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ETNA
Kent State University
etna@mcs.kent.edu

ON CONVERGENCE AND DIVERGENCE OF FOURIER–BESSEL SERIES *

KRZYSZTOF STEMPAK†

Abstract. We furnish another proof, based on an idea of Prestini [13], of a maximal inequality for the partial sum operators of Fourier–Bessel expansions proved by Guadalupe, Pérez, Ruiz and Varona [8]. Divergence results and mean convergence are also discussed.

Key words. Fourier–Bessel expansions, almost everywhere and norm convergence.

AMS subject classifications. 42C10.

1. Introduction and main results. Given $\nu > -1$ let $\{\lambda_n\}_{n=1}^{\infty}$ denote the sequence of successive positive zeroes of $J_\nu(x)$, the Bessel function of the first kind of order ν. Then

$$\int_0^1 J_\nu(\lambda_n x)J_\nu(\lambda_m x)x\,dx = \frac{1}{2}\Big(J_{\nu+1}(\lambda_n)\Big)^2\delta_{nm}, \qquad n,m = 1,2,\dots\ .$$

It follows that the functions

$$\varphi_n^\nu(x) = c_{n,\nu}\sqrt{x}J_\nu(\lambda_n x), \qquad n = 1,2,\dots,$$

where $c_{n,\nu} = \sqrt{2}/J_{\nu+1}(\lambda_n)$, form an orthonormal system in $L^2((0,1),dx) = L^2(dx)$ (all Lebesgue spaces we consider live on $(0,1)$ and dx denotes the Lebesgue measure there). The system is complete. In particular,

$$\varphi_n^{-1/2}(x) = \sqrt{2}\cos(\pi(n-1/2)x)\quad , \qquad \varphi_n^{1/2}(x) = \sqrt{2}\sin(\pi n x)$$

for $n = 1,2,\dots$. On the other hand the functions

$$\psi_n^\nu(x) = c_{n,\nu}J_\nu(\lambda_n x)/x^\nu, \qquad n = 1,2,\dots\ ,$$

form a complete orthonormal system in $L^2(x^{2\nu+1}dx)$ while the functions

$$\phi_n^\nu(x) = c_{n,\nu}J_\nu(\lambda_n x), \qquad n = 1,2,\dots\ ,$$

form a complete orthonormal system in $L^2(xdx)$. Various aspects of harmonic analysis of expansions with respect to the three systems were considered in the literature, cf., for instance, papers by Wing [17], Benedek and Panzone [1], [2], [3], Guadalupe, Pérez, Ruiz and Varona [7], [8] (and, of course, chapter XVII of Watson's monograph [16]).

It is easily seen that for the L^p-inequalities, an estimate for one of the three systems gives a weighted (with a power weight) estimate for the two other. It seems that the most effective is the choice of $\{\varphi_n^\nu\}$ as a leading system, see [8], and, actually, this is also the system we decided to choose.

We use the notation $L^{p,\alpha}(dx)$, $1 \le p \le \infty$, $-\infty < \alpha < \infty$, for the weighted Lebesgue spaces of all measurable functions on $(0,1)$ for which the quantity

$$||f||_{p,\alpha} = \left(\int_0^1 |f(x)|^p x^\alpha dx\right)^{1/p}$$

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†Instytut Matematyki, Politechnika Wrocławska, Wyb.Wyspiańskiego 27, 50–370 Wrocław, Poland(stempak@im.pwr.wroc.pl).

ETNA
Kent State University
etna@mcs.kent.edu

is finite (to be precise, $||f||_{\infty,\alpha} = \text{ess sup}_{x\in(0,1)}|f(x)|$). With any reasonable function f on $(0,1)$ we will associate its Fourier–Bessel coefficients

$$a_n^\nu = a_n^\nu(f) = \int_0^1 f(x)\varphi_n^\nu(x)\,dx$$

provided the coefficients exist. Assuming f is in $L^{p,\alpha}(dx)$, $1 \le p < \infty$, it is easy to check (note that $J_\nu(x) \sim x^\nu$ as $x \to 0$) that the coefficients do exist provided $\alpha < (\nu + \frac{3}{2})p - 1$. In particular, in the case when $\nu \ge -1/2$, this happens for $\alpha's$ from the A_p-power weight exponent range $-1 < \alpha < p-1$. Given f and its Fourier–Bessel series $f(x) \sim \sum_{n=1}^\infty a_n^\nu \varphi_n^\nu(x)$ we consider partial sum operators

$$S_N f(x) = \sum_1^N a_n^\nu \varphi_n^\nu(x)$$

and the maximal function

$$S^* f(x) = \sup_{N\ge 1} |S_N f(x)|.$$

We then have the following result

THEOREM 1.1. *Let $\nu \ge -1/2$, $1 < p < \infty$ and $-1 < \alpha < p-1$. Then*

$$||S^* f||_{p,\alpha} \le C||f||_{p,\alpha}$$

with a constant $C > 0$ independent of $f \in L^{p,\alpha}(dx)$.

As an immediate consequence we obtain

COROLLARY 1.2. *Let ν, p and α be as in Theorem 1.1. Then*

$$S_N f(x) \to f(x), \qquad N \to \infty,$$

a.e. for every function f in $L^{p,\alpha}(dx)$.

As already mentioned, this theorem was proved in [8] (actually in a more general setting of A_p weights). It follows from a weighted version of a general Gilbert's maximal transference theorem, [8], Theorem 1, by transfering L^p–weighted Carleson–Hunt–Young inequalities for the trigonometric system to fairly general systems (Theorem 1 of [6] settles the unweighted case, i.e., the case $\alpha = 0$ in Theorem 1.1). An equiconvergence result, from which Corollary 1.2 follows, was proved in [3], Theorem 4.

Here, in our proof of Theorem 1.1, we use an idea of Prestini [13]. She proved a.e. convergence of partial sums for the modified Hankel transform. A closer examination of her proof reveals that, in some way, the weighted non-modified Hankel transform setting is better suited for arguments which she used. The system $\{\varphi_n^\nu\}$ seems to be a compact interval version of the non-modified Hankel transform and this supports another argument for our choice of $\{\varphi_n^\nu\}$ as a leading system.

Theorem 1.1 is also applicable for proving results on a.e. convergence of partial sums for the other two aforementioned orthogonal systems. We start with the system $\{\psi_n^\nu\}_{n=1}^\infty$. As before for any reasonable function g on $(0,1)$ we associate to g its Fourier-Bessel coefficients

$$b_n^\nu = b_n^\nu(g) = \int_0^1 g(x)\psi_n^\nu(x)\,x^{2\nu+1}dx\,.$$

It is possible to evaluate the coefficients $\{b_n^\nu\}$, $\nu > -1$, for any function g in $L^p(x^{2\nu+1}dx)$, $1 \le p \le \infty$, since ψ_n^ν is a bounded function on $(0,1)$, hence in $L^q(x^{2\nu+1}dx)$, $1 \le q \le \infty$.

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Kent State University
etna@mcs.kent.edu

Note, that here and later on the measure $x^{2\nu+1}dx$ should be thought of as a solid measure, not as a weighted Lebesgue measure with a power weight.

We consider the corresponding partial sum operators

$$\mathcal{S}_N g(x) = \sum_1^N b_n^\nu \psi_n^\nu(x)$$

and the maximal function

$$\mathcal{S}^* g(x) = \sup_{N\geq 1} |\mathcal{S}_N g(x)|.$$

Since $\langle g, \psi_n^\nu\rangle_{L^2(x^{2\nu+1}dx)} = \langle g\cdot x^{\nu+1/2}, \varphi_n^\nu\rangle_{L^2(dx)}$ we have

$$\mathcal{S}^* g(x) = x^{-(\nu+1/2)} S^*(g\cdot x^{\nu+1/2})(x).$$

Hence the inequality

$$||\mathcal{S}^* g||_{L^p(x^{2\nu+1}dx)} \leq C||g||_{L^p(x^{2\nu+1}dx)}$$

is equivalent with

$$\int_0^1 |S^* f(x)|^p x^{(2\nu+1)(1-p/2)} dx \leq C \int_0^1 |f(x)|^p x^{(2\nu+1)(1-p/2)} dx.$$

Since $-1 < (2\nu+1)(1-p/2) < p-1$ if and only if $4(\nu+1)/(2\nu+3) < p < 4(\nu+1)/(2\nu+1)$ we then have

COROLLARY 1.3. *Let* $\nu \geq -1/2$ *and* $4(\nu+1)/(2\nu+3) < p \leq \infty$. *Then*

$$\mathcal{S}_N g(x) \to g(x), \qquad N\to\infty,$$

a.e. for every function g *in* $L^p(x^{2\nu+1}dx)$.

It will be shown in Section 4 that this result (as well as Corollary 1.4 below) is sharp. An equiconvergence result for expansions with respect to the system $\{\psi_n^\nu\}$ can be found in [5], Theorem 2.10.

We now pass to the system $\{\phi_n^\nu\}_{n=1}^\infty$. Given h, a function on $(0,1)$, we associate to it its Fourier-Bessel coefficients

$$d_n^\nu = d_n^\nu(h) = \int_0^1 h(x)\phi_n^\nu(x)\, x\, dx\, .$$

In the case $\nu \geq 0$, it is possible to evaluate the coefficients $\{d_n^\nu\}$ for every function h in $L^p(xdx)$, $1\leq p\leq\infty$, since ϕ_n^ν is a bounded function on $(0,1)$, hence in $L^q(x\,dx)$, $1\leq q\leq\infty$. If $-1<\nu<0$ we have to restrict our attention to $2/(2+\nu) < p \leq \infty$ since only then ϕ_n^ν is in $L^{p'}(x\,dx)$ (as before, here and later on we should look at $x\,dx$ as at the solid measure).

As earlier we consider the partial sum operators

$$\widetilde{\mathcal{S}}_N h(x) = \sum_1^N d_n^\nu \phi_n^\nu(x)$$

and the maximal function

$$\widetilde{\mathcal{S}}^* h(x) = \sup_{N\geq 1} |\widetilde{\mathcal{S}}_N h(x)|.$$

Since $\langle h, \phi_n^\nu \rangle_{L^2(x\,dx)} = \langle h \cdot x^{1/2}, \varphi_n^\nu \rangle_{L^2(dx)}$ we have

$$\widetilde{\mathcal{S}}^* h(x) = x^{-1/2} S^*(h \cdot x^{1/2})(x).$$

Hence the inequality

$$||\widetilde{\mathcal{S}}^* h||_{L^p(x\,dx)} \le C||h||_{L^p(x\,dx)}$$

is equivalent with

$$\int_0^1 |S^* f(x)|^p x^{1-p/2} dx \le C \int_0^1 |f(x)|^p x^{1-p/2} dx.$$

Since $-1 < 1 - p/2 < p - 1$ if and only if $4/3 < p < 4$ we then have

COROLLARY 1.4. *Let* $\nu \ge -1/2$ *and* $4/3 < p \le \infty$. *Then*

$$\widetilde{\mathcal{S}}_N f(x) \to f(x), \qquad N \to \infty,$$

a.e. for every function f *in* $L^p(x\,dx)$.

Throughout the paper, unless otherwise stated, ν is an arbitrary fixed real number greater than -1 and, for a given p, $1 \le p \le \infty$, p' will denote its conjugate, $p' = p/(p-1)$. As usual on such occasions, a positive constant C may vary from line to line.

2. Auxiliary results. We start with collecting some facts used later on. An asymptotic form of λ_n's is given by

$$\lambda_n = \pi(n + B_\nu + O(n^{-1})), \qquad B_\nu = \nu/2 - 1/4,$$

cf. [16], p. 618. We will also use the asymptotic form of $J_\nu(t)$:

$$J_\nu(t) = \sqrt{2/\pi t}\,(\cos(t + D_\nu) + O(t^{-1})), \qquad t \ge 1, \tag{2.1}$$

where $D_\nu = -(\nu\pi/2 + \pi/4)$. Since

$$\sqrt{\lambda_n} J_{\nu+1}(\lambda_n) = \sqrt{2/\pi}\,(\cos(\lambda_n + D_{\nu+1}) + O(n^{-1}))$$

and $\lambda_n + D_{\nu+1} = (n-1)\pi + O(n^{-1})$, we have

$$|\sqrt{\lambda_n} J_{\nu+1}(\lambda_n)| = \sqrt{2/\pi}\,(1 + O(n^{-1})).$$

This and the fact that for $\nu \ge -1/2$, $\sqrt{t} J_\nu(t)$ is a bounded function on $(0,\infty)$ also shows that in the case $\nu \ge -1/2$, $\varphi_n^\nu(x)$ are uniformly bounded on $(0,1)$:

$$|\varphi_n^\nu(x)| \le C, \qquad n = 1, 2, \dots \quad , \quad 0 < x < 1.$$

In what follows $a_n \sim b_n$ for $a_n > 0$, $b_n > 0$ means that $C^{-1} \le a_n/b_n \le C$ for a constant $C > 0$.

LEMMA 2.1. *Let* $\nu > -1$, $1 \le p < \infty$ *and* $\beta > -1 - \nu p$. *Then*

$$\int_0^n |J_\nu(y)|^p y^\beta dy \sim \begin{cases} 1, & -1 - \nu p < \beta < p/2 - 1, \\ \log n, & \beta = p/2 - 1, \\ n^{\beta - p/2 + 1}, & p/2 - 1 < \beta < \infty. \end{cases}$$

ETNA
Kent State University
etna@mcs.kent.edu

Proof. In the first case, $-1-\nu p<\beta<p/2-1$, we use $J_\nu(t)=O(t^{-1/2})$, $t\to\infty$, to get

$$\int_0^1 |J_\nu(y)|^p y^\beta dy = C_1 < \int_0^n |J_\nu(y)|^p y^\beta dy < C_1 + C\int_1^\infty y^{-p/2+\beta}dy = C_2.$$

If $\beta=p/2-1$ we use (2.1) to show that $\int_\pi^{\pi n}|J_\nu(y)|^p y^\beta dy\sim\log n$. The integral coming from the remainder gives the bound

$$\int_\pi^{\pi n} y^{-p-1}dy = O(1).$$

For the integral coming from the main part we write

$$\int_\pi^{\pi n}|\cos(y+D_\nu)|^p y^{-1}dy = \int_0^\pi |\cos(y+D_\nu)|^p\left(\sum_{k=1}^{n-1}\frac{1}{y+\pi k}\right)dy.$$

The required asymptotics comes from the fact that

$$\sum_{k=1}^{n-1}\frac{1}{y+\pi k}\sim\log n$$

uniformly in $y\in(0,\pi)$. In the last case, $p/2-1<\beta<\infty$, we use the same argument except the fact that now

$$\sum_{k=1}^{n-1}\frac{1}{(y+\pi k)^{-\beta+p/2}}\sim\int_\pi^{n\pi} t^{\beta-p/2}dt = Cn^{\beta-p/2+1}.$$

uniformly in $y\in(0,\pi)$. ∎

The next two propositions give precise asymptotics of L^p-norms of functions for the three orthogonal systems we consider.

PROPOSITION 2.2. *Let* $\nu>-1$ *and* $1\le p<\infty$ *or* $\nu\ge-1/2$ *and* $1\le p\le\infty$. *Then*

$$||\varphi_n^\nu||_{L^p(dx)}\sim 1.$$

Proof. If $1\le p<\infty$ it is sufficient to change variables and take $\beta=p/2$ in Lemma 2.1. If $p=\infty$ we use the fact that $\sqrt{t}J_\nu(t)$ is bounded on $(0,\infty)$. ∎

PROPOSITION 2.3. *Let* $\nu>-1/2$ *and* $1\le p\le\infty$. *Then*

$$||\psi_n^\nu||_{L^p(x^{2\nu+1}dx)}\sim\begin{cases}1, & 1\le p<\frac{4(\nu+1)}{2\nu+1},\\ (\log n)^{\frac{2\nu+1}{4(\nu+1)}}, & p=\frac{4(\nu+1)}{2\nu+1},\\ n^{\frac{2\nu+1}{2}-\frac{2(\nu+1)}{p}}, & \frac{4(\nu+1)}{2\nu+1}<p\le\infty.\end{cases}$$

If $-1<\nu\le-1/2$ *then* $||\varphi_n^\nu||_{L^p(dx)}\sim1$ *for* $1\le p<\infty$ *and* $||\varphi_n^\nu||_{L^\infty(dx)}\sim n^{\nu+1/2}$.

Proof. If $1\le p<\infty$ we change variables and use Lemma 2.1 with $\beta=2\nu+1-\nu p$. If $p=\infty$ we use the fact that $J_\nu(t)/t^\nu$ is bounded on $(0,\infty)$. ∎

PROPOSITION 2.4. *Let* $\nu>-1$ *and* $1\le p<\infty$ *or* $\nu\ge0$ *and* $p=\infty$. *Then*

$$||\phi_n^\nu||_{L^p(xdx)}\sim\begin{cases}1, & 1\le\ p<4,\\ (\log n)^{1/4}, & p=4,\\ n^{1/2-2/p}, & 4<\ p\le\infty.\end{cases}$$

Proof. If $1\le p<\infty$ we change variables and use Lemma 2.1 with $\beta=1$. If $p=\infty$ and $\nu\ge0$ we use the fact that $J_\nu(t)$ is bounded on $(0,\infty)$. ∎

3. Almost everywhere convergence: proof of Theorem 1.1. In this section we restrict our attention to the case $\nu \geq -1/2$. We will use the representation of the Dirichlet kernel

$$K_N(x,y) = \sum_0^N \varphi_n^\nu(x)\varphi_n^\nu(y)$$

given by MacRoberts (see [16], Lemma 4.1):

$$K_N(x,y) = G_N(x,y) + R_N(x,y),$$

where

$$G_N(x,y) = \frac{A_N}{2}\sqrt{xy}\frac{J_\nu(A_N x)J_{\nu+1}(A_N y) - J_\nu(A_N y)J_{\nu+1}(A_N x)}{x-y},$$

$A_N = (N + \frac{\nu}{2} + \frac{1}{4})\pi$, and

$$|R_N(x,y)| \leq C\left(\frac{1}{x+y} + \frac{1}{2-x-y}\right).$$

We then have

$$S_N f(x) = \int_0^1 G_N(x,y)f(y)dy + \int_0^1 R_N(x,y)f(y)dy = G_N f(x) + R_N f(x)$$

and

$$S^* f(x) \leq G^* f(x) + R^* f(x),$$

where $G^* f(x) = \sup_N |G_N f(x)|$ and $R^* f(x) = \sup_N |R_N f(x)|$. To get the estimate

$$||R^* f||_{p,\alpha} \leq C||f||_{p,\alpha}, \tag{3.1}$$

we write

$$R^* f(x) \leq C \int_0^1 \left(\frac{1}{x+y} + \frac{1}{2-x-y}\right)|f(y)|dy.$$

It is now sufficient to know that the integral operators

$$T_i f(x) = \int_0^1 K_i(x,y)f(y)dy, \qquad i = 1,2,$$

with the kernels

$$K_1(x,y) = \left(\frac{x}{y}\right)^\beta \cdot \frac{1}{x+y}, \qquad K_2(x,y) = \left(\frac{x}{y}\right)^\beta \cdot \frac{1}{2-x-y}$$

are bounded on $L^p(dx)$ whenever $1 < p < \infty$ and $-\frac{1}{p} < \beta < \frac{1}{p'}$. This is the content of a homogeneous kernel lemma, see [14], p.228, and Lemma 3 in [2].
We now start proving the second estimate

$$||G^* f||_{p,\alpha} \leq C||f||_{p,\alpha}. \tag{3.2}$$

ETNA
Kent State University
etna@mcs.kent.edu

We will show that

$$G^*f(x) \le C(Mf(x) + Mf(2x) + H^*f(x) + C^*f(x))$$

where M is the maximal Hardy-Littlewood operator

$$Mf(x) = \sup_{a<x<b} \frac{1}{b-a}\int_a^b |f(t)|dt,$$

H^* is the maximal Hilbert transform operator

$$H^*f(x) = \sup_{0<E<\infty}\left|\int_{0<|x-y|<E}\frac{f(y)}{x-y}dy\right|,$$

and C^* is the maximal Carleson operator

$$C^*f(x) = \sup_{n\in\mathbb{N}}\sup_{0<E<\infty}\left|\int_{0<|x-y|<E}\frac{e^{-iny}f(y)}{x-y}dy\right|.$$

This, combined with the well known weighted inequalities for each of the operators M, H^* and C^* (for the operator C^* see, for instance, [13]) gives (3.2) and together with (3.1) finishes the proof of Theorem 1.1.
Fix x in $(0,1)$ and f in $L^{p,\alpha}(dx)$ and write

$$G_Nf(x) = \sum_{i=1}^3\int_{\mathbb{R}} G_N(x,y)f_i(y)dy,$$

where $f_1 = f\cdot\chi_{[0,x/2]}$, $f_2 = f\cdot\chi_{[x/2,3x/2]}$, $f_3 = f\cdot\chi_{[3x/2,\infty)}$ (we consider f, originally defined on $(0,1)$, as a function on $\mathbb{R}$ by putting $f=0$ outside $(0,1)$). Hence

$$|G_Nf(x)| \le \sum_{i=1}^3 |G_Nf_i(x)|.$$

We start with estimating $|G_Nf_1(x)|$ by considering, for instance, the term resulting from taking $J_\nu(A_Ny)J_{\nu+1}(A_Nx)$ in the numerator in $G_N(x,y)$. Then

$$\begin{aligned}&|\sqrt{A_Nx}J_{\nu+1}(A_Nx)|\int_{\mathbb{R}}|\sqrt{A_Ny}J_\nu(A_Ny)|\frac{|f_1(y)|}{|x-y|}dy\\ &\le C\int_0^{x/2}\frac{|f(y)|}{x-y}dy \le C\frac{1}{x}\int_0^{x/2}|f(y)|dy \le C\,Mf(x).\end{aligned}$$

Analogously we estimate $|G_Nf_3(x)|$ obtaining

$$\begin{aligned}&|\sqrt{A_Nx}J_{\nu+1}(A_Nx)|\int_{\mathbb{R}}|\sqrt{A_Ny}J_\nu(A_Ny)|\frac{|f_3(y)|}{|x-y|}dy\\ &\le C\int_{3x/2}^1\frac{|f(y)|}{y-x}dy \le C\frac{1}{x}\int_{3x/2}^1|f(y)|dy \le C\,Mf(2x).\end{aligned}$$

The estimate of $|G_Nf_2(x)|$ is much more delicate. We have

$$\begin{aligned}\left|\int_{\mathbb{R}} G_N(x,y)f_2(y)dy\right| &\le \frac{1}{2}\Bigg(|\sqrt{A_Nx}J_\nu(A_Nx)|\cdot\left|\int_{x/2}^{3x/2}\frac{\sqrt{A_Ny}J_{\nu+1}(A_Ny)}{x-y}f(y)dy\right|\\ &\qquad + |\sqrt{A_Nx}J_{\nu+1}(A_Nx)|\cdot\left|\int_{x/2}^{3x/2}\frac{\sqrt{A_Ny}J_\nu(A_Ny)}{x-y}f(y)dy\right|\Bigg)\\ &\le C\left(\left|\int_{x/2}^{3x/2}\frac{\sqrt{A_Ny}J_{\nu+1}(A_Ny)}{x-y}f(y)dy\right| + \left|\int_{x/2}^{3x/2}\frac{\sqrt{A_Ny}J_\nu(A_Ny)}{x-y}f(y)dy\right|\right).\end{aligned}$$

ETNA
Kent State University
etna@mcs.kent.edu

Since both above integrals are treated in the same way it is sufficient to deal with the second integral only. Let

$$I_N f(x) = \int_{x/2}^{3x/2} \frac{\sqrt{A_N y}J_\nu(A_N y)}{x-y} f(y)dy.$$

Consider first these N's for which $A_N x \le 1$. We have

$$J_\nu(A_N y) = (J_\nu(A_N y) - J_\nu(A_N x)) + J_\nu(A_N x), \tag{3.3}$$

hence the piece in $I_N f(x)$ resulting from the summand $J_\nu(A_N x)$ is bounded by

$$\begin{aligned} &|J_\nu(A_N x)| \cdot \left| \int_{x/2}^{3x/2} \frac{\sqrt{A_N y}}{x-y} f(y)dy \right| \\ &\le |J_\nu(A_N x)| \cdot \left(\left| \int_{x/2}^{3x/2} \frac{\sqrt{A_N y} - \sqrt{A_N x}}{x-y} f(y)dy \right| + \sqrt{A_N x} \left| \int_{x/2}^{3x/2} \frac{1}{x-y} f(y)dy \right| \right). \end{aligned}$$

The term resulting from the last integral is estimated by

$$C \left| \int_{x/2}^{3x/2} \frac{1}{x-y} f(y)dy \right| \le CH^* f(x).$$

The term resulting from the first integral is estimated by

$$\begin{aligned} \sqrt{A_N}|J_\nu(A_N x)| \cdot | \int_{x/2}^{3x/2} \frac{1}{\sqrt{x}+\sqrt{y}} f(y)dy| &\le C|\sqrt{A_N x} J_\nu(A_N x)| \cdot \frac{1}{x} \int_{x/2}^{3x/2} |f(y)|dy \\ &\le C\, Mf(x). \end{aligned}$$

We now come back to the piece resulting in $I_N f(x)$ from taking $J_\nu(A_N y) - J_\nu(A_N x)$ in (3.3). Using the facts that $J_\nu'(t) = \frac{\nu}{t} J_\nu(t) - J_{\nu+1}(t)$, $\sqrt{t} J_\nu(t)$ and $\sqrt{t} J_{\nu+1}(t)$ are bounded on $(0, \infty)$, and the assumption $A_N x \le 1$ we obtain (here ξ is a number between x and y, hence comparable with x)

$$\begin{aligned} \left| \int_{x/2}^{3x/2} \frac{J_\nu(A_N x) - J_\nu(A_N y)}{x-y} \sqrt{A_N y} f(y)dy \right| &\le A_N \int_{x/2}^{3x/2} |\sqrt{A_N y} J_\nu'(A_N \xi)| \cdot |f(y)|dy \\ &\le C \frac{1}{x} \int_{x/2}^{3x/2} |f(y)|dy \le C\, Mf(x). \end{aligned}$$

It remains to consider those N's for which $A_N x \ge 1$. This time we write

$$J_\nu(A_N y) = Q_1(A_N y) + Q_2(A_N y),$$

where $Q_1(t) = J_\nu(t) + \sqrt{2/\pi t}\sin(t - D_{\nu+1})$ and $Q_2(t) = -\sqrt{2/\pi t}\sin(t - D_{\nu+1})$. The piece in $I_N f(x)$ resulting from the summand $Q_2(A_N y)$ is bounded by

$$\sqrt{2/\pi} \left| \int_{x/2}^{3x/2} \frac{\sin(A_N y - D_{\nu+1})}{x-y} f(y)dy \right| \le C\, C^* f(x).$$

To estimate the piece in $I_N f(x)$ resulting from the summand $Q_1(A_N y)$ we further write

$$Q_1(A_N y) = (Q_1(A_N y) - Q_1(A_N x)) + Q_1(A_N x).$$

Taking into account the second summand, $Q_1(A_N x)$, we get the bound

$$\begin{aligned}
&\left|\int_{x/2}^{3x/2} \frac{\sqrt{A_N y}Q_1(A_N x)}{x-y} f(y)dy\right| \\
&= \left|\int_{x/2}^{3x/2} \frac{(\sqrt{A_N y}-\sqrt{A_N x})+\sqrt{A_N x})Q_1(A_N x)}{x-y} f(y)dy\right| \\
&\leq \sqrt{A_N}|Q_1(A_N x)|\cdot \int_{x/2}^{3x/2} \frac{|f(y)|}{\sqrt{x}+\sqrt{y}}dy + |\sqrt{A_N x}Q_1(A_N x)|\cdot\left|\int_{x/2}^{3x/2}\frac{f(y)}{x-y}dy\right| \\
&\leq |\sqrt{A_N x}Q_1(A_N x)|\cdot \frac{1}{x}\int_{x/2}^{3x/2}|f(y)|dy + C\left|\int_{x/2}^{3x/2}\frac{f(y)}{x-y}dy\right| \\
&\leq C(Mf(x)+H^*f(x)).
\end{aligned}$$

For the first summand, $Q_1(A_N y) - Q_1(A_N x)$, we write

$$\begin{aligned}
&\left|\int_{x/2}^{3x/2} \frac{\sqrt{A_N y}(Q_1(A_N y)-Q_1(A_N x))}{x-y} f(y)dy\right| \\
&\leq \int_{x/2}^{3x/2} |Q_1'(A_N\xi(y))\sqrt{A_N y}|f(y)|dy, \qquad (3.4)
\end{aligned}$$

where $\xi(y)$ is between x and y. It is easily seen that

$$Q_1'(t) = \frac{\nu}{t}J_\nu(t) + \frac{1}{\sqrt{2\pi}}t^{-3/2}\sin(t-D_{\nu+1}) + R(t),$$

where $R(t) = O(t^{-3/2})$ as $t \to \infty$. Hence, we also have $|Q_1'(t)| \leq Ct^{-3/2}$ for $t \geq 1$. Therefore using $A_N x \geq 1$ we further estimate (3.4) by

$$\begin{aligned}
&CA_N\int_{x/2}^{3x/2}|A_N\xi(y)|^{-3/2}\sqrt{A_N y}|f(y)|dy \\
&\leq CA_N\int_{x/2}^{3x/2}|A_N x|^{-1}\sqrt{A_N y}|f(y)|dy \leq CMf(x).
\end{aligned}$$

This finishes the estimate of $I_N f(x)$ under the assumption $A_N x \geq 1$, hence the proof of Theorem 1.1.

4. Divergence results. As it was already mentioned the results from Corollaries 1.3 and 1.4 are sharp. In the proof of the following two propositions we apply an argument which was used by Meaney [11] in the case of Jacobi expansions.

PROPOSITION 4.1. *Let* $\nu > -1/2$ *and* $p_o = 4(\nu+1)/(2\nu+3)$. *There exists a function* g_o *in* $L^{p_o}(x^{2\nu+1}dx)$ *whose Fourier-Bessel series* $\sum_1^\infty b_n(g_o)\psi_n^\nu(x)$ *is divergent almost everywhere on* $(0,1)$.

Proof. To simplify the notation we use $||\cdot||_p$ to denote the L^p-norm with respect to the measure $x^{2\nu+1}dx$ on $(0,1)$ and $\langle\cdot,\cdot\rangle$ to denote the scalar product in $L^2(x^{2\nu+1}dx)$. Let $g \in L^{p_o}(x^{2\nu+1}dx)$. Then, by Proposition 2.3,

$$|b_n(g)| = \left|\int_0^1 g(x)\psi_n^\nu(x)x^{2\nu+1}dx\right| \leq ||g||_{p_o}||\psi_n^\nu||_{p_o'} \leq C(\log n)^{1/p_o'}. \qquad (4.1)$$

ETNA
Kent State University
etna@mcs.kent.edu

Consider the sequence of functionals

$$T_n g = \langle g, \psi_n^\nu \rangle ||\psi_n^\nu||_{p_o'}^{-1/2}$$

on $L^{p_o}(x^{2\nu+1}dx)$. The norms of these functionals equal $||\psi_n^\nu||_{p_o'}^{1/2}$. Suppose that for every g in $L^{p_o}(x^{2\nu+1}dx)$,

$$\sup_n |\langle g, \psi_n^\nu \rangle| \cdot ||\psi_n^\nu||_{p_o'}^{-1/2} < \infty.$$

Then, by the Banach-Steinhaus theorem, we would have $\sup_n ||\psi_n^\nu||_{p_o'}^{1/2} < \infty$ which is not possible since, again by Proposition 2.3, $||\psi_n^\nu||_{p_o'}^{1/2} \sim (\log n)^{1/2p_o'}$. Therefore, there is an g_o in $L^{p_o}(x^{2\nu+1}dx)$ such that

$$\sup_n |\langle g_o, \psi_n^\nu \rangle| \cdot ||\psi_n^\nu||_{p_o'}^{-1/2} = \infty,$$

hence, also

$$\sup_n |b_n(g_o)| = \infty. \tag{4.2}$$

Assume that $\sum_1^\infty b_n(g_o)\psi_n^\nu(x)$ converges on a subset A of positive measure in $(0,1)$. Clearly, we can think that $A \subset (\varepsilon, 1)$ for a fixed $\varepsilon > 0$. Therefore

$$b_n(g_o)\psi_n^\nu(x) \to 0, \qquad x \in A.$$

Given x in A, consider large n (such that $\lambda_n \varepsilon \geq 1$). By using (2.1), we obtain

$$\psi_n^\nu(x) = c_{n,\nu} x^{-\nu} \left(\left(\frac{2}{\pi \lambda_n x} \right)^{1/2} \cos(\lambda_n x + D_\nu) + O((\lambda_n x)^{-3/2}) \right).$$

Since the remainder gives the $O(n^{-1})$ income and, by (4.1), $b_n(g_o) = O((\log n)^{1/p_o'})$ the remainder part in $b_n(g_o)\psi_n^\nu(x)$ is $o(n)$. Hence, for every x in A, a set of a positive measure, we have

$$b_n(g_o) \cos(\lambda_n x + D_\nu) \to 0.$$

By a variant of the Cantor-Lebesgue lemma, cf. [18], p.316 or [12], Lemma 4, this implies $b_n(g_o) \to 0$ which contradicts (4.2). $\square$

PROPOSITION 4.2. *Let $\nu > -1$. There exists a function h_o in $L^{4/3}(x\,dx)$ whose Fourier-Bessel series $\sum_1^\infty d_n(h_o)\phi_n^\nu(x)$ is divergent almost everywhere on $(0,1)$.*

Proof. This time, to simplify the notation, we use $||\cdot||_p$ to denote the L^p-norm with respect to the measure $x\,dx$ on $(0,1)$ and $\langle\cdot,\cdot\rangle$ to denote the scalar product in $L^2(x\,dx)$. Let $h \in L^{4/3}(x\,dx)$. Then, by Proposition 2.4,

$$|d_n(h)| = \left| \int_0^1 h(x)\phi_n^\nu(x) x\,dx \right| \leq ||h||_{4/3} ||\phi_n^\nu||_4 \leq C(\log n)^{1/4}. \tag{4.3}$$

Consider the sequence of functionals

$$T_n h = \langle h, \phi_n^\nu \rangle ||\phi_n^\nu||_4^{-1/2}$$

ETNA
Kent State University
etna@mcs.kent.edu

on $L^{4/3}(x\,dx)$. The norms of these functionals equal $||\phi_n^\nu||_4^{1/2}$. Suppose that for every h in $L^{4/3}(x\,dx)$,

$$\sup_n |\langle h, \phi_n^\nu\rangle| \cdot ||\phi_n^\nu||_4^{-1/2} < \infty.$$

Then, by the Banach-Steinhaus theorem, we would have $\sup_n ||\phi_n^\nu||_4^{1/2} < \infty$ which is not possible since, again by Proposition 2.4, $||\phi_n^\nu||_4^{1/2} \sim (\log n)^{1/8}$. Therefore, there is an h_o in $L^{4/3}(x\,dx)$ such that

$$\sup_n |\langle h_o, \phi_n^\nu\rangle| \cdot ||\phi_n^\nu||_4^{-1/2} = \infty,$$

hence, also

$$\sup_n |d_n(h_o)| = \infty. \tag{4.4}$$

Assume that $\sum_1^\infty d_n(h_o)\psi_n^\nu(x)$ converges on a subset A of positive measure in $(\varepsilon, 1)$. Therefore

$$d_n(h_o)\phi_n^\nu(x) \to 0, \qquad x \in A.$$

Given x in A, consider large n (such that $\lambda_n\varepsilon \geq 1$). By using (2.1), we obtain

$$\phi_n^\nu(x) = c_{n,\nu}\left(\left(\frac{2}{\pi\lambda_n x}\right)^{1/2}\cos(\lambda_n x + D_\nu) + O((\lambda_n x)^{-3/2})\right).$$

Since the remainder is $O(n^{-1})$ and, by (4.3), $d_n(h_o) = O((\log n)^{1/4})$ the remainder part in $d_n(h_o)\phi_n^\nu(x)$ is $o(n)$. Hence, for every x in A, a set of a positive measure, we have

$$d_n(h_o)\cos(\lambda_n x + D_\nu) \to 0.$$

Again $d_n(h_o) \to 0$ which contradicts (4.4). ☐

5. Norm convergence: necessity. The problem of the norm convergence of partial sums of expansions with respect to the three orthogonal systems discussed in the introduction has been widely discussed in the literature, cf. papers of Benedek and Panzone [1], [2], Gilbert [6] and Wing [17]. Applying a fairly standard argument, we take here the opportunity to show the necessity parts in the following two results.

PROPOSITION 5.1. *Let $\nu > -1$ and $1 \leq p \leq \infty$. Then*

$$||S_N f - f||_{L^p(dx)} \to 0, \qquad N \to \infty \tag{5.1}$$

for every f in $L^p(dx)$ if and only if $\nu \geq -1/2$ and $1 < p < \infty$ or $-1 < \nu < -1/2$ and $2/(2\nu+3) < p < -2/(2\nu+1)$.

Proof. The sufficiency is contained in [17] and [2]. For the neccesity assume first that $\nu = -1/2$. Then (5.1) gives uniform boundedness of the partial sum operators for cosine expansions on $L^p(dx)$ and it is well known that this does not hold for $p = 1$ (or, for its conjugate, $p = \infty$). The general case $\nu \geq -1/2$ then follows by Gilbert's transplantation theorem [6]. If $-1 < \nu < -1/2$, then there is nothing to prove since the condition $p > 2/(2\nu+3)$ is just to guarantee the existence of the Fourier-Bessel coefficients $a_n^\nu(f)$ for a general f in $L^p(dx)$ and $-2/(2\nu+1)$ is the conjugate to $2/(2\nu+3)$. ☐

ETNA
Kent State University
etna@mcs.kent.edu

PROPOSITION 5.2. *Let* $\nu > -1$ *and* $1 \le p \le \infty$. *Then*

$$||\mathcal{S}_N g - g||_{L^p(x^{2\nu+1}dx)} \to 0, \qquad N \to \infty \tag{5.2}$$

for every g *in* $L^p(x^{2\nu+1}dx)$ *if and only if*

$$\frac{4(\nu+1)}{2\nu+3} < p < \frac{4(\nu+1)}{2\nu+1}. \tag{5.3}$$

Proof. The sufficiency is contained in [1]. Assume now that (5.2) holds true. Then, by the Banach-Steinhaus theorem,

$$||\mathcal{S}_N g||_{L^p(x^{2\nu+1}dx)} \le M||g||_{L^p(x^{2\nu+1}dx)}$$

with a constant $M > 0$ independent of $N = 1, 2, \ldots$, which implies

$$||b_N(g)\psi_N^\nu||_{L^p(x^{2\nu+1}dx)} \le 2M||g||_{L^p(x^{2\nu+1}dx)}.$$

This means that the operators

$$P_N g = \langle g, \psi_N^\nu \rangle \psi_N^\nu$$

are bounded on $L^p(x^{2\nu+1}dx)$ uniformly in $N = 1, 2, \ldots$. But the norms of these operators equal $||\psi_N^\nu||_p ||\psi_N^\nu||_{p'}$ and, by Proposition 2.3,

$$\sup_n ||\psi_n^\nu||_p ||\psi_n^\nu||_{p'} < \infty$$

if and only if p satisfies (5.3). $\square$

PROPOSITION 5.3. *Let* $\nu > -1$ *and* $1 \le p \le \infty$. *Then*

$$||\widetilde{\mathcal{S}}_N h - h||_{L^p(x\,dx)} \to 0, \qquad N \to \infty$$

for every g *in* $L^p(x\,dx)$ *if and only if* $\nu \ge -1/2$ *and* $4/3 < p < 4$ *or* $-1 < \nu < -1/2$ *and* $2/(2+\nu) < p < -2/\nu$.

Proof. The sufficiency is shown in [1]. For the necessity, in the case $\nu \ge -1/2$, we apply the same argument as in the previous proof using now Proposition 2.4. If $-1 < \nu < -1/2$ then, again, there is nothing to prove since the condition $p > 2/(2+\nu)$ is just for guaranteeing the existence of the Fourier-Bessel coefficients $d_n^\nu(h)$ for a general h in $L^p(xdx)$ and $-2/\nu$ is the conjugate to $2/(2+\nu)$. $\square$

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