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ETNA
Kent State University
etna@mcs.kent.edu

# SOME NONSTANDARD FINITE ELEMENT ESTIMATES WITH APPLICATIONS TO 3D POISSON AND SIGNORINI PROBLEMS *

FAKER BEN BELGACEM † AND SUSANNE C. BRENNER ‡

**Abstract.** In this paper we establish several nonstandard finite element estimates involving fractional order Sobolev spaces, with applications to bubble stabilized mixed methods for the three-dimensional Poisson and Signorini problems.

**Key words.** fractional order Sobolev spaces, Poisson, Signorini, mixed method, bubble finite element stabilization.

**AMS subject classifications.** 65N30, 35J85.

**1. Introduction.** It is well-known that interpolation error estimates play an important role in the analysis of finite element methods. The simplest interpolation error estimates appear in the following form:

$$\sum_{j=0}^{m} h^j |u - \Pi_h u|_{H^j(\Omega)} \le C h^m |u|_{H^m(\Omega)}, \tag{1.1}$$

where $\Omega$ is a bounded polyhedral domain in $\mathbb{R}^d$ ($d = 1, 2, 3$), $m$ is an integer and $\Pi_h$ is an interpolation operator from $H^m(\Omega)$ to the finite element space $V_h$ associated with a regular triangulation $\mathcal{T}_h$ of $\Omega$ of mesh-size $h$, and the seminorm $|\cdot|_{H^k(\Omega)}$ for a nonnegative integer is defined by

$$|v|^2_{H^k(\Omega)} = \sum_{|\alpha|=k} \|\partial^\alpha v\|^2_{L^2(\Omega)}.$$

In the case where $\Pi_h$ is defined locally on each element, since all the seminorms in (1.1) are also local, the estimate (1.1) can be established by a purely local analysis (cf. [12], [8]).

For applications to problems whose solutions are not regular, it is important to have estimates for $(u - \Pi_h u)$ in fractional order Sobolev seminorms. Let $k$ be a nonnegative integer and $0 < \lambda < 1$. The seminorm $|\cdot|_{H^{k+\lambda}(\Omega)}$ is defined by (cf. [1], [22])

$$|v|^2_{H^{k+\lambda}(\Omega)} = \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{[\partial^\alpha v(x) - \partial^\alpha v(y)]^2}{|x-y|^{d+2\lambda}}\, dxdy, \tag{1.2}$$

and the norm $\|\cdot\|_{H^{k+\lambda}(\Omega)}$ is given by

$$\|v\|^2_{H^{k+\lambda}(\Omega)} = \|v\|^2_{H^k(\Omega)} + |v|^2_{H^{k+\lambda}(\Omega)} = \sum_{j=0}^{k} |v|^2_{H^j(\Omega)} + |v|^2_{H^{k+\lambda}(\Omega)}.$$

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†Mathématiques pour l'Industrie et la Physique, Unité Mixte de Recherche CNRS–UPS–INSAT–UT1 (UMR 5640), Université Paul Sabatier, 118 route de Narbonne, 31062 TOULOUSE CEDEX, FRANCE. E-mail: belgacem@mip.ups-tlse.fr

‡Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA. This research was supported in part by the National Science Foundation under Grant No. DMS-0074246. E-mail: brenner@math.sc.edu

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Kent State University
etna@mcs.kent.edu



Sometimes such estimates follow easily from the basic estimate (1.1) and the interpolation theory of Sobolev spaces (cf. [22]). For example, let $\Pi = \Pi_h^1$ be the nodal interpolation operator for the $\mathcal{P}_1$ finite element space and $0 < \lambda < 1$, then (1.1) with $m = 2$ implies that

$$\|u - \Pi_h^1 u\|_{L^2(\Omega)} \le Ch^2 \|u\|_{H^2(\Omega)} \qquad \forall\, u \in H^2(\Omega),$$
$$\|u - \Pi_h^1 u\|_{H^1(\Omega)} \le Ch \|u\|_{H^2(\Omega)} \qquad \forall\, u \in H^2(\Omega),$$

and hence

$$\|u - \Pi_h^1 u\|_{H^\lambda(\Omega)} \le C_\lambda h^{2-\lambda} \|u\|_{H^2(\Omega)} \qquad \forall\, u \in H^2(\Omega).$$

Replacing $u$ with $(u - p)$ for an appropriate first order polynomial $p$ so that

$$\int_\Omega \partial^\alpha (u - p)\, dx = 0 \quad \text{for} \quad |\alpha| \le 1,$$

we can then deduce from the preceding estimate and the Friedrichs inequality (cf. [19]) that

$$|u - \Pi_h^1 u|_{H^\lambda(\Omega)} \le C_\lambda h^{2-\lambda} |u|_{H^2(\Omega)}.$$

Note that throughout this paper we use the symbol $C$ (with or without subscripts) to represent generic positive constants which can take different values at different places.

On the other hand, in the case where $\Pi = \Pi_h^0$ is the piecewise $L^2$-orthogonal projection operator from $L^2(\Omega)$ to the $\mathcal{P}_0$ finite element space, the standard estimate (1.1) with $m = 1$ only gives

$$\|u - \Pi_h^0 u\|_{L^2(\Omega)} \le Ch |u|_{H^1(\Omega)}.$$

For $0 < \lambda < \frac{1}{2}$, the estimate

$$(1.3) \qquad |u - \Pi_h^0 u|_{H^\lambda(\Omega)} \le C_\lambda h^{1-\lambda} |u|_{H^1(\Omega)}$$

does not follow from the interpolation theory of Sobolev spaces.

Similarly, for $0 < \lambda < \frac{1}{2}$, the estimate

$$(1.4) \qquad |u - \Pi_h^1 u|_{H^{1+\lambda}(\Omega)} \le C_\lambda h^{1-\lambda} |u|_{H^2(\Omega)}$$

does not follow from (1.1) and interpolation.

Inverse estimates are also important in the analysis of finite element methods. A standard inverse estimate for a Lagrange finite element space $V_h(\Omega) \subset H^1(\Omega)$ takes the form

$$(1.5) \qquad |v|_{H^1(\Omega)} \le Ch^{-1} \|v\|_{L^2(\Omega)} \qquad \forall\, v \in V_h(\Omega),$$

and it can be obtained by a purely local analysis (cf. [12], [8]).

Let $\lambda \in (0, 1)$. From (1.5) and the trivial estimate

$$\|v\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)} \qquad \forall\, v \in V_h(\Omega),$$

we can obtain by interpolation the following inverse estimate for the seminorm $|\cdot|_{H^\lambda(\Omega)}$:

$$|v|_{H^\lambda(\Omega)} \le C_\lambda h^{-\lambda} \|v\|_{L^2(\Omega)} \qquad \forall\, v \in V_h(\Omega).$$

However, for $0 < \lambda < \frac{1}{2}$, the inverse estimate

$$(1.6) \qquad |v|_{H^{1+\lambda}(\Omega)} \le C_\lambda h^{-\lambda} |v|_{H^1(\Omega)} \qquad \forall\, v \in V_h(\Omega)$$

does not follow from (1.5) and interpolation.

In this paper we will establish certain finite element estimates in fractional order Sobolev seminorms which include in particular (1.3), (1.4) and (1.6), with applications to some three dimensional mixed finite element methods.

The main difficulty in dealing with estimates in fractional order Sobolev seminorms is due to the non-local nature of the definition (1.2). Our key observation is that the analysis of such estimates can be reduced to a purely local one by combining the ideas from [7] and the estimate

$$\int_{\Omega\setminus T} \frac{1}{|x-y|^{d+2\lambda}}\, dy \le \frac{C_\lambda}{\rho(x,\partial T)^{2\lambda}} \qquad \forall\, x\in T, \tag{1.7}$$

where $T$ is an element in $\mathscr{T}_h$ and $\rho(x,\partial T)=\inf_{y\in\partial T}|x-y|$ is the distance from $x$ to the boundary of $T$. Furthermore, the local analysis can be handled by the following estimate on a reference domain $\hat{T}$:

$$\int_{\hat{T}} \frac{u^2(x)}{\rho(x,\partial\hat{T})^{2\lambda}}\, dx \le C_{\hat{T},\lambda}\|u\|^2_{H^\lambda(\hat{T})} \qquad \forall\, u\in H^\lambda(\hat{T}) \quad \text{and} \quad 0<\lambda<\frac{1}{2}. \tag{1.8}$$

The estimate (1.7) can be obtained easily by using the regularity of $\mathscr{T}_h$ and a direct calculation. The estimate (1.8), which follows from the Hardy inequalities, comes from the theory of Sobolev spaces. A proof of it can be found in either [18] or [15].

The rest of the paper is organized as follows. The nonstandard finite element estimates in fractional order Sobolev norms are proved in Section 2. Applications to three-dimensional mixed finite element methods for the Poisson problem and the Signorini contact problem are then given in Section 3.

**2. Nonstandard finite element estimates in fractional order Sobolev norms.** Let $\Omega$ be a bounded polyhedral domain in $\mathbb{R}^d$ for $d=1,2,3$. Let $\mathscr{T}_h$ be a regular triangulation (cf. [12], [8]) of $\Omega$, where $h=\max_{T\in\mathscr{T}_h}\operatorname{diam} T$. We will first show how certain estimates for the globally defined fractional order Sobolev seminorms can be reduced to local estimates.

LEMMA 2.1. *Let $\lambda\in(0,1)$ and $w\in H^{k+\lambda}(\Omega)$. Then the following error estimate holds:*

$$|w|^2_{H^{k+\lambda}(\Omega)} \le C_\lambda \sum_{|\alpha|=k}\sum_{T\in\mathscr{T}_h}\left[|\partial^\alpha w|^2_{H^\lambda(T)} + \int_T \frac{[\partial^\alpha w(x)]^2}{\rho(x,\partial T)^{2\lambda}}\, dx\right].$$

*Proof.* By the definition (1.2), we have

$$\begin{aligned} |w|^2_{H^{k+\lambda}(\Omega)} &= \sum_{|\alpha|=k}\int_\Omega\int_\Omega \frac{[\partial^\alpha w(x)-\partial^\alpha w(y)]^2}{|x-y|^{d+2\lambda}}\, dxdy \\ &= \sum_{|\alpha|=k}\Bigg[\sum_{T\in\mathscr{T}_h}\int_T\int_T \frac{[\partial^\alpha w(x)-\partial^\alpha w(y)]^2}{|x-y|^{d+2\lambda}}\, dxdy \\ &\qquad + \sum_{\substack{T,T'\in\mathscr{T}_h\\ T\neq T'}}\int_T\int_{T'} \frac{[\partial^\alpha w(x)-\partial^\alpha w(y)]^2}{|x-y|^{d+2\lambda}}\, dxdy\Bigg]. \end{aligned} \tag{2.1}$$

Note that the first sum inside the bracket on the right-hand side of (2.1) equals $\sum_{T\in\mathscr{T}_h}|\partial^\alpha w|^2_{H^\lambda(T)}$. The second sum can be estimated following the ideas in [7]:

$$\sum_{\substack{T,T'\in\mathscr{T}_h\\ T\neq T'}}\int_T\int_{T'} \frac{[\partial^\alpha w(x)-\partial^\alpha w(y)]^2}{|x-y|^{d+2\lambda}}\, dxdy \le 2\Bigg[\sum_{\substack{T,T'\in\mathscr{T}_h\\ T\neq T'}}\int_T\int_{T'} \frac{[\partial^\alpha w(x)]^2}{|x-y|^{d+2\lambda}}\, dxdy \tag{2.2}$$

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Kent State University
etna@mcs.kent.edu



$$+ \sum_{\substack{T,T'\in\mathscr{T}_h\\ T\neq T'}} \int_T \int_{T'} \frac{[\partial^\alpha w(y)]^2}{|x-y|^{d+2\lambda}}\, dxdy \Bigg].$$

Let us focus on the first term on the right-hand side of (2.2), since the second one can be worked out in exactly the same way. We have, by (1.7),

$$\text{(2.3)}\quad \sum_{\substack{T,T'\in\mathscr{T}_h\\ T\neq T'}} \int_T \int_{T'} \frac{[\partial^\alpha w(x)]^2}{|x-y|^{d+2\lambda}}\, dxdy = \sum_{T\in\mathscr{T}_h} \int_T [\partial^\alpha w(x)]^2 \Big(\int_{\Omega\setminus T} \frac{1}{|x-y|^{d+2\lambda}}\, dy\Big)\, dx$$
$$\leq C_\lambda \sum_{T\in\mathscr{T}_h} \int_T \frac{[\partial^\alpha w(x)]^2}{\rho(x,\partial T)^{2\lambda}}\, dx.$$

The lemma follows from (2.1)–(2.3). ☐

REMARK 2.2. *Note that the right-hand side of the estimate in Lemma 2.1 is in general infinite when* $\lambda \in [\frac{1}{2}, 1)$. *Therefore Lemma 2.1 is of interest mainly for* $\lambda \in (0, \frac{1}{2})$.

Let us now consider estimates for the piecewise constant interpolation operator $\Pi_h^0$. Let $\mathcal{P}_0(T)$ stand for the set of constant functions defined on $T \in \mathscr{T}_h$. The $\mathcal{P}_0$ finite element subspace $M_h(\Omega)$ is defined by

$$M_h(\Omega) = \Big\{ v_h \in L^2(\Omega) : v_h\big|_T \in \mathcal{P}_0(T) \quad \forall\, T \in \mathscr{T}_h \Big\}.$$

The piecewise constant interpolation operator $\Pi_h^0$ is the orthogonal projection from $L^2(\Omega)$ into $M_h(\Omega)$ and is defined as follows:

$$\text{(2.4)}\qquad \int_\Omega (v - \Pi_h^0 v)\psi_h\, dx = 0 \qquad \forall\, v \in L^2(\Omega),\ \psi_h \in M_h(\Omega).$$

THEOREM 2.3. *For any* $\lambda \in (0, \frac{1}{2})$ *and* $\mu \in [\lambda, 1]$, *the following error estimate holds*:

$$\text{(2.5)}\qquad |v - \Pi_h^0 v|_{H^\lambda(\Omega)} \leq C_\lambda h^{\mu-\lambda} |v|_{H^\mu(\Omega)} \qquad \forall\, v \in H^\mu(\Omega).$$

*Proof*. Set $v_h = \Pi_h^0 v$. From Lemma 2.1 we have

$$\text{(2.6)}\qquad |v - v_h|^2_{H^\lambda(\Omega)} \leq C_\lambda \sum_{T\in\mathscr{T}_h} \Bigg[ |v - v_h|^2_{H^\lambda(T)} + \int_T \frac{[(v-v_h)(x)]^2}{\rho(x,\partial T)^{2\lambda}}\, dx \Bigg].$$

Hence the proof of (2.5) is reduced to a local estimate which can be handled by (1.8) and the usual scaling argument as follows.

Let $\hat{K}$ be the reference element and consider the transformation $\hat{x} \mapsto x = B_T\hat{x} + b$, where $B_T$ is an invertible matrix. By (1.2), (1.8), Friedrichs' inequality and the regularity of $\mathscr{T}_h$, we obtain

$$|v - v_h|^2_{H^\lambda(T)} + \int_T \frac{[(v-v_h)(x)]^2}{\rho(x,\partial T)^{2\lambda}}\, dx$$
$$\leq \|B_T^{-1}\|^{d+2\lambda} |\det B_T|^2 |\hat{v} - \hat{v}_h|^2_{H^\lambda(\hat{T})}$$
$$+ \|B_T^{-1}\|^{2\lambda}\, |\det B_T| \int_{\hat{T}} \frac{[(\hat{v}-\hat{v}_h)(\hat{x})]^2}{\rho(\hat{x},\partial\hat{T})^{2\lambda}}\, d\hat{x}$$

$$\begin{aligned}
&\le C_\lambda \|B_T^{-1}\|^{2\lambda} |\det B_T|\ \|\hat v - \hat v_h\|^2_{H^\lambda(\hat T)}\\
&\le C_\lambda \|B_T^{-1}\|^{2\lambda} |\det B_T|\ |\hat v|^2_{H^\lambda(\hat T)}\\
&\le C_\lambda \|B_T^{-1}\|^{2\lambda} |\det B_T|\ |\hat v|^2_{H^\mu(\hat T)}\\
&\le C_\lambda \|B_T^{-1}\|^{2\lambda} |\det B_T| \|B_T\|^{d+2\mu} |\det B_T^{-1}|^2 |v|^2_{H^\mu(T)}\\
&\le C_\lambda h^{2(\mu-\lambda)} |v|^2_{H^\mu(T)}.
\end{aligned}$$

In view of (2.6), we deduce that

$$|v - v_h|^2_{H^\lambda(\Omega)} \le C_\lambda h^{2(\mu-\lambda)} \sum_{T\in\mathscr{T}_h} |v|^2_{H^\mu(T)} \le C_\lambda h^{2(\mu-\lambda)} |v|^2_{H^\mu(\Omega)},$$

and the lemma follows. □

By a straight-forward argument (or cf. [13]), we also have

$$\|v - \Pi_h^0 v\|_{L^2(\Omega)} \le C_\lambda h^\mu |v|_{H^\mu(\Omega)} \qquad \forall\, v \in H^\mu(\Omega),\ \lambda \le \mu \le 1,$$

which together with Theorem 2.3 imply

$$\|v - \Pi_h^0 v\|_{H^\lambda(\Omega)} \le C_\lambda h^{\mu-\lambda} |v|_{H^\mu(\Omega)} \qquad \forall\, v \in H^\mu(\Omega),\ \lambda \le \mu \le 1. \tag{2.7}$$

For the applications in Section 3 we will need interpolation error estimates in the dual spaces of the Sobolev spaces. The operator $\Pi_h^0$ can be extended to the dual space $H^\lambda(\Omega)'$ for $\lambda \in (0, \frac{1}{2})$ as follows:

$$\langle v - \Pi_h^0 v, \psi_h\rangle_{\lambda,\Omega} = 0, \qquad \forall\, v \in H^\lambda(\Omega)',\ \psi_h \in M_h(\Gamma), \tag{2.8}$$

where $\langle\cdot,\cdot\rangle_{\lambda,\Omega}$ is the duality pairing between $H^\lambda(\Omega)'$ and $H^\lambda(\Omega)$ that generalizes the inner product $(\cdot,\cdot)_{L^2(\Omega)}$.

THEOREM 2.4. *For any $\lambda \in (0, \frac{1}{2})$ and $\mu \in [\lambda, 1]$, the following error estimate holds:*

$$\|v - \Pi_h^0 v\|_{H^\mu(\Omega)'} \le C_\lambda h^{\mu-\lambda} \|v\|_{H^\lambda(\Omega)'} \qquad \forall\, v \in H^\lambda(\Omega)'.$$

*Proof*. Let $v \in H^\lambda(\Omega)'$ be arbitrary. By duality and the definitions (2.4) and (2.8), we can write

$$\|v - \Pi_h^0 v\|_{H^\mu(\Omega)'} = \sup_{w\in H^\mu(\Omega)} \frac{\langle v - \Pi_h^0 v, w\rangle_{\mu,\Omega}}{\|w\|_{H^\mu(\Omega)}} = \sup_{w\in H^\mu(\Omega)} \frac{\langle v, w - \Pi_h^0 w\rangle_{\lambda,\Omega}}{\|w\|_{H^\mu(\Omega)}}. \tag{2.9}$$

From (2.7), we have

$$\langle v, w - \Pi_h^0 w\rangle_{\lambda,\Omega} \le \|v\|_{H^\lambda(\Omega)'} \|w - \Pi_h^0 w\|_{H^\lambda(\Omega)} \le C_\lambda h^{\mu-\lambda} \|v\|_{H^\lambda(\Omega)'} \|w\|_{H^\mu(\Omega)},$$

and the theorem follows from (2.9). □

REMARK 2.5. *Theorem 2.3 and Theorem 2.4 remain valid on a $d$-dimensional $(d = 1, 2)$ polyhedral surface $\Gamma$, with essentially identical proofs. Note that the space $H^\lambda(\Gamma)'$ can also be written as $H^{-\lambda}(\Gamma)$.*

Next, we consider the nodal interpolation operator $\Pi_h^1$ from $\mathscr{C}(\bar\Omega)$ to a $\mathscr{C}^0$ Lagrange (or tensorial) finite element space $V_h$ associated with $\mathscr{T}_h$ (cf. [12], [8]).

THEOREM 2.6. *Let $\lambda \in (0, \frac{1}{2})$. Then we have, for $d = 1, 2$,*

$$|v - \Pi_h^1 v|_{H^{1+\lambda}(\Omega)} \le C_\lambda h^{\mu-\lambda} |v|_{H^{1+\mu}(\Omega)} \qquad \forall\, v \in H^{1+\mu}(\Omega),\ \lambda \le \mu \le 1, \tag{2.10}$$

*and for* $d = 3$*,*

$$|v - \Pi_h^1 v|_{H^{1+\lambda}(\Omega)} \le C_{\lambda,\mu} h^{\mu-\lambda} |v|_{H^{1+\mu}(\Omega)} \quad \forall\, v \in H^{1+\mu}(\Omega),\ \frac{1}{2} < \mu \le 1. \tag{2.11}$$

*Proof*. We will follow the notation introduced in Theorem 2.3. First we consider the cases of $d = 1$ and $d = 2$. Let $\Pi_h^1 v = v_h$. From Lemma 2.1 we have

$$|v - v_h|^2_{H^{1+\lambda}(\Omega)} \le C \sum_{T \in \mathscr{T}_h} \left[ |v - v_h|^2_{H^{1+\lambda}(T)} + \int_T \frac{|\nabla v(x) - \nabla v_h(x)|^2}{\rho(x, \partial T)^{2\lambda}}\, dx \right]. \tag{2.12}$$

Using scaling, (1.2), (1.8) and the regularity of $\mathscr{T}_h$, we have

$$\begin{aligned} &|v - v_h|^2_{H^{1+\lambda}(T)} + \int_T \frac{|\nabla v(x) - \nabla v_h(x)|^2}{\rho(x, \partial T)^{2\lambda}}\, dx \\ &\quad \le \|B_T^{-1}\|^{d+2\lambda} \|B_T\|^2 |\det B_T|^2 |\hat{v} - \hat{v}_h|^2_{H^{1+\lambda}(\hat{T})} \\ &\qquad + \|B_T^{-1}\|^{2\lambda} \|B_T\|^2 |\det B_T| \int_{\hat{T}} \frac{|\nabla \hat{v}(x) - \nabla \hat{v}_h(x)|^2}{\rho(\hat{x}, \partial \hat{T})^{2\lambda}}\, d\hat{x} \\ &\quad \le C_\lambda \|B_T^{-1}\|^{2\lambda} \|B_T\|^2 |\det B_T| \big[ |\hat{v} - \hat{v}_h|^2_{H^1(\hat{T})} + |\hat{v} - \hat{v}_h|^2_{H^{1+\lambda}(\hat{T})} \big]. \end{aligned} \tag{2.13}$$

From the equivalence of norms on finite dimensional vector spaces and Sobolev's inequality (cf. [1], [19], [22]), it is easy to see that

$$\begin{aligned} |\hat{v} - \hat{v}_h|^2_{H^1(\hat{T})} + |\hat{v} - \hat{v}_h|^2_{H^{1+\lambda}(\hat{T})} &\le C \left[ |\hat{v}|^2_{H^1(\hat{T})} + |\hat{v}|^2_{H^{1+\lambda}(\hat{T})} + \|\hat{v}_h\|^2_{L^\infty(\hat{T})} \right] \\ &\le C \left[ |\hat{v}|^2_{H^1(\hat{T})} + |\hat{v}|^2_{H^{1+\lambda}(\hat{T})} + \|\hat{v}\|^2_{L^\infty(\hat{T})} \right] \\ &\le C_\lambda \|\hat{v}\|^2_{H^{1+\lambda}(\hat{T})}. \end{aligned} \tag{2.14}$$

Combining (1.2), (2.13), (2.14), the Bramble-Hilbert lemma (cf. [6], [13]) and scaling, we have

$$\begin{aligned} &|v - v_h|^2_{H^{1+\lambda}(T)} + \int_T \frac{|\nabla v(x) - \nabla v_h(x)|^2}{\rho(x, \partial T)^{2\lambda}}\, dx \\ &\le C_\lambda \|B_T^{-1}\|^{2\lambda} \|B_T\|^2 |\det B_T| \\ &\qquad \times \inf_{p \in \mathcal{P}_1(\hat{T})} \left[ |(\hat{v} - p) - (\hat{v} - p)_h|^2_{H^1(\hat{T})} + |(\hat{v} - p) - (\hat{v} - p)_h|^2_{H^{1+\lambda}(\hat{T})} \right] \\ &\le C_\lambda \|B_T^{-1}\|^{2\lambda} \|B_T\|^2 |\det B_T| \inf_{p \in \mathcal{P}_1(\hat{T})} \|\hat{v} - p\|^2_{H^{1+\lambda}(\hat{T})} \\ &\le C_\lambda \|B_T^{-1}\|^{2\lambda} \|B_T\|^2 |\det B_T| |\hat{v}|^2_{H^{1+\lambda}(\hat{T})} \\ &\le C_\lambda \|B_T^{-1}\|^{2\lambda} \|B_T\|^2 |\det B_T| |\hat{v}|^2_{H^{1+\mu}(\hat{T})} \\ &\le C_\lambda \|B_T^{-1}\|^{2\lambda} \|B_T\|^2 |\det B_T| \|B_T\|^{2d+2\mu} \|B_T^{-1}\|^2 |\det B_T^{-1}|^2 |v|^2_{H^\mu(T)} \\ &\le C_\lambda h^{2(\mu-\lambda)} |v|^2_{H^\mu(T)}. \end{aligned} \tag{2.15}$$

The estimate (2.10) follows from (2.12) and (2.15).

The proof of (2.11) is similar, except that (2.14) must be replaced by

$$|\hat{v} - \hat{v}_h|^2_{H^1(\hat{T})} + |\hat{v} - \hat{v}_h|^2_{H^{1+\lambda}(\hat{T})} \le C_\mu \|\hat{v}\|^2_{H^{1+\mu}(\hat{T})}.$$

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etna@mcs.kent.edu



The theorem now follows. □

REMARK 2.7. *The results of [7] can be recovered from (2.10) by taking* $\mu = \lambda$.

Finally, we turn to inverse estimates involving fractional order Sobolev norms. For this purpose we will assume that the triangulation $\mathscr{T}_h$ is quasi-uniform (cf. [12], [8]).

THEOREM 2.8. *Let* $\lambda \in (0, \frac{1}{2})$ *and* $\theta \in [0, \lambda]$. *Then the following estimate holds*:

$$|v|_{H^\lambda(\Omega)} \le C_\lambda h^{\theta-\lambda} |v|_{H^\theta(\Omega)} \qquad \forall\, v \in M_h(\Omega).$$

*Proof*. We will follow the notation in Theorem 2.3. Let $v \in M_h(\Omega)$ be arbitrary. From Lemma 2.1 we have

$$|v|^2_{H^\lambda(\Omega)} \le C \sum_{T \in \mathscr{T}_h} \left[ |v|^2_{H^\lambda(T)} + \int_T \frac{[v(x)]^2}{\rho(x, \partial T)^{2\lambda}}\, dx \right]. \tag{2.16}$$

Using the equivalence of norms on a finite dimensional vector space, we obtain, by scaling, (1.2), (1.8) and the quasi-uniformity of $\mathscr{T}_h$,

$$\begin{aligned} |v|^2_{H^\lambda(T)} + \int_T \frac{[v(x)]^2}{\rho(x,\partial T)^{2\lambda}}\, dx &\le \|B_T^{-1}\|^{d+2\lambda} |\det B_T|^2 |\hat{v}|^2_{H^\lambda(\hat{T})} \\ &\quad + \|B_T^{-1}\|^{2\lambda} |\det B_T| \int_{\hat{T}} \frac{[\hat{v}(\hat{x})]^2}{\rho(\hat{x}, \partial\hat{T})^{2\lambda}}\, d\hat{x} \\ &\le C_\lambda \|B_T^{-1}\|^{2\lambda} |\det B_T| \|\hat{v}\|^2_{L^2(\hat{T})} \\ &\le C_\lambda \|B_T^{-1}\|^{2\lambda} |\det B_T| |\det B_T^{-1}| \|v\|^2_{L^2(T)} \\ &\le C_\lambda h^{-2\lambda} \|v\|^2_{L^2(T)}. \end{aligned}$$

Combining the preceding estimate and (2.16) we have

$$|v|_{H^\lambda(\Omega)} \le C_\lambda h^{-\lambda} \|v\|_{L^2(\Omega)} \qquad \forall\, v \in M_h(\Omega). \tag{2.17}$$

In other words the theorem holds for $\theta = 0$.

The proof for $\theta \in (0, \lambda]$ is more complicated. For $T \in \mathscr{T}_h$, we denote by $\sigma_T$ the collection of elements in $\mathscr{T}_h$ which share at least one vertex with $T$, i.e.,

$$\sigma_T = \{T' \in \mathscr{T}_h : \quad \overline{T} \cap \overline{T'} \ne \emptyset\}.$$

The domain $S_T$ is defined by

$$\overline{S}_T = \bigcup_{T' \in \sigma_T} \overline{T}'. \tag{2.18}$$

We have

$$\begin{aligned} |v|^2_{H^\lambda(\Omega)} &= \sum_{T, T' \in \mathscr{T}_h} \int_T \int_{T'} \frac{[v(x) - v(y)]^2}{|x-y|^{d+2\lambda}}\, dx dy \\ &= \sum_{\substack{T, T' \in \mathscr{T}_h \\ T' \in \sigma_T}} \int_T \int_{T'} \frac{[v(x) - v(y)]^2}{|x-y|^{d+2\lambda}}\, dx dy \\ &\quad + \sum_{\substack{T, T' \in \mathscr{T}_h \\ T' \notin \sigma_T}} \int_T \int_{T'} \frac{[v(x) - v(y)]^2}{|x-y|^{d+2\lambda}}\, dx dy. \end{aligned} \tag{2.19}$$

There is an easy estimate for the second sum on the right-hand side of (2.19):

$$
\begin{aligned}
&\sum_{\substack{T,T'\in\mathscr{T}_h\\ T'\notin\sigma_T}} \int_T\int_{T'} \frac{[v(x)-v(y)]^2}{|x-y|^{d+2\lambda}}\,dxdy \\
(2.20)\qquad &\qquad \le Ch^{2(\theta-\lambda)} \sum_{\substack{T,T'\in\mathscr{T}_h\\ T'\notin\sigma_T}} \int_T\int_{T'} \frac{[v(x)-v(y)]^2}{|x-y|^{d+2\theta}}\,dxdy \\
&\qquad \le Ch^{2(\theta-\lambda)}|v|^2_{H^\theta(\Omega)}.
\end{aligned}
$$

In view of (1.2) and (2.18), we can bound the first sum on the right-hand side of (2.19) by

$$
(2.21)\qquad \sum_{\substack{T,T'\in\mathscr{T}_h\\ T'\in\sigma_T}} \int_T\int_{T'} \frac{[v(x)-v(y)]^2}{|x-y|^{d+2\lambda}}\,dxdy \le \sum_{T\in\mathscr{T}_h} |v|^2_{H^\lambda(S_T)}.
$$

Let $\hat{S}_T$ be a reference domain with unit diameter which is similar to $S_T$, $F$ be the affine transformation that maps $\hat{S}_T$ to $S_T$, and $\hat{v} = v\circ F$ be the pull-back of $v$ to $\hat{S}_T$. We obtain, by applying (2.17) to $\hat{S}_T$ and using the Bramble-Hilbert lemma (cf. [13]),

$$
\begin{aligned}
(2.22)\qquad |\hat{v}|_{H^\lambda(\hat{S}_T)} &= \inf_{p\in\mathcal{P}_0(\hat{S}_T)} |\hat{v}-p|_{H^\lambda(\hat{S}_T)} \\
&\le C_\lambda \inf_{p\in\mathcal{P}_0(\hat{S}_T)} \|\hat{v}-p\|_{L^2(\hat{S}_T)} \le C_\lambda |\hat{v}|_{H^\theta(\hat{S}_T)}.
\end{aligned}
$$

Combining (2.22) with a scaling argument, we have

$$
|v|_{H^\lambda(S_T)} \le C_\lambda h^{\theta-\lambda}|v|_{H^\theta(S_T)},
$$

which together with (2.21) imply

$$
\begin{aligned}
(2.23)\qquad \sum_{\substack{T,T'\in\mathscr{T}_h\\ T'\in\sigma_T}} \int_T\int_{T'} \frac{[v(x)-v(y)]^2}{|x-y|^{d+2\lambda}}\,dxdy &\le C_\lambda h^{2(\theta-\lambda)} \sum_{T\in\mathscr{T}_h} |v|^2_{H^\theta(S_T)} \\
&\le C_\lambda h^{2(\theta-\lambda)}|v|^2_{H^\theta(\Omega)}.
\end{aligned}
$$

The case for $\theta\in(0,\lambda]$ now follows from (2.19), (2.20) and (2.23). ☐

In exactly the same way one can prove the following theorem for a $\mathscr{C}^0$ Lagrange (or tensorial) finite element space $V_h(\Omega)$.

THEOREM 2.9. *Let* $\lambda\in(0,\frac{1}{2})$ *and* $\theta\in[0,\lambda]$. *Then the following estimate holds*:

$$
|v|_{H^{1+\lambda}(\Omega)} \le C_\lambda h^{\theta-\lambda}|v|_{H^{1+\theta}(\Omega)} \qquad \forall\, v\in V_h(\Omega).
$$

**3. Bubble stabilization of three-dimensional mixed finite element methods.** In this section we apply Theorem 2.4 to establish optimal error estimates for the numerical solution of some second order elliptic partial differential equations by the bubble stabilized finite element method, when the exact solution is not regular. First, we present the discretization of the 3D Poisson problem, where the Dirichlet condition is dualized à la Babuška. The second application deals with the approximation of the 3D unilateral contact problem known as the Signorini problem.

**3.1. The Poisson problem with Lagrange multipliers.** Let $\Omega$ be a bounded polyhedral domain in $\mathbb{R}^3$ with boundary $\Gamma = \partial\Omega$. Given $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma)$, the problem of interest is the Poisson equation:

$$-\Delta u = f \qquad \text{in } \Omega, \tag{3.1}$$

$$u = g \qquad \text{on } \Gamma. \tag{3.2}$$

By using Lagrange multipliers to enforce the Dirichlet condition (cf. [2]), we can formulate (3.1)–(3.2) as the following saddle point problem:

Find $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)'$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \langle \varphi, v \rangle_{\frac{1}{2},\Gamma} = \int_\Omega f v \, dx \qquad \forall\, v \in H^1(\Omega), \tag{3.3}$$

$$\langle \psi, u \rangle_{\frac{1}{2},\Gamma} = \langle \psi, g \rangle_{\frac{1}{2},\Gamma} \qquad \forall\, \psi \in H^{\frac{1}{2}}(\Gamma)'. \tag{3.4}$$

Let $\mathscr{T}_h^\Omega$ be a regular triangulation made of elements that are tetrahedra with a maximum size $h$ (the extension to the hexahedra does not create any technical difficulty). The trace of $\mathscr{T}_h^\Omega$ on the boundary $\Gamma$ results in a regular (2D) triangulation $\mathscr{T}_h^\Gamma$ made of triangular elements which are the faces of tetrahedral elements in $\mathscr{T}_h^\Omega$. Let $Y_h(\Omega)$ be the space defined by

$$Y_h(\Omega) = \Big\{ v_h \in \mathscr{C}^0(\overline{\Omega}) : v_h\big|_\kappa \in \mathcal{P}_1(\kappa) \quad \forall\, \kappa \in \mathscr{T}_h^{\Omega)} \Big\},$$

where $\mathcal{P}_1(\kappa)$ is the set of all affine functions over $\kappa$. In the stabilized finite element approach, the discrete space $Y_h(\Omega)$ is enriched by cubic bubble functions defined on the boundary $\Gamma$. Let $\{x_1, x_2, x_3\}$ be the vertices of the triangle $T \in \mathscr{T}_h^\Gamma$ which is a face of the tetrahedral element $\kappa_T$ in $\mathscr{T}_h^\Omega$. The vertices of $\kappa_T$ are $(x_i)_{1 \le i \le 4}$ and $\lambda_i$ $(1 \le i \le 4)$ is the barycentric coordinate associated with $x_i$. The bubble function we need to use is defined to be

$$\varphi_T(x) = \frac{60}{|T|} \lambda_1(x) \lambda_2(x) \lambda_3(x) \qquad \forall\, x \in \kappa_T,$$

and extended by zero elsewhere. Then the locally stabilized finite element space is given by

$$X_h(\Omega) = Y_h(\Omega) \oplus \Big( \bigoplus_{T \in \mathscr{T}_h^\Gamma} \mathbb{R} \varphi_T \Big).$$

The approximate Lagrange multipliers are piecewise constant functions with respect to the mesh $\mathscr{T}_h^\Gamma$, i.e.,

$$M_h(\Gamma) = \Big\{ \psi_h \in L^2(\Gamma) : \psi_h\big|_T \in \mathcal{P}_0(T) \quad \forall\, T \in \mathscr{T}_h^\Gamma \Big\},$$

and the discrete problem for (3.3)–(3.4) is:

Find $(u_h, \varphi_h) \in X_h(\Omega) \times M_h(\Gamma)$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, dx + \langle \varphi_h, v_h \rangle_{\frac{1}{2},\Gamma} = \int_\Omega f v_h \, dx \qquad \forall\, v_h \in X_h(\Omega), \tag{3.5}$$

$$\langle \psi_h, u_h \rangle_{\frac{1}{2},\Gamma} = \langle \psi_h, g \rangle_{\frac{1}{2},\Gamma} \qquad \forall\, \psi_h \in M_h(\Gamma). \tag{3.6}$$

The properties that allow for existence and uniqueness results are the coercivity of the form $(u_h, v_h) \mapsto (\nabla(\cdot), \nabla(\cdot))_{L^2(\Omega)}$ on a subspace of $X_h(\Omega)$ and the so-called *inf-sup* condition of the form $(v_h, \psi_h) \mapsto \langle \psi_h, v_h \rangle_{\frac{1}{2},\Gamma}$ on the spaces $X_h(\Omega) \times M_h(\Gamma)$. There is no

**ETNA**
Kent State University
etna@mcs.kent.edu



particular difficulty in checking that the seminorm $|\cdot|_{H^1(\Omega)}$ is equivalent to the $H^1$-norm in the space

$$\Big\{v_h \in X_h(\Omega) : \langle \psi_h, v_h\rangle_{\frac{1}{2},\Gamma} = 0 \quad \forall\, \psi_h \in M_h(\Gamma)\Big\}.$$

This proves the coercivity of the first form on this space. Moreover, following the lines of [3], the spaces $X_h(\Omega)$ and $M_h(\Gamma)$ satisfy the Babuška-Brezzi condition (also known as the *inf-sup* condition):

$$\inf_{\psi_h \in M_h(\Gamma)} \sup_{v_h \in X_h(\Omega)} \frac{\langle \psi_h, v_h\rangle_{\frac{1}{2},\Gamma}}{\|v_h\|_{H^1(\Omega)} \|\psi_h\|_{H^{\frac{1}{2}}(\Gamma)'}} > \gamma,$$

where the constant $\gamma$ does not depend on $h$. Therefore, using the saddle point theory (cf. [9]), one can prove that problem (3.5)–(3.6) has a unique solution $(u_h, \varphi_h) \in X_h(\Omega) \times M_h(\Gamma)$ which satisfies the following abstract error estimate:

$$\begin{aligned} &\|u - u_h\|_{H^1(\Omega)} + \|\varphi - \varphi_h\|_{H^{\frac{1}{2}}(\Gamma)'} \le \\ &\quad C\Big(\inf_{v_h \in X_h(\Omega)} \|u - v_h\|_{H^1(\Omega)} + \inf_{\psi_h \in M_h(\Gamma)} \|\varphi - \psi_h\|_{H^{\frac{1}{2}}(\Gamma)'}\Big). \end{aligned} \tag{3.7}$$

In the following theorem we apply Theorem 2.4 to derive from (3.7) the convergence rate of our mixed finite element solution of (3.5)–(3.6).

THEOREM 3.1. *Assume that the exact solution $u$ of Poisson's problem belongs to $H^{1+\lambda}(\Omega)$ $(0 < \lambda < \frac{1}{2})$. Then the following error estimate holds:*

$$\|u - u_h\|_{H^1(\Omega)} + \|\varphi - \varphi_h\|_{H^{\frac{1}{2}}(\Gamma)'} \le Ch^{\lambda}(\|u\|_{H^{1+\lambda}(\Omega)} + \|f\|_{L^2(\Omega)}).$$

*Proof.* It is sufficient to observe that, since $u \in H^{1+\lambda}(\Omega)$ and $\Delta u \in L^2(\Omega)$, $\varphi = \frac{\partial u}{\partial n} \in H^{\frac{1}{2}-\lambda}(\Gamma)'$ with

$$\|\varphi\|_{H^{\frac{1}{2}-\lambda}(\Gamma)'} \le C(\|u\|_{H^{1+\lambda}(\Omega)} + \|f\|_{L^2(\Omega)}).$$

We can then use Theorem 2.4 to obtain

$$\inf_{\psi_h \in M_h(\Gamma)} \|\varphi - \psi_h\|_{H^{\frac{1}{2}}(\Gamma)'} \le Ch^{\lambda}\|\varphi\|_{H^{\frac{1}{2}-\lambda}(\Gamma)'} \le Ch^{\lambda}(\|u\|_{H^{1+\lambda}(\Omega)} + \|f\|_{L^2(\Omega)}).$$

The bound

$$\inf_{v_h \in X_h(\Omega)} \|u - v_h\|_{H^1(\Omega)} \le Ch^{\lambda}\|u\|_{H^{1+\lambda}(\Omega)}$$

can be obtained by using a finite element interpolation for non-smooth functions (cf. [21], [5]). □

REMARK 3.2. *In two dimensions, there is no need for stabilization since the natural spaces $Y_h(\Omega)$ and $M_h(\Gamma)$ (built as described above with obvious modification) are compatible regarding the inf-sup condition (cf. [2], [20]). In three dimensions this condition is lost and is restored by bubble stabilization (cf. [10], [11]). Note, however, that even for non-regular two-dimensional solutions, the one-dimensional result in Theorem 2.4 is needed for proving optimal convergence results.*

**3.2. The unilateral contact variational inequality.** Assume again that $\Omega$ is a bounded polyhedral domain in $\mathbb{R}^3$. The boundary $\Gamma = \partial\Omega$ is a union of three non-overlapping sections $\Gamma_u, \Gamma_g$ and $\Gamma_C$. The part $\Gamma_u$ of nonzero measure is subject to the Dirichlet conditions, while on $\Gamma_g$ the Neumann condition is prescribed, and $\Gamma_C$ is the candidate to be in contact with a rigid obstacle. To avoid technicalities arising from the special Sobolev space $H^{\frac{1}{2}}_{00}(\Gamma_C)$, we assume that $\Gamma_u$ and $\Gamma_C$ do not touch.

For given data $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma_g)'$, the Signorini problem consists of finding $u$ such that

$$-\Delta u = f \qquad \text{in } \Omega, \tag{3.8}$$

$$u = 0 \qquad \text{on } \Gamma_u, \tag{3.9}$$

$$\frac{\partial u}{\partial n} = g \qquad \text{on } \Gamma_g, \tag{3.10}$$

$$u \geq 0, \qquad \frac{\partial u}{\partial n} \geq 0, \qquad u\frac{\partial u}{\partial n} = 0 \qquad \text{on } \Gamma_C, \tag{3.11}$$

where $n$ is the outward unit normal of $\partial\Omega$. This model is currently encountered in the conditioning field (where $u$ is a temperature) and in the hydrostatic domain (where $u$ is a pressure).

Sometimes, for practical reasons, one may want to use the mixed formulation where the condition $\varphi = \frac{\partial u}{\partial n} \geq 0$ is taken into account explicitly. In this approach the space for the displacement $u$ is the subspace $H^1_0(\Omega, \Gamma_u)$ of $H^1(\Omega)$ consisting of functions that vanish on $\Gamma_u$, and the flux (normal derivative) $\varphi = \frac{\partial u}{\partial n}\big|_{\Gamma_C}$ belongs to the closed convex set

$$M(\Gamma_C) = \Big\{\psi \in H^{\frac{1}{2}}(\Gamma_C)' : \psi \geq 0\Big\}.$$

Here the nonnegativity of a distribution $\psi \in H^{\frac{1}{2}}(\Gamma_C)'$ is to be understood in the sense that $\langle \psi, \chi\rangle_{\frac{1}{2},\Gamma_C} \geq 0$ for any nonnegative $\chi \in H^{\frac{1}{2}}(\Gamma_C)$. Then $(u, \varphi)$ is the solution of the following mixed variational inequality:

Find $(u, \varphi) \in H^1_0(\Omega, \Gamma_u) \times M(\Gamma_C)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \; dx + \langle \varphi, v\rangle_{\frac{1}{2},\Gamma_C} = \int_\Omega f v \; dx \qquad \forall\, v \in H^1_0(\Omega, \Gamma_u), \tag{3.12}$$

$$\langle \psi - \varphi, u\rangle_{\frac{1}{2},\Gamma_C} \geq 0 \qquad \forall\, \psi \in M(\Gamma_C). \tag{3.13}$$

A complete analysis of this mixed problem is provided in [17] (cf. also [16]) where an existence and uniqueness result is proven (cf. Theorem 3.14 therein). Moreover we have the following stability estimate:

$$\|u\|_{H^1(\Omega)} + \|\varphi\|_{H^{\frac{1}{2}}(\Gamma_C)'} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma_g)'}).$$

It is useful to note that if $K(\Omega)$ is the convex cone

$$K(\Omega) = \Big\{v \in H^1_0(\Omega, \Gamma_u) : v \geq 0\Big\},$$

then $u \in K(\Omega)$ is also the unique solution of the following primal problem:

Find $u \in K(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla(v - u) \; dx \geq \int_\Omega f(v - u) \; dx + \langle g, v - u\rangle_{\frac{1}{2},\Gamma_g} \qquad \forall\, v \in K(\Omega). \tag{3.14}$$

ETNA
Kent State University
etna@mcs.kent.edu



REMARK 3.3. *The Neumann and Signorini conditions (3.10) and (3.11) are taken into account in a weak sense in the primal variational inequality (3.14). Indeed, we have*

$$\langle \frac{\partial u}{\partial n}, v\rangle_{\frac{1}{2},\Gamma} - \langle g, v\rangle_{\frac{1}{2},\Gamma_g} \;\geq\; 0 \qquad \forall\, v \in H^{\frac{1}{2}}_{00}(\Gamma,\Gamma_u) \quad \textit{and} \quad v\big|_{\Gamma_C} \geq 0, \tag{3.15}$$

$$\langle \frac{\partial u}{\partial n}, u\rangle_{\frac{1}{2},\Gamma} - \langle g, u\rangle_{\frac{1}{2},\Gamma_g} = 0, \tag{3.16}$$

*where $H^{\frac{1}{2}}_{00}(\Gamma,\Gamma_u)$ is the subspace of $H^{\frac{1}{2}}(\Gamma)$ consisting of functions that vanish on $\Gamma_u$. Roughly speaking, (3.15) says that $\frac{\partial u}{\partial n} = g$ on $\Gamma_g$ and $\frac{\partial u}{\partial n} \geq 0$ on $\Gamma_C$ while (3.16) expresses the saturation condition $u\frac{\partial u}{\partial n} = 0$ on $\Gamma_C$.*

The approximation of the variational inequality (3.12)–(3.13) is obtained by generalizing the two-dimensional bubble stabilized mixed finite elements in [3] to three dimensions. The finite element tools are the same as those introduced in the previous section. We assume moreover that the mesh $\mathscr{T}_h^\Omega$ is compatible with the partition of the boundary, meaning that the trace of it to $\Gamma_u, \Gamma_g$ and to $\Gamma_C$ results in two dimensional triangulations. The triangulation of $\Gamma_C$ is denoted by $\mathscr{T}_h^C$.

Taking into account the Dirichlet boundary condition, we define

$$Y_h(\Omega) = \Big\{ v_h \in \mathscr{C}^0(\Omega) : v_h\big|_\kappa \in \mathcal{P}_1(\kappa) \quad \forall\, \kappa \in \mathscr{T}_h^\Omega \quad \text{and} \quad v_h\big|_{\Gamma_C} = 0 \Big\}.$$

The finite element space where $u_h$ is computed is then given by

$$X_h(\Omega) = Y_h(\Omega) \oplus \Big( \bigoplus_{T\in\mathscr{T}_h^C} \mathbb{R}\varphi_T \Big),$$

and the convex cone for the discrete Lagrange multipliers on $\Gamma_C$ is taken to be

$$M_h(\Gamma_C) = \Big\{ \psi_h \in L^2(\Gamma_C) : \psi_h\big|_T \in \mathcal{P}_0(T) \quad \forall\, T \in \mathscr{T}_h^C \quad \text{and} \quad \psi_h \geq 0 \Big\}.$$

We are now ready to set the discrete mixed variational inequality:

Find $(u_h, \varphi_h) \in X_h(\Omega) \times M_h(\Gamma_C)$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, dx + \langle \varphi_h, v_h\rangle_{\frac{1}{2},\Gamma_C} = \int_\Omega f v_h \, dx \qquad \forall\, v_h \in X_h(\Omega), \tag{3.17}$$

$$\langle \psi_h - \varphi_h, u_h\rangle_{\frac{1}{2},\Gamma_C} \geq 0 \qquad \forall\, \psi_h \in M_h(\Gamma_C). \tag{3.18}$$

Again the availability of the Babuška-Brezzi condition

$$\inf_{\psi_h\in M_h(\Gamma_C)} \sup_{v_h\in X_h(\Omega)} \frac{\langle \psi_h, v_h\rangle_{\frac{1}{2},\Gamma_C}}{\|v_h\|_{H^1(\Omega)}\|\psi_h\|_{H^{\frac{1}{2}}(\Gamma_C)'}} > \gamma$$

allows us to prove existence, uniqueness and stability results.

The analysis of the discretization error is based on the saddle point theory for variational inequalities (cf. [17], [16]). The methodology is to first obtain the convergence rate on the primal variable $u$ by analyzing (3.14) and its approximation. We have therefore to write down a variational problem for $u_h$ after suppressing the Lagrange multiplier. Let us then introduce the closed convex cone

$$K_h(\Omega) = \Big\{ v_h \in X_h(\Omega) : \langle \psi_h, v_h\rangle_{\frac{1}{2},\Gamma_C} = \int_{\Gamma_C} \psi_h v_h \, d\Gamma \geq 0 \quad \forall\, \psi_h \in M_h(\Gamma_C) \Big\}.$$

It is easy to see that $u_h$ is also the unique solution of the following variational inequality:

Find $u_h \in K_h(\Omega)$ such that

$$\int_\Omega \nabla u_h \cdot \nabla(v_h - u_h)\, dx \geq \int_\Omega f(v_h - u_h)\, dx + \langle g, v_h - u_h\rangle_{\frac{1}{2},\Gamma_g} \tag{3.19}$$

$\forall\, v_h \in K_h(\Omega)$.

Note that $K_h(\Omega) \not\subset K(\Omega)$ and hence problem (3.19) is a nonconforming discretization of (3.14). Applying Falk's lemma (cf. [14]) to (3.19) yields the following error estimate, the proof of which can be found for instance in [4].

LEMMA 3.4. *The following error estimate holds:*

$$\|u - u_h\|^2_{H^1(\Omega)} \leq C\Big[\inf_{v_h\in K_h(\Omega)}\big(\|u - v_h\|^2_{H^1(\Omega)} + < \frac{\partial u}{\partial n}, v_h >_{\frac{1}{2},\Gamma} - \langle g, v_h\rangle_{\frac{1}{2},\Gamma_g}\big) + \inf_{v\in K(\Omega)}\big(\langle \frac{\partial u}{\partial n}, v - u_h\rangle_{\frac{1}{2},\Gamma} - \langle g, v - u_h\rangle_{\frac{1}{2},\Gamma_g}\big)\Big]. \tag{3.20}$$

The first infimum of the bound given in Lemma 3.2 is the approximation error and the boundary term involved there is specifically generated by the discretization of variational inequalities. The second infimum is the consistency error, the price for the "variational crime" due to the nonconformity of the approximation. These two errors will be studied separately.

LEMMA 3.5. *Assume that for some* $\lambda$ $(0 < \lambda < \frac{1}{2})$ *we have* $g \in H^{\frac{1}{2}-\lambda}(\Gamma_g)'$ *and that the exact solution* $u$ *of Signorini's problem belongs to* $H^{1+\lambda}(\Omega)$. *Then the following estimate holds*:

$$\inf_{v_h\in K_h(\Omega)}\big(\|u - v_h\|^2_{H^1(\Omega)} + < \frac{\partial u}{\partial n}, v_h >_{\frac{1}{2},\Gamma} - \langle g, v_h\rangle_{\frac{1}{2},\Gamma_g}\big) \leq Ch^{2\lambda}\Big[\|u\|_{H^{1+\lambda}(\Omega)} + \|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}-\lambda}(\Gamma_g)'}\Big]\|u\|_{H^{1+\lambda}(\Omega)}.$$

*Proof*. It suffices to choose $v_h \in Y_h(\Omega)$ to be the Lagrange interpolant of $u$. It is checked that $v_h \in K_h(\Omega)$ since $v_h\big|_{\Gamma_C} \geq 0$. The estimate is derived following [4]. □

LEMMA 3.6. *Assume that for some* $\lambda$ $(0 < \lambda < \frac{1}{2})$ *we have* $g \in H^{\frac{1}{2}-\lambda}(\Gamma_g)'$ *and that the exact solution* $u$ *of Signorini's problem belongs to* $H^{1+\lambda}(\Omega)$. *Then the following estimate holds*:

$$\inf_{v\in K(\Omega)}\big(\langle \frac{\partial u}{\partial n}, v - u_h\rangle_{\frac{1}{2},\Gamma} - \langle g, v - u_h\rangle_{\frac{1}{2},\Gamma_g}\big) \leq Ch^{\lambda}\Big[\|u\|_{H^{1+\lambda}(\Omega)} + \|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}-\lambda}(\Gamma_g)'}\Big]\big(\|u - u_h\|_{H^1(\Omega)} + h^{\lambda}\|u\|_{H^{1+\lambda}(\Omega)}\big).$$

*Proof*. First of all, observe that since $g \in H^{\frac{1}{2}-\lambda}(\Gamma_g)'$ and $\frac{\partial u}{\partial n} \in H^{\frac{1}{2}-\lambda}(\Gamma)'$, a density argument and (3.15) imply

$$\langle \frac{\partial u}{\partial n}, v\rangle_{\frac{1}{2}-\lambda,\Gamma} - \langle g, v\rangle_{\frac{1}{2}-\lambda,\Gamma_g} \geq 0 \qquad \forall\, v \in H^{\frac{1}{2}-\lambda}(\Gamma) \quad \text{and} \quad v\big|_{\Gamma_C} \geq 0.$$

It follows that if $\psi_h = \Pi_h^0(\frac{\partial u}{\partial n}) \in M_h(\Gamma)$ is the piecewise constant interpolant of $\frac{\partial u}{\partial n}$, then we have

$$\psi_h\big|_T = \frac{1}{|T|}\langle \frac{\partial u}{\partial n}, 1_T\rangle_{\frac{1}{2}-\lambda,\Gamma} = \frac{1}{|T|}\langle g, 1_T\rangle_{\frac{1}{2}-\lambda,\Gamma_g} \qquad \forall\, T \in \mathscr{T}_h^C,$$

where $1_T \in H^{\frac{1}{2}-\lambda}(\Gamma)$ is the characteristic function of the set $T$. In particular, this means that $\psi_h\big|_{\Gamma_g} \in M_h(\Gamma_g)$ (respectively, $\psi_h\big|_{\Gamma_C} \in M_h(\Gamma_C)$) is the piecewise constant interpolant of $g$ on $\Gamma_g$ (respecttively, of $\varphi$ on $\Gamma_C$).

Now, to prove the result of the lemma we choose $v = u \in K(\Omega)$, then

$$\langle\frac{\partial u}{\partial n}, u - u_h\rangle_{\frac{1}{2},\Gamma} - \langle g, u - u_h\rangle_{\frac{1}{2},\Gamma_g} = \langle\psi_h, u - u_h\rangle_{\frac{1}{2}-\lambda,\Gamma} - \langle\psi_h, u - u_h\rangle_{\frac{1}{2}-\lambda,\Gamma_g} + \langle\frac{\partial u}{\partial n} - \psi_h, u - u_h\rangle_{\frac{1}{2}-\lambda,\Gamma} - \langle g - \psi_h, u - u_h\rangle_{\frac{1}{2}-\lambda,\Gamma_g}.$$

In view of the unilateral contact condition we have

$$\langle\psi_h, u_h\rangle_{\frac{1}{2}-\lambda,\Gamma} - \langle\psi_h, u_h\rangle_{\frac{1}{2}-\lambda,\Gamma_g} = \int_{\Gamma_C} \psi_h u_h \, d\Gamma \geq 0.$$

It follows that

$$\langle\psi_h, u - u_h\rangle_{\frac{1}{2}-\lambda,\Gamma} - \langle\psi_h, u - u_h\rangle_{\frac{1}{2}-\lambda,\Gamma_g} \leq \langle\psi_h, u\rangle_{\frac{1}{2}-\lambda,\Gamma} - \langle\psi_h, u\rangle_{\frac{1}{2}-\lambda,\Gamma_g}.$$

On account of the saturation condition (3.16) we have then

$$\begin{aligned}
&\langle\psi_h, u - u_h\rangle_{\frac{1}{2}-\lambda,\Gamma} - \langle\psi_h, u - u_h\rangle_{\frac{1}{2}-\lambda,\Gamma_g} \\
&\quad\leq \langle\psi_h - \frac{\partial u}{\partial n}, u\rangle_{\frac{1}{2}-\lambda,\Gamma} - \langle\psi_h - g, u\rangle_{\frac{1}{2}-\lambda,\Gamma_g} \\
&\quad\leq \|\frac{\partial u}{\partial n} - \psi_h\|_{H^{\frac{1}{2}+\lambda}(\Gamma)'}\|u\|_{H^{\frac{1}{2}+\lambda}(\Gamma)} + \|g - \psi_h\|_{H^{\frac{1}{2}+\lambda}(\Gamma_g)'}\|u\|_{H^{\frac{1}{2}+\lambda}(\Gamma_g)}.
\end{aligned}$$

As $g \in H^{\frac{1}{2}-\lambda}(\Gamma_g)'$ and $\frac{\partial u}{\partial n} \in H^{\frac{1}{2}-\lambda}(\Gamma)'$, an application of Theorem 2.4 produces

$$\langle\psi_h - \frac{\partial u}{\partial n}, u\rangle_{\frac{1}{2}-\lambda,\Gamma} - \langle\psi_h - g, u\rangle_{\frac{1}{2}-\lambda,\Gamma_g} \leq Ch^{2\lambda}\Big[\|u\|_{H^{1+\lambda}(\Omega)} + \|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}-\lambda}(\Gamma_g)}\Big]\|u\|_{H^{1+\lambda}(\Omega)}. \tag{3.21}$$

The remaining part is handled in the following way:

$$\langle\frac{\partial u}{\partial n} - \psi_h, u - u_h\rangle_{\frac{1}{2}-\lambda,\Gamma} - \langle g - \psi_h, u - u_h\rangle_{\frac{1}{2}-\lambda,\Gamma_g} \leq \|\frac{\partial u}{\partial n} - \psi_h\|_{H^{\frac{1}{2}}(\Gamma)'}\|u - u_h\|_{H^{\frac{1}{2}}(\Gamma)} + \|g - \psi_h\|_{H^{\frac{1}{2}}(\Gamma_g)'}\|u - u_h\|_{H^{\frac{1}{2}}(\Gamma)}.$$

Again by Theorem 2.4 we obtain

$$\langle\frac{\partial u}{\partial n} - \psi_h, u - u_h\rangle_{\frac{1}{2}-\lambda,\Gamma} - \langle g - \psi_h, u - u_h\rangle_{\frac{1}{2}-\lambda,\Gamma_g} \leq Ch^{\lambda}\left[\|u\|_{H^{1+\lambda}(\Omega)} + \|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}-\lambda}(\Gamma_g)'}\right]\|u - u_h\|_{H^1(\Omega)}. \tag{3.22}$$

Combining (3.21) and (3.22) yields the proof of the lemma. ☐

Assembling together the results of both lemmas and using a bootstrapping argument gives the final error estimate.

THEOREM 3.7. *Assume that for some $\lambda$ $(0 < \lambda < \frac{1}{2})$ we have $g \in H^{\frac{1}{2}-\lambda}(\Gamma_g)'$ and that the exact solution $u$ of Signorini's problem belongs to $H^{1+\lambda}(\Omega)$. Then the solution of (3.19) satisfies*

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^{\lambda}\Big[\|u\|_{H^{1+\lambda}(\Omega)} + \|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}-\lambda}(\Gamma_g)'}\Big].$$

**ETNA**
Kent State University
etna@mcs.kent.edu



Because of the *inf-sup* condition it is possible to establish a convergence rate for the Lagrange multipliers which is also optimal.

COROLLARY 3.8. *Assume that for some* $\lambda$ $(0 < \lambda < \frac{1}{2})$ *we have* $g \in H^{\frac{1}{2}-\lambda}(\Gamma_g)'$ *and that the exact solution* $u$ *of Signorini's problem belongs to* $H^{1+\lambda}(\Omega)$. *Then the following estimate holds:*

$$\|\varphi - \varphi_h\|_{H^{\frac{1}{2}}(\Gamma_C)'} \leq Ch^{\lambda}\Big[\|u\|_{H^{1+\lambda}(\Omega)} + \|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}-\lambda}(\Gamma_g)'}\Big].$$

*Proof*. The saddle point theory provides (cf. [17]) the estimate

$$\|\varphi - \varphi_h\|_{H^{\frac{1}{2}}(\Gamma_C)'} \leq \|u - u_h\|_{H^1(\Omega)} + \inf_{\psi_h \in M_h(\Gamma_C)} \|\varphi - \varphi_h\|_{H^{\frac{1}{2}}(\Gamma_C)'}.$$

Taking $\psi_h = \Pi_h^0(\frac{\partial u}{\partial n})|_{\Gamma_c} = \Pi_h^{0,C}\varphi$ and applying Theorem 2.4 gives the estimate. □

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