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A PARTITION OF THE UNIT SPHERE INTO REGIONS OF EQUAL AREA AND SMALL DIAMETER*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. The recursive zonal equal area sphere partitioning algorithm is a practical algorithm for partitioning higher dimensional spheres into regions of equal area and small diameter. This paper describes the partition algorithm and its implementation in Matlab, provides numerical results and gives a sketch of the proof of the bounds on the diameter of regions. A companion paper gives details of the proof.

Key words. sphere, partition, area, diameter, zone

AMS subject classifications. 11K38, 31-04, 51M15, 52C99, 74G65

1. Introduction. For dimension d, the unit sphere $\mathbb{S}^d$ embedded in $\mathbb{R}^{d+1}$ is

$$\mathbb{S}^d := \left\{ x \in \mathbb{R}^{d+1} \;\middle|\; \sum_{k=1}^{d+1} x_k^2 = 1 \right\}. \tag{1.1}$$

This paper describes a partition of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ which is here called the recursive zonal equal area (EQ) partition. The partition $\mathrm{EQ}(d, N)$ is a partition of the unit sphere $\mathbb{S}^d$ into N regions of equal area and small diameter. It is defined via the algorithm given in Section 3.

Figure 1.1 shows an example of the partition $\mathrm{EQ}(2, 33)$, the recursive zonal equal area partition of $\mathbb{S}^2$ into 33 regions.

For the purposes of this paper, we define an equal area partition of $\mathbb{S}^d$ in the following way.

DEFINITION 1.1. *An* equal area partition *of $\mathbb{S}^d$ is a nonempty finite set P of* regions, *which are closed Lebesgue measurable subsets of $\mathbb{S}^d$ such that*

1. *the regions cover $\mathbb{S}^d$, that is*

$$\bigcup_{R \in P} R = \mathbb{S}^d;$$

2. *the regions have equal area, with the Lebesgue area measure σ of each $R \in P$ being*

$$\sigma(R) = \frac{\sigma(\mathbb{S}^d)}{|P|},$$

where $|P|$ denotes the cardinality of P; and

3. *the boundary of each region has area measure zero, that is, for each $R \in P$, $\sigma(\partial R) = 0$.*

Note that conditions 1 and 2 above imply that the intersection of any two regions of P has measure zero. This in turn implies that any two regions of P are either disjoint or only have boundary points in common. Condition 3 excludes pathological cases which are not of interest in this paper.

This paper considers the Euclidean diameter of each region, defined as follows.

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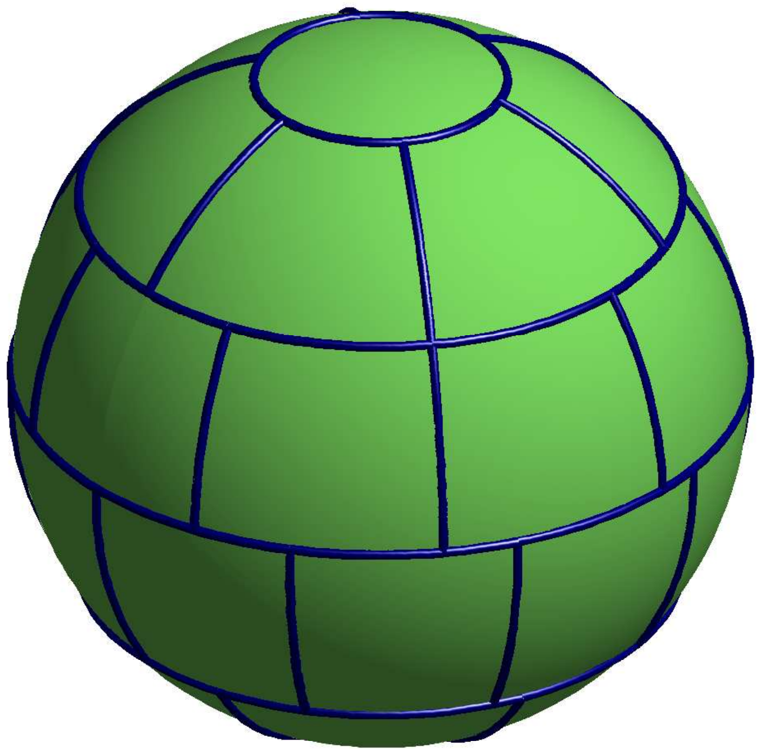

FIG. 1.1. *Partition* $\mathrm{EQ}(2, 33)$

DEFINITION 1.2. *The* diameter *of a region* $R \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$ *is*

$$\operatorname{diam} R := \sup\{e(x, y) \mid x, y \in R\},$$

where $e(x, y)$ *is the* $\mathbb{R}^{d+1}$ *Euclidean distance* $\|x - y\|$.

The following definitions are specific to the main theorems stated in this paper.

DEFINITION 1.3. *A set* Z *of partitions of* $\mathbb{S}^d$ *is said to be* diameter-bounded *with* diameter bound $K \in \mathbb{R}_+$ *if for all* $P \in Z$, *for each* $R \in P$,

$$\operatorname{diam} R \leqslant K \, |P|^{-1/d}.$$

DEFINITION 1.4. *The set of recursive zonal equal area partitions of* $\mathbb{S}^d$ *is defined as*

$$\mathrm{EQ}(d) := \{\mathrm{EQ}(d, N) \mid N \in \mathbb{N}_+\}.$$

where $\mathrm{EQ}(d, N)$ *denotes the recursive zonal equal area partition of the unit sphere* $\mathbb{S}^d$ *into* N *regions, which is defined via the algorithm given in Section 3.*

This paper claims that the partition defined via the algorithm given in Section 3 is an equal area partition which is diameter bounded. This is formally stated in the following

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theorems.

THEOREM 1.5. *For $d \geqslant 1$ and $N \geqslant 1$, the partition* $\mathrm{EQ}(d,N)$ *is an equal area partition of* $\mathbb{S}^d$.

THEOREM 1.6. *For $d \geqslant 1$,* $\mathrm{EQ}(d)$ *is diameter-bounded in the sense of Definition 1.3.*

These theorems are proved in detail in the companion paper [13]. The proof of Theorem 1.5 is straightforward, following immediately from the construction of Section 3. A sketch of the proof of Theorem 1.6 is given in Section 6 of this paper.

The construction for the recursive zonal equal area partition is based on Zhou's construction for $\mathbb{S}^2$ [27], as modified by Saff [16], and on Sloan's notes on the partition of $\mathbb{S}^3$ [18].

The existence of partitions of $\mathbb{S}^d$ into regions of equal area and small diameter is well known and has been used in a number of ways. Alexander [2, Lemma 2.4, p. 447] uses such a partition of $\mathbb{S}^2$ to derive a lower bound for the maximum sum of distances between points. The paper also suggests a construction for $\mathbb{S}^2$ [2, p. 447], which differs from Zhou's construction. For $6m^2$ regions, Alexander begins with a spherical cube which divides $\mathbb{S}^2$ into six regions, then divides each face into m slices by using a pencil of $m-1$ great circles with positions adjusted so that each slice has the same area. Finally, each slice is divided into m regions of equal area by another pencil of $m-1$ great circles, which may differ for each slice. Alexander then asserts that the diameters are the right magnitude and omits a proof. This construction has an obvious generalization for $\mathbb{S}^d$ with $2(d+1)m^d$ regions. Start with the appropriate spherical hypercube, then divide each face into m equal pieces, and so on. It is not clear that this partition of $\mathbb{S}^d$ is diameter-bounded in the sense of Definition 1.3.

The existence of a diameter bounded set of equal area partitions of $\mathbb{S}^d$ is used by Stolarsky [20], Beck and Chen [3] and Bourgain and Lindenstrauss [4], but no construction is given.

Stolarsky [20, p. 581] asserts the existence of such a set, saying simply,

> "Now clearly one can choose the A_i so that their Euclidean diameters are $\gg\ll N^{-1/(m-1)}$ for $1 \leqslant i \leqslant N$."

Here Stolarsky is discussing a partition of $\mathbb{S}^{m-1}$ into N regions labelled A_i. Stolarsky's notation $\gg\ll$ is equivalent to order notation, and his assertion can be restated as:

> There are constants $c, C > 0$ such that for any $N > 0$ one can choose the regions A_i so that their Euclidean diameters are bounded by $cN^{-1/(m-1)} \leqslant \operatorname{diam} A_i \leqslant CN^{-1/(m-1)}$ for $1 \leqslant i \leqslant N$.

The paper then uses this assertion to prove a theorem which relates the sum of distances between N points on $\mathbb{S}^{m-1}$ to a discrepancy which is defined in the paper.

Beck and Chen [3, pp. 237–238] essentially cites Stolarsky's result, asserting that

> "One can easily find a partition
>
> $$\mathbb{S}^d = \bigcup_{\ell=1}^{N} R_\ell$$
>
> such that for $1 \leqslant \ell \leqslant N$, $\sigma(R_\ell) = \sigma(\mathbb{S}^d)/N$ and $\operatorname{diam} R_\ell \ll N^{-1/d}$, where $\operatorname{diam} R_\ell$ is the diameter of R_ℓ."
>
> [With notation adjusted to match this paper.]

Bourgain and Lindenstrauss [4, p. 26] cite Beck and Chen [3] and use a diameter-bounded equal area partition of $\mathbb{S}^{n-1}$ to prove their Theorem 1 on the approximation of zonoids by zonotopes.

Stolarsky's assertion can be proven using the method used by Feige and Schechtman [7] to prove the following lemma.

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LEMMA 1.7. (Feige and Schechtman [7, Lemma 21, pp. 430–431]) *For each* $0 < \gamma < \pi/2$ *the sphere* $\mathbb{S}^{d-1}$ *can be partitioned into* $N = \big(O(1)/\gamma\big)^d$ *regions of equal area, each of diameter at most* γ.

Feige and Schechtman's proof is not fully constructive. The construction assumes the existence of an algorithm which creates a packing on the unit sphere having the maximum number of equal spherical caps of given spherical radius [25, p1091] [26, Lemma 1, p. 2112].

Wagner [23, p. 112] implies that a diameter-bounded sequence of equal area partitions of $\mathbb{S}^d$ can be constructed where each region is a rectangular polytope in spherical polar coordinates. For $\mathbb{S}^2$, this is the same form of partition as [27] and [16], and for $\mathbb{S}^d$, this is the same form as given in this paper.

Rakhmanov, Saff and Zhou [15], Zhou [27, 28] and Kuijlaars and Saff [17, 11] use the partition of $\mathbb{S}^2$ given by Zhou's construction to obtain bounds on the extremal energy of point sets.

Other constructions for equal area partitions of $\mathbb{S}^2$ have been used in the geosciences [10, 19] and astronomy [21, 5, 8], but these constructions do not have a proven bound on the diameter of regions. In particular, the regions of the "igloo" partitions of [5] have the same form as [27] and [16]. The paper [5] also discusses nesting schemes for "igloo" partitions.

This paper is organized as follows. Section 2 presents enough of the geometry of the unit sphere $\mathbb{S}^d$ to permit the description of the partition algorithm. Section 3 describes the partition algorithm. Section 4 presents an analysis of the regions of a partition. Section 5 proves a per-region bound on diameters. Section 6 sketches the proof of Theorem 1.6. Section 7 describes the Matlab implementation of the partition algorithm. Section 8 presents numerical results. Appendix A contains detailed proofs of lemmas.

2. The geometry of the unit sphere $\mathbb{S}^d$. This section describes some well known but essential aspects of the geometry of $\mathbb{S}^d$.

Spherical polar coordinates. Spherical polar coordinates describe a point $\mathbf{a}$ on $\mathbb{S}^d$ by using one longitude, $\alpha_1 \in \mathbb{R}$, considered modulo 2π, and $d-1$ colatitudes, α_k, for $k \in \{2,\ldots,d\}$, with $0 \leqslant \alpha_k \leqslant \pi$. The coordinates $(0,\alpha_2,\ldots,\alpha_d)$ and $(2\pi,\alpha_2,\ldots,\alpha_d)$ therefore describe the same point.

In these coordinates, for $d > 1$ the major colatitude is taken to be the last, α_d. Thus the North pole of $\mathbb{S}^d$ corresponds to $\alpha_d = 0$.

The sphere $\mathbb{S}^d$ defined by (1.1) is embedded in the vector space $\mathbb{R}^{d+1}$ with center at the origin.

A point $\mathbf{a} \in \mathbb{S}^d$ can therefore be described by its spherical polar coordinates or by its corresponding Cartesian coordinate vector.

DEFINITION 2.1. *We define the spherical polar to Cartesian coordinate map* $\mathbf{x}$ *by*

$$\mathbf{x} : \mathbb{R} \times [0,\pi]^{d-1} \to \mathbb{S}^d \subset \mathbb{R}^{d+1},$$
$$\mathbf{x}(\alpha_1,\alpha_2,\ldots,\alpha_d) = (a_1,a_2,\ldots,a_{d+1}),$$

where

$$a_1 := \cos\alpha_1 \prod_{j=2}^{d} \sin\alpha_j, \quad a_2 := \prod_{j=1}^{d} \sin\alpha_j,$$

$$a_k := \cos\alpha_{k-1} \prod_{j=k}^{d} \sin\alpha_j, \quad k \in \{3,\ldots,d+1\}.$$

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For example, if a point $\mathbf{a} \in \mathbb{S}^2$ has spherical polar coordinates (ϕ, θ), its Cartesian coordinates are $\mathbf{x}(\phi, \theta) = (\sin\theta\ \cos\phi, \sin\theta\ \sin\phi, \cos\theta)$.

Poles, parallels and meridians. The spherical polar coordinates for $\mathbb{S}^2$ can be described in terms of parallels of latitude and meridians of longitude. Here we generalize these concepts to $\mathbb{S}^d$.

The point $(0, \ldots, 1) = \mathbf{x}(0, \ldots, 0)$ is called the *North pole* and the point $(0, \ldots, -1) = \mathbf{x}(0, \ldots, 0)$ is called the *South pole*.

DEFINITION 2.2. *Let $\circ\mathbb{S}^d$ denote the unit sphere $\mathbb{S}^d$ excluding the North and South poles. For $\mathbf{a} := \mathbf{x}(\alpha_1, \ldots, \alpha_{d-1}, \alpha_d) \in \circ\mathbb{S}^d$ the* parallel *through $\mathbf{a}$ is*

$$\ominus(\mathbf{a}) := \{\mathbf{x}(\beta_1, \ldots, \beta_{d-1}, \alpha_d) \mid (\beta_1, \ldots, \beta_{d-1}) \in [0, 2\pi) \times [0, \pi]^{d-2}\} \tag{2.1}$$

and the meridian *through $\mathbf{a}$ is*

$$\oslash(\mathbf{a}) := \{\mathbf{x}(\alpha_1, \ldots, \alpha_{d-1}, \beta) \mid \beta \in (0, \pi)\}. \tag{2.2}$$

Euclidean and spherical distances. DEFINITION 2.3. *The spherical distance $s(\mathbf{a}, \mathbf{b})$ and the Euclidean distance $e(\mathbf{a}, \mathbf{b})$ between two points $\mathbf{a}, \mathbf{b} \in \mathbb{S}^d$ are*

$$s(\mathbf{a}, \mathbf{b}) := \cos^{-1}(\mathbf{a} \cdot \mathbf{b}), \quad e(\mathbf{a}, \mathbf{b}) := \|\mathbf{a} - \mathbf{b}\|.$$

We now recall a couple of well known elementary results.

LEMMA 2.4. *For $\mathbb{S}^d$,*

1. *Spherical distance is the arc length of an arc of a great circle, up to π.*
2. *For any two points $\mathbf{a}, \mathbf{b} \in \mathbb{S}^d$, the Euclidean and spherical distances are related by*

$$e(\mathbf{a}, \mathbf{b}) = \Upsilon\left(s(\mathbf{a}, \mathbf{b})\right),$$

where

$$\Upsilon(\theta) := \sqrt{2 - 2\cos\theta} = 2\sin\frac{\theta}{2}. \tag{2.3}$$

Spherical caps, collars and zones. For $d > 1$, for any point $\mathbf{a} \in \mathbb{S}^d$ and any angle $\theta \in [0, \pi]$, the closed *spherical cap* $S(\mathbf{a}, \theta)$ is

$$S(\mathbf{a}, \theta) := \{\mathbf{b} \in \mathbb{S}^d \ \mid\ s(\mathbf{a}, \mathbf{b}) \leqslant \theta\},$$

that is the set of points of $\mathbb{S}^d$ whose spherical distance to $\mathbf{a}$ is at most θ.

A closed spherical *collar* or annulus is the closure of the set difference between two spherical caps with the same center and different radii.

For $d > 1$, a *zone* is a closed subset of $\mathbb{S}^d$ which can be described by

$$Z(\alpha, \beta) := \left\{\mathbf{x}(\gamma_1, \ldots, \gamma_d) \in \mathbb{S}^d \ \mid\ \gamma_d \in [\alpha, \beta]\right\}, \tag{2.4}$$

where $0 \leqslant \alpha < \beta \leqslant \pi$.

$Z(0, \alpha)$ is a North polar cap, that is a spherical cap with center the North pole, and $Z(\alpha, \pi)$ is a South polar cap. If $0 < \alpha < \beta < \pi$, $Z(\alpha, \beta)$ is a collar.

Equatorial map. We define the *equatorial map* $\Pi : \circ\mathbb{S}^d \to \mathbb{S}^{d-1}$, using the following construction. Take any point $\mathbf{a} = \mathbf{x}(a_1, \ldots, a_d)$ of $\circ\mathbb{S}^d$ and find the intersection between the equator and $\oslash(\mathbf{a})$, the meridian through $\mathbf{a}$. This is the point $\mathbf{a}' = \mathbf{x}(a_1, \ldots, a_{d-1}, \frac{\pi}{2})$. Now identify the equator of $\mathbb{S}^d$ with the unit sphere $\mathbb{S}^{d-1}$ and so identify $\mathbf{a}' \in \mathbb{S}^d$ with $\Pi(\mathbf{a}) := \mathbf{x}(a_1, \ldots, a_{d-1}) \in \mathbb{S}^{d-1}$. We call $\Pi(\mathbf{a})$ the *equatorial image* of $\mathbf{a}$ in $\mathbb{S}^{d-1}$.

By a slight abuse of notation, for any $S \subset \mathbb{S}^d$ we define the equatorial image of S to be $\Pi\, S := \Pi(S \cap \circ\mathbb{S}^d)$. Thus the equatorial image of any zone of $\mathbb{S}^d$ is the whole of $\mathbb{S}^{d-1}$.

Regions which are rectilinear in spherical polar coordinates. To describe the recursive zonal equal area partition, we need to describe the regions which it produces. In general, these regions are rectilinear in spherical polar coordinates, and are of the form

$$R = \mathbf{x}\left([\tau_1, \upsilon_1] \times \ldots \times [\tau_d, \upsilon_d]\right), \tag{2.5}$$

where

$$\tau_1 \in [0, 2\,\pi), \quad \upsilon_1 \in (\tau_1, \tau_1 + 2\,\pi], \quad 0 \leqslant \tau_k < \upsilon_k \leqslant \pi, \quad k \in \{2, \ldots, d\}. \tag{2.6}$$

More specifically, for the pair of d-tuples $(\tau_1, \ldots, \tau_d), (\upsilon_1, \ldots, \upsilon_d) \in \mathbb{R} \times [0, \pi]^{d-1}$ satisfying (2.6) we define the region

$$\begin{aligned} \mathcal{R}\big((\tau_1, \ldots, \tau_d), (\upsilon_1, \ldots, \upsilon_d)\big) &:= \{\mathbf{x}(\alpha_1, \ldots, \alpha_d) \mid \alpha_k \in [\tau_k, \upsilon_k], k \in \{1, \ldots, d\}\} \\ &= \mathbf{x}\left([\tau_1, \upsilon_1] \times \ldots \times [\tau_d, \upsilon_d]\right). \end{aligned}$$

In this way, each region of $\mathbb{S}^d$ of the form (2.5) can be represented by the pair of d-tuples $(\tau_1, \ldots, \tau_d), (\upsilon_1, \ldots, \upsilon_d)$. In particular, for $d > 1$, a North polar cap of $\mathbb{S}^d$ can be described as

$$\mathcal{R}\big((0, 0, \ldots, 0, 0), (2\,\pi, \pi, \ldots, \pi, \upsilon_d)\big) = \mathbf{x}\left([0, 2, \pi] \times [0, \pi]^{d-2} \times [0, \upsilon_d]\right),$$

and a South polar cap of $\mathbb{S}^d$ can be described as

$$\mathcal{R}\big((0, 0, \ldots, 0, \tau_d), (2\,\pi, \pi, \ldots, \pi, \pi)\big) = \mathbf{x}\left([0, 2\,\pi] \times [0, \pi]^{d-2} \times [\tau_d, \pi]\right).$$

Each region of $\mathbb{S}^d$ of the form (2.5) has 2^d *pseudo-vertices*, each of which is a d-tuple in spherical polar coordinates $\mathbb{R} \times [0, \pi]^{d-1}$. The term "pseudo-vertex" is used because we may have degenerate cases where the points of $\mathbb{S}^d$ corresponding to two or more of these 2^d d-tuples coincide, as must happen when $\tau_1 = 0$ and $\upsilon_1 = 2\pi$. In these degenerate cases, the corresponding point of $\mathbb{S}^d$ may be an interior point of the region, or a point where the boundary of the region is smooth. Examples are:

1. The pair $\big((0, 0), (2\pi, \upsilon_2)\big)$ yields the four pseudo-vertices
$$\big\{(0, 0), (2\,\pi, 0), (0, \upsilon_2), (2\,\pi, \upsilon_2)\big\}$$
and the region $\mathcal{R}\big((0, 0), (2\pi, \upsilon_2)\big)$ which is a North polar cap of $\mathbb{S}^2$. The pseudo-vertices $(0, 0)$ and $(2\,\pi, 0)$ both correspond to $\mathbf{x}\left((0, 0)\right)$, which is the North pole, an interior point of $\mathcal{R}\big((0, 0), (2\pi, \upsilon_2)\big)$.
2. The pair $\big((0, 0, \tau_3), (2\pi, \upsilon_2, \upsilon_3)\big)$ yields the eight pseudo-vertices
$$\begin{aligned} \big\{&(0, 0, \tau_3), (2\pi, 0, \tau_3), (0, \upsilon_2, \tau_3), (2\pi, \upsilon_2, \tau_3), \\ &(0, 0, \upsilon_3), (2\pi, 0, \upsilon_3), (0, \upsilon_2, \upsilon_3), (2\pi, \upsilon_2, \upsilon_3)\big\}. \end{aligned}$$
and the region $\mathcal{R}\big((0, 0, \tau_3), (2\pi, \upsilon_2, \upsilon_3)\big)$ of $\mathbb{S}^3$ which is a descendant of a polar cap in $\mathbb{S}^2$.

We have the following elementary relationship between regions which are rectilinear in spherical polar coordinates.

LEMMA 2.5. *The equatorial image of a region of* $\mathbb{S}^d$ *which is rectilinear in spherical polar coordinates is a region of* $\mathbb{S}^{d-1}$ *which is also rectilinear in spherical polar coordinates. Specifically, we have*

$$\Pi\,\mathcal{R}\big((\tau_1,\dots,\tau_{d-1},\tau_d),(\upsilon_1,\dots,\upsilon_{d-1},\upsilon_d)\big) = \mathcal{R}\big((\tau_1,\dots,\tau_{d-1}),(\upsilon_1,\dots,\upsilon_{d-1})\big).$$

The area of spheres and spherical caps. For $d \geqslant 0$, the area of $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ is given by [14, p. 1]

$$\sigma(\mathbb{S}^d) = \frac{2\,\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}. \tag{2.7}$$

For all that follows, we will use the following abbreviations. For $d \geqslant 1$, we define

$$\omega := \sigma(\mathbb{S}^{d-1}) \quad \text{and} \quad \Omega := \sigma(\mathbb{S}^d). \tag{2.8}$$

The area of a spherical cap $S(\mathbf{a},\theta)$ of spherical radius θ and center $\mathbf{a}$ is [12, Lemma 4.1 p. 255]

$$\mathcal{V}(d,\theta) := \sigma\left(S(\mathbf{a},\theta)\right) = \omega \int_0^\theta (\sin\xi)^{d-1} d\xi. \tag{2.9}$$

It can be readily seen that $\mathcal{V}(2,\theta) = 4\pi\,\sin^2\frac{\theta}{2}$ and $\mathcal{V}(3,\theta) = \pi\,(2\theta - \sin(2\theta))$.

The area of a spherical cap can also be described using the incomplete Beta function.

LEMMA 2.6.

$$\frac{\mathcal{V}(d,\theta)}{\mathcal{V}(d,\pi)} = \frac{B\left(\sin^2\frac{\theta}{2};\frac{d}{2},\frac{d}{2}\right)}{B\left(\frac{d}{2},\frac{d}{2}\right)} =: I\left(\sin^2\frac{\theta}{2};\frac{d}{2},\frac{d}{2}\right),$$

where $B(x;a,b)$ *is the incomplete Beta function [6] and* $B(a,b)$ *is the Beta function.*

The function I of Lemma 2.6 is variously called the *incomplete Beta function ratio* [9, Chapter 25, p. 211], the *regularized Beta function* [24] or the cumulative distribution function of the *Beta distribution*.

To determine the spherical radius θ of a cap of area v we need to solve the equation $\mathcal{V}(d,\theta) = v$.

We note that $\mathcal{V}$ is a smooth non-negative monotonically increasing function of θ, with $\mathcal{V}(d,0) = 0$. It therefore has an inverse, which we will call Θ. We then have

$$\begin{aligned} \Theta\big(d,\mathcal{V}(d,\theta)\big) &= \theta, \quad \text{for} \quad \theta \in [0,\pi], \\ \mathcal{V}\big(d,\Theta(d,v)\big) &= v, \quad \text{for} \quad v \in [0,\Omega]. \end{aligned} \tag{2.10}$$

For brevity, the following notation omits the explicit dependence of $\mathcal{V}$ on d, ie. we will write $\mathcal{V}(\theta)$ for the area of a spherical cap of spherical radius θ.

3. The recursive zonal equal area partition. This section describes the recursive zonal equal area partition and recursive zonal equal area partition algorithm in some detail.

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3.1. The recursive zonal equal area partition algorithm: an outline. The recursive zonal equal area partition algorithm is recursive in dimension d. The pseudocode description for the algorithm for $\mathrm{EQ}(d, N)$ is as follows:

if $N = 1$ **then**
  There is a single region which is the whole sphere;
else if $d = 1$ **then**
  Divide the circle into N equal segments;
else
  Divide the sphere into zones, each the same area as an integer number of regions:
    1. Determine the colatitudes of polar caps,
    2. Determine an ideal collar angle,
    3. Determine an ideal number of collars,
    4. Determine the actual number of collars,
    5. Create a list of the ideal number of regions in each collar,
    6. Create a list of the actual number of regions in each collar,
    7. Create a list of colatitudes of each zone;
  Partition each spherical collar into regions of equal area,
    using the recursive zonal equal area partition algorithm for dimension $d - 1$;
endif.

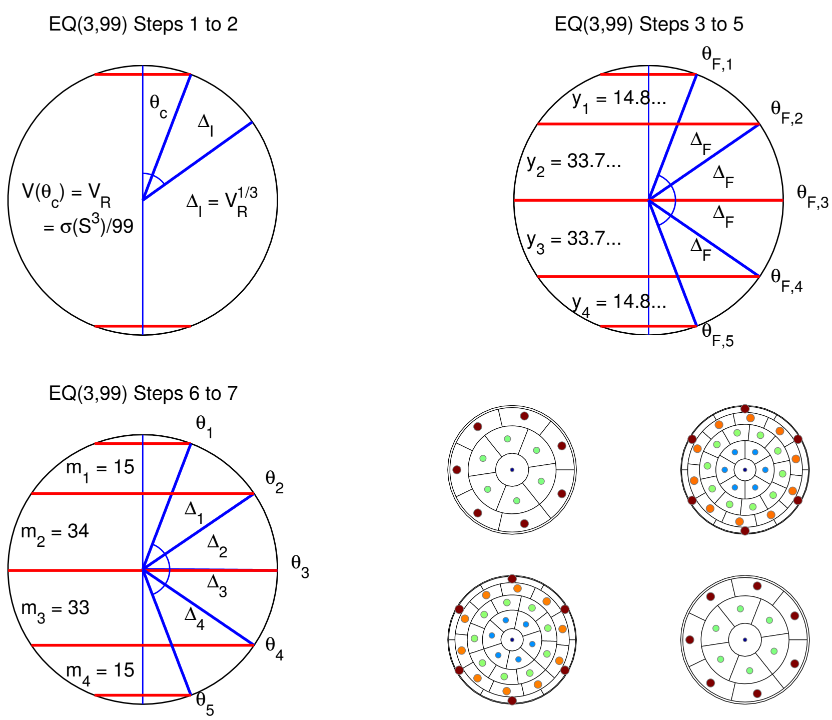

FIG. 3.1. *Partition algorithm for* $\mathrm{EQ}(3, 99)$

Figure 3.1 is an illustration of the algorithm for $\mathrm{EQ}(3, 99)$, with step numbers corresponding to the step numbers in the pseudocode. We now describe key steps of the algorithm in more detail.

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3.2. Dividing the sphere into zones. This is the key part of the algorithm, and is split into a number of steps. Each step is described in more detail below. For brevity, we assume $d > 1$ and $N > 1$ and we omit mentioning dependence on the variables d and N, where this can be done without confusion.

1. *Determining the colatitudes of polar caps.*
   Each polar cap is a spherical cap with the same area as that required for a region. For an N region partition of $\mathbb{S}^d$, the required area of a region R is

$$\mathcal{V}_R := \frac{\Omega}{N},$$

   where Ω is the area of $\mathbb{S}^d$, as per (2.8).
   The colatitude of the bottom of the North polar cap, θ_c is the spherical radius of a spherical cap of area $\mathcal{V}_R$. Therefore

$$\theta_c := \Theta(\mathcal{V}_R), \tag{3.1}$$

   where the function Θ is defined by (2.10). The colatitude of top of the South polar cap is then $\pi - \theta_c$.
2. *Determining an ideal collar angle.*
   As a result of Lemma 2.4, spherical distance approaches Euclidean distance as the distance goes to zero. We now use the idea that to keep the diameter bounded we want the shape of each region to approach a d-dimensional Euclidean hypercube as N goes to infinity. That way, the diameter approaches the diagonal length of the hypercube. The collar angle, the spherical distance between the top and bottom of a collar in the partition, therefore should approach $\mathcal{V}_R^{1/d}$ as N approaches infinity.
   We therefore define the *ideal collar angle* to be

$$\delta_I := \mathcal{V}_R^{1/d}.$$

3. *Determining an ideal number of collars.*
   Ideally, the sphere is to be partitioned into the North and South spherical caps, and a number of collars, all of which have angle δ_I. The ideal number of collars is therefore

$$n_I := \frac{\pi - 2\theta_c}{\delta_I}.$$

4. *Determining the actual number of collars.*
   We use a rounding procedure to obtain an integer n close to the ideal number of collars.
   If $N = 2$, then $n := 0$. Otherwise

$$n := \max\left(1, \operatorname{round}\left(n_I\right)\right), \tag{3.2}$$

   where, as usual, for $x \geqslant 0$,

$$\operatorname{round}(x) := \lfloor x + 0.5 \rfloor,$$

   where $\lfloor\rfloor$ is the floor (greatest integer) function.
   The number of collars is then n.

5. *Creating a list of the ideal number of regions in each collar.*
We number the zones southward from 1 for the North polar cap to $n+2$ for the South polar cap, and number the collars so that collar i is zone $i+1$.
We now assume $N > 2$. The *"fitting"* collar angle is

$$\delta_F := \frac{n_I}{n}\delta_I = \frac{\pi - 2\theta_c}{n}. \tag{3.3}$$

We use δ_F to produce an increasing list of "fitting" colatitudes of caps, defined by

$$\theta_{F,i} := \theta_c + (i-1)\delta_F, \tag{3.4}$$

for $i \in \{1, \ldots, n+1\}$.
The area of each corresponding "fitting" collar is given by successive colatitudes in this list. The ideal number of regions, y_i, in each collar$i \in \{1, \ldots, n\}$ is then

$$y_i := \frac{\mathcal{V}(\theta_{F,i+1}) - \mathcal{V}(\theta_{F,i})}{\mathcal{V}_R}.$$

6. *Creating a list of the actual number of regions in each collar.*
We use a rounding procedure similar to that of Zhou [27, Lemma 2.11, pp. 16–17]. With n the number of collars as defined by (3.2), we define m_i, the required number of regions in collar $i \in \{1, \ldots, n\}$ as follows.
Define the sequences a and m by starting with $a_0 := 0$, and for $i \in \{1, \ldots, n\}$,

$$m_i := \mathrm{round}(y_i + a_{i-1}), \quad a_i := \sum_{j=1}^{i}(y_j - m_j).$$

7. *Creating a list of colatitudes of each zone.*
We now define $\theta_0 := 0$, $\theta_{m+2} := \pi$ and for $i \in \{1, \ldots, n+1\}$, we define

$$\theta_i := \Theta\left(\left(1 + \sum_{j=1}^{i-1} m_j\right)\mathcal{V}_R\right).$$

For $i \in \{0, \ldots, n+1\}$, we use Z as per (2.4) to define zone $i+1$ to be $Z(\theta_i, \theta_{i+1})$.
Finally, for $i \in \{1, \ldots, n\}$, we define collar i to be zone $i+1$.

3.3. Partitioning a collar. We partition collar i of $\mathrm{EQ}(d, N)$ into m_i regions, each corresponding to a region of the partition $\mathrm{EQ}(d-1, m_i)$. We assume that each region of $\mathrm{EQ}(d-1, m_i)$ is rectilinear in spherical polar coordinates. If region $j \in \{1, \ldots, m_i\}$ of $\mathrm{EQ}(d-1, m_i)$ is $\mathcal{R}((\tau_1, \ldots, \tau_{d-1}), (\upsilon_1, \ldots, \upsilon_{d-1}))$, then we define the region R of collar i of $\mathrm{EQ}(d, N)$ corresponding to region j of $\mathrm{EQ}(d-1, m_i)$ to be

$$R := \mathcal{R}\big((\tau_1, \ldots, \tau_{d-1}, \theta_i), (\upsilon_1, \ldots, \upsilon_{d-1}, \theta_{i+1})\big). \tag{3.5}$$

REMARK 3.1. *The partition $EQ(d, N)$ is not fully specified by this algorithm. The algorithm instead specifies an equivalence class of partitions, unique up to rotations of the sectors of the partitions of $\mathbb{S}^1$. This means that the collars of $EQ(2, N)$ are free to rotate without changing diameters of the regions and without changing the colatitudes of the collars. The regions remain rectilinear in spherical polar coordinates.*

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4. Analysis of the recursive zonal equal area partition. The proofs of Theorems 1.5 and 1.6 proceed by induction on d, matching the recursion of the recursive zonal equal area partition algorithm. This section presents the preliminary analysis of the recursive zonal equal area partition, including the lemmas used in the proof of Theorem 1.6 [13]. Appendix A contains proofs of the lemmas.

By induction on the construction given in Section 3, we see that the regions produced by the recursive zonal equal area partition algorithm are rectilinear is spherical polar coordinates and for $d > 1$ each region R of collar i is of the form (3.5). Each such region therefore has an equatorial image of the form

$$\begin{aligned} \Pi\, R &= \mathbf{x}\left([\tau_1, \upsilon_1] \times [\tau_2, \upsilon_2] \times \ldots \times [\tau_{d-1}, \upsilon_{d-1}]\right) \\ &= \mathcal{R}((\tau_1, \ldots, \tau_{d-1}), (\upsilon_1, \ldots, \upsilon_{d-1})) \in \mathrm{EQ}(d-1, m_i) \end{aligned} \tag{4.1}$$

as per Lemma 2.5 and Section 3.3 above.

The following lemma on the diameter of the polar caps has an elementary proof, which is omitted.

LEMMA 4.1. *For $d > 1$ and $N > 1$, the diameter of each of the polar caps of the recursive zonal equal area partition $\mathrm{EQ}(d, N)$ is $2\,\sin\theta_c$, where θ_c is defined by (3.1).*

The following lemma leads to a bound on the diameter of a region contained in a collar.

LEMMA 4.2. *Given $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{S}^d$ where*

$$\begin{aligned} \mathbf{a} &:= \mathbf{x}(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, A), \\ \mathbf{b} &:= \mathbf{x}(\beta_1, \beta_2, \ldots, \beta_{d-1}, B), \\ \mathbf{c} &:= \mathbf{x}(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, B), \end{aligned} \tag{4.2}$$

with $\sin B \geqslant \sin A$, then the Euclidean $\mathbb{R}^{d+1}$ distance $e(\mathbf{a}, \mathbf{b})$ satisfies

$$e(\mathbf{a}, \mathbf{b}) \leqslant \sqrt{e(\mathbf{a}, \mathbf{c})^2 + e(\mathbf{c}, \mathbf{b})^2}.$$

The following definitions are of use in examining the diameter of R in terms of $e(\mathbf{a}, \mathbf{c})$ and $e(\mathbf{c}, \mathbf{b})$. For region R contained in collar i of $\mathrm{EQ}(d, N)$,

- the spherical distance between the top and bottom parallels of region R is

$$\delta_i := \theta_{i+1} - \theta_i, \tag{4.3}$$

- the maximum radius of collar i is

$$w_i := \max_{\xi \in [\theta_i, \theta_{i+1}]} \sin\xi = \begin{cases} \sin\theta_{i+1} & \text{if} \quad \theta_{i+1} < \pi/2, \\ \sin\theta_i & \text{if} \quad \theta_i > \pi/2, \\ 1 & \text{otherwise.} \end{cases} \tag{4.4}$$

We can now use Lemmas 2.4 and 4.2 to show that

LEMMA 4.3. *For region R contained in collar i of $\mathrm{EQ}(d, N)$ we have*

$$\begin{aligned} \operatorname{diam} R &\leqslant \sqrt{\Upsilon(\delta_i)^2 + w_i^2 (\operatorname{diam} \Pi\, R)^2} \\ &\leqslant \sqrt{\delta_i^2 + w_i^2 (\operatorname{diam} \Pi\, R)^2}, \end{aligned}$$

where δ_i and w_i are given by (4.3) and (4.4) respectively.

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5. A per-region bound on diameter. The following bound is not needed for the proof of Theorem 1.6, but is useful in checking the calculation of the diameters of individual regions.

DEFINITION 5.1. *The region diameter bound function* db *is defined on the regions of a partition* $\mathrm{EQ}(d, N)$ *as follows.*

For the whole sphere $\mathbb{S}^d$*,*

$$\mathrm{db}\,\mathbb{S}^d := 2.$$

For a region R *contained in* $\mathrm{EQ}(1, N)$*,*

$$\mathrm{db}\,R := \Upsilon\left(\frac{2\,\pi}{N}\right),$$

where Υ *is defined by (2.3).*

For $d > 1$*, for a spherical cap* R *with spherical radius* θ_c*,*

$$\mathrm{db}\,R := 2\,\sin\theta_c.$$

For $d > 1$*, for a region* R *contained in collar* $i \in \{1, \ldots, n\}$ *of a partition* $\mathrm{EQ}(d, N)$ *with* n *collars,*

$$\mathrm{db}\,R := \sqrt{\Upsilon(\delta_i)^2 + w_i^2(\mathrm{db}\,\Pi\,R)^2},$$

where $\Pi\,R$ *is defined by (4.1).*

THEOREM 5.2. *For any region* $R \in \mathrm{EQ}(d, N)$*,*

$$\mathrm{diam}\,R \leqslant \mathrm{db}\,R.$$

Proof. For the whole sphere $\mathbb{S}^d$, we have

$$\mathrm{diam}\,\mathbb{S}^d = 2 = \mathrm{db}\,\mathbb{S}^d.$$

The partition algorithm for $\mathrm{EQ}(1, N)$, with $N > 1$, divides $\mathbb{S}^1$ into N equal segments, as described in Section 3.1. For a region R contained in $\mathrm{EQ}(1, N)$, with $N > 1$, the region can be therefore be described by the pair of polar coordinates α, β. That is, $R = \mathcal{R}(\alpha, \beta)$. The spherical distance $s\big(\mathbf{x}(\alpha), \mathbf{x}(\beta)\big)$ is then given by

$$s\big(\mathbf{x}(\alpha), \mathbf{x}(\beta)\big) = \frac{2\pi}{N} \leqslant \pi.$$

Using Lemma 2.4, the diameter of R is then

$$\mathrm{diam}\,R = e\big(\mathbf{x}(\alpha), \mathbf{x}(\beta)\big) = \Upsilon\left(s\big(\mathbf{x}(\alpha), \mathbf{x}(\beta)\big)\right) = \Upsilon\left(\frac{2\,\pi}{N}\right) = \mathrm{db}\,R.$$

For $d > 1$, for a spherical cap R with spherical radius θ_c, by Lemma 4.1,

$$\mathrm{diam}\,R = 2\,\sin\theta_c = \mathrm{db}\,R.$$

For $d > 1$, for a region R contained in collar $i \in \{1, \ldots, n\}$ of a recursive zonal equal area partition of $\mathbb{S}^d$ with n collars, by Lemma 4.3, if $\mathrm{diam}\,\Pi\,R \leqslant \mathrm{db}\,\Pi\,R$ then

$$\begin{aligned}\mathrm{diam}\,R &\leqslant \sqrt{\Upsilon(\delta_i)^2 + w_i^2(\mathrm{diam}\,\Pi\,R)^2} \\ &\leqslant \sqrt{\Upsilon(\delta_i)^2 + w_i^2(\mathrm{db}\,\Pi\,R)^2} = \mathrm{db}\,R.\end{aligned}$$

The result follows by induction on d. □

6. Sketch of the proof of the Theorem 1.6. The proof of Theorem 1.6 proceeds by induction on the dimension d. The inductive step of the proof starts with the observation that if $d > 1$ and if the set $\mathrm{EQ}(d-1)$ has diameter bound κ, then for any region R of collar i of the partition $\mathrm{EQ}(d, N)$ we have

$$\operatorname{diam} \Pi\, R \leqslant \kappa m_i^{\frac{1}{1-d}},$$

and therefore from Lemma 4.3 we have

$$\operatorname{diam} R \leqslant \sqrt{\delta_i^2 + \kappa^2 p_i^2},$$

where the scaled $\mathbb{S}^{d-1}$ diameter bound p_i is

$$p_i := w_i m_i^{\frac{1}{1-d}}.$$

As a consequence, if $d > 1$ and if $\mathrm{EQ}(d-1)$ has diameter bound κ, then for any region R of the partition $\mathrm{EQ}(d, N)$

$$\operatorname{diam} R \leqslant \sqrt{(\max \delta)^2 + \kappa^2 (\max p)^2},$$

where

$$\max \delta := \max_{i \in \{1,\ldots,n\}} \delta_i, \quad \max p := \max_{i \in \{1,\ldots,n\}} p_i,$$

and n is the number of collars in the partition $\mathrm{EQ}(d, N)$.

Thus to prove the theorem it suffices to show that $\max \delta$ and $\max p$ are both of order $N^{-1/d}$. Since the Euclidean diameter of a region of $\mathbb{S}^d$ is always bounded above by 2, we need only prove that there is an $N_0 \geqslant 1$ such that for $N \geqslant N_0$ we have bounds of the right order. This is because for any $N_0 \geqslant 1$ and any $N \in [1, N_0]$ we have

$$2\, N_0^{1/d} N^{-1/d} \geqslant 2.$$

The key strategy in estimating $\max \delta$ and $\max p$ is to replace the integer variable i by a small number of real valued variables constrained to some feasible domain, replace δ and p with the equivalent functions of these real variables, and then to find and estimate continuous functions which dominate these equivalent functions.

To replace i, we first must model the rounding steps of the partition algorithm. We model the first rounding step by finding appropriate bounds for $\rho = n_I/n = \delta_F/\delta_I$, where δ_F is defined by (3.3).

The second rounding step takes the sequence y and produces the sequences m and a. To model this step, we first show that $a_i \in [-1/2, 1/2)$. This allows us to define the analog functions Y, M, Δ, W, P corresponding to y, m, δ, w, p respectively. These analog functions are defined on the real rounding variables τ and β and the angle variable θ, such that Y coincides with y, etc. when $\tau = -a_{i-1}$, $\beta = a_i$ and $\theta = \theta_{F,i}$, where $\theta_{F,i}$ is defined by (3.4).

We then define the feasible domain $\mathbb{D}$ such that the second rounding step always corresponds to a set of points in $\mathbb{D}$.

The final and longest part of the proof is to show that both Δ and P are asymptotically bounded of order $N^{-1/d}$ over the whole of $\mathbb{D}$. In this final part, we need estimates for the area function $\mathcal{V}$ and the inverse function Θ. Crude but very simple estimates of these functions yield bounds for Δ and P of the correct order.

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7. Implementation. The Recursive Zonal Equal Area (EQ) Sphere Partitioning Toolbox is a suite of Matlab [22] functions. These functions are intended for use in exploring different aspects of EQ sphere partitioning.

For $d \leqslant 2$, the area function $\mathcal{V}(d,\theta)$ uses the closed solution to the integral (2.9), and for $d > 3$ the area function uses the Matlab [22] function `BETAINC` to evaluate the regularized incomplete Beta function I of Lemma 2.6. For $d = 3$ the area function uses the closed solution for $\theta \in [\pi/6, 5\pi/6]$ and otherwise uses `BETAINC`.

The inverse function $\Theta(d, v)$ uses the closed solution to the inverse for $d \leqslant 2$, and otherwise uses the Matlab [22] function `FZERO` to find the solution. This loses some accuracy for area arguments near zero. In future, the inverse function may instead be based on an implementation of the inverse Beta distribution algorithm of Abernathy and Smith [1].

8. Numerical results.

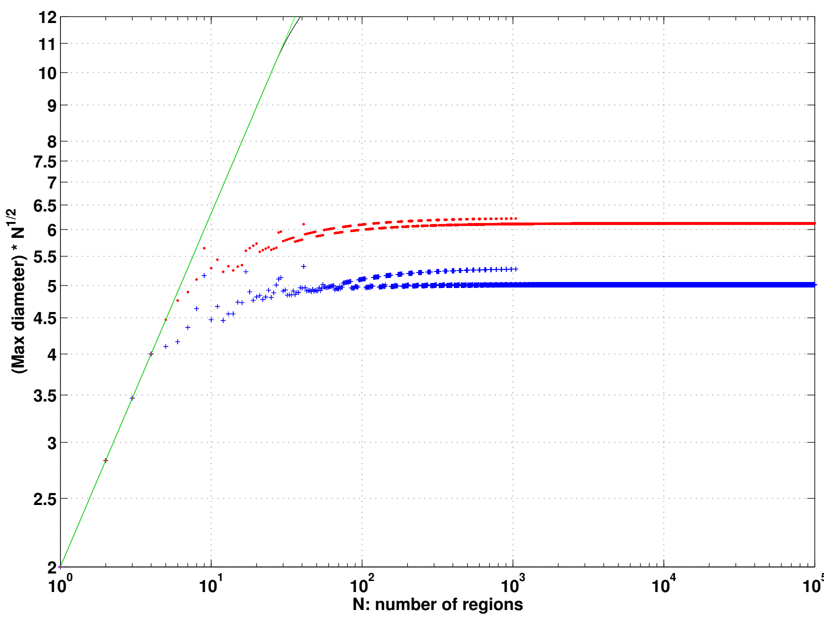

FIG. 8.1. *Maximum diameters of* $\mathrm{EQ}(2, N)$ *(log-log scale)*

Maximum diameters of regions. Figures 8.1, 8.2 and 8.3 are log-log plots corresponding to the recursive zonal equal area partitions of $\mathbb{S}^d$ for $d = 2$, $d = 3$ and $d = 4$ respectively. For each partition $\mathrm{EQ}(d, N)$, for N from 1 to 100 000, each figure shows the maximum per-region upper bound on diameter, as per Definition 5.1, depicted as red dots, and the maximum vertex diameter, depicted as blue $+$ signs.

The vertex diameter of a region is the maximum distance between pseudo-vertices of a region, except where a region spans 2π in longitude, in which case one of each pair of coincident pseudo-vertices is replaced by a point with the same colatitudes and a longitude increased by π. For low dimensions and for regions which do not straddle the equator, the vertex diameter provides a good lower bound on the diameter.

Only the upper and lower bounds on the maximum diameter are plotted, rather than the maximum diameter itself. This is because, for each region of each partition, the diameter is the solution of a constrained nonlinear optimization problem. It would therefore take quite a long time to calculate the maximum diameter of every partition for N from 1 to 100 000.

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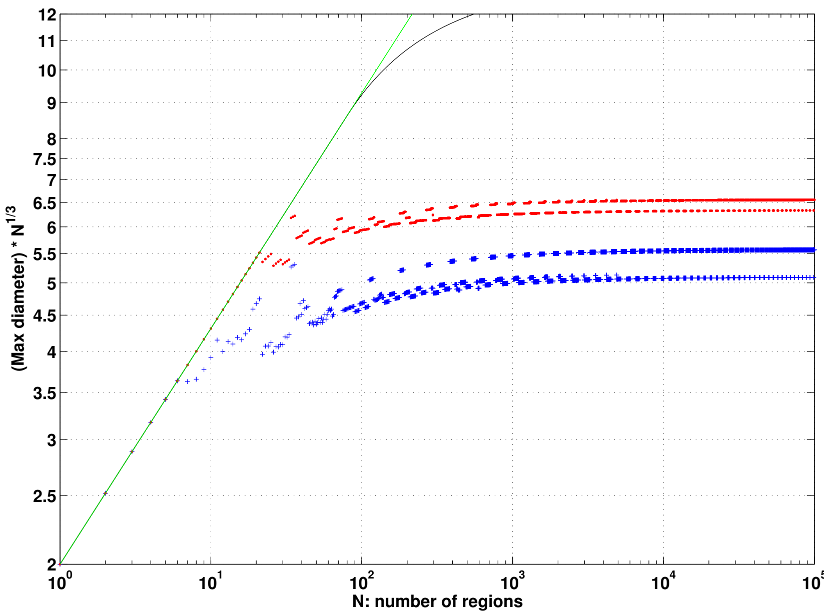

FIG. 8.2. *Maximum diameters of* $\mathrm{EQ}(3, N)$ *(log-log scale)*

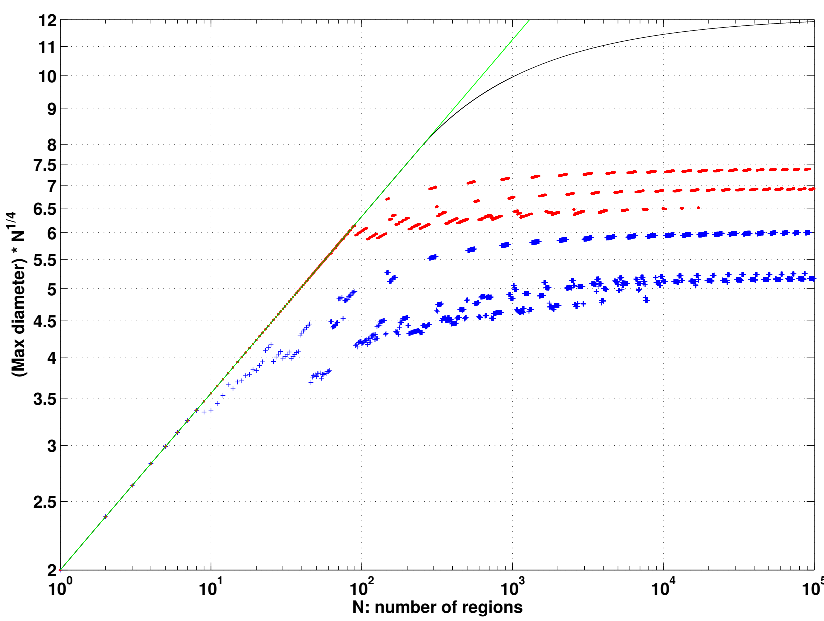

FIG. 8.3. *Maximum diameters of* $\mathrm{EQ}(4, N)$ *(log-log scale)*

Feige and Schechtman's construction yields the following upper bound on the smallest maximum diameter of an equal area partition of $\mathbb{S}^d$.

LEMMA 8.1. [7, Lemma 21, pp. 430–431] *For $d > 1$, $N > 2$, there is a partition $FS(d, N)$ of the unit sphere $\mathbb{S}^d$ into N regions, with each region $R \in FS(d, N)$ having area*

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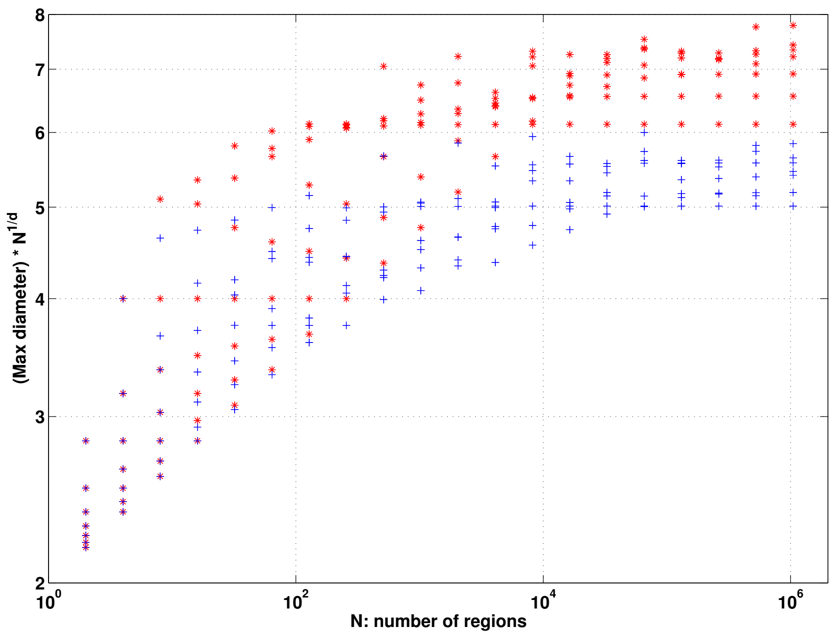

FIG. 8.4. *Maximum diameters of* $\mathrm{EQ}(d, N)$*, d from 2 to 8 (log-log scale)*

$\sigma(\mathbb{S}^d)/N$ and Euclidean diameter bounded above by

$$\operatorname{diam} R \leqslant \Upsilon\left(\min(\pi, 8\theta_c)\right), \tag{8.1}$$

with Υ defined by (2.3) *and θ_c defined by* (3.1).

A proof of Stolarsky's assertion using Lemma 8.1 and a proof of Lemma 8.1 itself are included in [13].

The black curve on each figure is the Feige-Schechtman bound (8.1). On each figure, this curve joins a straight line for which the maximum diameter of a region is 2.

Figures 8.1, 8.2, and 8.3 show that for $N \leqslant 100\,000$, we have $\operatorname{maxdiam}(2, N)N^{1/2} < 6.5$, $\operatorname{maxdiam}(3, N)N^{1/3} < 7$ and $\operatorname{maxdiam}(4, N)N^{1/4} < 7.5$ respectively.

Figure 8.4 plots the maximum per-region upper bound (Definition 5.1) depicted as red dots, and the maximum vertex diameter, depicted as blue $+$ signs, for the partitions $\mathrm{EQ}(d, 2^k)$, for d from 2 to 8, for k from 1 to 20. For the cases shown, we have $\operatorname{maxdiam}(d, 2^k) \times 2^{k/d} < 8$.

Running time. To benchmark the speed of the partition algorithm, the function `eq_regions`(d, N) was run for d from 1 to 11 and N from 2 to $2^{22} = 4194304$, in successive powers of 2, on a 2 GHz AMD Opteron processor, using Matlab 7.01 [22]. The benchmark was repeated a total of three times. For d from 2 to 11 and N from 8 to 2^{22}, the running time t was approximately

$$t(d, N) = (0.24 \pm 0.04)\, d^{1.90 \pm 0.07}\, N^{0.60 \pm 0.01} \text{ ms},$$

with the error bounds having 95% confidence level. Thus for this range of d and N, the running time of the partition algorithm is approximately $O(N^{0.6})$, which is faster than linear in N.

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Appendix A. Proofs of lemmas.

Proof of Lemma 2.6. From (2.7) we know that

$$\frac{\mathcal{V}(d,\theta)}{\mathcal{V}(d,\pi)} = \frac{\int_0^\theta \sin^{d-1}\xi\, d\xi}{\int_0^\pi \sin^{d-1}\xi\, d\xi}.$$

Now substitute $u = \sin^2(\xi/2)$. Then, since $\theta \in [0,\pi]$, by a well-known half angle formula we have $\sin\xi = 2u^{1/2}(1-u)^{1/2}$ and we also have $du = u^{1/2}(1-u)^{1/2} d\xi$, so

$$\begin{aligned}\frac{\mathcal{V}(d,\theta)}{\mathcal{V}(d,\pi)} &= \frac{\int_0^{\sin^2(\frac{\theta}{2})} u^{\frac{d}{2}-1}(1-u)^{\frac{d}{2}-1}\, du}{\int_0^1 u^{\frac{d}{2}-1}(1-u)^{\frac{d}{2}-1}\, du}\\ &= \frac{B\left(\sin^2\frac{\theta}{2}; \frac{d}{2}, \frac{d}{2}\right)}{B\left(\frac{d}{2}, \frac{d}{2}\right)}. \qquad \square\end{aligned}$$

Proof of Lemma 4.2. For any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^{d+1}$ we have

$$\begin{aligned} e(\mathbf{a},\mathbf{c})^2 + e(\mathbf{c},\mathbf{b})^2 - e(\mathbf{a},\mathbf{b})^2 &= (\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{c}) + (\mathbf{c}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{b})\\ &\quad - (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= 2\,\mathbf{a}\cdot\mathbf{b} - 2\,\mathbf{a}\cdot\mathbf{c} - 2\,\mathbf{c}\cdot\mathbf{b} + 2\,\mathbf{c}\cdot\mathbf{c} = 2\,(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c}).\end{aligned}$$

We therefore prove Lemma 4.2 by proving that $(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c}) \geqslant 0$.

First, note that rotations of $\mathbb{S}^d$ are isometries and therefore without loss of generality we may rotate the triangle **acb** to make calculation more convenient. Now note that we can apply a single $\mathbb{S}^{d-1}$ rotation to $\mathbb{S}^d$ while keeping the $\mathbb{S}^d$ colatitude fixed.

Therefore we can assume that

$$\mathbf{a} = \mathbf{x}(0,\ldots,0,0,A), \quad \mathbf{b} = \mathbf{x}(0,\ldots,0,C,B), \quad \mathbf{c} = \mathbf{x}(0,\ldots,0,0,B).$$

In Cartesian coordinates, for $d > 3$, we obtain

$$\begin{aligned}\mathbf{a} &= (0,\ldots,0,0,\sin A, \cos A),\\ \mathbf{b} &= (0,\ldots,0,\sin B\, \sin C, \sin B\, \cos C, \cos B),\\ \mathbf{c} &= (0,\ldots,0,0,\sin B, \cos B).\end{aligned}$$

Due to an unfortunate feature of the conventional mapping from spherical to Cartesian coordinates, for $\mathbb{S}^2 \subset \mathbb{R}^3$, we obtain

$$\mathbf{a} = (\sin A, 0, \cos A), \quad \mathbf{b} = (\sin B\, \cos C, \sin B\, \sin C, \cos B), \quad \mathbf{c} = (\sin B, 0, \cos B).$$

and for $\mathbb{S}^3 \subset \mathbb{R}^4$, we obtain

$$\begin{aligned}\mathbf{a} &= (0,0,\sin A, \cos A),\\ \mathbf{b} &= (\sin B\, \sin C, 0, \sin B\, \cos C, \cos B),\\ \mathbf{c} &= (0,0,\sin B, \cos B).\end{aligned}$$

In all three cases, we obtain

$$\begin{aligned}(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c}) &= (\sin A - \sin B)(\sin B\, \cos C - \sin B) + (0)(\sin B\, \sin C)\\ &\quad + (\cos A - \cos B)(0)\\ &= (\sin B - \sin A)\,(1-\cos C)\,\sin B \geqslant 0. \qquad \square\end{aligned}$$

To prove Lemma 4.3 we use the following results.

LEMMA A.1. *Let* $\mathbf{a}$, $\mathbf{c}$ *be points of region* R *in collar* i *of* $\mathrm{EQ}(d, N)$*, which additionally satisfy (4.2) with* $\sin B \geqslant \sin A$*, that is,*

$$\begin{aligned} R &= \mathcal{R}\big((\tau_1, \ldots, \tau_{d-1}, \theta_i), (\upsilon_1, \ldots, \upsilon_{d-1}, \theta_{i+1})\big), \\ \mathbf{a} &= \mathbf{x}(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, A), \quad \mathbf{c} = \mathbf{x}(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, B), \\ \alpha_k &\in [\tau_k, \upsilon_k], k \in \{1, \ldots, d-1\}, \quad A, B \in [\theta_i, \theta_{i+1}], \quad \sin B \geqslant \sin A. \end{aligned}$$

We then have $e(\mathbf{a}, \mathbf{c}) \leqslant \Upsilon(\delta_i) < \delta_i$*, where* δ_i *is given by (4.3).*

Proof. Since $\mathbf{a}$ and $\mathbf{c}$ differ only in colatitude we have

$$s(\mathbf{a}, \mathbf{c}) = |B - A| \leqslant \theta_{i+1} - \theta_i = \delta_i.$$

Using Lemma 2.4 we note that the function Υ increases monotonically with spherical distance, and for all $\theta \in (0, \pi]$ we have $\Upsilon(\theta) < \theta$. Therefore

$$e(\mathbf{a}, \mathbf{c}) = \Upsilon\left(s(\mathbf{a}, \mathbf{c})\right) < \Upsilon(\delta_i) < \delta_i. \qquad \square$$

LEMMA A.2. *Let* $\mathbf{b}$, $\mathbf{c}$ *be points of region* R *in collar* i *of* $\mathrm{EQ}(d, N)$*, which additionally satisfy (4.2), that is,*

$$\begin{aligned} R &= \mathcal{R}\big((\tau_1, \ldots, \tau_{d-1}, \theta_i), (\upsilon_1, \ldots, \upsilon_{d-1}, \theta_{i+1})\big), \\ \mathbf{b} &= \mathbf{x}(\beta_1, \beta_2, \ldots, \beta_{d-1}, B), \quad \mathbf{c} = \mathbf{x}(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, B), \\ \alpha_k, \beta_k &\in [\tau_k, \upsilon_k], k \in \{1, \ldots, d-1\}, \quad B \in [\theta_i, \theta_{i+1}]. \end{aligned}$$

We then have $e(\mathbf{c}, \mathbf{b}) \leqslant w_i \operatorname{diam} \Pi\, R$*, where* w_i *is given by (4.4).*

Proof. The points $\mathbf{b}$ and $\mathbf{c}$ both have colatitude B. Using the spherical polar coordinates of $\mathbf{b}$ and $\mathbf{c}$ and the mappings $\mathbf{x}$ and Π we see that if $\Pi\,\mathbf{b} = (b'_1, \ldots, b'_d)$ then

$$\mathbf{b} = (\sin B\, b'_1, \ldots, \sin B\, b'_d, \cos B),$$

and similarly for point $\mathbf{c}$. It follows that $e(\mathbf{c}, \mathbf{b}) = \sin B\, e(\Pi\,\mathbf{c}, \Pi\,\mathbf{b})$.

Since $\Pi\,(b), \Pi\,(c) \in \Pi\, R$, the Euclidean distance $e(\Pi\,(c), \Pi\,(b))$ is bounded by the diameter of $\Pi\, R$, so we have $e(\mathbf{c}, \mathbf{b}) \leqslant \sin B\ \operatorname{diam} \Pi\, R$. Since $B \in [\theta_i, \theta_{i+1}]$,

$$\sin B \leqslant w_i = \max_{\xi \in [\theta_i, \theta_{i+1}]} \sin \xi.$$

We therefore have $e(\mathbf{c}, \mathbf{b}) \leqslant w_i\ \operatorname{diam} \Pi\, R$. $\square$

We now use these results to prove Lemma 4.3.

Proof of Lemma 4.3. Let $\mathbf{a}, \mathbf{b}$ be points of region R such that $e(\mathbf{a}, \mathbf{b}) = \operatorname{diam} R$ and let

$$\mathbf{a} = \mathbf{x}(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, A), \quad \mathbf{b} = \mathbf{x}(\beta_1, \beta_2, \ldots, \beta_{d-1}, B),$$

with $\sin B \geqslant \sin A$. Now define $\mathbf{c} := \mathbf{x}(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, B)$.

By Lemmas A.1, A.2, 2.4 and 4.2, we then have

$$\begin{aligned} \operatorname{diam} R &= e(\mathbf{a}, \mathbf{b}) \\ &\leqslant \sqrt{e(\mathbf{a}, \mathbf{c})^2 + e(\mathbf{c}, \mathbf{b})^2} \\ &\leqslant \sqrt{\Upsilon(\delta_i)^2 + w_i^2 (\operatorname{diam} \Pi\, R)^2} \\ &\leqslant \sqrt{\delta_i^2 + w_i^2 (\operatorname{diam} \Pi\, R)^2}. \qquad \square \end{aligned}$$

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