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PROBABILITY AGAINST CONDITION NUMBER AND SAMPLING OF MULTIVARIATE TRIGONOMETRIC RANDOM POLYNOMIALS*

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Abstract. The difficult factor in the condition number $\|A\|\,\|A^{-1}\|$ of a large linear system $Ap = y$ is the spectral norm of A^{-1}. To eliminate this factor, we here replace worst case analysis by a probabilistic argument. To be more precise, we randomly take p from a ball with the uniform distribution and show that then, with a certain probability close to one, the relative errors $\|\Delta p\|$ and $\|\Delta y\|$ satisfy $\|\Delta p\| \le C\|\Delta y\|$ with a constant C that involves only the Frobenius and spectral norms of A. The success of this argument is demonstrated for Toeplitz systems and for the problem of sampling multivariate trigonometric polynomials on nonuniform knots. The limitations of the argument are also shown.

Key words. condition number, probability argument, linear system, Toeplitz matrix, nonuniform sampling, multivariate trigonometric polynomial

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1. The general setting. Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be an invertible linear operator. We equip the space $\mathbb{C}^n$ with the ℓ^2 norm $\|\cdot\|$ and denote the corresponding operator norm (= spectral norm) also by $\|\cdot\|$. Let $\varrho\mathbb{B}^n = \{q \in \mathbb{C}^n : \|q\| \le \varrho\}$, pick p in $\varrho_0\mathbb{B}^n \setminus \{0\}$ and put $y = Ap$. If $\widetilde{p} \in \widetilde{\varrho}\,\mathbb{B}^n$ is a perturbation to p, then $A(p+\widetilde{p}) = y + \widetilde{y}$ with $\widetilde{y} = A\widetilde{p}$. We have $\|\widetilde{p}\| \le \|A^{-1}\|\,\|\widetilde{y}\|$ and hence

$$\frac{\|\widetilde{p}\,\|}{\|p\|} \le \|A^{-1}\|\,\frac{\|\widetilde{y}\|}{\|y\|}\,\frac{\|y\|}{\|p\|},$$

which is usually written in the form

$$\|\Delta p\| \le \kappa^1(A,p)\|\Delta y\| \tag{1.1}$$

where $\kappa^1(A,p)$ is the condition number with respect to a vector defined by $\kappa^1(A,p) := \|A^{-1}\|\,\|Ap\|/\|p\|$ (see e.g. [1, p. 605]).

On the other hand, let $\widetilde{y} \in \widetilde{\varrho}\,\mathbb{B}^n$ be a perturbation to y. Then $\|\widetilde{y}\| \le \|A\|\,\|\widetilde{p}\|$ and thus

$$\frac{\|\widetilde{y}\,\|}{\|y\|} \le \|A\|\,\frac{\|\widetilde{p}\|}{\|y\|}\,\frac{\|A^{-1}y\|}{\|p\|}.$$

This can be written as

$$\|\Delta y\| \le \kappa^2(A,y)\|\Delta p\| \tag{1.2}$$

where now $\kappa^2(A,y)$ the condition number with respect to a vector given by $\kappa^2(A,y) := \|A\|\,\|A^{-1}y\|/\|y\|$ (see e.g. [3]). Note that always $\kappa^1(A,p) \le \kappa(A) := \|A\|\,\|A^{-1}\|$ and $\kappa^2(A,y) \le \kappa(A)$.

Now take an arbitrary linear (not necessarily invertible) operator $A : \mathbb{C}^n \to \mathbb{C}^m$ and suppose p and $\widetilde{p}$ are independently and randomly drawn from $\varrho_0\mathbb{B}^n$ and $\widetilde{\varrho}\,\mathbb{B}^n$ with the uniform distribution, respectively. Assume we know that

$$\mathbb{P}\Big(\alpha\|p\| \le \|Ap\| \le \beta\|p\|\Big) \ge 1-\theta, \quad \mathbb{P}\Big(\alpha\|\widetilde{p}\,\| \le \|A\widetilde{p}\,\| \le \beta\|\widetilde{p}\,\|\Big) \ge 1-\theta, \tag{1.3}$$

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where $\mathbb{P}(E)$ is the probability of the event E, $\theta \in [0,1)$, and $\alpha, \beta \in (0,\infty)$. Note that $\mathbb{P}(||p|| = 0) = 0$ and $\mathbb{P}(||Ap|| = 0) = 0$, so that these two cases can be ignored. It follows that

$$\frac{||\widetilde{p}||}{||p||} \le \frac{1}{\alpha}\,\frac{||A\widetilde{p}||}{||p||} \le \frac{\beta}{\alpha}\,\frac{||A\widetilde{p}||}{||Ap||} = \frac{\beta}{\alpha}\,\frac{||\widetilde{y}||}{||y||}$$

or, equivalently,

$$||\Delta p|| \le \frac{\beta}{\alpha}\,||\Delta y|| \tag{1.4}$$

with a probability of at least $(1-\theta)^2$. One similarly obtains that

$$||\Delta y|| \le \frac{\beta}{\alpha}\,||\Delta p|| \tag{1.5}$$

with a probability of at least $(1-\theta)^2$.

In many cases the estimation of $||A^{-1}||$ or $||A^{-1}y||$ is difficult, so that the basic quantities $\kappa^1(A,p)$ in (1.1) or $\kappa^2(A,y)$ in (1.2) are not available. In contrast to this, it is frequently easy to establish inequalities of the form (1.3). Consequently, we can invoke (1.4) or (1.5) to prove that the equation $Ap = y$ is well-posed at least with a certain probability.

In [3] it was shown that if $q \in R\mathbb{B}^n$ is randomly drawn from the uniform distribution, then

$$E\left(\frac{||Aq||^2}{||A||^2||q||^2}\right) = \frac{||A||_{\mathrm{F}}^2}{||A||^2 n}, \quad \sigma^2\left(\frac{||Aq||^2}{||A||^2||q||^2}\right) \le \frac{2}{n+2},$$

where E and σ^2 denote the expected value and the variance and $||\cdot||_{\mathrm{F}}$ stands for the Frobenius norm (= Hilbert-Schmidt norm). Consequently, Chebyshev's inequality yields

$$\mathbb{P}\left(\left|\frac{||Aq||^2}{||q||^2} - \frac{||A||_{\mathrm{F}}^2}{n}\right| \le \varepsilon ||A||^2\right) \ge 1 - \frac{2}{(n+2)\varepsilon^2},$$

whence

$$\mathbb{P}\left(\left(\frac{||A||_{\mathrm{F}}^2}{n} - \varepsilon||A||^2\right)||q||^2 \le ||Aq||^2 \le \left(\frac{||A||_{\mathrm{F}}^2}{n} + \varepsilon||A||^2\right)||q||^2\right) \ge 1 - \frac{2}{(n+2)\varepsilon^2}.$$

Notice that this estimate is independent of R. Thus, if $||A||_{\mathrm{F}}^2/n > \varepsilon||A||^2$, then (1.3) and (1.4) give

$$||\Delta p||^2 \le \frac{||A||_{\mathrm{F}}^2/n + \varepsilon||A||^2}{||A||_{\mathrm{F}}^2/n - \varepsilon||A||^2}\,||\Delta y||^2 \tag{1.6}$$

with a probability of at least

$$\left(1 - \frac{2}{(n+2)\varepsilon^2}\right)^2. \tag{1.7}$$

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2. Toeplitz systems. Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and pick $a \in L^\infty(\mathbb{T}^d)$. Put $E_n = \{1, \ldots, n\}$. The Toeplitz operator $T_n(a)$ is the linear operator on $\ell^2(E_n^d)$ defined by

$$(T_n(a)p)_j = \sum_{k \in E_n^d} a_{j-k} p_k \quad (j \in E_n^d),$$

where $\{a_m\}_{m \in \mathbb{Z}^d}$ is the sequence of the Fourier coefficients of a. The function a is usually referred to as the symbol of $T_n(a)$. The question of whether the condition numbers $\kappa(T_n(a)) = \|T_n(a)\| \, \|T_n^{-1}(a)\|$ remain bounded has been studied for a long time (see, e.g., [5], [6], [7]). Criteria are known for all dimensions d, but these are effectively verifiable for $d = 1$ only. Moreover, for smooth functions $a \in C(\mathbb{T})$ $(d = 1)$ it turns out that generically $\kappa(T_n(a))$ either remains bounded or grows exponentially fast to infinity (see [4] and [12]).

The following result tells a completely different story. It shows that from the probabilistic point of view the systems $T_n(a)p = y$ are asymptotically always well-conditioned. We put

$$\|a\|_\infty = \operatorname*{ess\,sup}_{\tau \in \mathbb{T}^d} |a(\tau)|, \quad \|a\|_2 = \left(\int_{\mathbb{T}^d} |a(\tau)|^2 \, \frac{|d\tau|}{(2\pi)^d} \right)^{1/2},$$

where $|d\tau| = |d(e^{i\theta_1}, \ldots, e^{i\theta_d})| = d\theta_1 \ldots d\theta_d$. Note that always $\|a\|_2 \le \|a\|_\infty$.

THEOREM 2.1. *If* $\varepsilon \|a\|_\infty^2 < \|a\|_2^2$, *then*

$$\mathbb{P}\left(\|\Delta p\|^2 \le \frac{2(\|a\|_2^2 + \varepsilon \|a\|_\infty^2)}{\|a\|_2^2 - \varepsilon \|a\|_\infty^2} \, \|\Delta y\|^2 \right) \ge \left(1 - \frac{2}{n^d \varepsilon^2}\right)^2$$

for all sufficiently large n. *In particular,*

$$\lim_{n \to \infty} \mathbb{P}\left(\|\Delta p\|^2 \le \frac{2(\|a\|_2^2 + \varepsilon \|a\|_\infty^2)}{\|a\|_2^2 - \varepsilon \|a\|_\infty^2} \, \|\Delta y\|^2 \right) = 1.$$

Proof. From (1.6) and (1.7) we deduce that

$$\mathbb{P}\left(\|\Delta p\|^2 \le \frac{\|T_n(a)\|_{\mathrm{F}}^2/n^d + \varepsilon \|T_n(a)\|^2}{\|T_n(a)\|_{\mathrm{F}}^2/n^d - \varepsilon \|T_n(a)\|^2} \, \|\Delta y\|^2 \right) \ge \left(1 - \frac{2}{(n^d+2)\varepsilon^2}\right)^2.$$

It is well known that $\|T_n(a)\|^2 \to \|a\|_\infty^2$, and the multivariate version of the Avram-Parter theorem (for which see [13]) says that $\|T_n(a)\|_{\mathrm{F}}^2/n^d \to \|a\|_2^2$. $\square$

3. Limitations. The argument employed here has of course its limitations. One of them is that passage to (1.6) requires that $\|A\|_{\mathrm{F}}^2/n > \varepsilon \|A\|^2$ and that $\varepsilon > 0$ cannot be made arbitrarily small because (1.7) must be a number less than one. In several cases of interest, $A = A_n$ is the principal $n \times n$ truncation of a bounded linear operator $\mathbb{A}$ on $\ell^2(\mathbb{N})$. In other words, $A = A_n$ is the compression of $\mathbb{A}$ to $\ell^2(\{1, \ldots, n\})$. It follows that $\|A_n\|$ converges to $\|\mathbb{A}\|$. The critical term is $\|A_n\|_{\mathrm{F}}^2/n$. If this term goes to zero sufficiently fast, we cannot guarantee the inequality $\|A_n\|_{\mathrm{F}}^2/n > \varepsilon \|A_n\|^2$ with an ε such that $n\varepsilon^2 \to \infty$.

Suppose, for example, $\mathbb{K}$ is a compact operator on $\ell^2(\mathbb{N})$. Then $K_n p = y$ is expected to be an asymptotically ill-posed equation. In [3, Theorem 5.1] it was shown that indeed $\|K_n\|_{\mathrm{F}}^2/n \to 0$, which confirms that expectation.

The Toeplitz matrices $T_n(a)$ make sense for every $a \in L^1(\mathbb{T})$. However, they are the principal $n \times n$ truncations of a bounded operator if and only if $a \in L^\infty(\mathbb{T})$. Theorem 2.1 for $d = 1$ says that $\|T_n(a)\|_{\mathrm{F}}^2/(n\|T_n(a)\|^2) \to \|a\|_2^2/\|a\|_\infty^2$ if $a \in L^\infty(\mathbb{T})$. In [3] it

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was proved that $\|T_n(a)\|_{\mathrm{F}}^2/(n\|T_n(a)\|^2) \to 0$ if $a \in L^1(\mathbb{T}) \setminus L^\infty(\mathbb{T})$. Thus, the systems $T_n(a)p = y$ are not tractable by our approach if $a \in L^1(\mathbb{T}) \setminus L^\infty(\mathbb{T})$. The same must be said of the principle $n \times n$ truncations of the Hankel matrix $\mathbb{A} = (a_{j+k-1})_{j,k=1}^\infty$ generated by the Fourier coefficients of a function $a \in L^1(\mathbb{T})$. In this case always $\|A_n\|_{\mathrm{F}}^2/(n\|A_n\|^2) \to 0$ (see [3, Theorem 5.2]). Notice that this also happens for $a \in L^\infty(\mathbb{T})$, in which case the Hankel matrix $\mathbb{A}$ induces a bounded operator on $\ell^2(\mathbb{N})$.

4. Sampling of trigonometric polynomials. Let $\Pi_M = \{-M, \dots, M\}$. We now embark on the problem of finding the coefficients p_k ($k \in \Pi_M^d$) of a trigonometric polynomial

$$p(x) = \sum_{k \in \Pi_M^d} p_k e^{2\pi i k \cdot x} \tag{4.1}$$

from the values $p(x_j)$ at given points $x_1, \dots, x_r \in [0,1]^d$. Thus, letting

$$p = (p_k)_{k \in \Pi_M^d}, \quad y = (p(x_j))_{j=1}^r,$$

we have to solve the linear equation $Up = y$ with

$$U = \left(e^{2\pi i k \cdot x_j}\right)_{j \in \{1,\dots,r\}, k \in \Pi_M^d}.$$

Put $n = (2M+1)^d$. Together with the $r \times n$ system $Up = y$ we also consider the $n \times n$ system $U^*Up = y$ and the $r \times r$ system $UU^*p = y$. In these three systems, p and y need not be the same vectors, but we always assume that p is randomly drawn from a ball of appropriate dimension with the uniform distribution.

In what follows we are concerned with asymptotic results. We therefore assume that d remains fixed but that $\{x_1, \dots, x_r\}$ is a set depending on M and that M and r go to infinity.

As for random sampling, we remark that Bass and Gröchenig [2] work with a randomly chosen set $\{x_1, \dots, x_r\}$ of sampling points and prove that there exist constants $A, B \in (0, \infty)$ depending only on M and d such that

$$\mathbb{P}\Big((1-\varepsilon)r\|q\|^2 \le \|Uq\|^2 \le (1+\varepsilon)r\|q\|^2\Big) \ge 1 - Ae^{-Br\varepsilon^2/(1+\varepsilon)}$$

for every $\varepsilon \in (0,1)$ and *all* trigonometric polynomials q of the form (4.1). Clearly, this implies that

$$\mathbb{P}\left(\|\Delta p\| \le \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\,\|\Delta y\|\right) \ge 1 - Ae^{-Br\varepsilon^2/(1+\varepsilon)}$$

and hence that if we fix a threshold $C > 1$, then $\mathbb{P}(\|\Delta p\| \le C\|\Delta y\|)$ converges to 1 exponentially fast as M remains fixed and r goes to infinity (oversampling). In contrast to this, we here pick $\{x_1, \dots, x_r\}$ deterministically and show that there are constants $\alpha, \beta \in (0, \infty)$ and $\theta \in (0,1)$ such that

$$\mathbb{P}\Big(\alpha\|q\| \le \|Uq\| \le \beta\|q\|\Big) \ge 1 - \theta$$

if q is randomly taken and that $\alpha/\beta \to 1$ and $\theta \to 1$ if r and M are appropriately adjusted and go to infinity. Our conclusion is that $\mathbb{P}(\|\Delta p\| \le C\|\Delta y\|) \to 1$ as the two appropriately adjusted parameters M and r go to infinity. The techniques employed here do not yield exponentially fast convergence.

5. The spectral norm. For our purposes we need upper bounds for the spectral norms of $U, T := U^*U$, and $S := UU^*$. Clearly, $\|U\|^2 = \|T\| = \|S\|$. The matrix T is a Toeplitz matrix,

$$T = (T_{k\ell})_{k,\ell \in \Pi_M^d}, \quad T_{k,\ell} = \sum_{j=1}^{r} e^{-2\pi i (k-\ell)\cdot x_j},$$

while S is of the form

$$S = (S_{hj})_{h,j=1}^{r}, \quad S_{hj} = K(x_h - x_j), \quad K(x) = \sum_{m \in \Pi_M^d} e^{2\pi i m\cdot x}.$$

In order to compensate for clusters in the sampling set $\{x_1, \ldots, x_r\}$, it is also useful to incorporate weights $w_j > 0$ into our problem, i.e., to consider the weighted approximation problem

$$\sum_{j=1}^{r} w_j |y_j - p(x_j)|^2 \xrightarrow{p} \min .$$

Letting $W := \operatorname{diag}(w_1, \ldots, w_r)$ we are therefore also interested in bounds for the spectral norm of the Toeplitz matrix T^w given by

$$T^w := U^*WU = (T^w_{k\ell})_{k,\ell \in \Pi_M^d}, \quad T^w_{k,\ell} = \sum_{j=1}^{r} w_j e^{-2\pi i (k-\ell)\cdot x_j}.$$

The norm $\|T\|$ can be estimated from above by the L^∞ norm of the symbol of the matrix. In the present case, this is the function

$$a(x) = \sum_{k \in \Pi_M^d} T_{k0} e^{2\pi i k\cdot x} = \sum_{k \in \Pi_M^d} \sum_{j=1}^{r} e^{2\pi i k\cdot (x - x_j)} = \sum_{j=1}^{r} K(x - x_j).$$

Consequently,

$$\|T\| \le \max_{x \in [0,1]^d} \left| \sum_{j=1}^{r} K(x - x_j) \right|. \tag{5.1}$$

On the other hand, to get an upper bound for the spectral norm of S, we can invoke Gershgorin's theorem. Since the diagonal entries of S are all n, we obtain that

$$\|S\| \le n + \max_j \sum_{h \ne j} |S_{hj}| = n + \max_j \sum_{h \ne j} |K(x_h - x_j)|. \tag{5.2}$$

Of course, it would suffice to work with only one of the estimates (5.1) and (5.2). We rather want to emphasise that two frequently used tools, estimates for Toeplitz matrices through the symbol on the one hand and estimates for diagonal dominant matrices via Gershgorin on the other, lead to essentially the same problem in the case at hand.

We consider two ways of characterising a sequence of knots $\{x_1, \ldots, x_r\}$ depending on M. First, for $x, y \in \mathbb{R}^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$ we define

$$\|z\|_\infty = \max(|z_1|, \ldots, |z_d|), \quad \operatorname{dist}(x, y) = \min_{k \in \mathbb{Z}^d} \|x - y + k\|_\infty .$$

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The separation distance μ and mesh norm δ of the set $\{x_1,\dots,x_r\} \subset [0,1]^d \subset \mathbb{R}^d$ are defined as

$$\mu = \min_{j\neq\ell} \mathrm{dist}\,(x_j,x_\ell), \qquad \delta = 2\max_{x\in\mathbb{T}^d}\min_{j=1,\dots,r} \mathrm{dist}\,(x,x_j),$$

respectively. Clearly, $\mu \le r^{-1/d} \le \delta$. In what follows, B_d always denotes a positive constant depending only on d, but not necessarily the same at each occurrence.

LEMMA 5.1. *If* $\min_j \|x-x_j\|_\infty \ge \mu$, *then*

$$\sum_{j=1}^{r} |K(x-x_j)| \le B_d\left(\left(M+\frac{1}{\mu}\log\frac{1}{\mu}\right)^d - M^d\right).$$

Proof. This can be shown by using standard arguments (see, for example, [9]) and the estimates

$$|K(x)| = |K(\xi_1,\dots,\xi_d)| \le \frac{(2M+1)^{d-1}}{2\,|\xi_s|} \quad \text{if} \quad \mu \le |\xi_s| \le \frac{1}{2},$$
$$|K(x)| = |K(\xi_1,\dots,\xi_d)| \le \frac{(2M+1)^{d-2}}{4\,|\xi_s|\,|\xi_t|} \quad \text{if} \quad \mu \le |\xi_s| \le \frac{1}{2},\ \mu \le |\xi_t| \le \frac{1}{2},$$

and so on. ☐

A second way of generating a sequence of nonuniform sampling points is as follows. Let $m_1 = m_1(M),\dots,m_d = m_d(M)$ be natural numbers that go to infinity as $M \to \infty$ and divide the cube $[0,1]^d$ into $m_1 \dots m_d$ congruent boxes

$$\left[\frac{i_1}{m_1},\frac{i_1+1}{m_1}\right] \times \dots \times \left[\frac{i_d}{m_d},\frac{i_d+1}{m_d}\right]. \tag{5.3}$$

Suppose each of these boxes contains at most L points of $\{x_1,\dots,x_r\}$ where L is independent of M. Notice that such a distribution of the sampling points allows the separation distance μ to be arbitrarily small.

LEMMA 5.2. *Take one of the boxes (5.3) containing x and denote by E the union of this box and the 3^d-1 neighboring boxes. Then*

$$\sum_{j\notin E} |K(x-x_j)| \le B_d L\left(\left(M+\frac{\log m_1}{m_1}\right)\dots\left(M+\frac{\log m_d}{m_d}\right) - M^d\right).$$

Proof. Proceed as in the proof of Lemma 5.1. ☐

THEOREM 5.3. *We have*

$$\|T\| = \|S\| \le B_d\left(M+\frac{1}{\mu}\log\frac{1}{\mu}\right)^d, \tag{5.4}$$

$$\|T\| = \|S\| \le B_d L\left(M+\frac{\log m_1}{m_1}\right)\dots\left(M+\frac{\log m_d}{m_d}\right). \tag{5.5}$$

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First proof. We use (5.1) to estimate $\|T\|$. There are at most 3^d indices j such that $\|x - x_j\|_\infty < \mu$. Thus, by Lemma 5.1,

$$\begin{aligned}\|T\| &\le \sum_{\|x-x_j\|_\infty<\mu} |K(x-x_j)| + \sum_{\|x-x_j\|_\infty\ge\mu} |K(x-x_j)| \\ &\le 3^d(2M+1)^d + B_d\left(\left(M + \frac{1}{\mu}\log\frac{1}{\mu}\right)^d - M^d\right),\end{aligned}$$

which implies (5.4). Combining the same kind of argument with Lemma 5.2, we obtain (5.5). □

Second proof. This time we employ (5.2) to tackle $\|S\|$. Fix $j \in \{1, \dots, r\}$. There are at most 3^d indices h such that $\|x_h - x_j\|_\infty < \mu$, and hence Lemma 5.1 yields (5.4) as in the previous proof. Also as in the preceding proof one can deduce (5.5) from Lemma 5.2. □

6. The system $\boldsymbol{Up = y}$. We first consider a sequence of sampling knots governed by a constraint for the separation distance μ.

THEOREM 6.1. *If $\mu \ge \gamma_1 r^{-1/d}$ and $\gamma_2 M^{1/2+\eta} \le r^{1/d} \le \gamma_3 \exp\sqrt[3]{M}$ with positive constants γ_1, γ_2, γ_3, η, then there are two sequences $\{C_M\}_{M=1}^\infty$ and $\{P_M\}_{M=1}^\infty$ such that*

$$C_M > 1, \quad \lim_{M\to\infty} C_M = 1, \quad P_M < 1, \quad \lim_{M\to\infty} P_M = 1$$

and $\mathbb{P}\Big(\|\Delta p\| \le C_M\|\Delta y\|\Big) \ge P_M$.

Proof. In the case at hand, the constant in (1.6) is

$$\frac{\|U\|_{\mathrm{F}}^2/n + \varepsilon\|U\|^2}{\|U\|_{\mathrm{F}}^2/n - \varepsilon\|U\|^2}. \tag{6.1}$$

Obviously, $\|U\|_{\mathrm{F}}^2 = rn$. The quantity $\|U\|^2 = \|T\| = \|S\|$ can be estimated by (5.4). Thus, the constant (6.1) is not greater than

$$\frac{1 + B_d\varepsilon r^{-1}(M + (1/\mu)\log(1/\mu))^d}{1 - B_d\varepsilon r^{-1}(M + (1/\mu)\log(1/\mu))^d}. \tag{6.2}$$

By assumption,

$$\frac{1}{\mu}\log\frac{1}{\mu} = O(r^{1/d}\log r) = O(r^{1/d}M^{1/3}).$$

Choose $\varepsilon = M^{-d/2+\eta d/2}$. Then

$$\begin{aligned}&\varepsilon r^{-1}(M + (1/\mu)\log(1/\mu))^d \\ &= \left(M^{-1/2+\eta/2}r^{-1/d}M + M^{-1/2+\eta/2}r^{-1/d}O(r^{1/d}M^{1/3})\right)^d \\ &= \left(M^{1/2+\eta/2}r^{-1/d}M^{-1/6+\eta/2}O(1)\right)^d = o(1),\end{aligned}$$

since we may assume that $-1/6 + \eta/2 < 0$. Consequently, (6.2) is

$$\frac{1+o(1)}{1-o(1)} =: C_M^2.$$

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From (1.6) and (1.7) we now obtain that

$$\mathbb{P}\Big(\|\Delta p\|^2 \le C_M^2\|\Delta y\|^2\Big) \ge \left(1 - \frac{2}{(n+2)\varepsilon^2}\right)^2 =: P_M.$$

Since

$$\frac{2}{(n+2)\varepsilon^2} = O\left(\frac{1}{M^d}\,M^{d-\eta d}\right) = O\left(M^{-\eta d}\right) = o(1),$$

we finally see that $P_M \to 1$ as $M \to \infty$. $\square$

In the following theorem, we consider a sequence of knots as before Lemma 5.2.

THEOREM 6.2. *If $r \ge \gamma M^d$ with a constant $\gamma > 0$ then there are two sequences $\{C_M\}_{M=1}^\infty$ and $\{P_M\}_{M=1}^\infty$ such that*

$$C_M > 1, \quad C_M = 1 + O\left(M^{-d/3}\right), \quad P_M < 1, \quad P_M = 1 - O\left(M^{-d/3}\right)$$

and $\mathbb{P}\Big(\|\Delta p\| \le C_M\|\Delta y\|\Big) \ge P_M$.

Proof. By (5.5), the constant (6.1) is at most

$$\frac{1 + \varepsilon B_d L r^{-1}\Big(M + (\log m_1)/m_1\Big)\ldots\Big(M + (\log m_d)/m_d\Big)}{1 - \varepsilon B_d L r^{-1}\Big(M + (\log m_1)/m_1\Big)\ldots\Big(M + (\log m_d)/m_d\Big)}.$$

Taking into account that $r^{-1} = O(M^{-d})$ and choosing $\varepsilon = M^{-d/3}$, we see that this is

$$\frac{1 + O(M^{-d/3})}{1 - O(M^{-d/3})} =: C_M^2 \quad \text{with} \quad C_M = 1 + O(M^{-d/3}).$$

From (1.6) and (1.7) we infer that $\mathbb{P}\Big(\|\Delta p\| \le C_M\|\Delta y\|\Big) \ge P_M$ with

$$P_M := \left(1 - \frac{2}{[(2M+1)^d + 2]\,M^{-2d/3}}\right)^2 = 1 - O(M^{-d/3}). \quad \square$$

Theorems 6.1 and 6.2 are based on the hypothesis that $r \ge \gamma_2 n^{1/2+\eta}$ (with some $\eta > 0$) and $r \ge \gamma n$. Thus, r has to go to infinity with a rate that does not lag too much in comparison with n. This restriction is caused by the reason outlined in Section 3.

7. The system $\sqrt{W}Up = y$. We now consider a sequence of sampling knots controlled by a constraint for the mesh norm δ. Let w_j be the area (volume) of the Voronoi region of the point x_j.

THEOREM 7.1. *If $\delta \le \gamma M^{-1}$ with a positive constant γ then there are two sequences $\{C_M\}_{M=1}^\infty$ and $\{P_M\}_{M=1}^\infty$ such that*

$$C_M > 1, \quad C_M = 1 + O\left(M^{-d/2}\right), \quad P_M < 1, \quad P_M = 1 - O\left(M^{-d/2}\right)$$

and $\mathbb{P}\Big(\|\Delta p\| \le C_M\|\Delta y\|\Big) \ge P_M$.

Proof. From [8, Theorem 5] we obtain that $\|T^w\| \le \left(2+e^{2\pi d M\delta}\right)^2$ and since

$$\|\sqrt{W}U\|_{\mathrm{F}}^2 = \sum_{k\in\Pi_M^d}\sum_{j=1}^{r} w_j = \sum_{k\in\Pi_M^d} 1 = n,$$

the constant in (1.6) is at most

$$\frac{1+\varepsilon(2+e^{2\pi d M\delta})^2}{1-\varepsilon(2+e^{2\pi d M\delta})^2}\,.$$

Now take $\varepsilon = M^{-d/2}$ and proceed as in the proof of Theorem 6.1 $\square$.

8. The system $U^*Up = y$. We now consider the $n\times n$ system $U^*Up = y$ in which p and the perturbation $\widetilde{p}$ are taken from $\varrho\,\mathbb{B}^n$ and $\widetilde{\varrho}\,\mathbb{B}^n$ with the uniform distribution. We establish the analogues of Theorems 6.1 and 6.2.

THEOREM 8.1. *If $\mu \ge \gamma_1 r^{-1/d}$ and $\gamma_2 M^{3/4+\eta} \le r^{1/d} \le \gamma_3 \exp\sqrt[5]{M}$ with positive constants γ_1, γ_2, γ_3, η, then there are two sequences $\{C_M\}_{M=1}^{\infty}$ and $\{P_M\}_{M=1}^{\infty}$ such that*

$$C_M > 1,\quad \lim_{M\to\infty} C_M = 1,\quad P_M < 1,\quad \lim_{M\to\infty} P_M = 1$$

and $\mathbb{P}\Big(\|\Delta p\| \le C_M\|\Delta y\|\Big) \ge P_M$.

Proof. The constant in (1.6) is

$$\frac{\|U^*U\|_{\mathrm{F}}^2/n+\varepsilon\|U^*U\|^2}{\|U^*U\|_{\mathrm{F}}^2/n-\varepsilon\|U^*U\|^2} = \frac{\|T\|_{\mathrm{F}}^2/n+\varepsilon\|T\|^2}{\|T\|_{\mathrm{F}}^2/n-\varepsilon\|T\|^2}. \tag{8.1}$$

The entries on the main diagonal of the $n\times n$ matrix T are all equal to r. Hence $\|T\|_{\mathrm{F}}^2 \ge nr^2$ and thus (8.1) is at most

$$\frac{r^2+\varepsilon\|T\|^2}{r^2-\varepsilon\|T\|^2}. \tag{8.2}$$

Choosing $\varepsilon = M^{-d/2+\eta d}$ and estimating as in the proof of Theorem 6.1, we arrive at the assertion. $\square$

THEOREM 8.2. *If $r \ge \gamma M^d$ with a constant $\gamma > 0$ then there are two sequences $\{C_M\}_{M=1}^{\infty}$ and $\{P_M\}_{M=1}^{\infty}$ such that*

$$C_M > 1,\quad C_M = 1+O\left(M^{-d/3}\right),\quad P_M < 1,\quad P_M = 1-O\left(M^{-d/3}\right)$$

and $\mathbb{P}\Big(\|\Delta p\| \le C_M\|\Delta y\|\Big) \ge P_M$.

Proof. Start with (8.2), take $\varepsilon = M^{-d/3}$, and proceed as in the proof of Theorem 6.2. $\square$

9. The system $UU^*p = y$. In this section, we turn our attention to the $r\times r$ system $UU^*p = y$ where p and $\widetilde{p}$ are drawn from the balls $\varrho\,\mathbb{B}^r$ and $\widetilde{\varrho}\,\mathbb{B}^r$ with the uniform distribution.

THEOREM 9.1. *If $\mu \ge \gamma_1 r^{-1/d}$ and $r^{1/d} \le \gamma_2 M^{4/3-\eta}$ with positive constants γ_1, γ_2, η, then there are two sequences $\{C_M\}_{M=1}^{\infty}$ and $\{P_M\}_{M=1}^{\infty}$ such that*

$$C_M > 1,\quad \lim_{M\to\infty} C_M = 1,\quad P_M < 1,\quad \lim_{M\to\infty} P_M = 1$$

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and $\mathbb{P}\Big(\|\Delta p\| \le C_M \|\Delta y\|\Big) \ge P_M$.

Proof. Now the constant in (1.6) equals

$$\frac{\|UU^*\|_{\mathrm{F}}^2/r + \varepsilon\|UU^*\|^2}{\|UU^*\|_{\mathrm{F}}^2/r - \varepsilon\|UU^*\|^2} = \frac{\|S\|_{\mathrm{F}}^2/r + \varepsilon\|S\|^2}{\|S\|_{\mathrm{F}}^2/r - \varepsilon\|S\|^2}. \tag{9.1}$$

All entries of the main diagonal of the $r \times r$ matrix S are all equal to n, whence $\|S\|_{\mathrm{F}}^2 \ge rn^2$. Consequently, (9.1) does not exceed

$$\frac{n^2 + \varepsilon\|S\|^2}{n^2 - \varepsilon\|S\|^2}. \tag{9.2}$$

Our assumption implies that $M \ge \gamma_3 r^{3/(4d)+2\varrho}$ with some (arbitrarily small) $\varrho > 0$. Letting $\varepsilon = r^{-1/2+2\varrho d}$ and estimating as in the proof of Theorem 6.1 we arrive at the assertion. ☐

The sequence of sampling knots in the following theorem is again as before Lemma 5.2. Notice that although we make no assumption about the connection between r and M in this theorem, the nature of the knot sequence involves that necessarily $r \le Lm_1 \ldots m_d$, so that r is in fact majorized by M.

THEOREM 9.2. *There are two sequences* $\{C_M\}_{M=1}^{\infty}$ *and* $\{P_M\}_{M=1}^{\infty}$ *such that*

$$C_M > 1, \quad C_M = 1 + O\left(M^{-d/3}\right), \quad P_M < 1, \quad P_M = 1 - O\left(M^{-d/3}\right)$$

and $\mathbb{P}\Big(\|\Delta p\| \le C_M \|\Delta y\|\Big) \ge P_M$.

Proof. This can be obtained from (9.2) with the choice $\varepsilon = M^{-d/3}$ as in the proof of Theorem 6.2. ☐

Thus, while for the matrices U and U^*U the number r of sampling points has to be greater than a certain power of M to guarantee a good conditioning with high probability, the situation is reverse for the matrix UU^*: here r must not exceed a certain quantity increasing with M in order to ensure a good conditioning with a probability close to one.

10. Example. Nonuniform sampling problems are currently emerging in more and more applications. The matrix-vector multiplication with the Vandermonde-like matrix U can realized in an efficient way with the Fourier transform for nonequispaced data [11]. The irregular sampling problem of band-limited functions can be solved with an iterative method such as CGNR or CGNE. In practice, the numerical results are usually much better than the deterministic predictions [10]. As an example we consider the Fourier transform on a polar grid (and are thus in particular in the case $d = 2$). For the rather complicated deterministic analysis of the example see [2], [8], and [9], for example. The points of the polar grid lie on concentric circles around the origin, restricted to the unit square (see Figure 10.1, left). Instead of concentric circles, the nodes of the *linogram* or *pseudo-polar grid* lie on concentric squares around the origin. Thus, they are typically given by a slope and an intercept. Depending on the slope, we distinguish two sets of nodes,

$$x_{t,j}^{\mathrm{H}} := \left(\frac{j}{R}, \frac{4t}{T}\frac{j}{R}\right), \quad x_{t,j}^{\mathrm{V}} := \left(-\frac{4t}{T}\frac{j}{R}, \frac{j}{R}\right) \tag{10.1}$$

where $j = -R/2, \ldots, R/2 - 1$ and $t = -T/4, \ldots, T/4 - 1$ (see Figure 10.1, right). The number of points for this grid is about $r = TR$. We choose $R = 4M$ and $T = 8M$. This gives $\delta = 1/(4M)$. In the proof of Theorem 7.1 we established that

$$\|\Delta p\|^2 \le \frac{1 + \varepsilon(2 + e^{\pi})^2}{1 - \varepsilon(2 + e^{\pi})^2}\|\Delta y\|^2 \tag{10.2}$$

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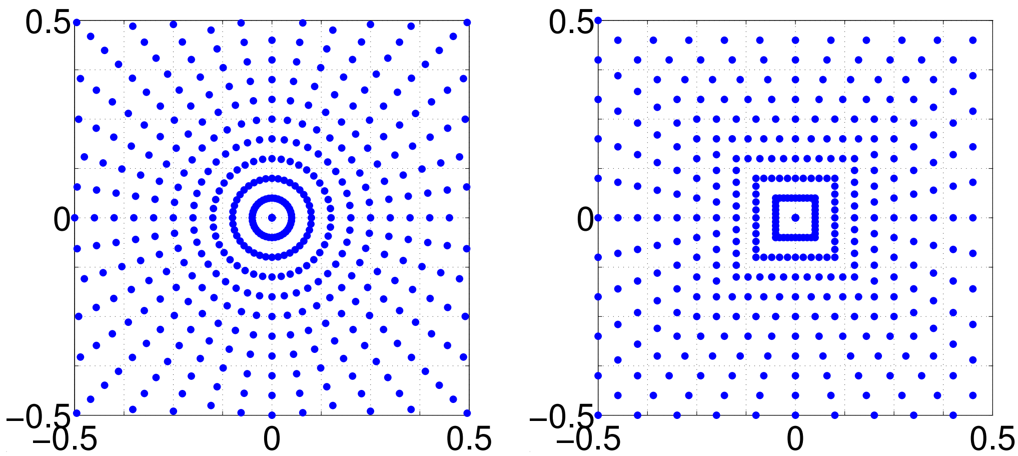

FIG. 10.1. *Polar grid (left) and linogram grid (right).*

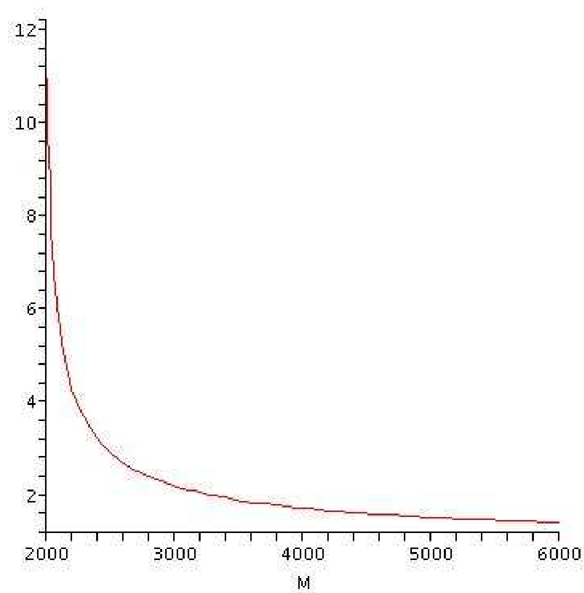

FIG. 10.2. *Estimate for the "probabilistic" condition number* $\|\Delta p\|/\|\Delta y\|$ *with a probability of at least 0.9.*

with a probability of at least

$$P := \left(1 - \frac{2}{(4M^2 + 4M + 3)\varepsilon^2}\right)^2 .$$

Figure 10.2 shows the constant in (10.2) and thus an estimate for the "probabilistic" condition number $\|\Delta p\|/\|\Delta y\|$ for the probability $P = 0.9$. Consequently, the reconstruction of a trigonometric polynomial from the values on the linogram grid can be realized very efficiently. Similar results can be obtained for the polar grid (Figure 10.1, left).

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