there exists a pair of coupled lower and upper solutions of* (AP_n) *for some* $K = (K_0, \dots, K_n) \in \mathbb{R}^{n+1}$. *If* f *is a continuous function that satisfies the inequalities*

$$\begin{aligned} &-\sum_{i=0}^{n} K_i\,(v_{k+i} - u_{k+i}) \leq \\ &f(k, v_k, v_{k+1}, \dots, v_{k+n}) - f(k, u_k, u_{k+1}, \dots, u_{k+n}) \\ &\qquad\qquad\qquad \leq \sum_{i=0}^{n} (M_i - K_i)\,(v_{k+i} - u_{k+i}), \end{aligned}$$

for all $k \in \{0, \dots, N-1\}$, *for every* $u,\ v \in \mathbb{R}^{N+n}$ *such that* $\alpha \leq u \leq v \leq \beta$ *in* $\{0, \dots, N+n-1\}$ *and* $M_i \geq 0$, $i = 0, \dots, n$, *such that*

$$\left(\sum_{i=0}^{n} M_i\right) \max_{k \in \{0, \dots, N+n-1\}} \left\{ \sum_{j=0}^{N-1} |G_K(k,j)| \right\} < 1,$$

then (AP_n) *has a unique solution in* $[\alpha, \beta]$.

Proof. Define the operator:

$$B[K] : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}^{N+n}$$

as

$$B[K](\eta, \xi) = A_n^+[K]\,\eta - A_n^-[K]\,\xi.$$

One can verify that

$$B[K]([\alpha, \beta] \times [\alpha, \beta]) \subset [\alpha, \beta],$$

and that

if $\alpha \leq \eta_1 \leq \eta_2 \leq \beta$ and $\alpha \leq \xi_2 \leq \xi_1 \leq \beta$, then $B[K](\eta_1, \xi_1) \leq B[K](\eta_2, \xi_2)$.

Defining $\varphi_0 = \alpha$, $\psi_0 = \beta$, $\varphi_m = B[K](\varphi_{m-1}, \psi_{m-1})$ and $\psi_m = B[K](\psi_{m-1}, \varphi_{m-1})$, we construct two monotone convergent sequences $\{\varphi_n\} \nearrow \varphi$ and $\{\Psi_n\} \searrow \Psi$, such that

$$\varphi = A_n^+[K]\varphi - A_n^-[K]\psi \ \text{ and } \ \psi = A_n^+[K]\psi - A_n^-[K]\varphi.$$

Now, from the inequality

$$\|\psi - \varphi\| \leq \|\psi - \varphi\| \quad (\sum_{i=0}^{n} M_i) \quad \| \sum_{j=0}^{N-1} |G_K(k,j)|\ \|,$$

we conclude that $\psi \equiv \varphi$ is the unique solution of (AP_n) in $[\alpha, \beta]$. $\square$

5. First and second order anti-periodic problems. As we have seen in the previous section, to give existence and uniqueness results for anti-periodic problems we must to study the Green's function related with operator $S_n[M_0, \dots, M_n]$. To do it, it is obtained in [2] the following expression for the Green's function of anti-periodic equations.

THEOREM 5.1. *Let* $K_0, \dots, K_{n-1} \in \mathbb{R}$ *be fixed such that there exists the operator* $T_n^{-1}[K_0, \dots, K_{n-1}, 0]$ *in* D_n *(*$T_n[M_0, \dots, M_n]$ *defined in Theorem 4.1 and* D_n *given in (4.1)). Then, if the following problem*

$$\begin{aligned} T_n[K_0, \dots, K_{n-1}, 0]\, u_k &= \sigma_k, & k &\in \{0, \dots, N-1\}, \\ u_i &= -u_{N+i}, & i &= 0, \dots, n-1, \end{aligned}$$

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has a unique solution for every $\sigma \in \mathbb{R}^N$, it is given by

$$u_k = \sum_{j=0}^{N-1} G_K(k,j)\,\sigma_j \ \ \textit{for all } k \in \{0,\dots,N+n-1\},$$

where $G_K : \{0,\dots,N+n-1\} \times \{0,\dots,N-1\} \to \mathbb{R}$ satisfies

$$G_K(k,j) = \begin{cases} z_{k-j-1}, & \textit{if} \quad 0 \le j \le k-1, \\ -z_{N+k-j-1}, & \textit{if} \quad k \le j \le N-1, \end{cases} \tag{5.1}$$

and z is the unique solution of

$$\begin{aligned} T_n[K_0,\dots,K_{n-1},0]\,z_k &= 0;\ k \ge 0, \\ z_i + z_{N+i} &= 0;\ i = 0,\dots,n-2, \\ z_{n-1} + z_{N+n-1} &= 1. \end{aligned}$$

Using this expression and defining the following constant

$$L_K = \begin{cases} \dfrac{(1+K)^N - 1}{(1+(1+K)^N)\,K}, & \text{if } K > -1 \text{ and } K \ne 0; \\ 1, & \text{if } K = -1; \\ \dfrac{N}{2}, & \text{if } K = -2 \text{ and } N \text{ even; or } K = 0; \\ \dfrac{1}{|2+K|}, & \text{if } K < -1 \text{ and } N \text{ odd;} \\ \dfrac{1-(1+K)^N}{(1+(1+K)^N)(2+K)}, & \text{if } K < -1 \text{ and } N \text{ even,} \end{cases} \tag{5.2}$$

the following two existence and uniqueness results for first order anti–periodic difference equations have been proved in [2].

THEOREM 5.2. *Assume that there exist coupled lower and upper solutions of*

$$\Delta x_k = f(k, x_{k+1}), \quad k \in \{0,\dots,N-1\}; \qquad x_0 = -x_N,$$

for $K \ne -2$ whenever N is odd, or $K \ne -1$. If f is a continuous function that satisfies the inequalities

$$-K\,(x-y) \le f(k,x) - f(k,y) \le (M-K)\,(x-y),$$

for every x, $y \in \mathbb{R}$ such that $\alpha_{k+1} \le y \le x \le \beta_{k+1}$, $k \in \{0,\dots,N-1\}$ and $M \ge 0$ such that $M\,L_K < 1$ (L_K given in (5.2)), then this problem has a unique solution in $[\alpha,\beta]$.

THEOREM 5.3. *Assume that there exist coupled lower and upper solutions of*

$$-\Delta x_k = f(k, x_k),\ k \in \{0,\dots,N-1\}; \qquad x_0 = -x_N,$$

for $K \ne -2$ whenever N is odd. If f is a continuous function that satisfies the inequalities

$$-K\,(x-y) \le f(k,x) - f(k,y) \le (M-K)\,(x-y),$$

for every x, $y \in \mathbb{R}$ such that $\alpha_k \le y \le x \le \beta_k$, $k \in \{0,\dots,N-1\}$ and $M \ge 0$ such that $M\,L_K < 1$ (L_K given in (5.2)), then this problem has a unique solution in $[\alpha,\beta]$.

If we consider the second order difference equation

$$(5.3)\quad -u(k+2)=f(k,u_k),\quad k\in\{0,\dots,N-1\};\qquad u_0=-u_N,\ u_1=-u_{N+1},$$

we can prove, by using Theorem 4.3, the existence of a unique solution in the sector formed by a pair of coupled lower and upper solutions. To this end we must solve the following linear problem for some $K>0$

$$-u(k+2)+K^2\,u_k=\sigma_k,\ k\in\{0,\dots,N-1\},\quad u_0=-u_N,\,u_1=-u_{N+1}.$$

In this case, function z introduced in Theorem 5.1 is given by this expression when $K\neq 1$

$$z_k=\begin{cases} 0, & \text{if } k \text{ and } N \text{ are even;}\\ -\dfrac{K^{k-1}}{1+K^N}, & \text{if } k \text{ is odd and } N \text{ is even;}\\ -\dfrac{K^{N+k-1}}{K^{2N}-1}, & \text{if } k \text{ is even and } N \text{ is odd;}\\ \dfrac{K^{k-1}}{K^{2N}-1}, & \text{if } k \text{ and } N \text{ are odd;}\end{cases}$$

and

$$z_k=\begin{cases} 0, & \text{if } k \text{ and } N \text{ are even;}\\ -1/2, & \text{if } k \text{ is odd and } N \text{ is even;}\\ 1/2, & \text{if } k \text{ is even and } N \text{ is odd;}\\ -1/2, & \text{if } k \text{ and } N \text{ are odd;}\end{cases}$$

when $K=1$.

As consequence, from the identity (5.1), one can verify that the Green's function G_{K^2} satisfies that

$$(5.4)\qquad R_K=\sum_{j=0}^{N-1}|G_{K^2}(k,j)|=\begin{cases}\frac{|K^N-1|}{(K^N+1)\,|K^2-1|}, & \text{if } N \text{ is even,}\\ \frac{1}{|K^2-1|}, & \text{if } N \text{ is odd,}\end{cases}$$

when $K\neq 1$, and

$$(5.5)\qquad R_K=\sum_{j=0}^{N-1}|G_{K^2}(k,j)|=\begin{cases}\frac{N}{4}, & \text{if } k \text{ and } N \text{ are even,}\\ \frac{N+1}{4}, & \text{if } k \text{ is odd and } N \text{ is even,}\\ \frac{N}{2}, & \text{if } N \text{ is odd,}\end{cases}$$

if $K=1$.

Thus, we obtain the following existence and uniqueness result for the second order problem (5.3).

THEOREM 5.4. *Assume that there exist coupled lower and upper solutions of problem (5.3) for some $K^2>0$. If f is a continuous function that satisfies the inequalities*

$$-K^2\,(x-y)\leq f(k,x)-f(k,y)\leq (M-K^2)\,(x-y),$$

for every $x,\,y\in\mathbb{R}$ such that $\alpha_k\leq y\leq x\leq\beta_k$, $k\in\{0,\dots,N-1\}$ and $M\geq 0$ such that $M\,R_K<1$ (R_K given in (5.4) and (5.5)), then this problem has a unique solution in $[\alpha,\beta]$.

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