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PIECEWISE LINEAR WAVELET COLLOCATION, APPROXIMATION OF THE BOUNDARY MANIFOLD, AND QUADRATURE *

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Abstract. In this paper we consider a piecewise linear wavelet collocation method for the solution of boundary integral equations of order $\mathbf{r} = 0, -1$ over a closed and smooth boundary manifold. The trial space is the space of all continuous and piecewise linear functions defined over a uniform triangular grid and the collocation points are the grid points. For the wavelet basis in the trial space we choose the three-point hierarchical basis together with a slight modification near the boundary points of the global patches of parametrization. We choose three, four, and six term linear combinations of Dirac delta functionals as wavelet basis in the space of test functionals. The usual compression results apply, i.e., for N degrees of freedom, the fully populated stiffness matrix of N^2 entries can be approximated by a sparse matrix with no more than $\mathcal{O}(N[\log N]^2)$ nonzero entries. The topic of the present paper, however, is to show that the parametrization can be approximated by low order piecewise polynomial interpolation and that the integrals in the stiffness matrix can be computed by quadrature, where the quadrature rules are combinations of product integration applied to non analytic factors of the integrand and of high order Gauß rules applied to the analytic parts. The whole algorithm for the assembling of the matrix requires no more than $\mathcal{O}(N[\log N]^4)$ arithmetic operations, and the error of the collocation approximation, including the compression, the approximate parametrization, and the quadratures, is less than $\mathcal{O}(N^{-1}[\log N]^2)$. Note that, in contrast to well-known algorithms by v.Petersdorff, Schwab, and Schneider, only a finite degree of smoothness is required.

Key words. boundary integral equation of order 0 and -1, piecewise linear collocation, wavelet algorithm, approximation of parametrization, quadrature.

AMS subject classifications. 45L10, 65D32, 65R20, 65N38.

1. Introduction. It is a well-known fact that conventional finite element discretizations of linear integral equations (e.g., of boundary integral equations) lead to systems of linear equations with fully populated matrices. Thus, even an iterative solution method requires a huge number of arithmetic operations and a large storage capacity. In order to improve upon these finite element approaches for integral equations, several algorithms have been developed. One of these consists of employing wavelet bases of finite element spaces. The basic idea goes back to Beylkin, Coifman, and Rokhlin [3], and has been thoroughly investigated by Dahmen, v.Petersdorff, Prößdorf, Schneider, and Schwab [14, 15, 35, 34, 33, 47] (cf. also the contributions by Alpert, Harten, Yad-Shalom, and the authors [1, 24, 41, 21]). In the present paper, we shall apply the wavelet technique to the piecewise linear collocation of two-dimensional boundary integral equations of order $\mathbf{r} = 0$ and $\mathbf{r} = -1$ corresponding to three-dimensional boundary value problems.

First, we shall recall the definition of a simple biorthogonal wavelet basis analyzed in [43] (cf. the familiar constructions in [26, 49, 29], and in [16, 17, 18, 6, 7, 8, 19]). The grids will be supposed to be uniform refinements of a coarse initial triangulation, and the basis will be the system of three-point hierarchical basis functions, i.e., each basis function will be a linear combination of no more than three finite element functions defined over the corresponding level of a grid hierarchy. In comparison to other bases of continuous wavelet functions our basis functions will have a rather small support, and we believe that this property

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is essential for the wavelet algorithm. Indeed, small supports lead to better compression rates, especially, for lower levels and to faster quadrature algorithms for the assembling of the stiffness matrix. For the basis in the test space spanned by Dirac delta functionals, we shall take the usual test functionals which can be considered as scaled versions of difference formulas (cf. the wavelet collocation methods by Dahmen, Prößdorf, Schneider, Harten, Yad-Shalom, and one of the authors [15, 24, 41, 40, 42]). Applying the wavelet basis functions of the trial and test space, we shall obtain the well-known compression results for trial wavelets with vanishing moments due to Dahmen, v.Petersdorff, Prößdorf, Schneider, and Schwab [15, 35, 47]. The compression for trial functions without vanishing moments is the same as in [40] (cf. also the univariate analogue for the Galerkin method treated in [35, 4]). In particular, to compute an approximate collocation solution with optimal asymptotic order of convergence, it is sufficient to compute and store $\mathcal{O}(N[\log N]^2)$ entries of the fully populated $N \times N$ stiffness matrix. Here N stands for the number of degrees of freedom.

In general, the stiffness matrix cannot be computed exactly. This is the case, for instance, if the boundary manifold is given only by a discrete set of points, or if no analytic formula is available to integrate the kernel and trial function. Therefore, we shall consider an algorithm for the approximation of the boundary surface and for the quadrature of the integrals. We emphasize that this is the most time consuming and the most difficult part of the wavelet method. To set up the stiffness matrix, we shall proceed as follows. Depending on the test functional, we shall define an appropriate partition of the supports of the trial basis functions. Over these subdomains we shall replace the parametrization of the boundary manifold by a quadratic or cubic interpolation. We shall assume that the kernel function is a finite sum of terms $(P,Q) \mapsto k(P,Q)p(P-Q)/|P-Q|^\alpha$, where $k(P,Q)$ is $2-\mathbf{r}$ times continuously differentiable, and where $p(P-Q)$ is a polynomial with constant coefficients. For the part $k(P,Q)$ of the kernel function, we shall apply a low order product integration rule with the weight function chosen as the product of $Q \mapsto p(P-Q)/|P-Q|^\alpha$ times the trial wavelet. The quadrature weights of the product rule, i.e., the integrals over the function $p(P-Q)/|P-Q|^\alpha$ times the trial wavelet will be computed by Gauß rules of order less than $\mathcal{O}(\log N)$. By doing this and using well-known ideas to treat singular integrals, we shall arrive at a fully discretized wavelet algorithm with $\mathcal{O}(N[\log N]^4)$ arithmetic operations to compute $\mathcal{O}(N[\log N]^2)$ entries of the stiffness matrix. Assuming that the collocation is stable, the asymptotic error of the exact collocation solution is known to be less than $\mathcal{O}(N^{-(2-\mathbf{r})/2})$, which is optimal for piecewise linear trial spaces. The fully discrete wavelet algorithm will also be shown to be stable, and to be convergent with an almost optimal error less than $\mathcal{O}(N^{-(2-\mathbf{r})/2}[\log N]^2)$ for $\mathbf{r}=0$ and less than $\mathcal{O}(N^{-(2-\mathbf{r})/2}[\log N]^{1.5})$ for $\mathbf{r}=-1$.

Note that alternative quadrature algorithms have been considered by Beylkin, Coifman, and Rokhlin [3] for integral operators with smooth kernels, and by v.Petersdorff, Schwab, and Schneider [35, 47] (cf. also the numerical implementation by Lage and Schwab [28]) for boundary integral operators with Green kernels over piecewise analytic boundaries. To our knowledge, the fully discrete algorithm of the present paper is the first which applies to boundary integral equations over surfaces with finite degree of smoothness. In fact, the required degree of smoothness for the geometry will be equal to the convergence order $2-\mathbf{r}$ increased by one, i.e., the same as for the conventional collocation algorithm. Moreover, besides the usual singular main part $p(P-Q)/|P-Q|^\alpha$ of Green kernels, the kernel function of the integral operator will be allowed to have an additional factor $k(P,Q)$ of finite smoothness degree $2-\mathbf{r}$. In the proof of the corresponding error estimates, we shall show that the techniques developed for the compression algorithm apply to the analysis of the discretization as well. The only thing to do is to replace the decay properties in the matrix entries due to the vanishing moments of the trial functions and the norm estimates due to the smoothness of the solution by error estimates of the approximate parameter mappings and of the quadrature rules, respectively.

The powers of the logarithms in the asymptotic convergence and complexity estimates are, of course, not optimal. Using the refined compression technique of Schneider [47], choosing wavelet basis functions with more vanishing moments, and applying higher order quadrature rules, the logarithmic powers can be dropped or, at least, their exponents can be reduced. Note, however, that the application of higher order moment conditions and quadratures requires additional smoothness assumptions. Furthermore, we believe that a simple algorithm like the one in the present paper is often more efficient than an asymptotically optimal method, since the number of degrees of freedom does not tend to infinity in realistic numerical computations.

The plan of the paper is as follows. In §2 we shall describe the boundary manifold, the integral equation, and the conventional piecewise linear collocation method. In §3 we shall introduce the three-point hierarchical wavelet functions of the piecewise linear trial space, the test wavelet functionals, and the corresponding compression algorithm. Section 4 will be devoted to the description of the interpolation of the parameter mappings and to the quadrature algorithm. All proofs will be deferred to §5 and §6. In particular, in §5 we shall recall some technical results from the compression estimates. The discretization including the approximation of the parametrizations and of the integration will be analyzed in §6. Finally, we present a numerical example in §7.

2. The piecewise linear collocation method.

2.1. The manifold. We suppose that the integral equation to be solved is given on a closed boundary manifold $\Gamma \subset \mathbb{R}^3$ with finite degree of smoothness. More exactly, we assume that Γ is the union of m_Γ triangular patches Γ_m, i.e.,

$$\begin{aligned} \Gamma &= \cup_{m=1}^{m_\Gamma} \Gamma_m, \quad \Gamma_m := \kappa_m(T), \\ T &:= \left\{(s,t) \in \mathbb{R}^2 :\ 0 \le s \le 1,\ 0 \le t \le \min\{s, 1-s\}\right\}. \end{aligned} \tag{2.1}$$

Here the κ_m denote parametrization mappings from the standard triangle T to the manifold Γ. We assume that the κ_m extend to mappings from a small neighbourhood of $T \subseteq \mathbb{R}^2$ to Γ and that these extensions are d_Γ times continuously differentiable. Here d_Γ is an integer which is assumed to be greater than or equal to three when dealing with zero order operators and greater than or equal to four when dealing with operators of order $\mathbf{r} = -1$. Furthermore, we suppose that the intersection of two patches Γ_m and $\Gamma_{m'}$ is either empty or a corner point for both patches or a whole side for Γ_m and $\Gamma_{m'}$. In the last case we assume that the two representations of $\Gamma_m \cap \Gamma_{m'}$

$$\left\{\kappa_m\Big(c_1 + \lambda(c_2 - c_1)\Big) :\ 0 \le \lambda \le 1\right\} = \left\{\kappa_{m'}\Big(c_1' + \lambda(c_2' - c_1')\Big) :\ 0 \le \lambda \le 1\right\}$$

satisfy the condition

$$\kappa_m\Big(c_1 + \lambda(c_2 - c_1)\Big) = \kappa_{m'}\Big(c_1' + \lambda(c_2' - c_1')\Big), \quad 0 \le \lambda \le 1. \tag{2.2}$$

Note that, for the numerical method, the parameter mappings κ_m need not be given for all points of T. We shall use only the values of κ_m at the points of a uniform grid over the triangle T.

Since the manifold is at least thrice continuously differentiable, for each $Q \in \Gamma$, there exists a unit vector n_Q normal to Γ at Q and pointing into the exterior domain bounded by

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Γ. The Sobolev spaces $H^s(\Gamma)$ over Γ can be defined in the usual way. We define the space $H^s(\Gamma_m)$ over Γ_m as the image of the Sobolev space over T, i.e.,

$$H^s(\Gamma_m) := \{f:\ f \circ \kappa_m \in H^s(T)\}\,.$$

2.2. The integral equation. Over Γ we consider a boundary integral operator A of order $\mathbf{r} = 0$ or $\mathbf{r} = -1$ mapping $H^{\mathbf{r}/2}$ into $H^{-\mathbf{r}/2}$. We suppose that A takes the form $A = K$ for $\mathbf{r} = -1$ and $A = aI + K$ for $\mathbf{r} = 0$, where aI stands for the operator of multiplication by a function a which may be zero, and the integral operator K is defined by

$$Ku(P) := \int_\Gamma k(P, Q, n_Q)\frac{p(P-Q)}{|P-Q|^\alpha}u(Q)\,\mathrm{d}_Q\Gamma. \tag{2.3}$$

The function p stands for a homogeneous polynomial of degree $\deg(p)$, the real number α is equal to $\mathbf{r} + 2 + \deg(p)$, and the kernel function k depends on the points $P, Q \in \Gamma$. This function need not be a restriction to $\Gamma \times \Gamma$ of a function defined on the space $\mathbb{R}^3 \times \mathbb{R}^3$. It may depend for instance on the unit normals n_P and n_Q pointing into the exterior or on any different kind of differentiable vector field over Γ. To simplify the notation, we assume a special dependence and take $k(P, Q) = k(P, Q, n_Q)$ with k defined on at least a neighbourhood of $\{(P, Q, n):\ \ P, Q \in \Gamma,\ n = n_Q\} \subset \Gamma \times \Gamma \times \mathbb{R}^3$. If $\mathbf{r} = 0$, then the integrand in (2.3) can be strongly singular and the integral is to be understood in the sense of a Cauchy principal value. To ensure the existence of this principal value, we assume that p is odd, i.e., $p(Q - P) = -p(P - Q)$. Note that in applications we often have a finite sum of integrals of the above type and additional terms of lower order. Only for simplicity of notation we restrict ourselves to the one term of (2.3).

For the operator A including the just defined integral operator K, we assume the continuity of the mapping $A:\ H^{s+\mathbf{r}}(\Gamma) \longrightarrow H^s(\Gamma)$ with $s = 0$ and $s = 1.1$ (or $s = 1.1$ replaced by a different s with $1 < s < 1.5$) and the invertibility of $A:\ H^{\mathbf{r}}(\Gamma) \longrightarrow H^0(\Gamma)$. Furthermore, we suppose a finite degree of smoothness, i.e., the function a and the kernel k are supposed to be d_k times continuously differentiable. More precisely, for any d_k-th order derivative $\partial_P^{d_k}$ taken with respect to the variable $P \in \Gamma$ and for any d_k-th order derivative $\partial_{Q,n}^{d_k}$ taken with respect to the variables $Q \in \Gamma$ and $n \in \mathbb{R}^3$, we require that $\partial_P^{d_k}\partial_{Q,n}^{d_k}k(P, Q, n_Q)$ be continuous. The degree of smoothness d_k is supposed to be greater than or equal to two for $\mathbf{r} = 0$ and to three for $\mathbf{r} = -1$. For an operator A which satisfies all these assumptions, we shall solve the operator equation $Au = v$ with known right-hand side v and unknown u. To get error estimates with optimal order $2 - \mathbf{r}$, we assume $u \in H^2(\Gamma)$.

Finally, we note that single and double layer operators and other boundary integral operators (cf., e.g., [30]) are examples of integral operators fulfilling all the assumptions of this section.

2.3. Grid and collocation points. Let us introduce a hierarchy of uniform grids over the standard triangle T. For the step sizes 2^{-l}, $l = 0, \ldots, L$, we set $\triangle_l^T := {}^1\!\triangle_l^T \cup {}^2\!\triangle_l^T$, where

$$\begin{aligned} {}^1\!\triangle_l^T &:= \left\{(i2^{-l}, j2^{-l}):\ 0 \le i \le 2^l,\ 0 \le j \le \min\{2^l - i, i\}\right\},\\ {}^2\!\triangle_l^T &:= \left\{(2^{-l-1}, 2^{-l-1}) + (i2^{-l}, j2^{-l}):\ 0 \le i < 2^l,\ 0 \le j < \min\{2^l - i, i+1\}\right\}, \end{aligned}$$

and we denote the grid points by $\tau = (s, t) \in \triangle_l^T$. The grid $\triangle_l^T$ is the restriction of the grid (cf. Figure 2.1)

$$\triangle_l^{\mathbb{R}^2} := \left\{(i2^{-l}, j2^{-l}):\ i, j \in \mathbb{Z}^2\right\} \cup \left\{(2^{-l-1}, 2^{-l-1}) + (i2^{-l}, j2^{-l}):\ i, j \in \mathbb{Z}^2\right\}$$

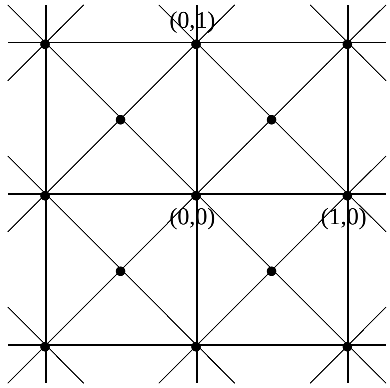

FIG. 2.1. *Grid* $\triangle_0^{\mathbb{R}^2}$.

to the triangle T. Using the parametrizations, we arrive at a grid hierarchy on Γ:

$$\triangle_l^\Gamma := \left\{\kappa_m(\tau) :\ m = 1, \dots, m_\Gamma,\ \tau \in \triangle_l^T\right\}.$$

Clearly, a grid point $P = \kappa_m(\tau)$ may have more than one representation. If P is in the interior of a side of the triangular patch Γ_m which is a common side with $\Gamma_{m'}$, then there are exactly two representations $P = \kappa_m(\tau)$ and $P = \kappa_{m'}(\tau')$. If P is a corner point of a patch, then there exist $k > 2$ representations $P = \kappa_{m_1}(\tau_1) = \kappa_{m_2}(\tau_2) = \dots = \kappa_{m_k}(\tau_k)$. We introduce ${}^i\triangle_l^\Gamma$ as the set of those $P \in \triangle_l^\Gamma$ whose representation $P = \kappa_m(\tau)$ with the smallest m satisfies $\tau \in {}^i\triangle_l^T$, and arrive at $\triangle_l^\Gamma = {}^1\triangle_l^\Gamma \cup {}^2\triangle_l^\Gamma$. The points of $\triangle_l^\Gamma$ will be denoted by capital letters like P and Q.

To each grid $\triangle_l^\Gamma$ there corresponds a partition of Γ into triangular pieces. Indeed, to get an index set for the partition triangles, let us introduce the sets of centroids

$$\begin{aligned}
\Box_0^{\mathbb{R}^2} &:= \left\{ \left(\frac{1}{2}, \frac{1}{6}\right) + k,\ \left(\frac{1}{2}, \frac{5}{6}\right) + k,\ \left(\frac{1}{6}, \frac{1}{2}\right) + k,\ \left(\frac{5}{6}, \frac{1}{2}\right) + k :\ k \in \mathbb{Z}^2 \right\}, \\
\Box_l^{\mathbb{R}^2} &:= \left\{ 2^{-l}\tau :\ \tau \in \Box_0^{\mathbb{R}^2} \right\}, \quad \Box_l^T \ := \ T \cap \Box_l^{\mathbb{R}^2}, \\
\Box_l^\Gamma &:= \left\{ \kappa_m(\tau) :\ \tau \in \Box_l^T,\ m = 1, 2, \dots, m_\Gamma \right\}.
\end{aligned}$$

For each point $\tau \in \Box_l^T$, there exist three uniquely defined neighbour points τ_1, τ_2, and τ_3 such that $\tau_1, \tau_2, \tau_3 \in \triangle_l^T$, that the triangle T_τ spanned by the three corners τ_1, τ_2, and τ_3 is of square measure $2^{-2l}/4$, and that τ is the centroid of T_τ. We arrive at the triangulation $\{T_\tau :\ \tau \in \Box_l^T\}$ of T. Note that, for $l' > l$, the centroids in $\Box_l^T$ are located at the boundaries of the smaller triangles $T_{\tau'}$ with $\tau' \in \Box_{l'}^T$. Hence there is a one to one correspondence between the triangles T_τ over several levels and the centroids in $\cup_{l=0}^L \Box_l^T$. Similarly to the triangulation over T, we define the triangulation $\{T_\tau :\ \ \tau \in \Box_l^{\mathbb{R}^2}\}$ of $\mathbb{R}^2$. For Γ and a point $Q = \kappa_m(\tau) \in \Box_l^\Gamma$, we set $\Gamma_Q := \{\kappa_m(\sigma) :\ \ \sigma \in T_\tau\}$ and arrive at the triangulation $\{\Gamma_Q :\ \ Q \in \Box_l^\Gamma\}$. Further, we denote the level l of the points $Q \in \Box_l^\Gamma$ by $l(Q)$. Notice that each partition triangle $\Gamma_Q,\ Q \in \Box_l^\Gamma$, of the generation l splits into four subtriangles of the generation $l + 1$. We call Γ_Q the father of the four subtriangles and, for $Q \in \Box_l^\Gamma,\ l > 0$, we denote the father of Γ_Q by Γ_{Q^F}.

Beside the grids $\triangle_l^\Gamma$ we introduce the difference grids $\nabla_l^\Gamma := \triangle_{l+1}^\Gamma \setminus \triangle_l^\Gamma$ for $l = 0, \dots, L-1$ and $\nabla_l^\Gamma := \triangle_0^\Gamma$ for $l = -1$, and obtain $\triangle_L^\Gamma = \bigcup_{l=-1}^{L-1} \nabla_l^\Gamma$. For $P \in \triangle_L^\Gamma$,

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we denote the unique level l for which $P \in \nabla_l^\Gamma$ by $l(P)$. Analogously to ∇_l^Γ, we define the difference grids and the point levels over T and $\mathbb{R}^2$ and get $\triangle_L^T = \bigcup_{l=-1}^{L-1} \nabla_l^T$ as well as $\triangle_L^{\mathbb{R}^2} = \bigcup_{l=-1}^{L-1} \nabla_l^{\mathbb{R}^2}$. Finally, in accordance with the splitting $\triangle_l^T = {}^1\!\triangle_l^T \cup {}^2\!\triangle_l^T$, we introduce ${}^i\nabla_l^T = \nabla_l^T \cap {}^i\!\triangle_{l+1}^T$ for $i = 1, 2$ and get $\nabla_l^T = {}^1\nabla_l^T \cup {}^2\nabla_l^T$ as well as ${}^2\nabla_l^T = {}^2\!\triangle_{l+1}^T$. Similarly, we define ${}^i\nabla_l^{\mathbb{R}^2}$ and ${}^i\nabla_l^\Gamma$.

Now the set of collocation points will be the grid $\triangle_L^\Gamma$, i.e., the test functionals of the collocation scheme are the Dirac delta functionals δ_P with $P \in \triangle_L^\Gamma$. The test space Dir_L^Γ is the span of all these δ_P.

2.4. The trial functions. To prepare for the introduction of linear spaces, we first define two-dimensional hat functions for the grid $\triangle_0^{\mathbb{R}^2}$.

$$ {}^1\varphi\,(s,t) := \max\Big\{0, 1 - \max\{|s-t|, |s+t|\}\Big\},\ {}^2\varphi\,(s,t) := \max\Big\{0, 1 - 2\max\{|s|, |t|\}\Big\}. $$

Clearly, the function ${}^2\varphi$ shifted to the point $(0.5, 0.5)$ and the function ${}^1\varphi$ are piecewise linear hat functions subordinate to the triangulation $\{T_\tau :\ \tau \in \square_0^{\mathbb{R}^2}\}$. Now, we get piecewise linear basis functions by dilating and shifting ${}^1\varphi$ and ${}^2\varphi$ to each grid point. More precisely, for each grid point $\tau \in {}^i\!\triangle_l^T$, we set $\varphi_\tau^l(\sigma) := {}^i\varphi\left(2^l(\sigma - \tau)\right)$. With the help of the parametrizations we introduce the piecewise linear (with respect to the parametrization) hat functions over Γ. For each grid point $P \in \triangle_l^\Gamma$, we set

$$ \varphi_P^l(Q) := \begin{cases} \varphi_\tau^l(\sigma) & \text{if there exist } m, \tau, \sigma \text{ s.t. } Q = \kappa_m(\sigma),\ P = \kappa_m(\tau), \\ 0 & \text{otherwise.} \end{cases} \tag{2.4} $$

Due to the assumptions on the parametrizations (cf. (2.2)) the basis functions are well defined. We denote the span of the functions $\varphi_P^l,\ P \in \triangle_l^\Gamma$ by Lin_l^Γ. Obviously, this is the space of all continuous and piecewise linear functions defined over the triangulation $\{\Gamma_Q :\ Q \in \square_l^\Gamma\}$ corresponding to the grid $\triangle_l^\Gamma$, where linearity is understood with respect to the parametrization. The space Lin_L^Γ will be the set of trial functions for the collocation.

2.5. The collocation scheme. Now the collocation method seeks an approximate solution u_L for the exact solution u of $Au = v$. This is sought in the trial space Lin_L^Γ by solving the system of collocation equations $Au_L(P) = v(P),\ P \in \triangle_L^\Gamma$. Using the representation $u_L = \sum_{P \in \triangle_L^\Gamma} \xi_P \varphi_P^L$, the collocation equations can be written in the form of a matrix equation $A_L\xi = \eta$, where we set $\xi := (\xi_P)_{P \in \triangle_L^\Gamma}$, $\eta := (\eta_P)_{P \in \triangle_L^\Gamma}$, and $\eta_P := v(P)$. The matrix of the linear system is the so called stiffness matrix given by $A_L := (a_{P',P})_{P',P \in \triangle_L^\Gamma}$ and $a_{P',P} := (A\varphi_P^L)(P')$. Moreover, using the interpolation projection R_L defined by $R_L f := \sum_{P \in \triangle_L^\Gamma} f(P)\varphi_P^L$, the collocation can be treated as a projection equation of the form $R_L A u_L = R_L v$.

Throughout this paper we shall assume that the collocation method applied to the operator equation $Au = v$ is stable. For the exact definition of stability and some remarks we refer the reader to §5.3. If the collocation is stable, if the exact solution u is in $H^2(\Gamma)$, and if $h \sim 2^{-L}$ denotes the step size of the discretization, then the approximate solution u_L satisfies the well-known optimal convergence estimates

$$ \|u - u_L\|_{H^{\mathbf{r}}(\Gamma)} \ \leq Ch^{2-\mathbf{r}}\|u\|_{H^2(\Gamma)}, \quad \mathbf{r} = 0, -1. \tag{2.5} $$

3. The wavelet algorithm.

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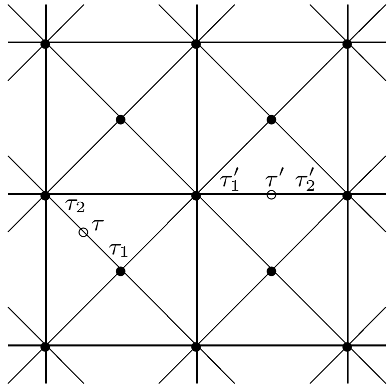

FIG. 3.1. *Neighbours τ_1 and τ_2.*

3.1. The wavelet basis of the trial space. Now we introduce the three-point hierarchical basis of [43] for the space of piecewise linear functions. Similar functions have been considered in [26, 49, 29] and are called three-point hierarchical basis functions. More precisely, for the plane and for any point $\tau \in \triangle_L^{\mathbb{R}^2}$, we set

$$\psi_\tau := \begin{cases} \varphi_\tau^0 & \text{if } \tau \in \nabla_{-1}^{\mathbb{R}^2}, \\ \varphi_\tau^{l+1} - \frac{1}{2}\left\{\varphi_{\tau_1}^{l+1} + \varphi_{\tau_2}^{l+1}\right\} & \text{if } \tau \in {}^1\nabla_l^{\mathbb{R}^2} \text{ with } l = l(\tau) \in \{0,\dots,L-1\}, \\ \varphi_\tau^{l+1} - \frac{1}{4}\left\{\varphi_{\tau_1}^{l+1} + \varphi_{\tau_2}^{l+1}\right\} & \text{if } \tau \in {}^2\nabla_l^{\mathbb{R}^2} \text{ with } l = l(\tau) \in \{0,\dots,L-1\}. \end{cases} \tag{3.1}$$

Here τ_1 and τ_2 denote the uniquely defined neighbours of τ on $\triangle_{l+1}^{\mathbb{R}^2}$ (cf. Figure 3.1). Indeed any difference grid point $\tau \in {}^2\nabla_l^{\mathbb{R}^2} \subset \triangle_{l+1}^{\mathbb{R}^2}$ has exactly two neighbour points τ_1 and τ_2 at minimal distance which belong to $\triangle_l^{\mathbb{R}^2} \subset \triangle_{l+1}^{\mathbb{R}^2}$. Any difference grid point $\tau' \in {}^1\nabla_l^{\mathbb{R}^2} \subset \triangle_{l+1}^{\mathbb{R}^2}$ has exactly two neighbour points τ_1' and τ_2' at minimal distance which belong to ${}^1\triangle_l^{\mathbb{R}^2} \subset \triangle_{l+1}^{\mathbb{R}^2}$.

The wavelet functions ψ_τ on the manifold Γ are slight modifications of (3.1). For the details and the properties of the basis we refer the reader to [43]. The final definition of the three-point hierarchical wavelet functions over the manifold Γ is

$$\psi_P := \begin{cases} \varphi_P^0 & \text{if } P \in \nabla_{-1}^{\Gamma}, \\ \varphi_P^{l+1} - \frac{1}{2}\left\{\varepsilon^{P,P_1}\varphi_{P_1}^{l+1} + \varepsilon^{P,P_2}\varphi_{P_2}^{l+1}\right\} & \text{if } P \in {}^1\nabla_l^{\Gamma} \\ & \text{with } l \in \{0,\dots,L-1\}, \\ \varphi_P^{l+1} - \frac{1}{4}\left\{\varepsilon^{P,P_1}\varphi_{P_1}^{l+1} + \varepsilon^{P,P_2}\varphi_{P_2}^{l+1}\right\} & \text{if } P \in {}^2\nabla_l^{\Gamma} \\ & \text{with } l \in \{0,\dots,L-1\}, \end{cases} \tag{3.2}$$

$$
(3.3)\qquad \varepsilon^{P,P'} := \begin{cases}
1 & \text{if} \quad \text{there is a parametrization patch } \Gamma_m \text{ such that } P \text{ and } P' \\
 & \qquad \text{belong to the interior of the triangle } \Gamma_m \text{ or there exists} \\
 & \qquad \text{a side } \Gamma_m \cap \Gamma_{m'} \text{ of a parametrization patch such that} \\
 & \qquad P \text{ and } P' \text{belong to the interior of the side } \Gamma_m \cap \Gamma_{m'}, \\
2 & \text{if} \quad \text{there exists a side } \Gamma_m \cap \Gamma_{m'} \text{ of a parametrization patch} \\
 & \qquad \text{such that } m < m', \text{ that } P \text{ is an interior point of } \Gamma_m, \\
 & \qquad \text{and that } P' \text{ belongs to the interior of the side } \Gamma_m \cap \Gamma_{m'} \\
 & \qquad \text{or } P' = \cap_{i=1}^k \Gamma_{m_i} \text{ is a corner of a parametrization patch,} \\
 & \qquad P' \in {}^2\!\triangle_0^\Gamma, \text{ the point } P \text{ is an interior point of a side} \\
 & \qquad \Gamma_{m_1} \cap \Gamma_{m_2}, \text{ and } \; m_1 < m_i, \; i = 2, \ldots, k, \\
4 & \text{if} \quad P' = \cap_{i=1}^k \Gamma_{m_i} \text{ is a corner of a parametrization patch,} \\
 & \qquad P' \in {}^1\!\triangle_0^\Gamma, \text{ the point } P \text{ is an interior point of a side} \\
 & \qquad \Gamma_{m_1} \cap \Gamma_{m_2}, \text{ and } \; m_1 < m_i, \; i = 2, \ldots, k \text{ or } P' = \cap_{i=1}^k \Gamma_{m_i} \\
 & \qquad \text{is a corner of a parametrization patch, } P' \in {}^2\!\triangle_0^\Gamma, \\
 & \qquad \text{the point } P \text{ is an interior point of the face } \Gamma_{m_1}, \text{ and} \\
 & \qquad m_1 < m_i, \; i = 2, \ldots, k, \\
0 & \text{otherwise,}
\end{cases}
$$

where P_1 and P_2 are the uniquely defined neighbours on $\triangle_{l+1}^\Gamma$ of $P \in \nabla_l^\Gamma$, i.e., $P_1 = \kappa_m(\tau_1)$ and $P_2 = \kappa_m(\tau_2)$ if $P = \kappa_m(\tau)$ is the representation with the minimal $m \in \{1, \ldots, m_\Gamma\}$ and if $\tau_1, \; \tau_2$ are the neighbours of τ. The coefficients $\varepsilon^{P,P'}$ are equal to one in almost all cases. Only if the point $P' = P_1, P_2$ is at the boundary of a parametrization patch, then a value of $\varepsilon^{P,P'}$ different from one is needed. The basis $\{\psi_P : \; P \in \triangle_L^\Gamma\}$ spans the trial space Lin_L^Γ. The function ψ_P with $P \in \nabla_l^\Gamma, \; l = 0, \ldots, L-1$ and with $\operatorname{supp} \psi_P$ contained in the interior of a single parametrization patch has two vanishing moments, i.e., it is orthogonal to the set of all functions that are constant or linear with respect to the parametrization. Orthogonality means here orthogonality with respect to the L^2 scalar product in the parameter domain.

3.2. The wavelet basis of the test space. Let us retain the definition of neighbour points $P_1, P_2 \in \triangle_l^\Gamma$ of $P \in \nabla_l^\Gamma, \; l = 0, \ldots, L-1$ from the last subsection, and recall that δ_P stands for the Dirac delta functional at point P. With this notation, we introduce the functionals

$$
(3.4)\qquad \vartheta_P := \begin{cases} \delta_P & \text{if } P \in \nabla_{-1}^\Gamma, \\ \delta_P - \frac{1}{2}\{\delta_{P_1} + \delta_{P_2}\} & \text{if } P \in \nabla_l^\Gamma \text{ with } l = l(P) \in \{0, \ldots, L-1\}. \end{cases}
$$

The set $\{\vartheta_P : \; P \in \triangle_L^\Gamma\}$ is a hierarchical basis of the test space Dir_L^Γ (cf. §2.3 and §5.2). For any $P \in \nabla_l^\Gamma, \; l = 0, \ldots, L-1$, the functional ϑ_P has two vanishing moments, i.e., it vanishes over the set of all functions that are constant or linear with respect to the parametrization. To simplify the notation, some times we shall write $f(\vartheta_P)$ for $\vartheta_P(f)$.

The basis $\{\vartheta_P\}$ will be suitable for the collocation applied to operators of order $\mathbf{r} = 0$. For $\mathbf{r} = -1$, a basis with more vanishing moments is needed (cf. [15, 47]). Following a general technique of Harten and Yad-Shalom, this wavelet basis $\{\vartheta_P^+ : \; P \in \triangle_L^\Gamma\}$ is given by

$$
(3.5)\qquad \vartheta_P^+ := \begin{cases} \delta_P & \text{if } P \in \nabla_{-1}^\Gamma \cup \nabla_0^\Gamma, \\ \delta_P + \frac{1}{8}\delta_{\tilde P_1} + \frac{1}{8}\delta_{\tilde P_2} - \frac{1}{4}\delta_{\tilde P_3} - \frac{1}{2}\delta_{\tilde P_4} - \frac{1}{2}\delta_{\tilde P_5} & \text{if } P \in \nabla_l^\Gamma \cap \operatorname{int}\Gamma_Q \text{ s.t. } l \geq 1 \\ & \text{and } Q \in \square_{l-1}^\Gamma, \\ \delta_P - \frac{3}{4}\delta_{\tilde P_1} - \frac{3}{8}\delta_{\tilde P_2} + \frac{1}{8}\delta_{\tilde P_3} & \text{if } P \in \nabla_l^\Gamma \cap \partial\Gamma_Q \text{ s.t. } l \geq 1 \\ & \text{and } Q \in \square_{l-1}^\Gamma. \end{cases}
$$

Here the points $\tilde P_i, \; i = 1, \ldots, 5$ are defined by their parameter values $\sigma_i := \kappa_m^{-1}(\tilde P_i)$ as follows. If $P = \kappa_m(\tau) \in \nabla_l^\Gamma$ is at the boundary $\partial\Gamma_Q$ of a triangle Γ_Q with $Q \in \square_{l-1}^\Gamma$, then τ

is located at a side S of the triangle $T_\sigma = \kappa_m^{-1}(\Gamma_Q)$ with $\sigma = \kappa_m^{-1}(Q) \in \square_{l-1}^{\mathbb{R}^2}$, the parameter point σ_1 is the mid-point of this side S, the point σ_2 is the end point of S closest to τ, and σ_2 is the other endpoint of S. If P is in the interior $\operatorname{int}\Gamma_Q$, then there is a unique straight line segment containing τ and connecting a corner point of T_σ with the mid-point of the opposite side S. In this case, σ_3 is the mid-point of S, the other mid-points of the sides of T_σ are σ_4 and σ_5, and σ_1 and σ_2 are the endpoints of the side S (cf. Figure 3.2).

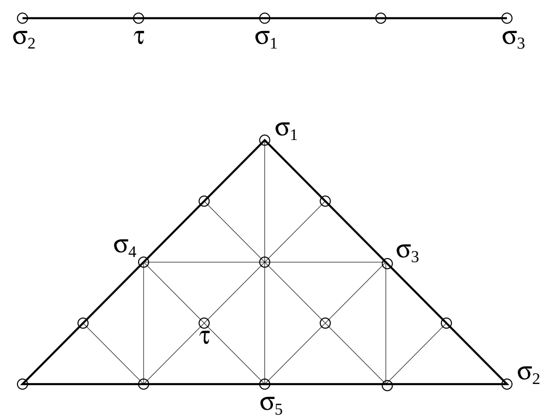

FIG. 3.2. *Points for* ϑ_P^+.

3.3. Wavelet transforms. For the trial space Lin_L^Γ we have two different systems of basis functions $\{\varphi_P^L\}$ and $\{\psi_P\}$ at our disposal. We denote the basis transform by $\mathcal{T}_A$ (lower index A stands for ansatz), i.e. the matrix $\mathcal{T}_A$ maps the coefficient vector $\xi^L := (\xi_P^L)_{P\in\triangle_L^\Gamma}$ of the representation $u_L = \sum_{P\in\triangle_L^\Gamma} \xi_P^L \varphi_P^L$ into the coefficient vector $\beta := (\beta_P)_{P\in\triangle_L^\Gamma}$ of the representation $u_L = \sum_{P\in\triangle_L^\Gamma} \beta_P \psi_P$. This transform can be determined by a pyramid type algorithm which is called fast wavelet transform (cf., e.g., [20]). Similarly, the inverse transform $\mathcal{T}_A^{-1}$ can also be realized by a pyramid type algorithm. Moreover, analogous to the trial space, we have two different bases in the test space. By $\mathcal{T}_T$ (lower index T stands for test space) we denote the linear transform mapping the vector $\gamma = (\gamma_P)_{P\in\triangle_L^\Gamma} := (\vartheta_P(f))_{P\in\triangle_L^\Gamma}$ of functionals applied to a function f into the vector of function values $\eta = (\eta_P)_{P\in\triangle_L^\Gamma} := (\delta_P(f))_{P\in\triangle_L^\Gamma} = (f(P))_{P\in\triangle_L^\Gamma}$. Again, the transform can be realized by a fast wavelet algorithm. Due to (3.4) and (3.5), the inverse $\mathcal{T}_T^{-1}$ is simply a multiplication by a sparse matrix.

3.4. Wavelet algorithm. Analogous to the stiffness matrix A_L in §2.5 we can set up a matrix with respect to the wavelet basis. We introduce A_L^w by $A_L^w := (a_{P',P}^w)_{P',P\in\triangle_L^\Gamma}$ and $a_{P',P}^w := \vartheta_{P'}(A\psi_P)$. Note that $A_L = \mathcal{T}_T A_L^w \mathcal{T}_A$. It will turn out that most of the entries $a_{P',P}^w$ are so small that they can be neglected. Thus in the next subsection we will give an a priori matrix pattern $\mathcal{P} \subset \triangle_L^\Gamma \times \triangle_L^\Gamma$ with no more than $\mathcal{O}(2^{2L}L^2)$ elements. We will replace A_L^w by the sparse matrix obtained by the compression

$$A_L^{w,c} := \left(a_{P',P}^{w,c}\right)_{P',P\in\triangle_L^\Gamma}, \quad a_{P',P}^{w,c} := \vartheta_{P'}(a\psi_P) + \begin{cases} \vartheta_{P'}(K\psi_P) & \text{if } (P',P)\in\mathcal{P}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

In the numerical computation the entries have to be computed by approximating the parametrization and by quadrature. We denote the approximate value for $a_{P',P}^{w,c}$ by $a_{P',P}^{w,c,q}$ and set

$$A_L^{w,c,q} := \left(a_{P',P}^{w,c,q}\right)_{P',P\in\triangle_L^\Gamma}, \quad A_L^c := \mathcal{T}_T A_L^{w,c} \mathcal{T}_A, \quad A_L^{c,q} := \mathcal{T}_T A_L^{w,c,q} \mathcal{T}_A. \tag{3.7}$$

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With this notation we can describe two variants of the wavelet algorithm which differ in the iterative solution of the discretized linear systems. The first is designed for integral operators of arbitrary order $\mathbf{r}$, the second for operators of order $\mathbf{r} = 0$.

First wavelet algorithm

(3.8)
- i) compute the right-hand side $\gamma := (\vartheta_P(v))_P = \mathcal{T}_T^{-1}(v(P))_P$
- ii) compute the sparsity pattern $\mathcal{P}$
- iii) assemble $A_L^{w,c,q}$ by a quadrature algorithm
- iv) solve $A_L^{w,c,q}\beta = \gamma$ iteratively, e.g., by the diagonally preconditioned GMRes method
- v) compute $\xi = \mathcal{T}_A^{-1}\beta$
- vi) post processing of the values $u(P) \approx \xi_P$, e.g., computation of linear functionals of the solution u

Second wavelet algorithm

(3.9)
- i) compute the right-hand side $\eta := (v(P))_P$
- ii) compute the sparsity pattern $\mathcal{P}$
- iii) assemble $A_L^{w,c,q}$ by a quadrature algorithm
- iv) solve $A_L\xi = \eta$ iteratively, e.g., by the GMRes method, whenever a multiplication by matrix A_L is required, then multiply by $\mathcal{T}_A$, by $A_L^{w,c,q}$, and by $\mathcal{T}_T$
- v) post processing of the values $u(P) \approx \xi_P$, e.g., computation of linear functionals of the solution u

The GMRes algorithm is described in [45], and the diagonal preconditioner for the algorithm (3.8) will be derived in §5.3 (cf. (5.10)).

3.5. The compression algorithm. In order to introduce the compression pattern $\mathcal{P}$ which is convenient for the quadrature algorithm, we need some notation. Let us retain the definition of ∇_l^Γ and $\triangle_L^\Gamma$ from §2.3. For $P \in \triangle_L^\Gamma$, recall that $l(P)$ is the level of P (cf. the end of §2.3). By Ψ_P we denote the support of the function ψ_P and by Θ_P the convex hull of the support of the test functional ϑ_P, i.e., $\Theta_P := \kappa_m(\mathrm{conv}(\kappa_m^{-1}(\mathrm{supp}\,\vartheta_P)))$. Now we take a constant $d \geq 1$ and define the set $\mathcal{P}$ as the set of all $(P', P) \in \triangle_L^\Gamma \times \triangle_L^\Gamma$ such that Ψ_P is completely contained in the interior of a single parameter patch Γ_m and

$$\mathrm{dist}\,(\Psi_P, \Theta_{P'}) \leq d2^{L-l(P')-l(P)}, \tag{3.10}$$

or such that Ψ_P contains points of at least two parameter patches and

$$\mathrm{dist}\,(\Psi_P, \Theta_{P'}) \leq d2^{1.5[L-l(P')]-l(P)}. \tag{3.11}$$

In numerical computations the compression parameter $d \geq 1$ should be determined by experiments. The well-known proof techniques of [15, 34, 47, 40] yield

THEOREM 3.1. *For the pattern $\mathcal{P}$, the number of nonzero entries $N_{\mathcal{P}}$ is less than the number $CL^2 2^{2L} \sim N[\log N]^2$, where $N \sim 2^{2L}$ is the number of degrees of freedom. If the piecewise linear collocation is stable and if d is sufficiently large, then the collocation method with compression is also stable. Moreover, the asymptotic error estimates (2.5) become*

$$\|u - u_L\|_{H^{\mathbf{r}}(\Gamma)} \leq Ch^{2-\mathbf{r}} \log h^{-1} \|u\|_{H^2(\Gamma)}, \quad \mathbf{r} = 0, -1. \tag{3.12}$$

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Clearly, the number of necessary arithmetic operations of all steps in the algorithms (3.8) and (3.9) except the steps iii) and iv) is less than $C\,N_{\mathcal{P}}$. Step iv) requires $C\,N_{\mathcal{P}}\log N$ operations. However, if we solve the systems successively over the grids $\triangle_l^\Gamma,\ l=0,\dots,L$ and if the initial solution for the grid $\triangle_{l+1}^\Gamma$ is the final solution from the coarser grid $\triangle_l^\Gamma$, then the number of necessary iterations is uniformly bounded. This cascading iteration method requires no more than $C\,N_{\mathcal{P}}$ operations. The key point for a fast algorithm, however, is the implementation of step iii). Usually, this is the most time consuming part of the numerical computation. For its realization and complexity, we refer to the results in §4 and the proofs in §6. Further details for the implementation of the wavelet algorithm can be found in [28, 39].

4. Approximation of the parametrization mappings and quadrature.

4.1. Parametrization and quadrature for the far field. Now we consider the computation of the matrix entries $a^{w,c,q}_{P',P}$ (cf. §3.4). Obviously, the terms $\vartheta_{P'}(a\psi_P)$ (cf. (3.6)) can be computed without difficulty, and the corresponding number of arithmetic operations is less than $\mathcal{O}(N\log N)$. Therefore, we only have to deal with the computation of $\vartheta_{P'}(K\psi_P)$ corresponding to the integral operator K. First we shall indicate the assembling of those entries for which $\operatorname{dist}(\Psi_P,\Theta_{P'})$ is large in a certain sense. We shall fix P' and define a quadrature partition in dependence on P' in order to apply a composite quadrature rule of low order. Clearly, if a trial function ψ_P has discontinuous first order derivatives over a subdomain, then the standard low order quadrature rules are not very accurate. Therefore, the quadrature partition will be finer than the partition into the patches of linearity, i.e., all trial functions ψ_P with (P',P) in the sparsity pattern $\mathcal{P}$ (cf. §3.5) will not only be piecewise linear but linear with respect to the parametrization κ_m on each quadrature subdomain. In the class of all partitions, we shall choose the coarsest partition with the just mentioned property. Over the subdomains of this partition we shall approximate the parametrizations κ_m by a low order polynomial interpolation and apply a composite quadrature rule.

Let us give the precise definition of the partition. For $l=0,\dots,L$, we introduce the set Qua_l^Γ as the set of all $Q\in\square_l^\Gamma$ such that:

i) There is a $P\in\nabla_{l-1}^\Gamma$ such that $(P',P)\in\mathcal{P}$ and that the support Ψ_P intersects the father Γ_{Q^F} of Γ_Q.
ii) If $l<L$, then we suppose that, for any $P\in\nabla_l^\Gamma$ with $(P',P)\in\mathcal{P},\ \Gamma_Q\cap\Psi_P=\emptyset$.

The quadrature partition is $\{\Gamma_Q:\ Q\in\cup_{l=0}^L Qua_l^\Gamma\}$. Clearly, condition i) means that in the partition of Γ the subset Γ_Q cannot be substituted by a larger $\Gamma_{Q'}$ without violating the linearity property, and condition ii) means that it is not necessary to divide Γ_Q further into smaller subdomains since already all the trial basis function ψ_P with $(P',P)\in\mathcal{P}$ are linear over Γ_Q. Indeed, if i) holds and if Γ_Q would be replaced by $\Gamma_{Q'}$, then $\Gamma_{Q^F}\subseteq\Gamma_{Q'}$ and the function ψ_P with $(P',P)\in\mathcal{P}$ and with $\operatorname{supp}\,\psi_P\cap\Gamma_{Q^F}\neq\emptyset$ (cf. condition i)) has a discontinuous first derivative over $\Gamma_{Q'}$. On the other hand, due to ii) the wavelet functions of level l with $(P',P)\in\mathcal{P}$ vanish over Γ_Q, and, due to the definition of $\mathcal{P}$ in (3.10), (3.11), the higher level wavelet functions with $(P',P)\in\mathcal{P}$ also vanish over Γ_Q. The lower level wavelets, however, are linear on Γ_Q.

LEMMA 4.1. *The set $\{\Gamma_Q:\ Q\in\cup_{l=0}^L Qua_l^\Gamma\}$ is a partition of Γ. For all P with $(P',P)\in\mathcal{P}$ and for all $Q\in\cup_{l=0}^L Qua_l^\Gamma$, the restriction of ψ_P to Γ_Q is linear with respect to the parametrization. Moreover, the partition $\{\Gamma_Q:\ Q\in\cup_{l=0}^L Qua_l^\Gamma\}$ is the coarsest partition with this linearity property and with $\{\Gamma_Q:\ Q\in\cup_{l=0}^L Qua_l^\Gamma\}\subseteq\{\Gamma_Q:\ Q\in\cup_{l=0}^L\square_l^\Gamma\}$.*

Proof. The proof is obvious from the following construction of the partition. □

The partition $\{\Gamma_Q:\ Q\in\cup_{l=0}^L Qua_l^\Gamma\}$ can be determined as follows. For each P', we have to determine the sets Qua_l^Γ with $l=0,\dots,L$. We do this for each level l separately. First we set up Qua_0^Γ. Then, if the subsets $Qua_{l'}^\Gamma,\ l'=0,\dots,l-1$ are determined, the search for the $Q \in \square_l^\Gamma$ satisfying the conditions i) and ii) can be restricted to all $Q \in \square_l^\Gamma$ with Γ_Q contained in the difference set Γ minus the union $\cup_{l'=0}^{l-1} \cup_{R \in Qua_{l'}^\Gamma} \Gamma_R$. Doing this for all $l = 1, \ldots, L$ and for all $P' \in \triangle_L^\Gamma$, only $\mathcal{O}(N_\mathcal{P})$ of the $\mathcal{O}(N^2)$ domains Γ_Q have to be checked to see whether or not they satisfy conditions i) and ii).

LEMMA 4.2. *There are constants $c_o > 0$ and $C_0 > 1$ such that, for all $\Gamma_Q \in Qua_l^\Gamma$ of level $l(Q) = l$ which are contained in a single parametrization patch Γ_m and the distance of which to the boundary $\partial\Gamma_m$ is greater than $c_0 2^{-l}$,*

$$d2^{L-l(P')-l(Q)} \leq \operatorname{dist}(\Gamma_Q, \Theta_{P'}) \;\leq\; C_0 d2^{L-l(P')-l(Q)}. \tag{4.1}$$

If (4.1) does not hold, then the distance of Γ_Q to $\partial\Gamma_m$ is less than $c_0 2^{-l}$, and we at least get the estimate

$$d2^{1.5[L-l(P')]-l(Q)} \leq \operatorname{dist}(\Gamma_Q, \Theta_{P'}) \;\leq\; C_0 d2^{1.5[L-l(P')]-l(Q)}. \tag{4.2}$$

Proof. In view of (3.10) and (3.11), condition i) is equivalent to the existence of a $P \in \nabla_{l-1}^\Gamma$ such that $\Psi_P \cap \Gamma_{Q^F} \neq \emptyset$ and that $\operatorname{dist}(\Psi_P, \Theta_{P'})$ is either less than $d2^{L-l(P')-(l-1)}$ for Ψ_P contained in the interior of a single parametrization patch Γ_m or less than $d2^{1.5[L-l(P')]-(l-1)}$ for Ψ_P not contained in the interior of a single parametrization patch. On the other hand, for an appropriate constant $c_L > 0$ depending on the Lipschitz constants of the inverse parametrization mappings, the diameter of $\Psi_P,\ P \in \nabla_{l-1}^\Gamma$ is less than $c_L 2^{-(l-1)}$. Hence, the bounds for $\operatorname{dist}(\Psi_P, \Theta_{P'})$ imply the upper estimates in (4.1) and (4.2).

Condition ii) is satisfied, if and only if, for any $P \in \nabla_l^\Gamma$ with $\Gamma_Q \cap \Psi_P \neq \emptyset$ and with Ψ_P contained in the interior of a single parametrization patch Γ_m, the distance $\operatorname{dist}(\Psi_P, \Theta_{P'})$ is greater than $d2^{L-l(P')-l}$, and if, for any $P \in \nabla_l^\Gamma$ with $\Gamma_Q \cap \Psi_P \neq \emptyset$ and with Ψ_P not contained in the interior of a single parametrization patch Γ_m, the distance $\operatorname{dist}(\Psi_P, \Theta_{P'})$ is greater than $d2^{1.5[L-l(P')]-l}$. However, Γ_Q is covered by the Ψ_P with $\Psi_P \cap \Gamma_Q \neq \emptyset$. Hence, the lower estimates for $\operatorname{dist}(\Psi_P, \Theta_{P'})$ ensure the lower bounds in (4.1) and (4.2). ☐

Having in mind the estimates (4.1) and (4.2), we shall call the quadrature subdomains of $\cup_{l=0}^{L-1} \{\Gamma_Q :\ Q \in Qua_l^\Gamma\}$ the far field subdomains corresponding to the functional $\vartheta_{P'}$. The domains $\{\Gamma_Q :\ Q \in Qua_L^\Gamma\}$ will be referred to as near field subdomains. In accordance with (3.6) and (2.3), we shall introduce quadrature approximations $a_{P',P,Q}^{w,c,q}$ for

$$\vartheta_{P'}\left(\int_{\Gamma_Q} k(\cdot, R, n_R) \frac{p(\cdot - R)}{|\cdot - R|^\alpha} \psi_P(R)\, \mathrm{d}_R\Gamma \right). \tag{4.3}$$

Here the functional $\vartheta_{P'}$ is applied to the function in brackets depending on the variable indicated by a dot. Using these $a_{P',P,Q}^{w,c,q}$, we define the entries $a_{P',P}^{w,c,q}$ by

$$a_{P',P}^{w,c,q} := \vartheta_{P'}(a\psi_P) + \begin{cases} 0 & \text{if } (P',P) \notin \mathcal{P}, \\ \sum_{l=0}^{L} \ \sum_{Q \in Qua_l^\Gamma :\ \Gamma_Q \subset \operatorname{supp} \psi_P} a_{P',P,Q}^{w,c,q} & \text{if } (P',P) \in \mathcal{P}. \end{cases} \tag{4.4}$$

We shall defer the definition of the near field terms $a_{P',P,Q}^{w,c,q},\ Q \in Qua_L^\Gamma$ to §4.2 and §4.3. In this subsection we introduce the far field terms $a_{P',P,Q}^{w,c,q}$ with $Q \in Qua_l^\Gamma$ and l running from 0 to $L-1$.

Let us fix a far field subdomain Γ_Q with $Q = \kappa_m(\tau) \in Qua_l^\Gamma$. Using the parametrisation κ_m over $T_\tau = \kappa_m^{-1}(\Gamma_Q)$, we write the integral of (4.3) in the form

$$\vartheta_{P'}\left(\int_{T_\tau} k(\cdot, \kappa_m(\sigma), n_{\kappa_m(\sigma)}) \frac{p\Big(\cdot - \kappa_m(\sigma)\Big)}{|\cdot - \kappa_m(\sigma)|^\alpha} \tilde{\psi}_P(\sigma) \mathcal{J}_m(\sigma)\, \mathrm{d}\sigma \right), \tag{4.5}$$

where $\mathcal{J}_m(\sigma) := |\partial_{\sigma_1}\kappa_m(\sigma) \times \partial_{\sigma_2}\kappa_m(\sigma)|$ is the Jacobian determinant of the transformation κ_m at $\sigma = (\sigma_1, \sigma_2) \in T_\tau$ and where $\tilde{\psi}_P(\sigma)$ stands for the factor $\psi_P(R) = \psi_P(\kappa_m(\sigma))$ which is independent of the parametrization κ_m (cf. (3.2) and (2.4)). We derive the approximation $a^{w,c,q}_{P',P,Q}$ for (4.5) in three steps.

In the **first step**, we replace the parametrization κ_m over T_τ by a polynomial interpolation κ'_m of degree $\mathbf{m} := 2 - \mathbf{r}$, i.e., we use a cubic interpolation with ten interpolation knots for $\mathbf{r} = -1$ and a quadratic interpolation with six knots for $\mathbf{r} = 0$. For instance, to get the quadratic interpolation, we denote by $\tau_i,\ i = 1, 2, 3$ the three corner points and by $\tau_i,\ i = 4, 5, 6$ the mid-points of the three sides of the triangle $T_\tau = \kappa_m^{-1}(\Gamma_Q)$. We set $\kappa'_m(\sigma) = \sum_{i=1}^{6} \kappa_m(\tau_i)\mathcal{L}_i(\sigma)$, where $\mathcal{L}_i$ is the quadratic Lagrange polynomial defined by $\mathcal{L}_i(\tau_j) = \delta_{j,i},\ i, j = 1, \ldots, 6$. For explicit formulas see, e.g., (5.1.18) of [2]. Hence, we approximate (4.5) by

$$\vartheta_{P'}\left(\int_{T_\tau} k(\cdot, \kappa_m(\sigma), n'_{\kappa'_m(\sigma)}) \frac{p\Big(\cdot - \kappa'_m(\sigma)\Big)}{|\cdot - \kappa'_m(\sigma)|^\alpha} \tilde{\psi}_P(\sigma)\mathcal{J}'_m(\sigma)\,\mathrm{d}\sigma\right), \tag{4.6}$$

where $\mathcal{J}'_m(\sigma) := |\partial_{\sigma_1}\kappa'_m(\sigma) \times \partial_{\sigma_2}\kappa'_m(\sigma)|$ is the Jacobian determinant of the transformation κ'_m at $\sigma = (\sigma_1, \sigma_2) \in T_\tau$. The symbol $n'_{\kappa'_m(\sigma)}$ in the last formula stands for the unit vector at the point $\kappa'_m(\sigma)$ which is normal to the approximating surface $\kappa'_m(T_\tau)$.

In the **second step**, we split the integrand of (4.6) into the product $f(\sigma)\tilde{\varrho}(\sigma)$

$$f(\sigma) := k(\cdot, \kappa_m(\sigma), n_{\kappa'_m(\sigma)})\mathcal{J}'_m(\sigma), \quad \tilde{\varrho}(\sigma) \ := \ \varrho\Big(\kappa'_m(\sigma)\Big) \ = \ \frac{p\Big(\cdot - \kappa'_m(\sigma)\Big)}{|\cdot - \kappa'_m(\sigma)|^\alpha}\tilde{\psi}_P(\sigma).$$

Note that f is globally $\mathbf{m}$ times differentiable by assumption whereas ϱ is singular at the points of $\operatorname{supp}\vartheta_{P'}$. We apply a product quadrature with weight $\tilde{\varrho}$ and of order $\mathbf{m}$ to the integral in (4.6). If $\mathbf{r} = -1$, then we choose the six point rule based upon the quadratic interpolation which has been used before. In case $\mathbf{r} = 0$ we take the three point rule. To simplify the notation, however, we write all the following formulae explicitly for the three point rule. The modifications for the corresponding formulae including the six point rule are straightforward. In the estimates and the convergence results, we always suppose that a quadrature of order $\mathbf{m}$ is in use. The product quadrature rule takes the form

$$\int_{T_\tau} f(\sigma)\tilde{\varrho}(\sigma)\,\mathrm{d}\sigma \approx \sum_{\upsilon=1}^{3} f(\tau_\upsilon)\int_{T_\tau}\tilde{\phi}_{Q,\,\upsilon}(\sigma)\tilde{\varrho}(\sigma)\,\mathrm{d}\sigma, \tag{4.7}$$

where $\tilde{\phi}_{Q,\,\upsilon}$ is the linear function on T_τ defined by $\tilde{\phi}_{Q,\,\upsilon}(\tau_{\upsilon'}) = \delta_{\upsilon,\upsilon'}$. In other words, the integral (4.6) is approximated by

$$\vartheta_{P'}\left(\sum_{\upsilon=1}^{3} k(\cdot, Q_\upsilon, n'_{Q'_\upsilon})\mathcal{J}'_m(\tau_\upsilon)b^{w,c,q}_{P,Q,\upsilon}(\cdot)\right), \tag{4.8}$$

$$b^{w,c,q}_{P,Q,\upsilon}(R) \ := \ \int_{T_\tau}\tilde{\phi}_{Q,\,\upsilon}(\sigma)\frac{p\Big(R - \kappa'_m(\sigma)\Big)}{|R - \kappa'_m(\sigma)|^\alpha}\tilde{\psi}_P(\sigma)\,\mathrm{d}\sigma, \tag{4.9}$$

where $Q_\upsilon := \kappa_m(\tau_\upsilon)$ and $Q'_\upsilon := \kappa'_m(\tau_\upsilon)$ denote the corner points of the triangles $\Gamma_Q = \kappa_m(T_\tau)$ and $\kappa'_m(T_\tau)$, respectively. The symbol $n'_{Q'_\upsilon}$ in the last formula stands for the unit vector at the point $Q'_\upsilon = \kappa'_m(\tau_\upsilon)$ which is normal to the approximating surface $\kappa'_m(T_\tau)$.

In the **third and last step** we have to compute the quadrature weights $b^{w,c,q}_{P,Q,\upsilon}$ of the product rule, i.e., the integrals over T_τ of $g(\sigma) := \tilde{\phi}_{Q,\,\upsilon}(\sigma)\varrho(\kappa'_m(\sigma))$. In some applications these

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integrals can be computed analytically. For the general case, we have to compute them by quadrature. Note that the weight ϱ is a smooth function on Γ_Q with singularities sufficiently far from Γ_Q. Under these circumstances, the integral of g can be approximated, e.g., by panel clustering or multipole techniques (cf. [44, 23]). We, however, describe a third alternative following [22, 25, 34, 47]. To get a quadrature rule over T_τ, we start from the Gauß-Legendre rule over $[0,1]$, i.e., from the interpolatory rule including the zeros $\sigma_G^k,\ k=1,\ldots,n_G$ of the Legendre polynomial as quadrature knots,

$$\int_0^1 F \approx \sum_{k=1}^{n_G} F(\sigma_G^k)\omega_G^k. \tag{4.10}$$

The order n_G will be specified later. Introducing Duffy's coordinates and applying the Gauß type tensor product rule to the resulting double integral, we arrive at

$$\begin{aligned}
\int_{T_\tau} g(\sigma)\,\mathrm{d}\sigma &= \int_0^1\int_0^1 g\Big(\tau_3+\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\sigma_1^D\,\mathrm{d}\sigma_2^D\,\mathrm{d}\sigma_1^D\cdot 2\,|T_\tau| \\
&\approx \sum_{k_1=1}^{n_G}\sum_{k_2=1}^{n_G} g\Big(\tau_3+\sigma_G^{k_1}(\tau_1-\tau_3)+\sigma_G^{k_1}\sigma_G^{k_2}(\tau_2-\tau_3)\Big)\sigma_G^{k_1}\omega_G^{k_1}\omega_G^{k_2}\cdot 2\,|T_\tau| \\
&=: \sum_{k=1}^{n_G^2} g(\sigma_\tau^k)\omega_\tau^k. \qquad (4.11)
\end{aligned}$$

Note that, for the numerical implementation, one could try to replace the rule (4.11) by triangular rules of high order or, e.g., by Stroud's conical product rule (cf. [51]), which is a slight modification of (4.11).

Thus the formulae (4.8), (4.9), and (4.11) together yield

$$\begin{aligned}
a_{P',P,Q}^{w,c,q} := \vartheta_{P'}\Bigg(&\sum_{\upsilon=1}^{3} k(\cdot,Q_\upsilon,n'_{Q'_\upsilon})\mathcal{J}'_m(\tau_\upsilon) \\
&\cdot\sum_{k=1}^{n_G^2}\tilde\phi_{Q,\upsilon}(\sigma_\tau^k)\left[\frac{p\Big(\cdot-\kappa'_m(\sigma_\tau^k)\Big)}{|\cdot-\kappa'_m(\sigma_\tau^k)|^\alpha}\tilde\psi_P(\sigma_\tau^k)\right]\omega_\tau^k\Bigg).
\end{aligned} \tag{4.12}$$

For $Q\in Qua_l^\Gamma$, we choose the quadrature order n_G in the last formula by

$$n_G := n_A + n_B\left[\frac{l}{1+{}^2\log\left(\dfrac{\operatorname{dist}(\Theta_{P'},\Gamma_Q)}{2^{-l}}\right)}\right], \tag{4.13}$$

where the integers $n_A>0$ and $n_B>0$ have to be determined by numerical experiments. In §6.1 we shall prove the existence of positive integers n_A and n_B such that the additional error due to the far field quadrature is, roughly speaking, less than the error of the exact collocation. Analogous error estimates are true also for the approximation of the near field and the singular integrals in the §4.2 and §4.3. More precisely, we get

THEOREM 4.1. *Suppose we use the compression pattern $\mathcal{P}$ of Theorem 3.1. If the exact collocation described in §2.5 is stable, if the compression parameter d (cf. (3.10) and (3.11)) and the quadrature parameters $n_A,\ n_B,\ n_C,\ n_D,\ n_E$, and n_F for the determination of the Gauß order n_G (cf. (4.12), (4.16), (4.21), and (4.28)) are sufficiently large, then the* *compressed collocation together with approximation of the boundary and with the quadrature of §§4.1 - 4.3 is also stable. The error for the collocation solution u_L, including compression, approximation of the parameter mappings, and quadrature, satisfies*

$$\|u - u_L\|_{H^{\mathbf{r}}(\Gamma)} \leq C\, h^{2-\mathbf{r}} \begin{cases} [\log h^{-1}]^2 & \text{if } \mathbf{r} = 0 \\ [\log h^{-1}]^{1.5} & \text{if } \mathbf{r} = -1. \end{cases} \tag{4.14}$$

The number of quadrature knots and the number of necessary arithmetic operations for the computation of the stiffness matrix $A_L^{w,c,q}$ is less than $C\,N[\log N]^4$.

Proof. Stability and error estimates will be a consequence of Remark 5.1 and Lemmata 6.1, 6.3, and 6.5. The complexity bound will be shown in Lemmata 6.2, 6.4, and 6.6. □

REMARK 4.1. *A fast code for the computation of $a_{P',P,Q}^{w,c,q}$ computes first, for fixed $\vartheta_{P'}$ and Q, the quadratures in (4.12) with $\psi_P \circ \kappa_m$ replaced by the three linear basis functions $\phi_{Q,\iota}$, $\iota = 1,2,3$ over T_τ. Then, in a loop over all P with $\Gamma_Q \subset \operatorname{supp}\psi_P$, the values $a_{P',P,Q}^{w,c,q}$ are evaluated as a linear combination of the three quadratures over the basis functions, and $a_{P',P,Q}^{w,c,q}$ is updated to the actual value of the sum (4.4).*

4.2. Parametrization and quadrature for the nonsingular near field. Let us fix a test functional $\vartheta_{P'}$ and a $Q \in Qua_L^\Gamma$, and let us consider the integral (4.3) for which we seek the quadrature $a_{P',P,Q}^{w,c,q}$. Recall from §3.2 that the test functional $\vartheta_{P'}$ is a linear combination of point evaluation functionals. Thus there are points P_λ and uniformly bounded coefficients $\mu_\lambda = \mu_\lambda^{P'}$ such that $\vartheta_{P'}(f) = \sum_{\lambda=1}^{\lambda_{P'}} \mu_\lambda f(P_\lambda)$. In correspondence with this, we can split the unknown quadrature expression $a_{P',P,Q}^{w,c,q}$ into

$$\begin{aligned} a_{P',P,Q}^{w,c,q} &= \sum_{\lambda=1}^{\lambda_{P'}} \mu_\lambda\, a_{P',\lambda,P,Q}^{w,c,q}, \\ a_{P',\lambda,P,Q}^{w,c,q} &\approx \int_{\Gamma_Q} k(P_\lambda, R, n_R) \frac{p(P_\lambda - R)}{|P_\lambda - R|^\alpha} \psi_P(R)\, \mathrm{d}_R\Gamma. \end{aligned} \tag{4.15}$$

We distinguish two cases. If P_λ is in Γ_Q, then the integral (4.15) is singular, and we defer the definition of the singular quadrature $a_{P',\lambda,P,Q}^{w,c,q}$ to §4.3. For $P_\lambda \notin \Gamma_Q$, the integral (4.15) is not singular and the corresponding nonsingular near field quadrature $a_{P',\lambda,P,Q}^{w,c,q}$ is treated now. We apply the technique of the previous subsection (cf. the quadrature rule of (4.12)) to (4.15) and get

$$\begin{aligned} a_{P',\lambda,P,Q}^{w,c,q} := &\sum_{\upsilon=1}^{3} k(P_\lambda, Q_\upsilon, n'_{Q'_\upsilon}) \mathcal{J}'_m(\tau_\upsilon) \\ &\cdot \sum_{k=1}^{n_G^2} \tilde{\phi}_{Q,\upsilon}(\sigma_\tau^k) \left[\frac{p\Big(P_\lambda - \kappa'_m(\sigma_\tau^k)\Big)}{|P_\lambda - \kappa'_m(\sigma_\tau^k)|^\alpha} \tilde{\psi}_P(\sigma_\tau^k) \right] \omega_\tau^k, \end{aligned} \tag{4.16}$$

where this time the order n_G is chosen to be $n_G := n_C + L n_D$. In practical computations the integers $n_C > 0$ and $n_D > 0$ have to be determined by experiments.

4.3. Parametrization and quadrature for the singular near field.

4.3.1. What is left from §4.1 and §4.2 is to derive quadrature approximations $a_{P',\lambda,P,Q}^{w,c,q}$ for the integral (4.15) with $Q \in \square_L^\Gamma$ and $P_\lambda \in \Gamma_Q$. For this quadrature standard techniques can be used (cf., e.g., [27, 22, 46]). We present some of the well-known techniques here. First, we consider the case of weakly singular integrals. This occurs if $\mathbf{r} = -1$ or if $\mathbf{r} = 0$ and the kernel function depending on the variables P and R contains a factor $n_P \cdot (P - R)$

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or $n_R \cdot (P - R)$. For definiteness, we restrict our consideration to the case of an additional factor $n_R \cdot (P - R)$. More precisely, we suppose that the kernel takes the form (cf. the double layer kernel in, e.g., [30])

$$k(P, R, n_R)\,\frac{p(P-R)}{|P-R|^{\alpha}} = \tilde{k}(P, R, n_R)\,\frac{\tilde{p}(P-R)[n_R \cdot (P-R)]^{1+\mathbf{r}}}{|P-R|^{\alpha}}\,. \tag{4.17}$$

For $\mathbf{r} = 0$, we assume that $\tilde{k}$ fulfills all the assumptions made for k in §2.2 and that $\tilde{p}$ is a homogeneous polynomial of degree $\deg(\tilde{p}) = \deg(p) - 1$, i.e., $\deg(\tilde{p}) - \alpha = -3$. Hence, for a suitable constant $C > 0$, we get $|n_R \cdot (P - R)| \le C|P - R|^2$ and

$$\left|\tilde{k}(P, R, n_R)\frac{\tilde{p}(P-R)[n_R \cdot (P-R)]^{1+\mathbf{r}}}{|P-R|^{\alpha}}\right| \le C\,|P-R|^{-1}\,,$$

and our kernel (4.17) is indeed weakly singular.

Now, we fix the test functional $\vartheta_{P'}$, a point $P_\lambda \in \operatorname{supp} \vartheta_{P'}$, and a triangle $\Gamma_Q = \kappa_m(T_\tau)$ with $Q = \kappa_m(\tau) \in \square_L^{\Gamma}$ and $P_\lambda \in \Gamma_Q$. Clearly, the grid point P_λ is one of the corner points of Γ_Q. We denote the three corners of T_τ by $\tau_\iota,\ \iota = 1, 2, 3$ and suppose $\kappa_m(\tau_3) = P_\lambda$. In the triangles T_τ and Γ_Q we introduce Duffy's coordinates

$$\delta\left(\sigma^D\right) := \delta\left(\sigma_1^D, \sigma_2^D\right) := \tau_3 + \sigma_1^D\left(\tau_1 - \tau_3\right) + \sigma_1^D \sigma_2^D\left(\tau_2 - \tau_3\right), \tag{4.18}$$

and set $\tilde{\kappa}_m(\sigma^D) := \kappa_m(\delta(\sigma^D))$. The Jacobian determinant corresponding to Duffy's coordinate in T_τ is given by $\mathcal{J}_\delta(\sigma^D) = |(\tau_1 - \tau_3) \times (\tau_2 - \tau_3)|\sigma_1^D = 2\,|T_\tau|\sigma_1^D$ and the Jacobian $\tilde{\mathcal{J}}_m(\sigma^D)$ of $\tilde{\kappa}_m$ is equal to the product $\mathcal{J}_m(\delta(\sigma^D))\mathcal{J}_\delta(\sigma^D)$. We seek an approximation $a^{w,c,q}_{P',\lambda,P,Q}$ for the integral

$$\begin{aligned}
&\int_{\Gamma_Q} \tilde{k}(P_\lambda, R, n_R)\frac{\tilde{p}(P_\lambda - R)[n_R \cdot (P_\lambda - R)]^{1+\mathbf{r}}}{|P_\lambda - R|^{\alpha}}\psi_P(R)\,\mathrm{d}_R\Gamma \\
&= \int_0^1\!\!\int_0^1 \Bigg\{ \tilde{k}(P_\lambda, \tilde{\kappa}_m(\sigma^D), n_{\tilde{\kappa}_m(\sigma^D)}) \\
&\qquad \cdot \frac{\tilde{p}\Big(P_\lambda - \tilde{\kappa}_m(\sigma^D)\Big)\Big[n_{\tilde{\kappa}_m(\sigma^D)} \cdot \Big(P_\lambda - \tilde{\kappa}_m(\sigma^D)\Big)\Big]^{1+\mathbf{r}}}{|P_\lambda - \tilde{\kappa}_m(\sigma^D)|^{\alpha}} \cdot \\
&\qquad \cdot \mathcal{J}_m\Big(\delta(\sigma^D)\Big)\mathcal{J}_\delta(\sigma^D)\tilde{\psi}_P^D(\sigma^D)\Bigg\}\,\mathrm{d}\sigma_2^D\,\mathrm{d}\sigma_1^D,
\end{aligned} \tag{4.19}$$

where $\tilde{\psi}_P^D(\sigma^D) := \psi_P(\tilde{\kappa}_m(\sigma^D))$. Due to the additional factor σ_1^D in $\mathcal{J}_\delta(\sigma^D)$, the weak singularity of the kernel function cancels.

We proceed in three steps. First, we replace the parametrization $\tilde{\kappa}_m$ by the approximate parametrization in Duffy coordinates $\tilde{\kappa}'_m := \kappa'_m \circ \delta$, where κ'_m is the polynomial interpolation to κ_m of polynomial degree $\mathbf{m} = 2 - \mathbf{r}$. We suppose that P_λ is one of the interpolation knots. Second, we apply a product rule of order $\mathbf{m}$. To this end the integrand in (4.19) with $\tilde{\kappa}_m$ replaced by $\tilde{\kappa}'_m$ is split into the product $f \cdot \varrho$ with

$$\begin{aligned}
f(\sigma^D) &:= \tilde{k}(P_\lambda, \tilde{\kappa}_m(\sigma^D), n'_{\tilde{\kappa}'_m(\sigma^D)})\mathcal{J}'_m\Big(\delta(\sigma^D)\Big), \\
\varrho(\sigma^D) &:= \frac{\tilde{p}\Big(P_\lambda - \tilde{\kappa}'_m(\sigma^D)\Big)\Big[n'_{\tilde{\kappa}'_m(\sigma^D)} \cdot \Big(P_\lambda - \tilde{\kappa}_m(\sigma^D)\Big)\Big]^{1+\mathbf{r}}}{|P_\lambda - \tilde{\kappa}_m(\sigma^D)|^{\alpha}}\mathcal{J}_\delta(\sigma^D)\tilde{\psi}_P^D(\sigma^D).
\end{aligned}$$

For $\mathbf{r} = -1$, the quadrature rule could be the tensor product variant of a quadratic interpolatory rule and, for $\mathbf{r} = 0$, we simply take the tensor product linear interpolatory rule.

$$\int_0^1 \int_0^1 f(\sigma^D)\varrho(\sigma^D)\,\mathrm{d}\sigma_2^D\,\mathrm{d}\sigma_1^D \approx \sum_{\upsilon=1}^{4} f(\tau_\upsilon^D) \int_0^1 \int_0^1 \tilde{\phi}_\upsilon^D(\sigma^D)\varrho(\sigma^D)\,\mathrm{d}\sigma_2^D\,\mathrm{d}\sigma_1^D,$$

where $\tau_\upsilon^D,\ \upsilon = 1,\dots,4$ denote the four corners of $[0,1]\times[0,1]$ and $\tilde{\phi}_\upsilon^D$ is the bilinear basis function defined by $\tilde{\phi}_\upsilon^D(\tau_{\upsilon'}^D) = \delta_{\upsilon,\upsilon'}$. Again, to simplify the notation we shall write the subsequent formulae with the linear interpolatory rule. The modifications for the tensor product of the quadratic interpolatory rule are straightforward. In the third and last step we apply the tensor product variant of the Gauß-Legendre rule of order n_G

$$\int_0^1 \int_0^1 g(\sigma^D)\,\mathrm{d}\sigma_2^D\,\mathrm{d}\sigma_1^D \approx \sum_{k_1=1}^{n_G}\sum_{k_2=1}^{n_G} g\Big(\sigma_G^{k_1},\sigma_G^{k_2}\Big)\omega_G^{k_1}\omega_G^{k_2} \;=: \sum_{k=1}^{n_G^2} g(\tilde{\sigma}^k)\tilde{\omega}^k, \tag{4.20}$$

with order $n_G = n_E + L\,n_F$ to compute an approximation to the integral of the function $g(\sigma^D) = \tilde{\phi}_\upsilon^D(\sigma^D)\varrho(\sigma^D)$. Finally, we arrive at

$$a_{P',\lambda,P,Q}^{w,c,q} := \sum_{\upsilon=1}^{4} \tilde{k}(P_\lambda, Q_\upsilon^D, n'_{R_\upsilon^D})\mathcal{J}'_m\left(\delta\left(\tau_\upsilon^D\right)\right)\cdot \tag{4.21}$$
$$\sum_{k=1}^{n_G^2} \tilde{\phi}_\upsilon^D(\tilde{\sigma}^k)\frac{\tilde{p}\Big(P_\lambda - \tilde{\kappa}'_m(\tilde{\sigma}^k)\Big)\Big[n'_{\tilde{\kappa}'_m(\tilde{\sigma}^k)}\cdot\Big(P_\lambda - \tilde{\kappa}'_m(\tilde{\sigma}^k)\Big)\Big]^{1+\mathbf{r}}}{|P_\lambda - \tilde{\kappa}'_m(\tilde{\sigma}^k)|^\alpha}\mathcal{J}_\delta(\tilde{\sigma}^k)\tilde{\psi}_P^D(\tilde{\sigma}^k)\tilde{\omega}^k.$$

Here we have set $Q_\upsilon^D := \tilde{\kappa}_m(\tau_\upsilon^D)$ and $R_\upsilon^D := \tilde{\kappa}'_m(\tau_\upsilon^D)$, and $n'_{Q''}$ denotes the unit normal to the approximate surface at Q''. Note that the Jacobian of $\tilde{\kappa}'_m$ takes the form $\mathcal{J}'_m(\delta(\sigma^D))\mathcal{J}_\delta(\sigma^D)$. The numbers n_E and n_F in the definition of n_G are to be determined by numerical experiments.

4.3.2. Now let us consider $\mathbf{r} = 0$ and suppose the integral operator is strongly singular. If the value $\psi_P(P_\lambda)$ vanishes, then this additional zero turns the strongly singular integral into a weakly singular, and we may apply the same procedure as for the weakly singular case treated before. For $\psi_P(P_\lambda) \neq 0$ or $\phi_{Q,\iota}(P_\lambda) \neq 0$ (cf. Remark 4.1), we substitute $\psi_P = \psi_P(P_\lambda) + (\psi_P - \psi_P(P_\lambda))$, respectively $\phi_{Q,\iota} = \phi_{Q,\iota}(P_\lambda) + (\phi_{Q,\iota} - \phi_{Q,\iota}(P_\lambda))$, into the singular integral. This way the integral splits into two parts, where the integral containing the functions $(\psi_P - \psi_P(P_\lambda))$, respectively $(\phi_{Q,\iota} - \phi_{Q,\iota}(P_\lambda))$, can be approximated like in the case $\psi_P(P_\lambda) = 0$. The only strongly singular case occurs if $\psi_P(P_\lambda) \neq 0$, respectively $\phi_{Q,\iota}(P_\lambda) \neq 0$, and if the function ψ_P, respectively $\phi_{Q,\iota}$, are replaced by the constants $\psi_P(P_\lambda)$, respectively $\phi_{Q,\iota}(P_\lambda)$. Without loss of generality we set these constants to one.

4.3.3. For the computation of the corresponding singular integrals, there exist several techniques (cf., e.g., [27, 46]). Here we shall present a quadrature algorithm similar to that in [9, 48] since this seems to require less assumptions on the smoothness. We consider a fixed singularity point P_λ. Since the singular integral is to be understood in the sense of Cauchy's principal value, we have to treat the quadrature for all Γ_Q with $P_\lambda \in \Gamma_Q$ simultaneously. Let m_0 stand for the smallest positive integer such that $P_\lambda \in \Gamma_{m_0}$. Beside m_0 we consider an arbitrary m and an arbitrary Γ_Q such that $P_\lambda \in \Gamma_Q \subseteq \Gamma_m$, i.e., $P_\lambda = \kappa_m(\tau_3)$ for a corner τ_3 of $T_\tau = \kappa_m^{-1}(\Gamma_Q)$. Note that the parameter value τ_3 in $P_\lambda = \kappa_m(\tau_3)$ depends, of course, on the parametrization κ_m and on the triangle Γ_Q. However, to simplify the notation, we do not indicate this dependence. By the assumption of §2.1 the parametrization κ_{m_0} mapping T

onto Γ_{m_0} extends to a neighbourhood of T. Hence, we can define

$$\begin{aligned} T(P_\lambda, m, \varepsilon) &:= \left\{\sigma :\ \left|\nabla\left(\kappa_{m_0}^{-1}\circ\kappa_m\right)(\tau_3)\cdot(\sigma-\tau_3)\right| \le \varepsilon\right\}, \\ \Gamma(P_\lambda,\varepsilon) &:= \bigcup_{m=1,\dots,m_\Gamma:\, P_\lambda\in\Gamma_m} \kappa_m\Big(T(P_\lambda,m,\varepsilon)\Big) \;\approx\; \{\kappa_{m_0}(\sigma):\ |\sigma-\tau_3|\le\varepsilon\}. \end{aligned}$$

By assumption the polynomial part p of the kernel function is odd. For such kernels, it is not hard to see that (cf. [31], Chapter XI, §1)

$$\left|\int_{\Gamma(P_\lambda,\varepsilon)} k(P_\lambda,R,n_R)\frac{p(P_\lambda-R)}{|P_\lambda-R|^\alpha}\,\mathrm{d}_R\Gamma\right| \le C\,\varepsilon. \tag{4.22}$$

We seek a quadrature with error less than $C\,2^{-2L}$. Therefore, the integral over Γ can be replaced by that over $\Gamma\setminus\Gamma(P_\lambda,2^{-2L})$, and it remains to approximate the integral

$$\begin{aligned} &\int_{\Gamma_Q\setminus\Gamma(P_\lambda,2^{-2L})} k(P_\lambda,R,n_R)\frac{p(P_\lambda-R)}{|P_\lambda-R|^\alpha}\,\mathrm{d}_R\Gamma \;= \\ &\sum_m \int_{T_\tau\setminus T(P_\lambda,m,2^{-2L})} k\left(\kappa_m(\tau_3),\kappa_m(\sigma),n_{\kappa_m(\sigma)}\right)\frac{p\Big(\kappa_m(\tau_3)-\kappa_m(\sigma)\Big)}{\left|\kappa_m(\tau_3)-\kappa_m(\sigma)\right|^\alpha}\mathcal{J}_m(\sigma)\,\mathrm{d}\sigma, \end{aligned} \tag{4.23}$$

for each Γ_Q with $P\in\Gamma_Q$. We replace the parametrization κ_m over $T_\tau\setminus T(P_\lambda,m,2^{-2L})$ by the quadratic interpolation κ'_m defined over T_τ, and it remains to compute

$$\int_{T_\tau\setminus T'(P_\lambda,m,2^{-2L})} k\left(\kappa_m(\tau_3),\kappa_m(\sigma),n'_{\kappa'_m(\sigma)}\right)\frac{p\Big(\kappa'_m(\tau_3)-\kappa'_m(\sigma)\Big)}{\left|\kappa'_m(\tau_3)-\kappa'_m(\sigma)\right|^\alpha}\mathcal{J}'_m(\sigma)\,\mathrm{d}\sigma, \tag{4.24}$$

$$T'(P_\lambda,m,\varepsilon) \;:=\; \left\{\sigma :\ \left|\nabla\left([\kappa'_{m_0}]^{-1}\circ\kappa'_m\right)(\tau_3)\cdot(\sigma-\tau_3)\right|\le\varepsilon\right\}. \tag{4.25}$$

Similar to the product rule in §4.1 and §4.3.1, we approximate the last integral over the domain $T_\tau\setminus T'(P_\lambda,m,2^{-2L})$ by a product rule with the integrand f and the product weight $\tilde{\varrho}$ given by

$$f(\sigma) := k(\kappa_m(\tau_3),\kappa_m(\sigma),n_{\kappa'_m(\sigma)})\mathcal{J}'_m(\sigma), \quad \tilde{\varrho}(\sigma) \;:=\; \frac{p\Big(\kappa'_m(\tau_3)-\kappa'_m(\sigma)\Big)}{|\,\kappa'_m(\tau_3)-\kappa'_m(\sigma)|^\alpha}.$$

This way we get

$$\begin{aligned} a^{w,c,q}_{P',\lambda,P,Q} &:= \sum_{\upsilon=1}^{3} k\left(\kappa_m(\tau_3),\kappa_m(\tau_\upsilon),n'_{\kappa'_m(\tau_\upsilon)}\right)\mathcal{J}'_m(\tau_\upsilon)\; b^{w,c,q}_{P',\lambda,Q,\upsilon}, \\ b^{w,c,q}_{P',\lambda,Q,\upsilon} &\approx \int_{T_\tau\setminus T'(P_\lambda,m,2^{-2L})} \tilde{\phi}_{Q,\upsilon}(\sigma)\frac{p\Big(\kappa'_m(\tau_3)-\kappa'_m(\sigma)\Big)}{\left|\kappa'_m(\tau_3)-\kappa'_m(\sigma)\right|^\alpha}\,\mathrm{d}\sigma. \end{aligned} \tag{4.26}$$

In contrast to the far field integrals where $b^{w,c,q}_{P',\lambda,Q,\upsilon}$ can be computed by simple analytic formulae, the quadrature weight $b^{w,c,q}_{P',\lambda,Q,\upsilon}$ of the present situation will be computed by introducing a geometric mesh and by applying high order quadrature rules over each subdomain. Fixing a grading parameter $0<q<1$, we denote the largest ι such that (for δ cf. (4.18))

$$T'(P_\lambda,m,2^{-2L}) \subseteq \left\{\delta(\sigma^D)\in T_\tau:\ 0\le\sigma_1^D\le q^{\iota-1},\ 0\le\sigma_2^D\le 1\right\}$$

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by ι_0. Clearly, $\iota_0 \sim L$. We divide the domain of integration $T_\tau \setminus T'(P_\lambda, m, 2^{-2L})$ into the union of the subdomains $T_{\tau,\iota},\ \iota = 1, \ldots, \iota_0$

$$
\begin{aligned}
(4.27)\quad T_{\tau,\iota} &:= \left\{\delta(\sigma^D) \in T_\tau :\ q^\iota < \sigma_1^D \le q^{\iota-1},\ 0 \le \sigma_2^D \le 1\right\},\\
T_{\tau,\iota_0} &:= \left\{\delta(\sigma^D) \in T_\tau :\ 0 \le \sigma_1^D \le q^{\iota_0-1},\ 0 \le \sigma_2^D \le 1\right\} \setminus T'(P_\lambda, m, 2^{-2L}).
\end{aligned}
$$

The optimal grading parameter q should be determined by numerical experiments. Note that for a different kind of integrals the choice $q = 0.15$ is optimal (cf., e.g., [48]). For fixed ι with $1 \le \iota \le \iota_0$, we observe that $T_{\tau,\iota} = \{\delta(\sigma^D) :\ 0 \le \sigma_2^D \le 1,\ S_a(\sigma_2^D) \le \sigma_1^D \le S_b\}$, where S_b is equal to $q^{\iota-1}$ and $S_a(\sigma_2^D) := q^\iota$ for $\iota < \iota_0$. The bound $S_a(\sigma_2^D)$ for $\iota = \iota_0$ is the solution σ_1^D of the equation $|\nabla([\kappa'_{m_0}]^{-1} \circ \kappa'_m)(\tau_3) \cdot (\delta(\sigma^D) - \tau_3)| = 2^{-2L}$, i.e., the boundary curve $\sigma_2^D \mapsto \delta(S_a(\sigma_2^D), \sigma_2^D)$ of the domain $T'(P_\lambda, m, 2^{-2L})$ is an ellipse. We may write the integral restricted to $T_{\tau,\iota}$ in the form

$$
\int_{T_{\tau,\iota}} \tilde\phi_{Q,\upsilon}(\sigma) \frac{p\Big(\kappa'_m(\tau_3) - \kappa'_m(\sigma)\Big)}{|\kappa'_m(\tau_3) - \kappa'_m(\sigma)|^\alpha}\, \mathrm{d}\sigma \ =
$$
$$
\int_0^1 \int_{S_a(\sigma_2^D)}^{S_b} \tilde\phi_{Q,\upsilon}\left(\delta(\sigma^D)\right) \frac{p\left(\kappa'_m(\tau_3) - \tilde\kappa'_m(\sigma^D)\right)}{|\kappa'_m(\tau_3) - \tilde\kappa'_m(\sigma^D)|^\alpha} \mathcal{J}_\delta(\sigma^D)\, \mathrm{d}\sigma_1^D\, \mathrm{d}\sigma_2^D.
$$

Applying the tensor product variant of the Gauß-Legendre rule (4.20) to the last integral, we complete the formula (4.26) by the quadrature

$$
\begin{aligned}
(4.28)\quad b^{w,c,q}_{P',\lambda,Q,\upsilon} &:= \sum_{\iota=1}^{\iota_0} \sum_{k_2=1}^{n_G} \sum_{k_1=1}^{n_G} \tilde\phi_{Q,\upsilon}\left(\delta(\sigma^D_{k_1,k_2})\right) \frac{p\left(\kappa'_m(\tau_3) - \tilde\kappa'_m(\sigma^D_{k_1,k_2})\right)}{\left|\kappa'_m(\tau_3) - \tilde\kappa'_m(\sigma^D_{k_1,k_2})\right|^\alpha} \cdot \\
&\qquad \mathcal{J}_\delta(\sigma^D_{k_1,k_2}) \left| S_b - S_a(\sigma_G^{k_2}) \right| \omega^G_{k_1} \omega^G_{k_2},\\
(4.29)\quad \sigma^D_{k_1,k_2} &:= \left(S_a(\sigma_G^{k_2}) + \sigma_G^{k_1}\left[S_b - S_a(\sigma_G^{k_2}) \right], \sigma_G^{k_2} \right).
\end{aligned}
$$

The order n_G in (4.29) is again chosen to be $n_G := n_E + Ln_F$. Finally, the quadrature approximation is given by (4.15), (4.26), and (4.28).

5. Preliminary results from the compression estimates.

5.1. The properties of the three-point hierarchical basis. Retain the notation of the basis from 3.1. From now on C stands for a generic constant the value of which varies from instance to instance. For two expressions E_1 and E_2, we write $E_1 \sim E_2$ if there is a constant independent of the parameters involved in E_1 and E_2 such that $E_1/C \le E_2 \le C\, E_1$. Under some additional technical assumptions, we infer the following lemma from [43].

LEMMA 5.1.

(i) *For $-0.5 < s < 1.5$, the basis $\{\psi_P :\ P \in \cup_{L=0}^\infty \triangle_L^\Gamma\}$ is a Riesz basis, i.e., for any L and for any vector of real numbers $(\xi_P)_P$, we get*

$$
(5.1)\qquad \left\| \sum_{P \in \triangle_L^\Gamma} \xi_P \psi_P \right\|_{H^s(\Gamma)} \sim \sqrt{\sum_{P \in \triangle_L^\Gamma} 2^{2l(P)(s-1)} |\xi_P|^2}.
$$

(ii) *For the interpolation projection R_L defined in §2.5, for $u \in H^t(\Gamma)$, and for the Sobolev space orders $0 \le s \le t \le 2,\ s < 1.5,\ t > 1$, we get*

$$
(5.2)\qquad \|u - R_L u\|_{H^s(\Gamma)} \ \le\ C 2^{-L(t-s)} \|u\|_{\oplus_{m=1}^{m_\Gamma} H^t(\Gamma_m)}.
$$

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(iii) *For the $L^2(\Gamma)$ orthogonal projection P_L and for the Sobolev space orders $-2 \le s \le t \le 2,\ s < 1.5,\ t > -1.5$, we get*

$$\|u - P_L u\|_{H^s(\Gamma)} \ \le\ C2^{-L(t-s)} \|u\|_{H^t(\Gamma)}. \tag{5.3}$$

(iv) *For the Sobolev space orders $s \le t < 1.5$, the functions u_L from Lin_L^Γ fulfill the inverse property (Bernstein inequality)*

$$\|u_L\|_{H^t(\Gamma)} \ \le\ C2^{L(t-s)} \|u_L\|_{H^s(\Gamma)}. \tag{5.4}$$

5.2. The properties of the wavelet basis in the test space. The properties of the basis of test wavelets introduced in §3.2 can be described using the predual basis. We simply define the classical hierarchical basis by $\chi_P := \varphi_P^{l+1}$ for $P \in \nabla_l^\Gamma$ and observe the duality property $\langle \vartheta_P, \chi_{P'} \rangle := \vartheta_P(\chi_{P'}) \ =\ \delta_{P,P'}$ as well as $\mathrm{span}\{\chi_P :\ P \in \triangle_L^\Gamma\} = Lin_L^\Gamma$. The interpolation projection can be represented as

$$R_L u \ =\ \sum_{P \in \triangle_L^\Gamma} u(P) \varphi_P^L \ =\ \sum_{P \in \triangle_L^\Gamma} \langle \vartheta_P, u \rangle\, \chi_P.$$

The approximation and inverse properties for the space predual to the test functionals are formulated in Lemma 5.1 ii)-iv). The following properties are well-known.

LEMMA 5.2.

(i) *For $1 < s < 1.5$, the basis $\{\chi_P :\ P \in \cup_{L=0}^\infty \triangle_L^\Gamma\}$ is a Riesz basis, i.e., for any L and for any vector of real numbers $(\xi_P)_P$, we get*

$$\left\| \sum_{P \in \triangle_L^\Gamma} \xi_P \chi_P \right\|_{H^s(\Gamma)} \ \sim\ \sqrt{\sum_{P \in \triangle_L^\Gamma} 2^{2l(P)(s-1)} |\xi_P|^2}. \tag{5.5}$$

(ii) *The finite element basis $\varphi_P^L,\ P \in \triangle_L^\Gamma$ satisfies the discrete norm equivalence*

$$\left\| \sum_{\tilde P \in \triangle_L^\Gamma} \xi_{\tilde P} \varphi_{\tilde P}^L \right\|_{L^2(\Gamma)} \ \sim\ \frac{1}{2^L} \sqrt{\sum_{\tilde P \in \triangle_L^\Gamma} |\xi_{\tilde P}|^2}.$$

In particular, we get

$$\left\| \sum_{P \in \triangle_L^\Gamma} \xi_P \chi_P \right\|_{L^2(\Gamma)} \sim \frac{1}{2^L} \sqrt{\sum_{\tilde P \in \triangle_L^\Gamma} | \sum_{P \in \triangle_L^\Gamma} \xi_P \chi_P(\tilde P)|^2}. \tag{5.6}$$

(iii) *Standard estimates yield the upper bound*

$$\left\| \sum_{P \in \triangle_L^\Gamma} \xi_P \chi_P \right\|_{L^2(\Gamma)} \le C \sqrt{L \sum_{l=-1}^{L-1} 2^{-2l} \sum_{\tilde P \in \nabla_l^\Gamma} |\xi_P|^2} \ \le\ C\,L \sup_{P \in \triangle_L^\Gamma} |\xi_P|\,. \tag{5.7}$$

Similar results hold for the basis predual to the functionals $\{\vartheta_P^+\}$. We only have to replace Lin_L^Γ by the space of continuous and piecewise quadratic functions over the partition $\{\Gamma_Q :\ Q \in \square_L^\Gamma\}$ and to substitute the hat functions $\chi_P := \varphi_P^{l(P)+1}$ by the piecewise quadratic interpolants.

5.3. General error estimates for the numerical solution and preconditioning. In this subsection we recall well-known error estimates for stable numerical methods. We discuss the assumptions on the stability and necessary conditions which ensure that the numerical methods, perturbed by compression and by boundary approximation and quadrature, admit the same asymptotic orders of convergence as the unperturbed methods. Moreover, we give necessary conditions which ensure the existence of diagonal preconditioners for the matrix $A^{w,c,q}$ of the compressed and approximated collocation method.

The collocation method for the equation $Au = v$ defines an approximate solution $u_L \in Lin_L^\Gamma$ by $R_L A u_L = R_L v$ (cf. §2.5). This method is called stable in the space $H^s(\Gamma)$ if the approximate operators $R_L A : \ Lin_L^\Gamma \longrightarrow Lin_L^\Gamma$ are invertible for sufficiently large L and if their inverses are bounded, i.e.,

$$\left\| \left(R_L A|_{Lin_L^\Gamma} \right)^{-1} w_L \right\|_{H^{s+\mathbf{r}}(\Gamma)} \leq C \, \|w_L\|_{H^s(\Gamma)}, \quad w_L \in Lin_L^\Gamma.$$

We suppose that the collocation method is stable for $s = 0$. Additionally, we suppose stability also for $s = 1.1$ (or for an arbitrary s with $1 < s < 1.5$ instead of 1.1). Note that stability is well-known for second kind integral operators including compact integral operators. In particular this is true for double layer operators over smooth boundaries (cf., e.g., [2]). For first kind operators and operators involving strongly singular integral operators, the question of stability is not yet solved. A first step toward the solution is done in [36, 37, 11, 14]. Note that, since our trial space Lin_L^Γ is generated by two scaling functions, the stability is needed for a multiwavelet space (cf. the univariate multiwavelet paper [38]). Though a rigorous proof of stability is missing, collocation methods are frequently used without observing any instability.

To simplify the notation, let us denote the operator $R_L A|_{Lin_L^\Gamma}$ by A_L, i.e., by the same symbol as for its matrix with respect to the basis $\{\varphi_P^L : \ P \in \triangle_L^\Gamma\}$ (cf. §2.5). Similarly, we denote by A_L^c and $A_L^{c,q}$ the operators in Lin_L^Γ the matrix of which with respect to $\{\varphi_P^L : \ P \in \triangle_L^\Gamma\}$ is A_L^c and $A_L^{c,q}$, respectively (cf. (3.7)). Using the L^2 orthogonal projection P_L, we represent the error $u - u_L$ of the fully discretized and compressed method $A_L^{c,q} u_L = R_L v$ as

$$\begin{aligned} & u - u_L \\ & = u - P_L u - (A_L^{c,q})^{-1} \left\{ R_L A u - A_L^{c,q} P_L u \right\} \\ & = u - P_L u - (A_L^{c,q})^{-1} \left\{ [A_L - A_L^{c,q}] P_L u + A(I - P_L) u - (I - R_L) A (I - P_L) u \right\}. \end{aligned}$$

We apply the boundedness assumption on A (cf. §2.2), assume the stability of $A_L^{c,q}$ for Sobolev index $s = 0$, and use Lemma 5.1 to get

$$\begin{aligned} \|u - u_L\|_{H^{\mathbf{r}}(\Gamma)} \leq \|u - P_L u\|_{H^{\mathbf{r}}(\Gamma)} + C \Big\{ & \|[A_L - A_L^{c,q}] P_L u\|_{H^0(\Gamma)} + \\ & \|(I - P_L) u\|_{H^{\mathbf{r}}(\Gamma)} + 2^{-1.1L} \, \|A(I - P_L) u\|_{H^{1.1}(\Gamma)} \Big\} \\ \leq C 2^{-(2-\mathbf{r})L} \, \|u\|_{H^2(\Gamma)} + C \, & \|[A_L - A_L^{c,q}] P_L u\|_{H^0(\Gamma)} \, . \end{aligned}$$

In other words, to ensure the optimal convergence order $2 - \mathbf{r}$, we need the estimate $\|[A_L - A_L^{c,q}] P_L u\|_{H^0(\Gamma)} \leq C_u 2^{-(2-\mathbf{r})L}$ and the stability of $A_L^{c,q}$. Since A_L is stable by assumption and since $A_L^{c,q} = A_L \{ I + A_L^{-1} [A_L^{c,q} - A_L] \}$, for the stability of $A_L^{c,q}$, it will be sufficient to require

$$\left\| A_L - A_L^{c,q} \right\|_{H^s(\Gamma) \leftarrow H^{s+\mathbf{r}}(\Gamma)} \leq \frac{1}{2} \left[\sup_{L' = L_0, L_0+1, \ldots} \|A_{L'}^{-1}\|_{H^{s+\mathbf{r}}(\Gamma) \leftarrow H^s(\Gamma)} \right]^{-1} \tag{5.8}$$

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for $s = 0$. In view of the inverse property iv) of Lemma 5.1 the last condition is a consequence of

$$\left\| [A_L - A_L^{c,q}]P_L u \right\|_{H^0(\Gamma)} \leq C 2^{-(s-\mathbf{r})L} \|u\|_{H^s(\Gamma)} \tag{5.9}$$

with the choice $s = 1.1$ if we show that the constant C in (5.9) can be made smaller than any prescribed positive number. The usual compression estimates prove the error estimates in (5.9) for $s = 2$ and $s = 1.1$ but with the difference $A_L - A_L^{c,q}$ replaced by $A_L - A_L^c$. We refer the reader to [15, 34, 47, 40] for the details. In the present paper it will be our task to prove the estimates (5.9) for $s = 2$ and $s = 1.1$ with $A_L - A_L^{c,q}$ replaced by $A_L^c - A_L^{c,q}$.

The issue of wavelet preconditioners has been addressed by many authors (cf., e.g., [13, 15, 29, 52]) and we will follow the same ideas. In the case $\mathbf{r} = 0$ the stability of $A_L^{c,q}$ implies that the matrix $A_L^{c,q}$ has a condition number which is already uniformly bounded with respect to L. Thus, for the algorithm (3.9), no preconditioning is needed, and we can restrict our consideration to algorithm (3.8). Unfortunately, the wavelet transform $\mathcal{T}_T^{-1}$ (cf. §3.3) does not have a uniformly bounded condition number with respect to Euclidean matrix norm. Therefore, preconditioning is needed even for $\mathbf{r} = 0$, and the preconditioner is to be derived from the stability for a different Sobolev index. We choose, e.g., $s = 1.1$. Let us consider an operator A of order $\mathbf{r} = 0, -1$ and suppose the stability of A_L in the Sobolev space $H^{1.1}(\Gamma)$. If we could prove (5.8) for $s = 1.1$, then $A_L^{c,q}$ is also stable in $H^{1.1}(\Gamma)$. From §3.1 and 5.2, we recall that $A_L^{w,c,q}$ is the matrix of the operator $A_L^{c,q}$ with respect to the bases $\{\psi_P : P \in \triangle_L^\Gamma\}$ and $\{\chi_P : P \in \triangle_L^\Gamma\}$. Under assumption (5.8), the assertions i) of Lemmata 5.2 and 5.1 imply that the matrices

$$\left(\delta_{P,P'} 2^{l(P')(1.1-1)}\right)_{P,P' \in \triangle_L^\Gamma} A_L^{w,c,q} \left(\delta_{P,P'} 2^{-l(P)(\mathbf{r}+1.1-1)}\right)_{P,P' \in \triangle_L^\Gamma} \tag{5.10}$$

have condition numbers which are uniformly bounded with respect to L, i.e., the matrix $A_L^{w,c,q}$ admits a diagonal preconditioning. The boundedness of the condition number ensures the fast convergence of the iterative solver in the wavelet algorithm (3.8). In other words, for the fast iterative solution of the linear systems $A_L^{w,c,q}\beta = \gamma$ (cf. part iv) of (3.8)) using preconditioning, we only have to prove (5.8). This, however, follows from the inverse property iv) in Lemma 5.1 and from (5.9) with $s = 1.1$ and with a sufficiently small constant C. Again, (5.9) is well-known for the difference $A_L - A_L^{c,q}$ replaced by $A_L - A_L^c$ (cf. [15, 34, 47, 40]). The estimate (5.8) with $A_L - A_L^{c,q}$ replaced by $A_L^c - A_L^{c,q}$ will be treated in the next section. All together, we get

REMARK 5.1. *For almost optimal rates of convergence, for stability, and for preconditioning, we only have to find an appropriate nonnegative constant κ and to prove*

$$\left\| [A_L^c - A_L^{c,w}]P_L u \right\|_{H^0(\Gamma)} \leq C\, L^\kappa 2^{-(2-\mathbf{r})L} \, \|u\|_{H^{1.1}(\Gamma)}\,, \quad u \in H^{1.1}(\Gamma). \tag{5.11}$$

To derive an estimate like (5.11), we shall use the following Schur lemma and the following estimate by the l^∞ matrix norm.

LEMMA 5.3. *Denote the entries of the compressed matrix of quadrature errors $[A_L^c - A_L^{c,q}]$ with respect to the wavelet bases $\{\chi_{P'}\}$ and $\{\psi_P\}$ by $a_{P',P} := a^{w,c}_{P',P} - a^{w,c,q}_{P',P}$. Choose an arbitrary real number x. Then the left hand side of (5.11) can be estimated as*

$$\left\| [A_L^c - A_L^{c,q}]P_L u \right\|_{L^2(\Gamma)} \leq C \|u\|_{H^{1.1}(\Gamma)} \sqrt{\Sigma_1\, \Sigma_2}, \tag{5.12}$$

$$\Sigma_1 := \sup_{P' \in \triangle_L^\Gamma} 2^{-xl(P')} \sum_{P \in \triangle_L^\Gamma} \frac{|a_{P',P}|}{2^{[1.1-x]l(P)}}, \quad \Sigma_2 := \sum_{l=-1}^{L-1} 2^{[0.9-x]l} \sup_{P \in \nabla_l^\Gamma} \sum_{P' \in \triangle_L^\Gamma} \frac{|a_{P',P}|}{2^{[2-x]l(P')}}.$$

Proof. In view of (5.6), we get, for $P_L u = \sum \xi_P \psi_P$,

$$\begin{aligned} \left\| [A_L^c - A_L^{c,q}] P_L u \right\|_{L^2(\Gamma)}^2 &= \left\| \sum_{P',P \in \triangle_L^\Gamma} a_{P',P} \xi_P \chi_{P'} \right\|_{L^2(\Gamma)}^2 \\ &\le C\, 2^{-2L} \sum_{\tilde{P} \in \triangle_L^\Gamma} \left| \sum_{P' \in \triangle_L^\Gamma} \sum_{P \in \triangle_L^\Gamma} a_{P',P} \xi_P \chi_{P'}(\tilde{P}) \right|^2 . \end{aligned}$$

Clearly, the function values $\chi_{P'}(\tilde{P})$ are nonnegative and less than one. We apply the Cauchy-Schwarz inequality and some easy calculations to arrive at

$$\begin{aligned} &\left\| [A_L^c - A_L^{c,q}] P_L u \right\|_{L^2(\Gamma)}^2 \\ &\le C\, 2^{-2L} \sum_{\tilde{P} \in \triangle_L^\Gamma} \Bigg[\sum_{P' \in \triangle_L^\Gamma} \sum_{P \in \triangle_L^\Gamma} |a_{P',P}| 2^{[x-1.1]l(P)} \\ &\qquad\qquad \times \sum_{P \in \triangle_L^\Gamma} |a_{P',P}| 2^{[1.1-x]l(P)} |\xi_P|^2 \, |\chi_{P'}(\tilde{P})|^2 \Bigg] \\ &\le C\, 2^{-2L} \Sigma_1 \sum_{\tilde{P} \in \triangle_L^\Gamma} \sum_{P' \in \triangle_L^\Gamma} 2^{xl(P')} \sum_{P \in \triangle_L^\Gamma} |a_{P',P}| 2^{[1.1-x]l(P)} |\xi_P|^2 |\chi_{P'}(\tilde{P})|^2 . \end{aligned}$$

Now we observe that, for a fixed P', the number of $\tilde{P} \in \triangle_L^\Gamma$ such that $\chi_{P'}(\tilde{P}) > 0$ is less than $C 2^{2[L-l(P')]}$. Using this as well as (5.1) valid for the wavelet expansion of $P_L u$, we continue

$$\begin{aligned} \left\| [A_L^c - A_L^{c,q}] P_L u \right\|_{L^2(\Gamma)}^2 &\le C\, \Sigma_1 \sum_{P \in \triangle_L^\Gamma} \sum_{P' \in \triangle_L^\Gamma} 2^{[x-2]l(P')} |a_{P',P}| 2^{[1.1-x]l(P)} |\xi_P|^2, \\ &\le C\, \Sigma_1 \sum_{l=-1}^{L-1} 2^{[0.9-x]l} \sup_{P \in \nabla_l^\Gamma} \left[\sum_{P' \in \triangle_L^\Gamma} 2^{[x-2]l(P')} |a_{P',P}| \right] \\ &\qquad \times \sum_{P \in \nabla_l^\Gamma} 2^{2[1.1-1]l(P)} |\xi_P|^2 \;\le\; C \|u\|_{H^{1.1}(\Gamma)}^2 \, \Sigma_1 \, \Sigma_2 . \qquad \square \end{aligned}$$

LEMMA 5.4. *Using the notation of Lemma 5.3, we get the l^∞ matrix norm bound*

$$\left\| [A_L^c - A_L^{c,q}] P_L u \right\|_{L^2(\Gamma)} \le C\, L \|u\|_{H^{1.1}(\Gamma)} \sup_{P' \in \triangle_L^\Gamma} \sum_{P \in \triangle_L^\Gamma} 2^{-0.05\, l(P)} |a_{P',P}| . \tag{5.13}$$

Proof. Using the estimate $\sup_{P \in \triangle_L^\Gamma} |2^{0.05\, l(P)} \xi_P| \le C \, \| \sum_{P \in \triangle_L^\Gamma} \xi_P \psi_P \|_{H^{1.1}(\Gamma)}$, which is a simple consequence of (5.1), and the upper bound (5.7), the assertion follows easily. $\square$

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6. The estimation of the errors due to the approximate parametrization and due to the quadrature.

6.1. The far field estimate. In this subsection we suppose that the near field and the singular integrations are performed exactly and derive the convergence estimates for the far field case. The error estimate for the near field and for the singular integrals will be considered in §6.2 and §6.3, respectively. In view of Remark 5.1, it remains to prove

LEMMA 6.1. *Suppose $A_L^c \in \mathcal{L}(Lin_L^\Gamma)$ is the approximate operator of the compressed collocation method including the sparsity pattern $\mathcal{P}$ (cf. §3.5). If $A_L^{c,q}$ is the operator of the compressed collocation method including the approximation of the parameter mappings and the quadrature of the far field (cf. §4.1), then we get (5.11) with $\kappa = 1.5$.*

Proof. **i)** It remains to estimate Σ_1 and Σ_2 (cf. Lemma 5.3). For the interpolation and quadrature, we shall prove the error estimate

$$(6.1)\quad |a_{P',P}| = |a^{w,c}_{P',P} - a^{w,c,q}_{P',P}| \le a^1_{P',P},$$
$$a^1_{P',P} := Cd^{-\mathbf{m}}2^{-\mathbf{m}L}\int_{\{R\in\operatorname{supp}\psi_P:\ \operatorname{dist}(R,\Theta_{P'})>2d2^{-l(P')}\}} \operatorname{dist}(R,\Theta_{P'})^{-2}\,\mathrm{d}_R\Gamma,$$

if the support of ψ_P is contained in the interior of a single parametrization patch Γ_m and

$$(6.2)\qquad |a_{P',P}| = |a^{w,c}_{P',P} - a^{w,c,q}_{P',P}| \le a^1_{P',P} + a^2_{P',P},$$
$$a^2_{P',P} := Cd^{-1-\mathbf{m}}2^{-\mathbf{m}L}\int_{\substack{\{R\in\operatorname{supp}\psi_P\cap\Gamma_m\cap\Gamma_{m'}:\\ \operatorname{dist}(R,\Theta_{P'})>2d2^{0.5L-1.5l(P')}\}}} \frac{\mathrm{d}_R[\Gamma_m\cap\Gamma_{m'}]}{\operatorname{dist}(R,\Theta_{P'})},$$

if the support of ψ_P intersects at least two parametrization patches. We introduce the numbers $\operatorname{dist} := \operatorname{dist}(\Theta_{P'},\Psi_P)$, $\mathrm{M}_1 := d2^{L-l(P)-l(P')}$, and $\mathrm{M}_2 := d2^{1.5[L-l(P')]-l(P)}$ (cf. the formulae (3.10) and (3.11)). Substituting the estimate (6.1) into the definition of Σ_1 and choosing $x=0$, we get

$$\begin{aligned}
\Sigma_1 &\le \sup_{P'\in\triangle_L^\Gamma}\sum_{l(P)=-1}^{L-1}2^{-1.1\,l(P)}\sum_{\substack{P\in\nabla^\Gamma_{l(P)}:\\ \operatorname{dist}\le\mathrm{M}_1}} Cd^{-\mathbf{m}}2^{-\mathbf{m}L}\int_{\{\dots\}}\operatorname{dist}(R,\Theta_{P'})^{-2}\,\mathrm{d}_R\Gamma\\
&\le Cd^{-\mathbf{m}}2^{-\mathbf{m}L}\sup_{P'\in\triangle_L^\Gamma}\sum_{l(P)=-1}^{L-1}2^{-1.1\,l(P)}\int_{\{R:\ 2d2^{-l(P')}<\operatorname{dist}(R,\Theta_{P'})<\mathrm{M}_1\}}\frac{\mathrm{d}_R\Gamma}{\operatorname{dist}(R,\Theta_{P'})^2}\\
&\le Cd^{-\mathbf{m}}L2^{-\mathbf{m}L}.
\end{aligned}$$

Similarly, we estimate Σ_2 including (6.1).

$$\begin{aligned}
\Sigma_2 &\le \sum_{l=-1}^{L-1}2^{0.9\,l}\sup_{P\in\triangle_l^\Gamma}\sum_{l'=-1}^{L-1}2^{-2l'}\sum_{\substack{P'\in\nabla^\Gamma_{l'}:\\ \operatorname{dist}\le\mathrm{M}_1}} Cd^{-\mathbf{m}}2^{-\mathbf{m}L}\int_{\{\dots\}}\operatorname{dist}(R,\Theta_{P'})^{-2}\,\mathrm{d}_R\Gamma\\
&\le Cd^{-\mathbf{m}}2^{-\mathbf{m}L}\sum_{l=-1}^{L-1}2^{0.9\,l}\sup_{P\in\triangle_l^\Gamma}\sum_{l'=-1}^{L-1}\int_{\{\dots\}}\left\{2^{-2l'}\sum_{\substack{P'\in\nabla^\Gamma_{l'}:\\ \operatorname{dist}\le\mathrm{M}_1}}\operatorname{dist}(R,\Theta_{P'})^{-2}\right\}\mathrm{d}_R\Gamma\\
&\le Cd^{-\mathbf{m}}2^{-\mathbf{m}L}\sum_{l=-1}^{L-1}2^{0.9\,l}\sup_{P\in\triangle_l^\Gamma}\left\{\sum_{l'=-1}^{L-1}\right.
\end{aligned}$$

$$\int_{\operatorname{supp}\psi_P}\int_{\{P':\,2d2^{-l(P')}<\operatorname{dist}(P',R)<\mathrm{M}_1\}}\frac{\mathrm{d}_{P'}\Gamma\,\mathrm{d}_R\Gamma}{\operatorname{dist}(R,P')^2}\Bigg\}$$

$$\leq Cd^{-\mathbf{m}}2^{-\mathbf{m}L}\sum_{l=-1}^{L-1}2^{0.9\,l}\sup_{P\in\triangle_l^\Gamma}\sum_{l'=-1}^{L-1}L\int_{\operatorname{supp}\psi_P}\mathrm{d}_R\Gamma\;\leq\;Cd^{-\mathbf{m}}L^2 2^{-\mathbf{m}L}.$$

If we substitute the estimate $a^2_{P',P}$ of (6.2) into the definition of Σ_1 and if we choose $x=0$, we get

$$\Sigma_1\leq\sup_{P'\in\triangle_L^\Gamma}\sum_{l=-1}^{L-1}2^{-1.1\,l}\sum_{\substack{P\in\nabla_l^\Gamma:\\ \operatorname{dist}\leq\mathrm{M}_2}}\Bigg\{Cd^{-1-\mathbf{m}}2^{-\mathbf{m}L}\cdot$$

$$\int_{\substack{\{R\in\operatorname{supp}\psi_P\cap\Gamma_m\cap\Gamma_{m'}:\\ \operatorname{dist}(R,\Theta_{P'})>2d2^{0.5L-1.5l(P')}\}}}\frac{\mathrm{d}_R[\Gamma_m\cap\Gamma_{m'}]}{\operatorname{dist}(R,\Theta_{P'})}\Bigg\}$$

$$\leq Cd^{-1-\mathbf{m}}2^{-\mathbf{m}L}\sup_{l(P')=-1,\dots,L-1}\sum_{l=-1}^{L-1}\Bigg\{2^{-1.1\,l}\cdot$$

$$\int_{\substack{\{R\in\Gamma_m\cap\Gamma_{m'}:\,2d2^{0.5L}\cdot\\ 2^{-1.5l(P')}<\operatorname{dist}(R,\Theta_{P'})<\mathrm{M}_2\}}}\frac{\mathrm{d}_R[\Gamma_m\cap\Gamma_{m'}]}{\operatorname{dist}(R,\Theta_{P'})}\Bigg\}$$

$$\leq Cd^{-1-\mathbf{m}}L2^{-\mathbf{m}L}.$$

Similarly, we estimate Σ_2 including $a^2_{P',P}$ of (6.2).

$$\Sigma_2\leq\sum_{l=-1}^{L-1}2^{0.9\,l}\sup_{P\in\triangle_l^\Gamma}\sum_{l'=-1}^{L-1}2^{-2l'}\sum_{\substack{P'\in\nabla_{l'}^\Gamma:\\ \operatorname{dist}\leq\mathrm{M}_2}}Cd^{-1-\mathbf{m}}2^{-\mathbf{m}L}\int_{\{\dots\}}\frac{\mathrm{d}_R[\Gamma_m\cap\Gamma_{m'}]}{\operatorname{dist}(R,\Theta_{P'})}$$

$$\leq Cd^{-1-\mathbf{m}}2^{-\mathbf{m}L}\sum_{l=-1}^{L-1}2^{0.9\,l}\sup_{P\in\triangle_l^\Gamma}\Bigg\{\sum_{l'=-1}^{L-1}$$

$$\int_{\{\dots\}}\Bigg\{2^{-2l'}\sum_{\substack{P'\in\nabla_{l'}^\Gamma:\\ \operatorname{dist}\leq\mathrm{M}_2}}\operatorname{dist}(R,\Theta_{P'})^{-1}\Bigg\}\mathrm{d}_R[\Gamma_m\cap\Gamma_{m'}]\Bigg\}$$

$$\leq Cd^{-1-\mathbf{m}}2^{-\mathbf{m}L}\sum_{l=-1}^{L-1}2^{0.9\,l}\sup_{P\in\triangle_l^\Gamma}\Bigg\{\sum_{l'=-1}^{L-1}$$

$$\int_{\operatorname{supp}\psi_P}\int_{\substack{\{P':\,2d2^{0.5L-1.5l(P')}\\ <\operatorname{dist}(R,P')\leq\mathrm{M}_2\}}}\frac{\mathrm{d}_{P'}\Gamma\,\mathrm{d}_R[\Gamma_m\cap\Gamma_{m'}]}{\operatorname{dist}(R,P')}\Bigg\}$$

$$\leq Cd^{-1-\mathbf{m}}2^{-\mathbf{m}L}\sum_{l=-1}^{L-1}2^{0.9\,l}\sup_{P\in\triangle_l^\Gamma}\sum_{l'=-1}^{L-1}\int_{\operatorname{supp}\psi_P}\mathrm{d}_R[\Gamma_m\cap\Gamma_{m'}]\;\leq\;Cd^{-1-\mathbf{m}}L2^{-\mathbf{m}L}.$$

ii) Let us prove (6.1) and (6.2). We shall prove that the quadrature error over a small triangle Γ_Q from the quadrature partition of Ψ_P is less than

$$C2^{-\mathbf{m}l(P')}2^{-[4-\mathbf{r}]l(Q)}\operatorname{dist}\left(\Gamma_Q,\Theta_{P'}\right)^{-2-\mathbf{m}}. \tag{6.3}$$

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Now observe that the number of triangles Γ_Q contained in Ψ_P with distance to the boundary $\partial\Gamma_m$ greater than $c_o 2^{-l(Q)}$ is less than (cf. (4.1))

$$\begin{aligned}
\sum_{l(Q)=l(P)}^{L-1} \sum_{\substack{\Gamma_Q\in Qua^{\Gamma}_{l(Q)} \\ \Gamma_Q\subseteq \operatorname{supp}\psi_P}} 1 \;&\leq\; \sum_{l(Q)=l(P)}^{L-1} \sum_{\substack{\Gamma_Q\in Qua^{\Gamma}_{l(Q)} \\ \Gamma_Q\subseteq \operatorname{supp}\psi_P}} 2^{-2l(Q)} \left[2^{-l(Q)}\right]^{-2} \\
&\leq C \sum_{l(Q)=l(P)}^{L-1} \sum_{\substack{\Gamma_Q\in Qua^{\Gamma}_{l(Q)} \\ \Gamma_Q\subseteq \operatorname{supp}\psi_P}} 2^{-2l(Q)} \left[d^{-1}2^{-L+l(P')}\operatorname{dist}(\Gamma_Q,\Theta_{P'})\right]^{-2} \\
&\leq Cd^2 2^{2L-2l(P')} \int_{\{R\in \operatorname{supp}\psi_P:\, \operatorname{dist}(R,\Theta_{P'})>2d2^{-l(P')}\}} \operatorname{dist}(R,\Theta_{P'})^{-2}\, \mathrm{d}_R\Gamma.
\end{aligned}$$

Consequently, the quadrature error $|a_{P',P}|$ over the corresponding part $\cup\Gamma_Q$ of Ψ_P contained completely in a single parametrization patch Γ_m is the sum of the terms in (6.3) and can be estimated by

$$\begin{aligned}
|a_{P',P}| &\leq C \sum_{l(Q)=l(P)}^{L-1} \sum_{\substack{\Gamma_Q\in Qua^{\Gamma}_{l(Q)} \\ \Gamma_Q\subseteq \operatorname{supp}\psi_P}} \frac{2^{-\mathbf{m}l(P')}2^{-[4-\mathbf{r}]l(Q)}}{\left[d2^{L-l(Q)-l(P')}\right]^{2+\mathbf{m}}} \\
&\leq C\, \frac{2^{-\mathbf{m}l(P')}}{\left[d2^{L-l(P')}\right]^{2+\mathbf{m}}}\, d^2 2^{2L-2l(P')} \int_{\{\dots\}} \operatorname{dist}(R,\Theta_{P'})^{-2}\, \mathrm{d}_R\Gamma \\
&\leq Cd^{-\mathbf{m}}2^{-\mathbf{m}L} \int_{\{R\in \operatorname{supp}\psi_P:\, \operatorname{dist}(R,\Theta_{P'})>2d2^{-l(P')}\}} \operatorname{dist}(R,\Theta_{P'})^{-2}\, \mathrm{d}_R\Gamma.
\end{aligned}$$

Analogously, we observe that the number of triangles Γ_Q contained in Ψ_P with distance to the boundary $\partial\Gamma_m = \Gamma_m \cap \Gamma_{m'}$ less than $c_o 2^{-l(Q)}$ is less than (cf. (4.2))

$$\begin{aligned}
\sum_{l(Q)=l(P)}^{L-1} \sum_{\substack{\Gamma_Q\in Qua^{\Gamma}_{l(Q)} \\ \Gamma_Q\subseteq \operatorname{supp}\psi_P}} 1 \;&\leq\; \sum_{l(Q)=l(P)}^{L-1} \sum_{\substack{\Gamma_Q\in Qua^{\Gamma}_{l(Q)} \\ \Gamma_Q\subseteq \operatorname{supp}\psi_P}} 2^{-l(Q)} \left[2^{-l(Q)}\right]^{-1} \\
&\leq C \sum_{l(Q)=l(P)}^{L-1} \sum_{\substack{\Gamma_Q\in Qua^{\Gamma}_{l(Q)} \\ \Gamma_Q\subseteq \operatorname{supp}\psi_P}} 2^{-l(Q)} \left[d^{-1}2^{-1.5[L-l(P')]}\operatorname{dist}(\Gamma_Q,\Theta_{P'})\right]^{-1} \\
&\leq Cd2^{1.5[L-l(P')]} \int_{\substack{\{R\in \operatorname{supp}\psi_P\cap\Gamma_m\cap\Gamma_{m'}: \\ \operatorname{dist}(R,\Theta_{P'})>2d2^{0.5L-1.5l(P')}\}}} \frac{\mathrm{d}_R[\Gamma_m\cap\Gamma_{m'}]}{\operatorname{dist}(R,\Theta_{P'})}.
\end{aligned}$$

Consequently, the quadrature error $|a_{P',P}|$ over the corresponding part $\cup\Gamma_Q$ of Ψ_P intersecting at least two parametrization patches Γ_m and $\Gamma_{m'}$ is the sum of the terms in (6.3) and can be estimated by

$$|a_{P',P}| \leq C \sum_{l(Q)=l(P)}^{L-1} \sum_{\substack{\Gamma_Q\in Qua^{\Gamma}_{l(Q)} \\ \Gamma_Q\subseteq \operatorname{supp}\psi_P}} \frac{2^{-\mathbf{m}l(P')}2^{-[4-\mathbf{r}]l(Q)}}{\left[d2^{1.5[L-l(P')]-l(Q)}\right]^{2+\mathbf{m}}}$$

$$\leq C\,\frac{2^{-\mathbf{m}l(P')}}{\left[d2^{1.5[L-l(P')]}\right]^{2+\mathbf{m}}}\,Cd2^{1.5[L-l(P')]}\,\cdot$$

$$\int_{\substack{\{R\in\operatorname{supp}\psi_P\cap\Gamma_m\cap\Gamma_{m'}:\\ \operatorname{dist}(R,\Theta_{P'})>2d2^{0.5L-1.5l(P')}\}}}\frac{\mathrm{d}_R[\Gamma_m\cap\Gamma_{m'}]}{\operatorname{dist}(R,\Theta_{P'})}$$

$$\leq Cd^{-1-\mathbf{m}}2^{-\mathbf{m}L}\int_{\substack{\{R\in\operatorname{supp}\psi_P\cap\Gamma_m\cap\Gamma_{m'}:\\ \operatorname{dist}(R,\Theta_{P'})>2d2^{0.5L-1.5l(P')}\}}}\frac{\mathrm{d}_R[\Gamma_m\cap\Gamma_{m'}]}{\operatorname{dist}(R,\Theta_{P'})}.$$

In other words (6.1) and (6.2) are proved if we can show (6.3).

iii) Let us prove (6.3). This, however, is a consequence of $\operatorname{dist}(\Theta_{P'},\Gamma_Q)<C$ and of the stronger (respectively, equivalent) estimate

$$\tilde{a}_{P',(Q,\iota)}\leq C2^{-\mathbf{m}l(P')}2^{-(4-\mathbf{r})l(Q)}\operatorname{dist}(\Theta_{P'},\Gamma_Q)^{-\mathbf{r}-2-\mathbf{m}}, \tag{6.4}$$

where $\tilde{a}_{P',(Q,\iota)}$ is the absolute value of the approximation error to

$$\vartheta_{P'}\left(\int_\Gamma k(\cdot,R,n_R)\frac{p(\cdot-R)}{|\cdot-R|^\alpha}\phi_{Q,\iota}(R)\,\mathrm{d}_R\Gamma\right), \tag{6.5}$$

and where $\phi_{Q,\iota}(\kappa_m(\sigma))=\tilde{\phi}_{Q,\iota}(\sigma)$ is the Lagrange basis function used in (4.7). It remains to derive (6.4). The approximation to (6.5) (cf. (4.12)) is obtained by interpolating the parametrization κ_m, by applying a $2-\mathbf{r}$ order product rule to the integral over T_τ of the integrand $\sigma\mapsto k(\cdot,\kappa_m(\sigma),n'_{\kappa'_m(\sigma)})\mathcal{J}'_m(\sigma)$, and by applying an n_G order quadrature to the integrals of the weight functions $\sigma\mapsto\tilde{\phi}_{Q,\upsilon}(\sigma)p(\cdot-\kappa'_m(\sigma))\,|\cdot-\kappa'_m(\sigma)|^{-\alpha}\,\phi_{Q,\iota}(\kappa_m(\sigma))$ (cf. Remark 4.1). Let us make this more precise. It is not hard to see that the test functional $\vartheta_{P'}$ is a scaled version of a difference formula and that it satisfies a certain Leibniz rule of the form $\vartheta_{P'}(fg)=\sum_{i=1}^{i_{P'}}\vartheta_{P',1,i}(f)\vartheta_{P',2,i}(g)$, where the $\vartheta_{P',j,i}$ are, just like the $\vartheta_{P'}$, finite linear combination of Dirac delta functionals with bounded coefficients and with $\operatorname{supp}\vartheta_{P',j,i}\subseteq\operatorname{supp}\vartheta_{P'}$. Moreover, the sum $\mathbf{m}_{P',1,i}+\mathbf{m}_{P',2,i}$ of the vanishing moments $\mathbf{m}_{P',j,i}$ for $\vartheta_{P',j,i}$ is equal to the number $\mathbf{m}:=2-\mathbf{r}$ of vanishing moments for $\vartheta_{P'}$. Applying the Leibniz rule to (6.5), we get the integrand

$$\sum_{i=1}^{i_{P'}}\int_{\Gamma_Q}k(\vartheta_{P',1,i},R,n_R)\vartheta_{P',2,i}\left(\frac{p(\cdot-R)}{|\cdot-R|^\alpha}\right)\phi_{Q,\iota}(R)\,\mathrm{d}_R\Gamma.$$

Consequently, the term $\tilde{a}_{P',(Q,\iota)}$ is the sum over i of errors due to replacing the parameter mapping κ_m by its interpolation κ'_m, due to applying a $2-\mathbf{r}$ order product rule to the integral over T_τ of the integrand $\sigma\mapsto k(\vartheta_{P',1,i},\kappa_m(\sigma),n'_{\kappa'_m(\sigma)})\mathcal{J}'_m(\sigma)$, and due to applying a tensor product variant of Gauß quadrature of order n_G to the integrals of the corresponding weight functions $\sigma\mapsto\tilde{\phi}_{Q,\upsilon}(\sigma)\vartheta_{P',2,i}(p(\cdot-\kappa'_m(\sigma))\,|\cdot-\kappa'_m(\sigma)|^{-\alpha})\,\tilde{\phi}_{Q,\iota}(\sigma)$ for $\upsilon=1,2,3$. Indeed, this splitting according to the Leibniz rule into a sum over $i=1,\dots,i_{P'}$ has to be included into the derivation of formula (4.12). We have not mentioned this since the splitting is not seen explicitly in the final formula and since we did not want to overload the presentation in §4.1 by these technical details.

Clearly, concerning the replacement of κ_m, we get

$$|\kappa_m(\sigma)-\kappa'_m(\sigma)|\leq C\,2^{-(\mathbf{m}+1)l(Q)},$$

for $\sigma\in T_\tau=\kappa_m^{-1}(\Gamma_Q)$ and $|\nabla_\sigma\kappa_m(\sigma)-\nabla_\sigma\kappa'_m(\sigma)|\leq C\,2^{-\mathbf{m}l(Q)}$, if ∇_σ is the gradient with respect to σ. From the smoothness assumptions on κ_m in §2.1 and on the integral kernel in

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§2.2, we conclude

$$|\mathcal{J}_m(\sigma) - \mathcal{J}'_m(\sigma)| \le C2^{-\mathbf{m}l(Q)}, \quad |\mathcal{J}_m(\sigma)| \le C, \quad |\mathcal{J}'_m(\sigma)| \le C,$$

$$\begin{aligned}
\left| k\left(\vartheta_{P',1,i}, \kappa_m(\sigma), n_{\kappa_m(\sigma)}\right) - k\left(\vartheta_{P',1,i}, \kappa_m(\sigma), n'_{\kappa'_m(\sigma)}\right)\right| &\le C2^{-\mathbf{m}l(Q)} 2^{-\mathbf{m}_{P',1,i}\, l(P')},\\
\left| k\left(\vartheta_{P',1,i}, \kappa_m(\sigma), n_{\kappa_m(\sigma)}\right)\right| &\le C2^{-\mathbf{m}_{P',1,i}\, l(P')},\\
\left| k\left(\vartheta_{P',1,i}, \kappa_m(\sigma), n'_{\kappa'_m(\sigma)}\right)\right| &\le C2^{-\mathbf{m}_{P',1,i}\, l(P')},
\end{aligned}$$

(6.6)

$$\begin{aligned}
\left| \vartheta_{P',2,i}\left(\frac{p\Big(\cdot - \kappa_m(\sigma)\Big)}{|\cdot - \kappa_m(\sigma)|^\alpha}\right) - \vartheta_{P',2,i}\left(\frac{p\Big(\cdot - \kappa'_m(\sigma)\Big)}{|\cdot - \kappa'_m(\sigma)|^\alpha}\right)\right| &\le C\frac{2^{-(\mathbf{m}+1)l(Q)}\; 2^{-\mathbf{m}_{P',2,i}\, l(P')}}{\operatorname{dist}^{2+\mathbf{r}+\mathbf{m}_{P',2,i}+1}}\\
&\le C\frac{2^{-\mathbf{m}l(Q)}\; 2^{-\mathbf{m}_{P',2,i} l(P')}}{\operatorname{dist}^{2+\mathbf{r}+\mathbf{m}_{P',2,i}}}\;,\\
\left| \vartheta_{P',2,i}\left(\frac{p\Big(\cdot - \kappa_m(\sigma)\Big)}{|\cdot - \kappa_m(\sigma)|^\alpha}\right)\right| &\le C\frac{2^{-\mathbf{m}_{P',2,i}\, l(P')}}{\operatorname{dist}^{2+\mathbf{r}+\mathbf{m}_{P',2,i}}},\\
\left| \vartheta_{P',2,i}\left(\frac{p\Big(\cdot - \kappa'_m(\sigma)\Big)}{|\cdot - \kappa'_m(\sigma)|^\alpha}\right)\right| &\le C\frac{2^{-\mathbf{m}_{P',2,i}\, l(P')}}{\operatorname{dist}^{2+\mathbf{r}+\mathbf{m}_{P',2,i}}},
\end{aligned}$$

where we have used the notation $\operatorname{dist} := \operatorname{dist}(\Theta_{P'}, \Gamma_Q)$ and the estimate $\operatorname{dist} > 2^{-l(Q)}$ (cf. (4.1) and (4.2)). Hence, we arrive at

$$\begin{aligned}
&\Bigg| k\left(\vartheta_{P',1,i}, \kappa_m(\sigma), n_{\kappa_m(\sigma)}\right) \vartheta_{P',2,i}\left(\frac{p\Big(\cdot - \kappa_m(\sigma)\Big)}{|\cdot - \kappa_m(\sigma)|^\alpha}\right) \mathcal{J}_m(\sigma)\phi_{\tau,\iota}(\sigma) -\\
&\quad k\left(\vartheta_{P',1,i}, \kappa_m(\sigma), n'_{\kappa'_m(\sigma)}\right) \vartheta_{P',2,i}\left(\frac{p\Big(\cdot - \kappa'_m(\sigma)\Big)}{|\cdot - \kappa'_m(\sigma)|^\alpha}\right) \mathcal{J}'_m(\sigma)\phi_{\tau,\iota}(\sigma)\Bigg|\\
&\qquad \le C\frac{2^{-\mathbf{m}l(Q)}\; 2^{-\mathbf{m}l(P')}}{\operatorname{dist}^{2+\mathbf{r}+\mathbf{m}}}\;,
\end{aligned}$$

and the integral over T_τ of this difference is less than the right-hand side of (6.4).

On the other hand, the error of the product rule can be estimated by the supremum norm interpolation error of the integrand multiplied by the weighted measure of the integration domain. Using the smoothness assumptions on κ_m from §2.1 and on the kernel function k from §2.2 as well as the definition of κ'_m as an $\mathbf{m}+1 = 3-\mathbf{r}$ order interpolation to κ_m, we observe that the interpolation error due to the product integration is less than $2^{-(2-\mathbf{r})l(Q)}$. Note that, again, from the rate of convergence $\mathcal{O}(2^{-(3-\mathbf{r})l(Q)})$ for the approximation of the geometry a factor $2^{-l(Q)}$ is lost since the integrand contains first order derivatives. Estimating the integrals over the weight functions of the product rule with the help of (6.6), we get an upper estimate $C2^{-\mathbf{m}_{P',2,i}l(P')}2^{-2l(Q)}\operatorname{dist}^{-\mathbf{r}-2-\mathbf{m}_{P',2,i}}$ for them, and the error of the product rule is less or equal to the right-hand side of (6.4).

iv) Let us turn to the quadrature error of the n_G-th order quadrature applied to the integral over the weight function and show that this is also less than the right-hand side of (6.4). To deduce an error estimate for (4.11), we start from a univariate estimate for the Gauß rule. Let σ_G^k, ω_G^k denote the nodes and the weights of the Gauß-Legendre quadrature rule on $[0,1]$, and define

$$R_{n_G}[F] \;=\int_0^1 F(x)\,dx - \sum_{k=1}^{n_G} \omega_G^k F(\sigma_G^k).$$

It is well-known that the error bound (see, e.g., [5, p.149]; in this reference, the interval of integration is $[-1,1]$),

$$\left|R_{n_G}[F]\right| \;\le\; \frac{\pi_{n_G}}{2^{4n_G+1}(2n_G)!}\,\left\|F^{(2n_G)}\right\|_{C[0,1]}, \quad \pi_{n_G} \;:=\; \frac{2^{4n_G+1}\,n_G!^4}{(2n_G+1)\,(2n_G)!^2} \quad\le\quad \pi,$$

is best possible for every $n_G \in N$, and $\lim_{n_G\to\infty} \pi_{n_G} = \pi$. For any bivariate function $(\sigma_1^D,\sigma_2^D) \to \tilde f(\sigma_1^D,\sigma_2^D)$, we conclude (cf. Chapter 5 of [50])

$$\begin{aligned}
(6.7)\qquad &\left|\int_0^1\int_0^1 \tilde f(\sigma_1^D,\sigma_2^D)d\sigma_1^D\,d\sigma_2^D \;-\; \sum_{k_1=1}^{n_G}\sum_{k_2=1}^{n_G}\omega_G^{k_1}\omega_G^{k_2}\tilde f(\sigma_G^{k_1},\sigma_G^{k_2})\right| \\
&\le \left|\int_0^1\left(\int_0^1 \tilde f(\sigma_1^D,\sigma_2^D)\sigma_2^D - \sum_{k_2=1}^{n_G}\omega_G^{k_2}\tilde f(\sigma_1^D,\sigma_G^{k_2})\right)d\sigma_1^D\right| \\
&\quad + \left|\sum_{k_2=1}^{n_G}\omega_G^{k_2}\left(\int_0^1 \tilde f(\sigma_1^D,\sigma_G^{k_2})d\sigma_1^D - \sum_{k_1=1}^{n_G}\omega_G^{k_1}\tilde f(\sigma_G^{k_1},\sigma_G^{k_2})\right)\right| \\
&\le \sup_{\sigma_1^D\in[0,1]}\left|R_{n_G}[\tilde f(\sigma_1^D,\cdot)]\right| + \sup_{\sigma_2^D\in[0,1]}\left|R_{n_G}[\tilde f(\cdot,\sigma_2^D)]\right| \\
&\le \frac{\pi}{2^{4n_G+1}(2n_G)!}\left\{\sup_{[0,1]^2}\left|\partial^{2n_G}_{\sigma_1^D}\tilde f\right| + \sup_{[0,1]^2}\left|\partial^{2n_G}_{\sigma_2^D}\tilde f\right|\right\}.
\end{aligned}$$

In particular, setting $\tilde f(\sigma_1^D,\sigma_2^D) := 2|T_\tau| f(\tau_3+\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3))\sigma_1^D$, the rule (4.11) applied to function f is the tensor product Gauß rule applied to $\tilde f$, and we get

$$\left|\int_{T_\tau} f - \sum_{k=1}^{n_G^2} f(\sigma_\tau^k)\omega_\tau^k\right| \le \frac{\pi}{2^{4n_G+1}(2n_G)!}\left\{\sup\left|\partial^{2n_G}_{\sigma_1^D}\tilde f\right| + \sup\left|\partial^{2n_G}_{\sigma_2^D}\tilde f\right|\right\},$$

$$\begin{aligned}
\partial^{2n_G}_{\sigma_2^D}\tilde f(\sigma^D) &= 2|T_\tau|\,\partial^{2n_G}_{\sigma^+} f\left(\tau_3+\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\right)\sigma_1^D\left[\sigma_1^D(\tau_2-\tau_3)\right]^{2n_G}, \\
\partial^{2n_G}_{\sigma_1^D}\tilde f(\sigma^D) &= 2|T_\tau|\,\partial^{2n_G}_{\sigma^\dagger} f\left(\tau_3+\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\right)\sigma_1^D\;\cdot \\
&\qquad\qquad \left[(\tau_1-\tau_3)+\sigma_2^D(\tau_2-\tau_3)\right]^{2n_G} \\
&\quad +2n_G\cdot 2|T_\tau|\,\partial^{2n_G-1}_{\sigma^\dagger} f\left(\tau_3+\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\right)\;\cdot \\
&\qquad\qquad \left[(\tau_1-\tau_3)+\sigma_2^D(\tau_2-\tau_3)\right]^{2n_G-1},
\end{aligned}$$

where ∂_{σ^+} and $\partial_{\sigma^\dagger}$ stand for the derivatives in the directions of $(\tau_2-\tau_3)/|\tau_2-\tau_3|$ and

$$\frac{(\tau_1-\tau_3)+\sigma_2^D(\tau_2-\tau_3)}{\left|(\tau_1-\tau_3)+\sigma_2^D(\tau_2-\tau_3)\right|},$$

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respectively. Hence, using the relations $|\tau_2 - \tau_3| \sim 2^{-l(Q)}$ and $|(\tau_1 - \tau_3) + \sigma_2^D(\tau_2 - \tau_3)| \sim 2^{-l(Q)}$, we conclude

$$\left| \int_{T_\tau} f - \sum_{k=1}^{n_G^2} f(\sigma_\tau^k)\omega_\tau^k \right|$$
$$\leq C \frac{\pi}{2^{4n_G+1}(2n_G)!}\, 2n_G\, 2|T_\tau| \sup_{\substack{n=2n_G-1,2n_G \\ \tilde\sigma=\sigma^+,\sigma^\dagger}} \left[\sup_{T_\tau} |\partial_{\tilde\sigma}^n f|\ 2^{-nl(Q)} \right].$$

Now, consider the weight function to which we apply the tensor product Gauß rule, i.e., we consider (cf. Remark 4.1)

$$f(\sigma) := \tilde\phi_{Q,\upsilon}(\sigma)\, \vartheta_{P',2,i} \left(\frac{p\Big(\cdot - \kappa_m'(\sigma)\Big)}{|\cdot - \kappa_m'(\sigma)|^\alpha} \right) \tilde\phi_{Q,\iota}(\sigma). \tag{6.8}$$

We shall show next that the directional derivative of order n to f is less than the expression $C2^{-\mathbf{m}_{P',2,i}l(P')}2^{2l(Q)}[\varepsilon\mathrm{dist}]^{-\mathbf{r}-\mathbf{m}_{P',2,i}-n}$ including a small fixed constant $\varepsilon > 0$. Using $2^{-l(Q)} \leq \mathrm{dist}$ (cf. (4.1) and (4.2)), we arrive at a quadrature error of at most

$$\frac{C}{2^{4n_G+1}(2n_G)!}\, 2n_G\, 2^{-2l(Q)}\, 2^{-\mathbf{m}_{P',2,i}l(P')}2^{2l(Q)}[\varepsilon\mathrm{dist}]^{-\mathbf{r}-\mathbf{m}_{P',2,i}-(2n_G-1)}\, 2^{-(2n_G-1)l(Q)}.$$

The last expression multiplied by the bound $C2^{-\mathbf{m}_{P',1,i}l(P')}$ resulting from the integrand factor $k(\vartheta_{P',1,i}, \kappa_m(\sigma), n'_{\kappa_m'(\sigma)})$ is less than the right-hand side of (6.4) if

$$2^{4n_G}(2n_G-1)! \left[\frac{\varepsilon\mathrm{dist}}{2^{-l(Q)}} \right]^{2n_G-3} \geq C2^{(2-\mathbf{r})l(Q)}.$$

Passing to the logarithms and using Stirling's formula for the logarithm of $(2n_G-1)!$, we get the sufficient condition

$$\begin{aligned} \log 2 \cdot 4n_G \ + \left(2n_G - \frac{1}{2}\right) \log(2n_G-1) - (2n_G-1) + (2n_G-3)\log\varepsilon \\ +(2n_G-3)\log\left[\frac{\mathrm{dist}}{2^{-l(Q)}}\right] \geq \log 2 \left\{ C + (2-\mathbf{r})l(Q) \right\}. \end{aligned} \tag{6.9}$$

Choosing n_A sufficiently large in (4.13), the Gauß order n_G is large and we can replace the first part

$$\log 2 \cdot 4n_G \ + \left(2n_G - \frac{1}{2}\right) \log(2n_G-1) - (2n_G-1) + (2n_G-3)\log\varepsilon$$

on the left-hand side of (6.9) by the smaller term $(2n_G-3)\log 2$. This leads to the sufficient condition

$$(2n_G-3)\left\{ 1 + {}^2\!\log\left[\frac{\mathrm{dist}}{2^{-l(Q)}}\right] \right\} \geq C + (2-\mathbf{r})l(Q). \tag{6.10}$$

In other words, choosing n_A sufficiently large and setting $n_B = 1 - \mathbf{r}/2$ in (4.13), the number n_G fulfills (6.10), and the estimate (6.4) is proved if only the upper estimate for the derivative to the function in (6.8) holds.

v) Let us show the estimate $C2^{-\mathbf{m}_{P',2,i}l(P')}2^{2l(Q)}[\varepsilon\text{dist}]^{-\mathbf{r}-\mathbf{m}_{P',2,i}-n}$ for the n-th order derivative of the function in (6.8). To simplify the notation we prove the estimate for the directional derivatives only for the partial derivative with respect to the coordinate t_1 of $\sigma = (t_1, t_2) \in T_\tau$. Clearly, due to the linearity, the absolute value of a j-th order derivative of $\tilde{\phi}_{Q,\iota}$ with $Q \in Qua_l^\Gamma$ is bounded by $C2^{lj}$ for $j = 0, 1$, and is zero for $j > 1$. To show the uniform boundedness of the derivatives to $\sigma \mapsto \vartheta_{P',2,i}(p(\cdot - \kappa_m'(\sigma)) \mid \cdot - \kappa_m'(\sigma)|^{-\alpha})$, we fix a t_2 and consider the function

$$I \ni t_1 \mapsto \frac{p\left(P_\lambda - \kappa_m'(t_1,t_2)\right)}{|\,P_\lambda - \kappa_m'(t_1,t_2)|^\alpha} \quad =: \frac{p\Big(p_2(t_1)\Big)}{|p_2(t_1)|^\alpha}, \quad I \;:=\; \{t_1 :\; (t_1,t_2) \in T_\tau\} \tag{6.11}$$

and its extension to the complex plane. We fix a point $t_I \in I$. For the polynomial p_2 of degree $\deg(p_2)$ less than or equal to the degree $2 - \mathbf{r}$ of the interpolation, the standard estimates for interpolation imply

$$\left(\frac{\partial}{\partial t_1}\right)^k \Big(P_\lambda - \kappa_m'(t_I,t_2)\Big) \sim \left(\frac{\partial}{\partial t_1}\right)^k \Big(P_\lambda - \kappa_m(t_I,t_2)\Big), \quad k = 0,1,\ldots,\deg(p_2),$$

$$\left|\left(\frac{\partial}{\partial t_1}\right)^k p_2(t_I)\right| \sim \begin{cases} |P_\lambda - \kappa_m(t_I,t_2)| & \text{if } k = 0 \\ \left|\left(\frac{\partial}{\partial t_1}\right)^k \kappa_m(t_I,t_2)\right| & \text{if } k = 1,\ldots,\deg(p_2) \end{cases}$$

$$\sim \begin{cases} \text{dist} & \text{if } k = 0 \\ C & \text{if } k = 1,\ldots,\deg(p_2). \end{cases}$$

Consequently, for any complex t_1 with $\text{dist}(t_1, I) \le \varepsilon\text{dist}$ and with a constant $\varepsilon > 0$ sufficiently small, we get

$$p_2(t_1) \;=\; \sum_{k=0}^{\deg(p_2)} \frac{\partial_{t_1}^k p_2(t_I)}{k!}(t_1 - t_I)^k,$$

$$|p_2(t_1)| \ge |p_2(t_I)| - \sum_{k=1}^{\deg(p_2)} \frac{\left|\partial_{t_1}^k p_2(t_I)\right|}{k!}\,|t_1 - t_I|^k \;\ge\; \frac{1}{C}\text{dist} - \mathcal{O}(\varepsilon\text{dist}) \;\ge\; \frac{1}{2C}\text{dist},$$

as well as $|p_2(t_1)| \le C\text{dist}$. In other words, the function $p(p_2(t_1))|p_2(t_1)|^{-\alpha}$ is analytic for t_1 with $\text{dist}(t_1, I) < \varepsilon\text{dist}$, and, using the estimate $p(p_2(t_1)) \le \text{dist}^{\deg(p)}$, we conclude

$$\left|\frac{p\Big(p_2(t_1)\Big)}{|p_2(t_1)|^\alpha}\right| \le C\,\text{dist}^{-2-r}. \tag{6.12}$$

If we apply the functional $\vartheta_{P',2,i}$ to $p(\cdot - \kappa_m'(\sigma)) \mid \cdot - \kappa_m'(\sigma)|^{-\alpha}$, then we apply a difference formula with a scaling factor of order $\sim 2^{-l(P')\mathbf{m}_{P',2,i}}$. Since the difference scheme can be represented as a derivative taken at an intermediate point, we can write the function $\vartheta_{P',2,i}(p(\cdot - \kappa_m'(\sigma)) \mid \cdot - \kappa_m'(\sigma)|^{-\alpha})$ as a sum of functions similar to that in (6.11). Analogous to (6.12), we arrive at the estimate

$$\left|\vartheta_{P',2,i}\left(\frac{p\Big(\,\cdot - \kappa_m'(t_1,t_2)\Big)}{|\cdot - \kappa_m'(t_1,t_2)|^\alpha}\right)\right| \le C\;2^{-l(P')\mathbf{m}_{P',2,i}}\;\text{dist}^{-2-r-\mathbf{m}_{P',2,i}} \tag{6.13}$$

valid for the complex extension to all t_1 with $\mathrm{dist}(t_1, I) < \varepsilon\mathrm{dist}$. Now, we represent the analytic function by Cauchy's integral over a closed contour C around I with distance $\varepsilon\mathrm{dist}$ to I, i.e., by

$$\vartheta_{P',2,i}\left(\frac{p\,(\cdot - \kappa'_m(t_1,t_2))}{|\cdot - \kappa'_m(t_1,t_2)|^\alpha}\right) = \frac{1}{2\pi\mathbf{i}}\int_C \left\{\vartheta_{P',2,i}\left(\frac{p\,(\cdot - \kappa'_m(t,t_2))}{|\cdot - \kappa'_m(t,t_2)|}\right)\right\}\frac{1}{t-t_1}\,\mathrm{d}t\;.$$

Differentiating this equation with respect to t_1, restricting t_1 to I, and using (6.13), we get

$$\begin{aligned}&\left|\frac{\partial^k}{\partial t_1^k}\vartheta_{P',2,i}\left(\frac{p\,(\cdot - \kappa'_m(t_1,t_2))}{|\cdot - \kappa'_m(t_1,t_2)|^\alpha}\right)\right|\\ &\le C\,2^{-l(P')\mathbf{m}_{P',2,i}}\,[\varepsilon\mathrm{dist}]^{-2-r-\mathbf{m}_{P',2,i}-k},\quad (t_1,t_2)\in T_\tau.\end{aligned}$$

This together with the estimate $C\,2^{l(Q)j}$ for the j-th derivatives of the functions $\phi_{Q,\,\iota}$ and $\phi_{Q,\,\upsilon}$, and with $\mathrm{dist}^{-1} \le 2^{l(Q)}$ (cf. (4.1) and (4.2)) proves that the n-th order derivatives of the function f in (6.8) are indeed less than

$$C2^{-\mathbf{m}_{P',2,i}l(P')}2^{2l(Q)}[\varepsilon\mathrm{dist}]^{-\mathbf{r}-\mathbf{m}_{P',2,i}-n}.\ \square$$

LEMMA 6.2. *The number of necessary arithmetic operations for setting up the far field part of the stiffness matrix $A_L^{w,c,q}$, including the sparsity pattern $\mathcal{P}$, is less than $CL^4 2^{2L}$.*

Proof. Clearly, if the test functional $\vartheta_{P'}$ and the domain of integration Γ_Q is fixed, then the number of operations is less than a constant multiple of the number of quadrature knots plus the number of trial functions ψ_P with $\Gamma_Q \subseteq \Psi_P$. Thus, for fixed $\vartheta_{P'}$ and Γ_Q, no more than $C\,L^2$ operations are needed. The number of all arithmetic operations for the quadrature is less than $C\,L^2$ times $\sum_{P'}\sum_l \#Qua_l^\Gamma$, where $\#Qua_l^\Gamma$ is the number of domains Γ_Q in Qua_l^Γ. We only have to count the number of domains Γ_Q in Qua_l^Γ. The estimates (4.1) (cf. (3.10)) and (4.2) (cf. (3.11)) together with the proof of the complexity bound in Theorem 3.1 (cf. [15, 34, 47, 40]) lead to the bound $Cd^2L^2 2^{2L}$ for $\sum_{P'}\sum_l \#Qua_l^\Gamma$. This implies our assertion. $\square$

6.2. The estimates for the first part of the nonsingular near field. In this subsection we suppose that the far field integration and the integration of the singular integrals are performed exactly and derive the convergence estimates for the nonsingular near field case. In view of Remark 5.1 it remains to prove

LEMMA 6.3. *Suppose that $A_L^c \in \mathcal{L}(Lin_L^\Gamma)$ is the approximate operator of the compressed collocation method including the sparsity pattern $\mathcal{P}$ and that $A_L^{c,q}$ is the operator of the compressed collocation method including the approximation of the parameter mappings and the quadrature of §4.2. Then we get the estimate (5.11) with $\kappa = 1 + \mathbf{r}$.*

Proof. **i)** For the interpolation and quadrature, we shall prove the error estimate

$$\begin{aligned}|a_{P',P}| &= |a^{w,c}_{P',P} - a^{w,c,q}_{P',P}|\\ &\le C2^{-\mathbf{m}L}\int_{\mathrm{supp}\,\psi_P}\left[2^{-L} + \mathrm{dist}\,(\mathrm{supp}\,\vartheta_{P'},R)\right]^{-\mathbf{r}-2}\mathrm{d}_R\Gamma.\end{aligned}\tag{6.14}$$

Substituting the estimate (6.14) into the l^∞ matrix norm in (5.13), we get

$$\begin{aligned}\sup_{P'\in\triangle_L^\Gamma}\sum_{P\in\triangle_L^\Gamma}\frac{|a_{P',P}|}{2^{0.05\,l(P)}} \le \sup_{P'\in\triangle_L^\Gamma}\sum_{l(P)=-1}^{L-1}2^{-0.05\,l(P)}\sum_{P\in\nabla_{l(P)}^\Gamma}C2^{-\mathbf{m}L}\times\\ \int_{\mathrm{supp}\,\psi_P}\left[2^{-L}+\mathrm{dist}\,(\mathrm{supp}\,\vartheta_{P'},R)\right]^{-\mathbf{r}-2}\mathrm{d}_R\Gamma.\end{aligned}$$

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$$\leq C 2^{-\mathbf{m}L} \sup_{P' \in \triangle_L^\Gamma} \sum_{l(P)=-1}^{L-1} 2^{-0.05\, l(P)} \, .$$
$$\int_\Gamma \Big[2^{-L} + \mathrm{dist}\,(\mathrm{supp}\, \vartheta_{P'}, R) \Big]^{-\mathbf{r}-2} \mathrm{d}_R\Gamma$$
$$\leq C L^{\delta_{\mathbf{r},0}} 2^{-\mathbf{m}L}.$$

It remains to prove the basic estimate (6.14).

ii) Clearly, for the Γ_Q which are disjoint to $\Theta_{P'}$, we get the estimate (6.4). In the general case including the case $\Gamma_Q \cap \Theta_{P'} \neq \emptyset$ but with Γ_Q disjoint to the finite set of points $\mathrm{supp}\, \vartheta_{P'}$, we only have

$$\tilde{a}_{P',\,(Q,\,\iota)} \leq C 2^{-(4-\mathbf{r})l(Q)} \Big[2^{-L} + \mathrm{dist}\,(\mathrm{supp}\, \vartheta_{P'}, \Gamma_Q) \Big]^{-\mathbf{r}-2} . \tag{6.15}$$

Here we have $l(Q) = L$ for the near field. Hence, we arrive at

$$|a_{P',P}| \leq C \sum_{\Gamma_Q \subseteq \mathrm{supp}\, \psi_P :\, \Gamma_Q \cap \mathrm{supp}\, \vartheta_{P'} \neq \emptyset} 2^{-(4-\mathbf{r})L} \Big[2^{-L} + \mathrm{dist}\,(\mathrm{supp}\, \vartheta_{P'}, \Gamma_Q) \Big]^{-\mathbf{r}-2}$$
$$\leq C 2^{-\mathbf{m}L} \int_{\mathrm{supp}\, \psi_P} \Big[2^{-L} + \mathrm{dist}\,(\mathrm{supp}\, \vartheta_{P'}, R) \Big]^{-\mathbf{r}-2} \mathrm{d}_R\Gamma. \ \square$$

LEMMA 6.4. *The number of necessary arithmetic operations for setting up the near field part of the stiffness matrix $A_L^{w,c,q}$ treated in §4.2 is less than $CL^4 2^{2L}$.*

Proof. Similar to §6.1, the number of operations is less than $C\, L^2$ times the number of domains Γ_Q in Qua_L^Γ. Thus, we only have to count the number of domains Γ_Q in Qua_L^Γ. In view of (4.1) and (4.2), the proof of Theorem 3.1 (cf. [15, 34, 47, 40]) implies our assertion. $\square$

6.3. The estimates for the singular near field. In this subsection we suppose that the far field integration and the integration of the nonsingular near field integrals are performed exactly and derive the convergence estimates for the singular near field case. In view of Remark 5.1, it remains to prove

LEMMA 6.5. *Suppose $A_L^c \in \mathcal{L}(Lin_L^\Gamma)$ is the approximate operator of the compressed collocation method including the sparsity pattern $\mathcal{P}$ (cf. §3.5). If $A_L^{c,q}$ is the operator of the compressed collocation method including the approximation of the parameter mappings and the quadrature of §4.3, then, for $\mathbf{r} = -1$ and for the case of $\mathbf{r} = 0$ with weakly singular kernels of the form (4.17), the estimates (5.11) hold with $\kappa = 1$. For the strongly singular case, (5.11) holds with $\kappa = 2$.*

Proof. **i)** Without loss of generality we suppose $\tau_3 = 0$ and $P_\lambda = \kappa_m(0)$ in the formulae of §4.3. We derive the analogue to (6.15) which takes the form $\tilde{a}_{P',\,(Q,\,\iota)} \leq 2^{-[2-2\mathbf{r}]L}$. First, we consider the case of weakly singular integrals and consider the error for fixed $\vartheta_{P'}$, fixed $P_\lambda \in \mathrm{supp}\, \vartheta_{P'}$, and fixed (Q,ι) with $P_\lambda \in \Gamma_Q$ and $Q \in \square_L^\Gamma$, i.e., we consider the error for the integral in (4.19) with $\tilde{\psi}_P^D$ replaced by $\tilde{\Phi}_{Q,\,\iota} := \phi_{Q,\,\iota} \circ \tilde{\kappa}_m$ (cf. Remark 4.1). We shall show that the error of approximation is less than $\mathcal{O}(2^{-\mathbf{m}L})$. To this end we consider the errors due to the approximation of κ_m, due to the product integration, and due to the approximation of the quadrature weights separately.

ii) To estimate the error due to the replacement of κ_m by κ_m' in this integral, we need a few technical inequalities (cf. the subsequent formulae (6.16)-(6.26)). We observe

$$\tilde{\kappa}_m(\sigma^D) - \tilde{\kappa}_m(0) \tag{6.16}$$

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$$
\begin{aligned}
&= \int_0^1 \nabla\tilde\kappa_m(\lambda\sigma^D)\,\mathrm{d}\lambda\cdot\sigma^D\\
&= \int_0^1 \Bigg\{\nabla\kappa_m\Big(\lambda\sigma_1^D(\tau_1-\tau_3)+\lambda^2\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\cdot\\
&\qquad\qquad \Big((\tau_1-\tau_3)+\lambda\sigma_2^D(\tau_2-\tau_3)\,,\ \lambda\sigma_1^D(\tau_2-\tau_3)\Big)\Bigg\}\,\mathrm{d}\lambda\cdot\begin{pmatrix}\sigma_1^D\\ \sigma_2^D\end{pmatrix}\\
&= \int_0^1 \Bigg\{\nabla\kappa_m\Big(\lambda\sigma_1^D(\tau_1-\tau_3)+\lambda^2\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\cdot\\
&\qquad\qquad \Big((\tau_1-\tau_3)+2\lambda\sigma_2^D(\tau_2-\tau_3)\Big)\Bigg\}\,\mathrm{d}\lambda\,\sigma_1^D.
\end{aligned}
$$

This and the corresponding relation for $\tilde\kappa_m$ replaced by $\tilde\kappa_m'$ imply

$$
\big|\,\tilde\kappa_m(\sigma^D)-\tilde\kappa_m(0)\,\big|\sim 2^{-L}\sigma_1^D,\ \big|\,\tilde\kappa_m'(\sigma^D)-\tilde\kappa_m'(0)\,\big|\sim 2^{-L}\sigma_1^D, \tag{6.17}
$$

$$
\begin{aligned}
\big|\,\tilde p\left(\tilde\kappa_m(0)-\tilde\kappa_m(\sigma^D)\right)\big| &\sim \left[2^{-L}\sigma_1^D\right]^{\deg(\tilde p)},\\
\big|\,\tilde p\left(\tilde\kappa_m'(0)-\tilde\kappa_m'(\sigma^D)\right)\big| &\sim \left[2^{-L}\sigma_1^D\right]^{\deg(\tilde p)}.
\end{aligned} \tag{6.18}
$$

By assumption, we get that $\mathcal{J}_m\circ\delta$ and k are bounded. Since κ_m' approximates κ_m over T_τ with order $\mathbf{m}+1$ and since the gradient $\nabla\kappa_m'$ approximates $\nabla\kappa_m$ over T_τ with order $\mathbf{m}=2-\mathbf{r}$, formula (6.16) leads us to

$$
\begin{aligned}
\big|\,\tilde\kappa_m(\sigma^D)-\tilde\kappa_m'(\sigma^D)\,\big| \le C\,2^{-(3-\mathbf{r})L}\sigma_1^D,\ \Big|\,\mathcal{J}_m\Big(\delta(\sigma^D)\Big)-\mathcal{J}_m'\Big(\delta(\sigma^D)\Big)\Big|&\\
\le C\,2^{-(2-\mathbf{r})L},&
\end{aligned} \tag{6.19}
$$

$$
\Big|\,k(P_\lambda,\tilde\kappa_m(\sigma^D),n_{\tilde\kappa_m(\sigma^D)})-k(P_\lambda,\tilde\kappa_m(\sigma^D),n'_{\tilde\kappa_m'(\sigma^D)})\Big|\le C\,2^{-(2-\mathbf{r})L}. \tag{6.20}
$$

Moreover, from (6.17) and (6.19) it is not hard to conclude that

$$
\begin{aligned}
\big|\,\tilde p\left(\tilde\kappa_m(0)-\tilde\kappa_m(\sigma^D)\right)-\tilde p\left(\tilde\kappa_m'(0)-\tilde\kappa_m'(\sigma^D)\right)\big| &\le C\,2^{-(3-\mathbf{r})L}\sigma_1^D\left[2^{-L}\sigma_1^D\right]^{\deg(\tilde p)-1}\\
&\le C\,2^{-(2-\mathbf{r})L}\left[2^{-L}\sigma_1^D\right]^{\deg(\tilde p)},
\end{aligned} \tag{6.21}
$$

$$
\begin{aligned}
\Big|\,\big|\tilde\kappa_m(0)-\tilde\kappa_m(\sigma^D)\big|^{-\alpha}-\big|\tilde\kappa_m'(0)-\tilde\kappa_m'(\sigma^D)\big|^{-\alpha}\Big| &\le C\,2^{-(3-\mathbf{r})L}\sigma_1^D\left[2^{-L}\sigma_1^D\right]^{-\alpha-1}\\
&\le C\,2^{-(2-\mathbf{r})L}\left[2^{-L}\sigma_1^D\right]^{-\alpha}.
\end{aligned} \tag{6.22}
$$

To estimate $n_{\tilde\kappa_m(\sigma^D)}\cdot(\tilde\kappa_m(0)-\tilde\kappa_m(\sigma^D))$, we observe that $n_{\tilde\kappa_m(\sigma^D)}\cdot\nabla\kappa_m(\delta(\sigma^D))=0$, and that equation (6.16) leads us to

$$
\begin{aligned}
&n_{\tilde\kappa_m(\sigma^D)}\cdot\Big(\tilde\kappa_m(\sigma^D)-\tilde\kappa_m(0)\Big)\\
&= n_{\tilde\kappa_m(\sigma^D)}\cdot\int_0^1\Bigg\{\Bigg[\nabla\kappa_m\Big(\lambda\sigma_1^D(\tau_1-\tau_3)+\lambda^2\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\\
&\qquad\qquad -\nabla\kappa_m\Big(\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\Bigg]\\
&\qquad\qquad\qquad \cdot\Big((\tau_1-\tau_3)+2\lambda\sigma_2^D(\tau_2-\tau_3)\Big)\Bigg\}\,\mathrm{d}\lambda\,\sigma_1^D.
\end{aligned} \tag{6.23}
$$

Analogous to (6.16), we write

$$\begin{aligned}
&\nabla\kappa_m\Big(\lambda\sigma_1^D(\tau_1-\tau_3)+\lambda^2\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)-\nabla\kappa_m\Big(\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\\
&=\int_0^1\nabla^2\kappa_m\Big([1+\mu(\lambda-1)]\sigma_1^D(\tau_1-\tau_3)+[1+\mu(\lambda^2-1)]\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\,\mathrm{d}\mu\\
&\quad\cdot\big[(\lambda-1)(\tau_1-\tau_3)+(\lambda^2-1)\sigma_2^D(\tau_2-\tau_3)\big]\cdot\sigma_1^D,
\end{aligned} \tag{6.24}$$

and, from inserting this into the representation of $n_{\tilde\kappa_m(\sigma^D)}\cdot(\tilde\kappa_m(0)-\tilde\kappa_m(\sigma^D))$ as well as from the analogous formula for the expression $n'_{\tilde\kappa'_m(\sigma^D)}\cdot(\tilde\kappa'_m(0)-\tilde\kappa'_m(\sigma^D))$, we obtain

$$\left|n_{\tilde\kappa_m(\sigma^D)}\cdot\Big(\tilde\kappa_m(0)-\tilde\kappa_m(\sigma^D)\Big)\right|\le C\,\left[2^{-L}\sigma_1^D\right]^2, \tag{6.25}$$

$$\begin{aligned}
&\left|n_{\tilde\kappa_m(\sigma^D)}\cdot\Big(\tilde\kappa_m(0)-\tilde\kappa_m(\sigma^D)\Big)-n'_{\tilde\kappa'_m(\sigma^D)}\cdot\Big(\tilde\kappa'_m(0)-\tilde\kappa'_m(\sigma^D)\Big)\right|\\
&\qquad\le C\,2^{-(1-\mathbf{r})L}\left[2^{-L}\sigma_1^D\right]^2.
\end{aligned} \tag{6.26}$$

Now, using (6.16)-(6.26), the error due to the replacement of κ_m by κ'_m can be represented as the sum of the errors corresponding to the replacements in the several factors of the integrand in (4.19). These factors are $\tilde k(P_\lambda,\tilde\kappa_m(\sigma^D),n_{\tilde\kappa_m(\sigma^D)})$, $\tilde p(\tilde\kappa_m(0)-\tilde\kappa_m(\sigma^D))$, $|\tilde\kappa_m(0)-\tilde\kappa_m(\sigma^D)|^{-\alpha}$, $[n_{\tilde\kappa_m(\sigma^D)}\cdot(\tilde\kappa_m(0)-\tilde\kappa_m(\sigma^D))]^{1+\mathbf{r}}$, and $\mathcal{J}_m(\delta(\sigma^D))$, respectively. The last factor $\mathcal{J}_\delta(\sigma^D)\tilde\Phi_{Q,\iota}(\sigma^D)$ needs no replacement of κ_m. We arrive at the estimate

$$\begin{aligned}
&C\int_0^1\int_0^1\Bigg\{\Big[2^{-(2-\mathbf{r})L}\,[2^{-L}\sigma_1^D]^{\deg(\tilde p)}\,[2^{-L}\sigma_1^D]^{-\alpha}\,[2^{-L}\sigma_1^D]^{2(1+\mathbf{r})}\,C\Big]\\
&\quad+\Big[C\,2^{-(2-\mathbf{r})L}[2^{-L}\sigma_1^D]^{\deg(\tilde p)}\,[2^{-L}\sigma_1^D]^{-\alpha}\,[2^{-L}\sigma_1^D]^{2(1+\mathbf{r})}\,C\Big]\\
&\quad+\Big[C\,[2^{-L}\sigma_1^D]^{\deg(\tilde p)}\,2^{-(2-\mathbf{r})L}[2^{-L}\sigma_1^D]^{-\alpha}\,[2^{-L}\sigma_1^D]^{2(1+\mathbf{r})}\,C\Big]\\
&\quad+\Big[C\,[2^{-L}\sigma_1^D]^{\deg(\tilde p)}\,[2^{-L}\sigma_1^D]^{-\alpha}\,[2^{-(1-\mathbf{r})L}]^{1+\mathbf{r}}[2^{-L}\sigma_1^D]^{2(1+\mathbf{r})}\,C\Big]\cdot\delta_{\mathbf{r},0}\\
&\quad+\Big[C\,[2^{-L}\sigma_1^D]^{\deg(\tilde p)}\,[2^{-L}\sigma_1^D]^{-\alpha}\,[2^{-L}\sigma_1^D]^{2(1+\mathbf{r})}\,2^{-(2-\mathbf{r})L}\Big]\Bigg\}2^{-2L}\,\sigma_1^D\,\mathrm{d}\sigma_2^D\,\mathrm{d}\sigma_1^D\\
&\le C\begin{cases}2^{-4L} & \text{if } \mathbf{r}=-1\\ 2^{-2L} & \text{if } \mathbf{r}=0.\end{cases}
\end{aligned} \tag{6.27}$$

This completes the estimate for the first step in approximating the integral.

iii) The second step is the product integration of order $\mathbf{m}=2-\mathbf{r}$. Analogous to the derivation of (6.27) from (6.16)-(6.26), we conclude that the integral over the weight function $\tilde\phi_r^D\tilde p|\ldots|^{-\alpha}[\ldots]^{1+\mathbf{r}}J_\delta\tilde\phi_{Q,\iota}$ is less than 2^{-L}. Hence, it remains to estimate the interpolation error for the $\mathbf{m}$-th order interpolation which defines the product rule. Clearly, the interpolation error is less than a constant times the supremum of the derivatives to the integrand function $\tilde k(P_\lambda,\tilde\kappa_m(\sigma^D),n'_{\tilde\kappa'_m(\sigma^D)})\;\mathcal{J}'_m(\sigma^D)$ if the derivatives are taken with respect to σ_1^D or σ_2^D up to the $\mathbf{m}$-th order. Since our product rule relies upon tensor product interpolation, mixed derivatives need not to be considered. The integrand is a composite function of the outer functions $\tilde k$, κ'_m, and $\mathcal{J}'_m$ and of the inner function δ. By assumption (cf. §2.1 and §2.2)

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the corresponding derivatives of κ'_m, $\mathcal{J}'_m$, and $\tilde{k}$ do exist and they are uniformly bounded. For the inner function δ, each order of derivative with respect to σ_1^D and σ_2^D brings a factor $(\tau_1-\tau_3)+\sigma_2^D(\tau_2-\tau_3)$ and $\sigma_1^D(\tau_2-\tau_3)$, respectively. Thus the derivatives of order $\mathbf{m}$ are less than $2^{-\mathbf{m}L}$, and the estimate on the right-hand side of (6.27) is an upper bound also for the error of product integration in the second step of approximation. We even get the better bound 2^{-3L} for $\mathbf{r}=0$.

iv) To analyze the third step, we introduce the notation

$$H(\lambda,\mu) := \lambda(\tau_1-\tau_3)+\mu(\tau_2-\tau_3), \quad \tilde{H}(\lambda,\mu) \ := \ \lambda\frac{\tau_1-\tau_3}{|\tau_1-\tau_3|}+\mu\frac{\tau_2-\tau_3}{|\tau_1-\tau_3|}.$$

In this last step an n_G-th order rule is applied to the integral of the weight function from the previous step, i.e., to

$$\begin{aligned}
&\int_0^1\int_0^1\Bigg\{\ \tilde{\phi}^D_{Q,\,\upsilon}(\sigma^D)\frac{\tilde{p}\Big(\tilde{\kappa}'_m(0)-\tilde{\kappa}'_m(\sigma^D)\Big)}{|\tilde{\kappa}'_m(0)-\tilde{\kappa}'_m(\sigma^D)|^\alpha}\Big[n_{\tilde{\kappa}'_m(\sigma^D)}\cdot\big(\tilde{\kappa}'_m(0)-\tilde{\kappa}'_m(\sigma^D)\big)\Big]^{1+\mathbf{r}}\cdot \\
&\qquad\qquad\qquad\qquad \mathcal{J}_\delta(\sigma^D)\tilde{\Phi}_{Q,\,\iota}(\sigma^D)\ \Bigg\}\,\mathrm{d}\sigma_1^D\,\mathrm{d}\sigma_2^D \\
&=\int_0^1\int_0^1\Bigg\{\ \tilde{\phi}^D_{Q,\,\upsilon}(\sigma^D)\frac{\tilde{p}\Big(\int_0^1\nabla\kappa'_m\left(H(\lambda\sigma_1^D,\lambda^2\sigma_1^D\sigma_2^D)\right)\cdot H(1,2\lambda\sigma_2^D)\,\mathrm{d}\lambda\Big)}{\Big|\int_0^1\nabla\kappa'_m\Big(H(\lambda\sigma_1^D,\lambda^2\sigma_1^D\sigma_2^D)\Big)\cdot H(1,2\lambda\sigma_2^D)\,\mathrm{d}\lambda\Big|^\alpha}\cdot \\
&\qquad\Bigg[n_{\tilde{\kappa}'_m(\sigma^D)}\cdot\int_0^1\Bigg\{\int_0^1\nabla^2\kappa'_m\Bigg(H\left([1+\mu(\lambda-1)]\sigma_1^D,[1+\mu(\lambda^2-1)]\sigma_1^D\sigma_2^D\right)\Bigg)\,\mathrm{d}\mu \\
&\qquad H\left(\lambda-1,(\lambda^2-1)\sigma_2^D\right)H\left(1,2\lambda\sigma_2^D\right)\Bigg\}\,\mathrm{d}\lambda\Bigg]^{1+\mathbf{r}}2|T_\tau|\tilde{\Phi}_{Q,\,\iota}(\sigma^D)\ \Bigg\}\,\mathrm{d}\sigma_1^D\,\mathrm{d}\sigma_2^D \\
&=\frac{2|T_\tau|}{|\tau_1-\tau_3|}\int_0^1\int_0^1\Bigg\{\ \tilde{\phi}^D_{Q,\,\upsilon}(\sigma^D)\frac{\tilde{p}\Big(\int_0^1\nabla\kappa'_m\left(H(\lambda\sigma_1^D,\lambda^2\sigma_1^D\sigma_2^D)\right)\cdot\tilde{H}(1,2\lambda\sigma_2^D)\,\mathrm{d}\lambda\Big)}{\Big|\int_0^1\nabla\kappa'_m\Big(H(\lambda\sigma_1^D,\lambda^2\sigma_1^D\sigma_2^D)\Big)\cdot\tilde{H}(1,2\lambda\sigma_2^D)\,\mathrm{d}\lambda\Big|^\alpha}\cdot \\
&\qquad\Bigg[n_{\tilde{\kappa}'_m(\sigma^D)}\cdot\int_0^1\Bigg\{\int_0^1\nabla^2\kappa'_m\Bigg(H\left([1+\mu(\lambda-1)]\sigma_1^D,[1+\mu(\lambda^2-1)]\sigma_1^D\sigma_2^D\right)\Bigg)\,\mathrm{d}\mu \\
&\qquad\tilde{H}\left(\lambda-1,(\lambda^2-1)\sigma_2^D\right)\tilde{H}\left(1,2\lambda\sigma_2^D\right)\Bigg\}\,\mathrm{d}\lambda\Bigg]^{1+\mathbf{r}}\tilde{\Phi}_{Q,\,\iota}(\sigma^D)\ \Bigg\}\,\mathrm{d}\sigma_1^D\,\mathrm{d}\sigma_2^D,
\end{aligned}\tag{6.28}$$

where the equalities $\mathcal{J}_\delta(\sigma^D)=2|T_\tau|\sigma_1^D$, (6.16), (6.23), and (6.24) have been substituted into the first integral. The last integrand is a function which can be treated as the integrand in part v) of the proof to Lemma 6.1. Indeed, to apply (6.7), we need an estimate for the derivatives. Without loss of generality, we consider the derivative with respect to σ_1^D. For the k-th order derivatives of $\tilde{\phi}^D_{Q,\,\upsilon}$ and $\tilde{\Phi}_{Q,\iota}$, we get the bound $C2^{kL}$ if $k=0,1$ and the bound zero if $k\geq 2$. Similar to (6.11), we fix σ_2^D and set

$$\begin{aligned}
p_2(\sigma_1^D) &:= \int_0^1\nabla\kappa'_m\left(H(\lambda\sigma_1^D,\lambda^2\sigma_1^D\sigma_2^D)\right)\cdot\tilde{H}(1,2\lambda\sigma_2^D)\,\mathrm{d}\lambda, \\
p_3(\sigma_1^D) &:= \Bigg[n_{\tilde{\kappa}'_m(\sigma^D)}\cdot\int_0^1\Bigg\{\int_0^1\nabla^2\kappa'_m\Bigg(H\left([1+\mu(\lambda-1)]\sigma_1^D,[1+\mu(\lambda^2-1)]\sigma_1^D\sigma_2^D\right)\Bigg)\,\mathrm{d}\mu
\end{aligned}$$

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$$\tilde{H}\left(\lambda-1,(\lambda^2-1)\sigma_2^D\right)\tilde{H}\left(1,2\lambda\sigma_2^D\right)\Big\}\,\mathrm{d}\lambda\Big]^{1+\mathbf{r}},$$

and consider

$$[0,1]\ni\sigma_1^D \;\mapsto\; \frac{\tilde{p}(p_2(\sigma_1^D))}{|p_2(\sigma_1^D)|^\alpha}p_3(\sigma_1^D) \tag{6.29}$$

together with its extension to the complex plane. Since the parametrizations κ_m are injective mappings, we get $\|\kappa_m(\sigma)\xi\|\geq\|\xi\|,\ \forall\xi\in\mathbb{R}^2$ and

$$\begin{aligned} p_2(\tilde{\sigma}_1^D) &\sim \int_0^1 \nabla\kappa_m\left(H(\lambda\tilde{\sigma}_1^D,\lambda^2\tilde{\sigma}_1^D\sigma_2^D)\right)\cdot\tilde{H}(1,2\lambda\sigma_2^D)\,\mathrm{d}\lambda \\ &\sim \nabla\kappa_m\Big(H(0,0)\Big)\int_0^1\tilde{H}(1,2\lambda\sigma_2^D)\,\mathrm{d}\lambda, \end{aligned}$$

as well as $|p_2(\tilde{\sigma}_1^D)|\geq 1/C$ for a $\tilde{\sigma}_1^D$ such that $0\leq\tilde{\sigma}_1^D\leq 1$. On the other hand, the k-th order derivative of the interpolation κ_m' to κ_m is bounded by $C2^{kL}$ if k is less than or equal to the total degree of the polynomial κ_m', and the k-th order derivative of $H(\cdot,\cdot)$ is less than $C2^{-kL}$. Consequently, the k-th order derivative of p_2 at σ_1^D with $k\leq\deg(p_2)$ and $0\leq\sigma_1^D\leq 1$ is less than a constant. We obtain

$$\begin{aligned} p_2(\sigma_1^D) &= \sum_{k=0}^{\deg(p_2)}\frac{\partial^k_{\sigma_1^D}p_2(\tilde{\sigma}_1^D)}{k!}(\sigma_1^D-\tilde{\sigma}_1^D)^k, \\ \left|p_2(\sigma_1^D)\right| &\geq \left|p_2(\tilde{\sigma}_1^D)\right| - \sum_{k=1}^{\deg(p_2)}\frac{\left|\partial^k_{\sigma_1^D}p_2(\tilde{\sigma}_1^D)\right|}{k!}\left|\sigma_1^D-\tilde{\sigma}_1^D\right|^k \\ &\geq 1/C-\sum_{k=1}^{\deg(p_2)}C\left|\sigma_1^D-\tilde{\sigma}_1^D\right|^k, \end{aligned}$$

where $\tilde{\sigma}_1^D$ with $0\leq\tilde{\sigma}_1^D\leq 1$ can be chosen such that $|\sigma_1^D-\tilde{\sigma}_1^D|\leq\operatorname{dist}(\sigma_1^D,[0,1])$. Hence, we can take a sufficiently small $\varepsilon>0$ and observe that $|p_2(\sigma_1^D)|\geq 1/(2C)$ for any complex σ_1^D with $\operatorname{dist}(\sigma_1^D,[0,1])\leq\varepsilon$. Similarly, we obtain $|p_2(\sigma_1^D)|\leq C$ and $|p_3(\sigma_1^D)|\leq C$. Analogous to part v) of the proof to Lemma 6.1, we arrive at the estimate $C\varepsilon^{-(k+1)}$ for the k-th order derivative of (6.29) and at the bound $C2^{2L}\varepsilon^{-(2n_G-1)}$ for the $2n_G$-th order derivative of the integrand in (6.28). The estimate $C2^{-L}$ for the factor $2|T_\tau|\,|\tau_1-\tau_3|^{-1}$ and the error estimate (6.7) applied to the quadrature approximation of (6.28) yield the bound

$$C\frac{\pi}{2^{4n_G}(2n_G)!}2^{-L}2^{2L}\varepsilon^{-(2n_G-1)}\leq C2^{L-{}^2\log\varepsilon\,[2n_G-1]-4n_G+{}^2\log e\,[-(2n_G+\frac{1}{2})\log 2n_G+2n_G]}.$$

The last bound is less than $2^{-(3-\mathbf{r})L}$ if we set $n_F:=2-\mathbf{r}/2$ and choose n_E sufficiently large in $n_G=n_E+L\,n_F$. Hence, we get the estimate on the right-hand side of (6.27) for the quadrature error of the Gauß rules. We even get the better bound 2^{-3L} for $\mathbf{r}=0$.

v) Now let us estimate the entries in the case of strongly singular integral operators. We assume $\mathbf{r}=0$ and distinguish the two cases $\phi_{Q,\iota}(P_\lambda)=0$ and $\phi_{Q,\iota}(P_\lambda)\neq 0$. If $\phi_{Q,\iota}(P_\lambda)=\tilde{\Phi}_{Q,\iota}(0,0)=0$, then we can repeat the estimate from above. Indeed, the obvious estimate $|\phi_{Q,\iota}(R)|\leq C\,2^L\,|R-P_\lambda|$ provides us with a factor $|R-P_\lambda|$ which cancels one factor $|R-P_\lambda|$ from the denominator $|R-P_\lambda|^\alpha$. Though we have $\mathbf{r}=0$, there

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is no factor $n_R \cdot (R - P_\lambda)$ this time. Hence, we get the estimate $C2^{-3L}$ in (6.27) which is to be multiplied by the factor 2^L from the estimate $|\phi_{Q,\iota}(R)| \le C\,2^L\,|R - P_\lambda|$. In other words, the final estimate for the matrix entries is again $C\,2^{-2L}$.

Finally, we turn to the case $\phi_{Q,\iota}(P_\lambda) \neq 0$ and consider the error of the approximation (4.26) and (4.28). The first part of the error is due to restricting the domain of integration from T_τ to $T_\tau \setminus T'(P_\lambda, m, 2^{-2L})$. This is less than $C2^{-2L}$ by (4.22). The second part of the error is caused by the replacement of the parametrization in the kernel function. Writing the difference of (4.23) and (4.24) in Duffy's coordinates and using the equations (6.19)-(6.22) with the polynomial $\tilde{p}$ replaced by p, we obtain the bound

$$C\,2^{-2L} \int_{\delta^{-1}[T\setminus T(P_\lambda,m,2^{-2L})]} |\sigma_1^D|^{-1}\,\mathrm{d}\sigma^D \le C\,2^{-2L} \int_{2^{-2L}}^{1} |\sigma_1^D|^{-1}\,\mathrm{d}\sigma_1^D \le C\,L\,2^{-2L}. \tag{6.30}$$

By simple estimates analogous to those in [31], Chapter XI, §1, the third part of the error due to the change of the parametrization in the integration domain $T_\tau \setminus T'(P_\lambda, m, 2^{-2L})$ is less than $C2^{-2L}$. The error bound (6.30) for the fourth part due to product integration follows as in the case $\mathbf{r} = -1$. Finally, it remains to estimate the error of the tensor product Gauß rule in (4.28). This however can be treated as in parts iv) and v) of the proof to Lemma 6.1 and as in part iv) of the present proof since the ratio of the diameter of $T_{\tau,\iota}$ to the distance of $T_{\tau,\iota}$ to the singularity point τ_3 is bounded from below and since the variable integration bound $S_a(\sigma_2^D)$ for the inner integration is analytic. Indeed, the function $S_a(\sigma_2^D)$ for $\iota = \iota_0$ depending on the parameter $\varepsilon = 2^{-2L}$ (cf. (4.25)) is of the form $S_a(\sigma_2^D) = 2^{-2L}S(2^{2L}\sigma_2^D)$ with an S such that $\sigma \mapsto \delta(S(\sigma), \sigma)$ describes the boundary curve of an ellipse. The summation over all ι from one to $\iota_0 = \mathcal{O}(L)$ leads to an additional factor $C\,L$.

vi) In other words, for the algorithms (3.8) and (3.9), we have the estimate (6.15) with $\mathrm{dist}(\mathrm{supp}\,\vartheta_{P'}, \Gamma_Q)$ replaced by 2^{-L}, i.e., the same estimate like for the almost singular entries with $\mathrm{dist}(\mathrm{supp}\,\vartheta_{P'}, \Gamma_Q) \sim 2^{-L}$. Only for the strong singular case we have an additional factor L. Hence, the proof to Lemma 6.3 completes the proof of the corresponding assertions of Lemma 6.5. ◻

LEMMA 6.6. *If $\mathbf{r} = -1$ or if $\mathbf{r} = 0$ and the operator has a kernel function of the form (4.17), then the number of necessary arithmetic operations for setting up the singular near field part of the stiffness matrix $A_L^{w,c,q}$ including $\mathcal{P}$ is less than $CL^2 2^{2L}$. If $\mathbf{r} = 0$ and if the kernel function is strongly singular, then no more than $CL^3 2^{2L}$ arithmetic operations are required.*

Proof. First, we consider the case where the kernel function is weakly singular and is of the form (4.17). Then the number of all P_λ is less than $C\,2^{2L}$, and for each point there is only a bounded number of Q with $P_\lambda \in \Gamma_Q$ and $l(Q) = L$. For each Γ_Q, there are no more than $C\,L^2$ quadrature knots in Γ_Q and no more than $C\,L$ functions ψ_P such that $\Gamma_Q \subseteq \mathrm{supp}\,\psi_P$. Thus the number of operations is less than $C\,L^2 2^{2L}$. In the case where the operator has a strongly singular kernel, Γ_Q is divided in $\iota_0 \sim L$ subdomains, and the number of quadrature knots is bounded by $C\,L^2$ for each subdomain. Thus the whole number of knots is bounded by $C\,L^3 2^{2L}$. ◻

7. Application to a boundary integral equation of geodesy. In [43] the numerical performance of our algorithm was reported for the case of the double layer equation. Now we apply it to the computation of the gravity field of the earth from data measured at the surface of the earth. Thus we have to solve the boundary value problem

$$\triangle w(P) = 2\omega^2,\ P \in \Omega_a, \qquad |\nabla w(P)| = g(P),\ P \in \Gamma,$$

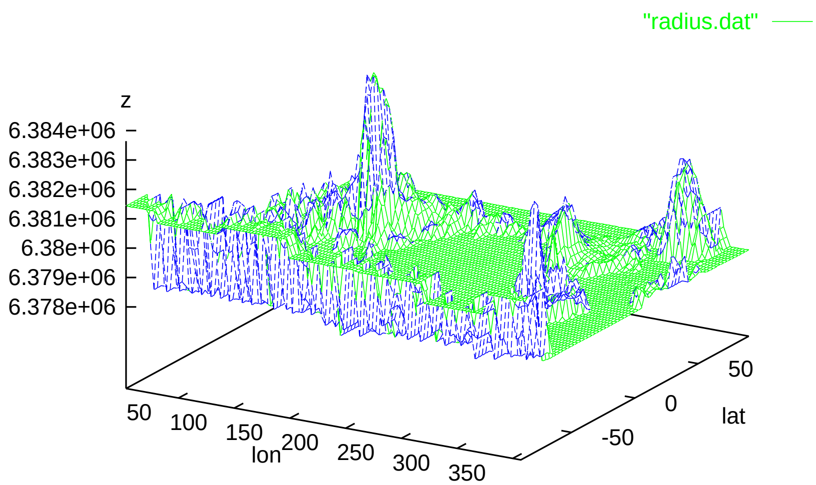

FIG. 7.1. *Resolution of surface of the earth.*

where Ω_a is the domain exterior to the earth, Γ the surface of the earth, ω the rotational velocity, w the yet unknown potential of gravity. Introducing the well approximating reference potential $w_0 \approx w$ corresponding to the mass concentrated at the center of the earth and neglecting higher order terms, the problem turns into the following oblique derivative boundary value problem for the unknown difference potential $\delta w = w - w_0$.

$$\triangle[\delta w](P) = 0,\ P \in \Omega_a, \qquad \partial_{l(P)}[\delta w](P) \;=\; \delta g(P),\ P \in \Gamma. \tag{7.1}$$

Here we have set $\delta g = (g^2 - g_0^2)/2|g_0|$, $g_0 = \nabla w_0$, and the direction of the oblique derivative is $l = g_0/|g_0|$. We seek δw in the form of a single layer potential with unknown layer function u defined over Γ, namely,

$$\delta w(Q) = \frac{1}{4\pi} \int_\Gamma \frac{u(P)}{|Q - P|}\, \mathrm{d}_P\Gamma. \tag{7.2}$$

Substituting this representation into the boundary value condition (7.1) and using the jump relation, we arrive at the singular boundary integral equation

$$-2\pi \cdot \cos[n(Q), l(Q)] \cdot u(Q) + p.v. \int_\Gamma \frac{\cos[l(Q), P - Q]}{|P - Q|^2} \cdot u(P) \cdot \mathrm{d}_P\Gamma = v(Q), \tag{7.3}$$

where n denotes the outward pointing normal at the surface of the earth. We have to solve this equation numerically for u. Substituting the approximation of u into (7.2), we get the unknown difference potential.

We approximate our surface by a piecewise quadratic interpolation over a triangulation which is obtained from the uniform partition of four initial triangles into 2^L, $L = 8$ subtriangles (to get an impression of the resolution, cf. Figure 7.1, where the distance of the surface points to the center of the earth is plotted depending on the latitude and longitude). The z-component (z-direction from south pole to north pole) of the gradient appearing in the

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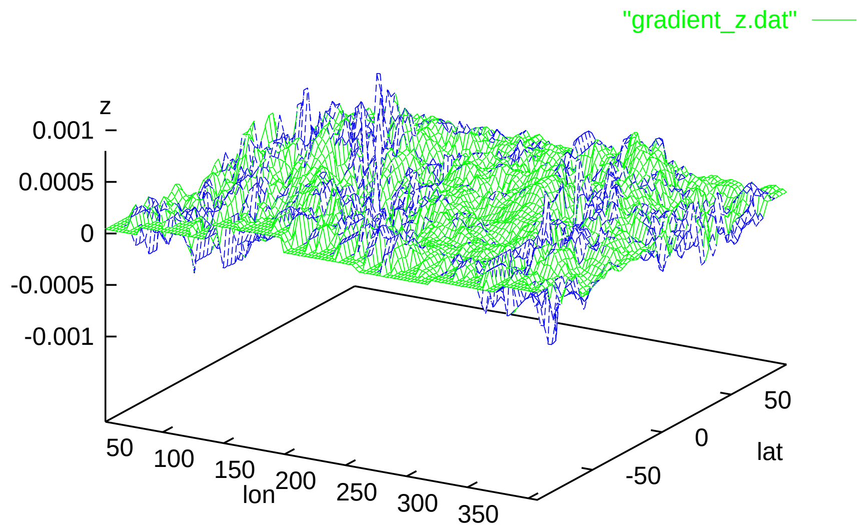

FIG. 7.2. *Right-hand side δg.*

right-hand δg of the equation (7.3) and the potential solution δw at the surface of the earth corresponding to data simulated from a realistic model are presented in the Figures 7.2 and 7.3. As expected, the degree of smoothness is not very high.

The approximate solution obtained by the numerical algorithm of the present paper looks the same as the solution in Figure 7.3. The corresponding numerical error is presented in Figure 7.4. It turns out that the relative error of the potential w is about $0.8 \cdot 10^{-7}$. Table 7.1 contains the errors and compression rates for several levels. Here L denotes the level and DOF the degrees of freedom. The L^2 error for the layer solution u is denoted by $LE = LE_L$, and $DE = DE_L$ stands for the supremum error of the potential solution computed at some points of the exterior Ω_a close to the surface Γ. By LO and DO we have denoted the approximate convergence orders $LE \sim h^{LO}$ and $DE \sim h^{DO}$ given by

$$LO_L := \frac{\log LE_{L+1} - \log LE_L}{\log h_{L+1} - \log h_L}, \quad DO_L := \frac{\log DE_{L+1} - \log DE_L}{\log h_{L+1} - \log h_L}.$$

The number IT is the number of GMRes iteration necessary to get a solution of the linear system of equations with an estimated accuracy less than 10^{-14}. Finally, RA is the compression ratio, i.e., the quotient of the number of all matrix entries divided by the number of entries in the compressed matrix. For our computations, we have used the compression criteria

$$\mathrm{dist}\,(\Psi_P, \Theta_{P'}) \le \max\left\{0.25\, L^{0.125} 2^{L-l(P')-l(P)}, 2 \cdot 2^{l(P')}, 2 \cdot 2^{l(P)}\right\}$$

and

$$\mathrm{dist}\,(\Psi_P, \Theta_{P'}) \le \max\left\{0.25\, L^{0.25} 2^{1.66\,[L-l(P')]-1.666\,l(P)}, 2 \cdot 2^{l(P')}, 2 \cdot 2^{l(P)}\right\}$$

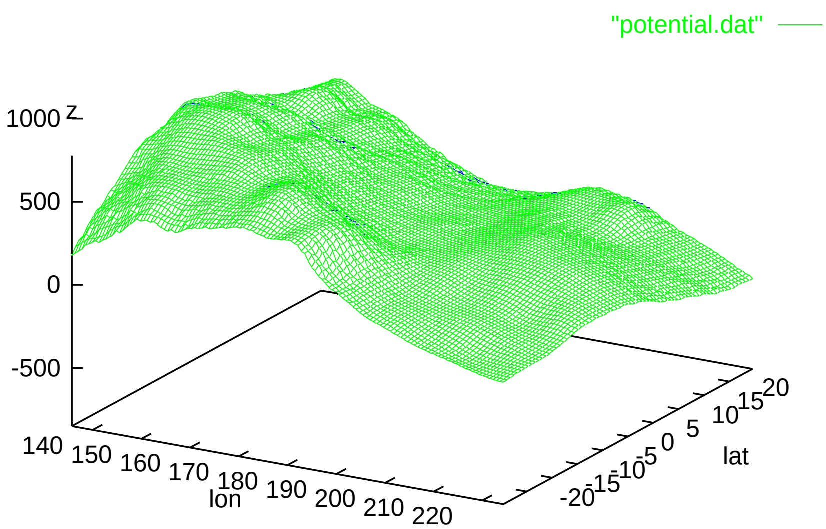

FIG. 7.3. *Potential solution δw.*

L	DOF	LE	LO	DE	DO	IT	RA
4	514	$1.9{\cdot}10^{-4}$	0.33	$3.2{\cdot}10^{2}$	0.54	33	3.73
5	2 050	$1.0{\cdot}10^{-4}$	0.89	$1.0{\cdot}10^{2}$	1.61	35	10.97
6	8 194	$3.7{\cdot}10^{-5}$	1.42	$5.3{\cdot}10^{1}$	0.98	36	36.02
7	32 770	$1.7{\cdot}10^{-5}$	1.16	$1.5{\cdot}10^{1}$	1.78	37	124.39
8	131 074			$5.1{\cdot}10^{0}$	1.58	45	440.76

TABLE 7.1
Errors and compression rates.

instead of (3.10) and (3.11), respectively. Note that the high compression rates makes it possible to solve such large systems with a dense matrix on a work station with 0.5 Gb main memory.

The constants in the quadrature computation have been chosen as $n_A = n_C := 1$, $n_B = n_D := 1.5$, and $q := 2$ (cf. §4). The Gauß order n_G has been set to ten for the singular integral in §4.3. For the computation of the potential (cf. (7.2)), we have used the same quadrature technique but with $n_A := 2$ and $n_B := 1.5$. With this choice, the quadrature error is less then the error due to the linear collocation scheme. In fact, the application of higher order Gauß rules and additional uniform partitioning of the quadrature subdomains did not change the results significantly. The main part of the computation time is spent for the assembling of the stiffness matrix and only about five per cent for the iterative solution. We did not take much time to optimize our code. For the nonoptimized code, however, we have observed that halving the step size leads to a factor between five and six for the computing time. Note that, for a conventional boundary element code, halving the step size results in a factor sixteen for the computing time.

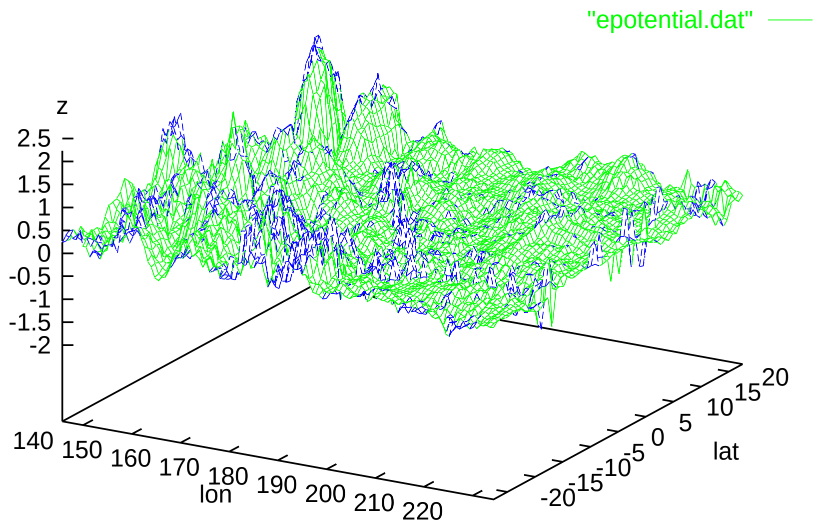

FIG. 7.4. *Error of potential solution δw.*

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