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ETNA
Kent State University
etna@mcs.kent.edu

GERŠGORIN-TYPE EIGENVALUE INCLUSION THEOREMS AND THEIR SHARPNESS*

RICHARD S. VARGA†

Dedicated to Professor John Todd, on the occasion of his 90th birthday, May 16, 2001.

Abstract. Here, we investigate the relationships between $\mathcal{G}(A)$, the union of Geršgorin disks, $\mathcal{K}(A)$, the union of Brauer ovals of Cassini, and $\mathcal{B}(A)$, the union of Brualdi lemniscate sets, for eigenvalue inclusions of an $n \times n$ complex matrix A. If $\sigma(A)$ denotes the spectrum of A, we show here that

$$\sigma(A) \subseteq \mathcal{B}(A) \subseteq \mathcal{K}(A) \subseteq \mathcal{G}(A)$$

is valid for any weakly irreducible $n \times n$ complex matrix A with $n \geq 2$. Further, it is evident that $\mathcal{B}(A)$ can contain the spectra of related $n \times n$ matrices. We show here that the spectra of these related matrices can fill out $\mathcal{B}(A)$. Finally, if $\mathcal{G}^{\mathcal{R}}(A)$ denotes the minimal Geršgorin set for A, we show that

$$\mathcal{G}^{\mathcal{R}}(A) \subseteq \mathcal{B}(A).$$

Key words. Geršgorin disks, Brauer ovals of Cassini, Brualdi lemniscate sets, minimal Geršgorin sets.

AMS subject classification. 15A18.

1. Geršgorin Disks and Ovals of Cassini. For any $n \geq 2$, let A be any $n \times n$ complex matrix (written $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$), and let $\sigma(A)$ denote its **spectrum** (i.e., $\sigma(A) := \{\lambda \in \mathbb{C} : \det[A - \lambda I_n] = 0\}$). A familiar result of Geršgorin [3] is that if

$$G_i(A) := \left\{ z \in \mathbb{C} : |z - a_{i,i}| \leq r_i(A) := \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}| \right\} \quad (1 \leq i \leq n) \tag{1.1}$$

denotes the i-th **Geršgorin disk** for A, then the union of these n Geršgorin disks contains all eigenvalues of A:

$$\sigma(A) \subseteq \mathcal{G}(A) := \bigcup_{i=1}^{n} G_i(A). \tag{1.2}$$

A less familiar result of Brauer [1] is that if

$$K_{i,j}(A) := \{ z \in \mathbb{C} : |z - a_{i,i}| \cdot |z - a_{j,j}| \leq r_i(A) \cdot r_j(A) \} \quad (1 \leq i, j \leq n; i \neq j) \tag{1.3}$$

denotes the (i,j)-th **Brauer oval of Cassini** for A, then similarly

$$\sigma(A) \subseteq \mathcal{K}(A) := \bigcup_{\substack{i,j=1 \\ i \neq j}}^{n} K_{i,j}(A), \tag{1.4}$$

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† Institute for Computational Mathematics, Department of Mathematics and Computer Science, Kent State University, Kent, Ohio 44242 (U.S.A.), Research supported under NSF Grant DMS-9707359. varga@msc.kent.edu

where $\mathcal{K}(A)$ now depends on $\binom{n}{2} = \frac{n(n-1)}{2}$ sets $K_{i,j}(A)$.

It is interesting that both $\mathcal{G}(A)$ and $\mathcal{K}(A)$ are defined solely from the *same* $2n$ numbers,

$$\{a_{i,i}\}_{i=1}^{n} \text{ and } \{r_i(A)\}_{i=1}^{n}, \tag{1.5}$$

determined from the matrix A, and it is natural to ask which of the sets $\mathcal{G}(A)$ and $\mathcal{K}(A)$ is smaller, as the smaller set would give a "tighter" estimate for the spectrum $\sigma(A)$. That $\mathcal{K}(A) \subseteq \mathcal{G}(A)$ holds in all cases is a result, not well known, which was stated by Brauer [1], and, as the idea of the proof is simple and will be used later in this paper, its proof is given here.

Theorem A. *For any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$ with $n \geq 2$,*

$$\mathcal{K}(A) \subseteq \mathcal{G}(A). \tag{1.6}$$

Proof. Fix any i and j, with $1 \leq i,\ j \leq n$ and $i \neq j$, and let z be any point of $K_{i,j}(A)$, so that from (1.3),

$$|z - a_{i,i}| \cdot |z - a_{j,j}| \leq r_i(A) \cdot r_j(A) \quad (1 \leq i, j \leq n; i \neq j). \tag{1.7}$$

If $r_i(A) \cdot r_j(A) = 0$, then $z = a_{i,i}$ or $z = a_{j,j}$. But, as $a_{i,i} \in G_i(A)$ and $a_{j,j} \in G_j(A)$ from (1.1), then $z \in G_i(A) \cup G_j(B)$. If $r_i(A) \cdot r_j(A) > 0$, we have from (1.7) that

$$\left(\frac{|z - a_{i,i}|}{r_i(A)}\right) \cdot \left(\frac{|z - a_{j,j}|}{r_j(A)}\right) \leq 1. \tag{1.8}$$

As the factors on the left of (1.8) cannot both exceed unity, then at least one of these factors is at most unity, i.e., $z \in G_i(A)$ or $z \in G_j(z)$. Hence, in either case, it follows that $z \in G_i(A) \cup G_j(A)$, so that

$$K_{i,j}(A) \subseteq G_i(A) \cup G_j(A) \quad (1 \leq i, j \leq n, i \neq j). \tag{1.9}$$

With (1.9), it follows from (1.2) and (1.4) that

$$\mathcal{K}(A) := \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} K_{i,j}(A) \subseteq \bigcup_{\substack{i,j=1\\ i\neq j}}^{n} \{G_i(A) \cup G_j(A)\} = \bigcup_{\ell=1}^{n} G_\ell(A) =: \mathcal{G}(\mathcal{A}),$$

the desired result of (1.6) □

We remark that the case of equality in the inclusion of (1.9) is covered (cf. [7]) in

$$\begin{cases} K_{i,j}(A) = G_i(A) \cup G_j(A) \text{ only if} \\ r_i(A) = r_j(A) = 0, \text{ or } r_i(A) = r_j(A) > 0 \text{ and } a_{i,i} = a_{j,j}. \end{cases} \tag{1.10}$$

As mentioned above, $\mathcal{G}(A)$ and $\mathcal{K}(A)$ depend solely on the same $2n$ numbers of (1.5) which are derived from the matrix A, but there is a continuum of matrices (for $n \geq 2$) which give rise to the same numbers in (1.5). More precisely, following the notations of [6], let

$$\Omega(A) := \left\{B = [b_{i,j}] \in \mathbb{C}^{n\times n} : b_{i,i} = a_{i,i} \text{ and } r_i(B) = r_i(A), 1 \le i \le n\right\}, \tag{1.11}$$

and let

$$\hat{\Omega}(A) := \left\{B = [b_{i,j}] \in \mathbb{C}^{n\times n} : b_{i,i} = a_{i,i} \text{ and } r_i(B) \le r_i(A), \quad 1 \le i \le n\right\}, \tag{1.12}$$

so that $\Omega(A) \subseteq \hat{\Omega}(A)$. We note, from the final inequality in (1.3), that the first inclusion of (1.4) is then valid for all matrices in $\Omega(A)$ or $\hat{\Omega}(A)$, i.e., with (1.6) and with the definitions of

$$\sigma(\Omega(A)) := \bigcup_{B\in\Omega(A)} \sigma(B), \quad \text{and } \sigma(\hat{\Omega}(A)) := \bigcup_{B\in\hat{\Omega}(A)} \sigma(B), \tag{1.13}$$

it follows that

$$\sigma(\Omega(A)) \subseteq \sigma(\hat{\Omega}(A)) \subseteq \mathcal{K}(A) \subseteq \mathcal{G}(A). \tag{1.14}$$

Recently in Varga and Krautstengl [7], the following was established. (For notation, if T is a set in the complex plane $\mathbb{C}$, then $\overline{T}$ denotes its closure, $T' := \mathbb{C}\backslash T$ its complement, and $\partial T := \overline{T} \bigcap \overline{(T')}$ its boundary.)

Theorem B. *For any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$ with $n \ge 2$,*

$$\sigma(\Omega(A)) = \begin{cases} \partial\mathcal{K}(A) = \partial K_{1,2}(A) \text{ if } n = 2, \text{ and} \\ \mathcal{K}(A) \text{ if } n \ge 3, \end{cases} \tag{1.15}$$

and, in general, for any $n \ge 2$,

$$\sigma(\hat{\Omega}(A)) = \mathcal{K}(A). \tag{1.16}$$

In other words, for $n \ge 3$, each point of the Brauer ovals of Cassini $\mathcal{K}(A)$ is an eigenvalue of *some* matrix in $\Omega(A)$ or $\hat{\Omega}(A)$, and, given only the data of (1.5), $\mathcal{K}(\mathcal{A})$ does a *perfect job* of estimating the spectra of *all* matrices in $\Omega(A)$ or $\hat{\Omega}(A)$.

2. Lemniscate and Brualdi Lemniscate Sets. Given an $n \times n$ complex matrix $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$, let $\{i_j\}_{j=1}^m$ be any m distinct positive integers from $N := \{1, 2, \cdots, n\}$, so that $n \ge m$. Then, the **lemniscate**[1] of **order m**, derived from $\{i_j\}_{j=1}^m$ and the $2n$ numbers $\{a_{i,i}\}_{i=1}^n$ and $\{r_i(A)\}_{i=1}^n$, is the compact set in $\mathbb{C}$ defined by

$$\ell_{i_1,\cdots,i_m}(A) := \left\{ z \in \mathbb{C} : \prod_{j=1}^m \left|z - a_{i_j,i_j}\right| \le \prod_{j=1}^m r_{i_j}(A) \right\}, \tag{2.1}$$

and their union, denoted by

[1] The classical definition of a lemniscate (cf. Walsh [8, p. 54]) is the curve, corresponding to the case of equality in (2.1). The above definition of a lemniscate then is the union of this curve and its interior.

$$\mathcal{L}_{(m)}(A) := \bigcup_{1 \le i_1, i_2, \cdots, i_m \le n} \ell_{i_1, i_2, \cdots, i_m}(A) \quad (\{i_j\}_{j=1}^m \text{ are distinct in } N), \tag{2.2}$$

is over all $\binom{n}{m}$ such choices of $\{i_j\}_{j=1}^m$ from N. As special cases, the Geršgorin disks $G_i(A)$ of (1.1) are lemniscates of order 1, while the Brauer ovals of Cassini $K_{i,j}(A)$ of (1.3) are lemniscates of order 2, so that with (1.2) and (1.4)), we have

$$\mathcal{L}_{(1)}(A) = \mathcal{G}(A) \text{ and } \mathcal{L}_{(2)}(A) = \mathcal{K}(A). \tag{2.3}$$

When one considers the proof of Geršgorin's result (1.2) or the proof of Brauer's result (1.4), the difference is that the former focuses on **one** row of the matrix A, while the latter focuses on **two** distinct rows of the matrix A. But, from the result of (1.6) of Theorem A, this would seem to suggest that "using more rows in A gives better eigenvalue inclusion results for the spectrum of A". Alas, it turns out that $\mathcal{L}_{(m)}(A)$, as defined in (2.2), **fails**, in general for $m > 2$, to give a set in the complex plane which contains the spectrum of each A in $\mathbb{C}^{n \times n}, n \ge m$, as the following example (attributed to Morris Newman in Marcus and Minc [4]) shows. It suffices to consider the 4×4 matrix

$$B := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ where } \sigma(B) = \{0, 1, 1, 2\}, \tag{2.4}$$

where $\{b_{i,i} = 1\}_{i=1}^4$ and where $r_1(B) = r_2(B) = 1; r_3(B) = r_4(B) = 0$. On choosing $m = 3$ in (2.1), then, for any choice of three distinct integers $\{i_1, i_2, i_3\}$ from $\{1, 2, 3, 4\}$, the product $r_{i_1}(B) \cdot r_{i_2}(B) \cdot r_{i_3}(B)$ is zero, and the associated lemniscate in (2.1), for B of (2.4), always reduces to the set of points z for which $|z - 1|^3 = 0$, so that $z = 1$ is its sole point. Hence, with (2.2), $\mathcal{L}_{(3)}(B) = \{1\}$, which does **not** contain $\sigma(B)$. (The same argument also gives $\mathcal{L}_{(4)}(B) = \{1\}$, and this can be extended to all $n > 2$.)

This brings us to the penetrating work of Brualdi [2], which shows how the union of higher-order lemniscates for a general matrix A (these lemniscates not necessarily being of the same order) **can** give compact sets in the complex plane $\mathbb{C}$ which contain $\sigma(A)$, thereby circumventing the counterexample of (2.4). Brualdi's construction depends on a very clever use of circuits, from the directed graph of A, which we now describe. Given $A = [a_{i,j}] \in \mathbb{C}^{n \times n}$, $n \ge 2$, then $\Gamma(A)$ is the **directed graph** on n distinct vertices $\{v_i\}_{i=1}^n$ for the matrix A, consisting of a (directed) **arc** $\overrightarrow{v_i v_j}$, from vertex v_i to vertex v_j, only if $i \ne j$ and if $a_{i,j} \ne 0$. (This omits the usual use of loops when $a_{i,i} \ne 0$.) A **path** π from vertex v_i to vertex v_j is a sequence $i = i_0, i_1, \cdots, i_k = j$ of vertices for which $\overrightarrow{v_{i_0} v_{i_1}}, \overrightarrow{v_{i_1} v_{i_2}}, \cdots, \overrightarrow{v_{i_{k-1}} v_{i_k}}$ are abutting arcs, and the **length** of the path π is said to be k. Then, the directed graph $\Gamma(A)$ is said to be **strongly connected** if, for each ordered pair (i, j) of vertices v_i and v_j, there is a path from v_i to v_j. (As is well-known, A is **irreducible** if and only if $\Gamma(A)$ is strongly connected.) A **circuit** γ of $\Gamma(A)$ is a path corresponding to the sequence $i_1, i_2, \cdots, i_p, i_{p+1} = i_1$ (where $p \ge 2$), where $i_1, i_2, \cdots, i_p$ are all distinct, and where $\overrightarrow{v_{i_1} v_{i_2}}, \cdots, \overrightarrow{v_{i_p} v_{i_1}}$ are arcs of $\Gamma(A)$. (The length of this circuit γ is p.) Then, $\mathcal{C}(A)$ denotes the set of all circuits γ in $\Gamma(A)$. Following Brualdi [2], a matrix A is said to be **weakly irreducible** if each vertex v_i of $\Gamma(A)$ belongs to **some** circuit γ in $\mathcal{C}(A)$. (Obviously, A irreducible implies A is weakly irreducible.)

Next, for $n \ge 2$, we define the set

$$\mathcal{P}_n := \{ \text{ all cycles, of length at least two, from the integers } (1, 2, \cdots, n) \},$$

where it can be verified that the cardinality of $\mathcal{P}_n$ is given by

$$\text{card } (\mathcal{P}_n) = \sum_{k=2}^{n} \binom{n}{k} (k-1)! \ . \tag{2.5}$$

Then, a circuit γ of $\Gamma(A)$, given by the sequence $i_1, i_2, \cdots, i_p, i_{p+1}$ with $i_{p+1} = i_1$, can be associated with an element in $\mathcal{P}_n$, i.e.,

$$\gamma = (i_1, i_2, \cdots, i_p) \in \mathcal{P}_n, \text{ where } 2 \leq p \leq n \text{ and where } i_{p+1} = i_1. \tag{2.6}$$

With the above notations and definitions, a result of Brualdi [2, Cor. 2.4], is

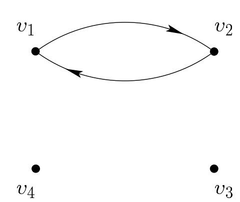
Theorem C. *For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2$, for which A is weakly irreducible, then*

$$\sigma(A) \subseteq \bigcup_{\gamma \in \mathcal{C}(A)} \left\{ z \in \mathbb{C} : \prod_{i \in \gamma} |z - a_{i,i}| \leq \prod_{i \in \gamma} r_i(A) \right\} =: \mathcal{B}(A). \tag{2.7}$$

We have introduced in (2.7) the quantity $\mathcal{B}(A)$, which we call the **Brualdi lemniscate set** for A, as it is the union of lemniscates (see (2.1)), of possibly *different* orders, derived from the matrix A.

For the matrix $B = [b_{i,j}]$ of (2.4), its directed graph is

$\Gamma(B)$:



and, as $\mathcal{C}(B)$ consists solely of the circuit $(1, 2)$, then B is not weakly irreducible, as there are no circuits through vertices v_3 or v_4. However, on considering the near-by matrix B_ϵ, defined by

$$B_\epsilon := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \epsilon \\ 0 & 0 & \epsilon & 1 \end{bmatrix} (\epsilon > 0), \text{ with } \sigma(B_\epsilon) = \{0, 1-\epsilon, 1+\epsilon, 2\},$$

its directed graph is

Then, $\mathcal{C}(B_\epsilon) = (1, 2) \cup (3, 4)$, and B_ϵ is thus weakly irreducible for any $\epsilon > 0$. Applying Theorem C to B_ϵ gives a valid eigenvalue inclusion for B_ϵ:

$$\sigma(B_\epsilon) \subseteq \left\{ z \in \mathbb{C} : |z - 1|^2 \leq 1 \right\} \cup \left\{ z \in \mathbb{C} : |z - 1|^2 \leq \epsilon^2 \right\}.$$

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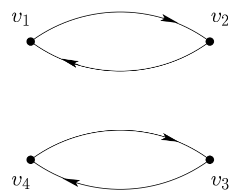
Kent State University
etna@mcs.kent.edu

$\Gamma(B_\epsilon)$:



We note that the eigenvalue inclusion of Brualdi's Theorem C, applied to a weakly irreducible matrix $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$ with $n \geq 2$, now depends on all the quantities of

$$\{a_{i,i}\}_{i=1}^{n}, \ \{r_i(A)\}_{i=1}^{n}, \text{ and } \mathcal{C}(A). \tag{2.8}$$

As (2.8) requires *more* information from the matrix A, in order to form the Brualdi lemniscate set $\mathcal{B}(A)$, then is required by the Brauer ovals of Cassini $\mathcal{K}(A)$ or the Geršgorin disks $\mathcal{G}(A)$, one might expect that $\mathcal{B}(A)$ is a set, which is *no larger*, in the complex plane, than $\mathcal{K}(A)$ or $\mathcal{G}(A)$. This will be shown to be true in Theorem 1 of the next section. In addition, one can ask, in the spirit of Theorem B, if the union of the spectra of all matrices, which match the data of (2.8), fills out the Brualdi lemniscate set $\mathcal{B}(A)$ of (2.7). This will be precisely answered in Theorem 2 of Section 4.

3. Comparison of Brauer's Ovals of Cassini and Brualdi's Lemniscate Sets. Our new result here is very much in the spirit of the proof of Theorem A. (See also [6].)

Theorem 1. *For any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}, n \geq 2$, which is weakly irreducible, then, with the definitions of* (1.4) *and* (2.7)*,*

$$\mathcal{B}(A) \subseteq K(A). \tag{3.1}$$

Remark. This establishes that the Brualdi lemniscate set for a matrix A is always no larger than the union of the Brauer ovals of Cassini for this matrix.

Proof. Consider any circuit γ in $\mathcal{C}(A)$. If this circuit has length two, i.e., $\gamma = (i_1, i_2)$, where $i_3 = i_1$, it follows from (2.7) that

$$\mathcal{B}_\gamma(A) := \{z \in \mathbb{C} : |z - a_{i_1,i_1}| \cdot |z - a_{i_2,i_2}| \leq r_{i_1}(A) \cdot r_{i_2}(A)\}\,, \tag{3.2}$$

which, from (1.3), is exactly $K_{i_1,i_2}(A)$, i.e.,

$$\mathcal{B}_\gamma(A) = K_{i_1,i_2}(A). \tag{3.3}$$

Next, assume that γ has length $p > 2$, i.e., $\gamma = (i_1, i_2, \cdots, i_p)$, where $i_{p+1} = i_1$. Since A is weakly irreducible, each vertex of $\Gamma(A)$ has a circuit passing through it, so that $r_\ell(A) > 0$ for each ℓ in N. From (2.7), we define

$$\mathcal{B}_\gamma(A) := \left\{ z \in \mathbb{C} : \prod_{j=1}^{p} \left|z - a_{i_j,i_j}\right| \leq \prod_{j=1}^{p} r_{i_j}(A) \right\}. \tag{3.4}$$

Let z be any point of $\mathcal{B}_\gamma(A)$. On squaring the inequality in (3.4), we have

$$|z-a_{i_1,i_1}|^2 \cdot |z-a_{i_2,i_2}|^2 \cdots |z-a_{i_p,i_p}|^2 \le r_{i_1}^2(A)\cdot r_{i_2}^2(A)\cdots r_{i_p}^2(A).$$

As these $r_{i_j}(A)$'s are all positive, we can equivalently express the above inequality as

$$\left(\frac{|z-a_{i_1,i_1}|\cdot|z-a_{i_2,i_2}|}{r_{i_1}(A)\cdot r_{i_2}(A)}\right)\cdot\left(\frac{|z-a_{i_2,i_2}|\cdot|z-a_{i_3,i_3}|}{r_{i_2}(A)\cdot r_{i_3}(A)}\right)\cdots \cdot\left(\frac{|z-a_{i_p,i_p}|\cdot|z-a_{i_1,i_1}|}{r_{i_p}(A)\cdot r_{i_1}(A)}\right)\le 1. \tag{3.5}$$

As the factors on the left of (3.5) cannot all exceed unity, then at least one of the factors is at most unity. Hence, there is an ℓ with $1\le \ell\le p$ such that

$$|z-a_{i_\ell,i_\ell}|\cdot|z-a_{i_{\ell+1},i_{\ell+1}}|\le r_{i_\ell}(A)\cdot r_{i_{\ell+1}}(A)$$

(where if $\ell=p$, then $i_{\ell+1}=i_1$). But from the definition in (1.3), we see that $z\in K_{i_\ell,i_{\ell+1}}(A)$, and, as a consequence, it follows that

$$\mathcal{B}_\gamma(A)\subseteq\bigcup_{j=1}^{p}K_{i_j,i_{j+1}}(A)\quad(\text{where } i_{p+1}=i_1). \tag{3.6}$$

Thus, from (2.7) and (3.6),

$$\mathcal{B}(A):=\bigcup_{\gamma\in\mathcal{C}(A)}\mathcal{B}_\gamma(A)\subseteq\bigcup_{\substack{i,j=1\\ i\neq j}}^{n}K_{i,j}(A):=\mathcal{K}(A),$$

the desired result of (3.1). □

We next show that there are many cases where equality holds in (3.1). Consider any matrix $A=[a_{i,j}]\in\mathbb{C}^{n\times n}$, with $n>2$, for which every non-diagonal entry $a_{i,j}$ of A is nonzero. The matrix A is then clearly irreducible, and hence, weakly irreducible. Moreover, *every* partition, of length ≥ 2, of $\mathcal{P}_n$ of (2.5) can be associated with a circuit γ of $\mathcal{C}(A)$, so that the total number of these circuits is, from (2.5),

$$\sum_{k=2}^{n}\binom{n}{k}(k-1)!\ . \tag{3.7}$$

In this case, as each Brauer oval of Cassini $K_{i,j}(A), i\neq j$, corresponds to a circuit (of length 2) in $\mathcal{B}(A)$, it follows from (2.7) that $\mathcal{K}(A)\subseteq\mathcal{B}(A)$, but as the reverse inclusion holds in (3.1), then

$$\mathcal{B}(A)=\mathcal{K}(A). \tag{3.8}$$

In other words, the Brualdi lemniscate set $\mathcal{B}(A)$ need not, in general, be a proper subset of the Brauer ovals of Cassini $\mathcal{K}(A)$. We remark that for a 10×10 complex matrix A, all of whose off-diagonal entries are nonzero, there are, from (3.7), 1,112,073 distinct circuits in $\mathcal{C}(A)$. But because of (3.8), only $\binom{10}{2}=45$ of these circuits, corresponding to the Brauer ovals of Cassini, are needed to determine $\mathcal{B}(A)$.

ETNA
Kent State University
etna@mcs.kent.edu

4. The Sharpness of Brualdi Lemniscate Sets. Given $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$, with $n \geq 2$ for which A is weakly irreducible, we have from (2.7) of Theorem C that

$$\sigma(A) \subseteq \mathcal{B}(A), \tag{4.1}$$

where the associated Brualdi lemniscate set $\mathcal{B}(A)$ is determined, in (2.7), from the quantities

$$\{a_{i,i}\}_{i=1}^{n}, \quad \{r_i(A)\}_{i=1}^{n}, \text{ and } \mathcal{C}(A). \tag{4.2}$$

It is again evident that any matrix $B = [b_{i,j}] \in \mathbb{C}^{n\times n}$, having the identical quantities of (4.2), has its eigenvalues also in $\mathcal{B}(A)$, i.e., with notations similar to (1.11) and (1.12), if

$$\Omega_{\mathcal{B}}(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n\times n} : b_{i,i} = a_{i,i}, r_i(B) = r_i(A), 1 \leq i \leq n, \text{ and } \mathcal{C}(B) = \mathcal{C}(A)\}, \tag{4.3}$$

where $\sigma(\Omega_{\mathcal{B}}(A)) := \bigcup_{B\in\Omega_{\mathcal{B}}(A)} \sigma(B)$, and if

$$\hat{\Omega}_{\mathcal{B}}(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n\times n} : b_{i,i} = a_{i,i}, 0 < r_i(B) \leq r_i(A), 1 \leq i \leq n, \text{ and } \mathcal{C}(B) = \mathcal{C}(A)\}, \tag{4.4}$$

where $\sigma(\hat{\Omega}_{\mathcal{B}}(A)) := \bigcup_{B\in\hat{\Omega}_{\mathcal{B}}(A)} \sigma(B)$, it follows, in analogy with (1.14), that

$$\sigma(\Omega_{\mathcal{B}}(A)) \subseteq \sigma(\hat{\Omega}_{\mathcal{B}}(A)) \subseteq \mathcal{B}(A). \tag{4.5}$$

(We note, since A is weakly irreducible, that $r_i(A) > 0$ for all $1 \leq i \leq n$.)

It is natural to ask if equality can hold throughout in (4.5). The answer, in general, is **no**, as the following simple example shows. Consider the matrix

$$D = \begin{bmatrix} +1 & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & +1 \end{bmatrix}, \tag{4.6}$$

so that

$$r_1(D) = r_2(D) = 1, \text{ and } r_3(D) = \frac{1}{2}.$$

The directed graph $\Gamma(D)$ is then
so that D is irreducible, and the circuit set of D is

$$\mathcal{C}(D) = (1,2) \cup (1,2,3).$$

Now, any matrix E in $\Omega_{\mathcal{B}}(D)$ can be expressed, from (4.3), as

$$E = \begin{bmatrix} 1 & e^{i\theta_1} & 0 \\ (1-s)e^{i\theta_2} & -1 & se^{i\theta_3} \\ \frac{1}{2}e^{i\theta_4} & 0 & 1 \end{bmatrix}, \tag{4.7}$$

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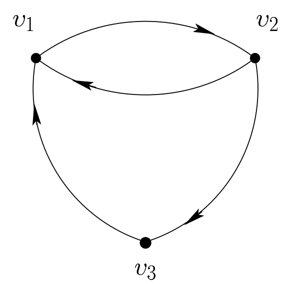
etna@mcs.kent.edu

$\Gamma(D)$:



where s satisfies $0 < s < 1$ and where $\{\theta_i\}_{i=1}^4$ are any real numbers. (Note that letting $s = 0$ or $s = 1$ in (4.7) does *not* preserve the circuit sets of $\mathcal{C}(D)$.) With $\gamma_1 := (1,2)$ and $\gamma_2 := (1,2,3)$, we see that

$$(4.8) \qquad \begin{cases} \mathcal{B}_{\gamma_1}(D) = \{z \in \mathbb{C} : |z-1| \cdot |z+1| \leq 1\}, \text{ and} \\ \mathcal{B}_{\gamma_2}(D) = \left\{z \in C : |z-1|^2 \cdot |z+1| \leq 1/2\right\}, \end{cases}$$

where the set $\mathcal{B}_{\gamma_2}(D)$ consists of two disjoint components. These sets are shown in Figure 1.

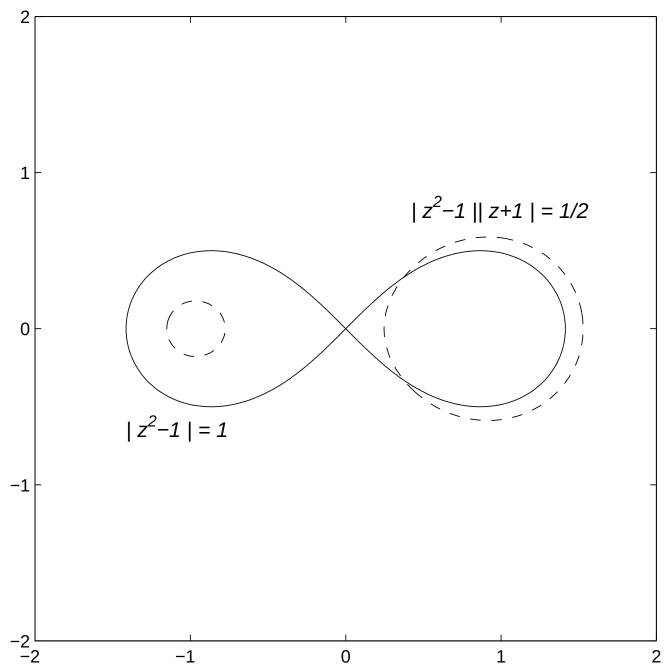


Figure 1

It can be seen, from (4.8) and from Figure 1, that $z = 0$ is a boundary point of the compact sets $\mathcal{B}_{\gamma_1}(D)$ and $\mathcal{B}(D) := \mathcal{B}_{\gamma_1}(D) \cup \mathcal{B}_{\gamma_2}(D)$. Suppose that we can find an s with $0 < s < 1$ and real values of $\{\theta_1\}_{i=1}^4$ for which an associated matrix E of (4.7) has eigenvalue 0. This implies that $\det E = 0$, which, by direct calculations with (4.7), gives

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Kent State University
etna@mcs.kent.edu

$$0 = \det E = -1 - (1-s)^{i(\theta_1+\theta_2)} + \frac{1}{2} s e^{i(\theta_1+\theta_3+\theta_4)}, \text{ or}$$

$$1 = \left\{ -(1-s) e^{i(\theta_1+\theta_2)} + \frac{1}{2} s e^{i(\theta_1+\theta_3+\theta_4)} \right\}. \tag{4.9}$$

But as $0 < s < 1$, the right side of (4.9) is in modulus at most

$$(1-s) + \frac{1}{2} s = \frac{2-s}{2} < 1,$$

so that $\det E \neq 0$ for any E in $\Omega_{\mathcal{B}}(D)$, i.e., $0 \notin \sigma(\Omega_{\mathcal{B}}(D))$. A similar argument shows (cf. (4.4)) that $0 \notin \sigma(\hat{\Omega}_{\mathcal{B}}(D))$. But as $0 \in \mathcal{B}(D)$, we have

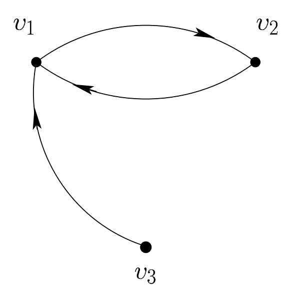
$$\sigma(\Omega_{\mathcal{B}}(D)) \subseteq \sigma(\hat{\Omega}_{\mathcal{B}}(D)) \stackrel{\subset}{\neq} \mathcal{B}(A). \tag{4.10}$$

But, in order to achieve equality in (4.10), suppose that we *allow* s to be zero, noting from (4.8) that the parameter s plays *no role* in $\mathcal{B}(D) = \mathcal{B}_{\gamma_1}(D) \cup \mathcal{B}_{\gamma_2}(D)$. Then, on setting $s = 0$ in (4.7), the matrix E of (4.7) becomes

$$\hat{E} = \begin{bmatrix} 1 & e^{i\theta_1} & 0 \\ e^{i\theta_2} & -1 & 0 \\ \frac{1}{2} e^{i\theta_4} & 0 & 1 \end{bmatrix}, \tag{4.11}$$

and on choosing $\theta_1 = 0$ and $\theta_2 = \pi$, then $z = 0$ is an eigenvalue of $\hat{E}$, where $\hat{E}$ is the *limit* of matrices E in (4.7) when $s \downarrow 0$. We note that the directed graph of $\Gamma(\hat{E})$ is

$\Gamma(\hat{E}):$



so that $\mathcal{C}(\hat{E}) \neq \mathcal{C}(D)$, but $\hat{E}$ remains an element of $\Omega(A)$ of (1.11).

This example suggests that we consider the **closures** of the sets $\Omega_{\mathcal{B}}(A)$ and $\hat{\Omega}_{\mathcal{B}}(A)$ of (4.3) and (4.4), where $A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2$, is weakly irreducible:

$$\overline{\Omega}_{\mathcal{B}}(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n \times n} : \text{ there is a sequence of matrices } \{E_j\}_{j=1}^{\infty} \text{ in } \Omega_{\mathcal{B}}(A), \text{ for which } B = \lim_{j \to \infty} E_j\}. \tag{4.12}$$

and

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Kent State University
etna@mcs.kent.edu

$$\hat{\overline{\Omega}}_{\mathcal{B}}(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n\times n} : \text{ there is a sequence of matrices } \{E_j\}_{j=1}^{\infty} \text{ in } \hat{\Omega}_{\mathcal{B}}(A), \text{ for which } B = \lim_{j\to\infty} E_j.\} \tag{4.13}$$

This brings us to the new result of

Theorem 2. *For any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}, n \geq 2$, which is weakly irreducible, then*

$$\partial\mathcal{B}(A) \subseteq \sigma(\overline{\Omega}_{\mathcal{B}}(A)) \subseteq \sigma(\hat{\overline{\Omega}}_{\mathcal{B}}(A)) = \mathcal{B}(A), \tag{4.14}$$

i.e., each boundary point of $\mathcal{B}(A)$ is an eigenvalue of some matrix in $\overline{\Omega}_{\mathcal{B}}(A)$, and each point of $\mathcal{B}(A)$ is an eigenvalue of some matrix in $\hat{\overline{\Omega}}_{\mathcal{B}}(A)$.

Remark. This establishes the sharpness of the Brualdi set $\mathcal{B}(A)$ for the matrix A, as the final equality in (4.14) gives that the spectra of matrices in $\hat{\Omega}_{\mathcal{B}}(A)$ are dense in $\mathcal{B}(A)$.

Proof. Since $\sigma(\Omega_{\mathcal{B}}(A)) \subseteq \sigma(\hat{\Omega}_{\mathcal{B}}(A))$ from (4.5), it follows that their closures of (4.12) and (4.13) necessarily satisfy $\sigma(\overline{\Omega}_{\mathcal{B}}(A)) \subseteq \sigma(\hat{\overline{\Omega}}_{\mathcal{B}}(A)$, giving the middle inclusion of (4.14). It suffices to establish the first inclusion and the final equality in (4.14).

Consider any circuit γ in $\mathcal{C}(A)$. From our discussion in Section 2, we can express γ as an element of $\mathcal{P}_n$ of (2.5), i.e.,

$$\gamma = (i_1, i_2, \cdots, i_p) \text{ where } i_{p+1} := i_1, \text{ and where } 2 \leq p \leq n. \tag{4.15}$$

Without loss of generality, we can assume, after a suitable permutation of the rows and columns of A, that

$$\gamma = (1, 2, \cdots, p), \tag{4.16}$$

noting that this permutation leaves unchanged the collection of diagonal entries, row sums, and circuits of A. This permutated matrix, also called A, then has the partitioned form

$$A = \left[\begin{array}{ccc|ccc} a_{1,1} & \cdots & a_{1,p} & a_{1,p+1} & \cdots & a_{1,n} \\ \vdots & & & & & \vdots \\ a_{p,1} & & a_{p,p} & a_{p,p+1} & & a_{p,n} \\ \hline a_{p+1,1} & & a_{p+1,p} & a_{p+1,p+1} & & a_{p+1,n} \\ \vdots & & & & & \vdots \\ a_{n,1} & \cdots & a_{n,p} & a_{n,p+1} & \cdots & a_{n,n} \end{array}\right] = \left[\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & A_{2,2} \end{array}\right], \tag{4.17}$$

where the matrices $A_{1,2}, A_{2,1}$, and $A_{2,2}$ are not present in (4.17) if $p = n$. Our aim below is to construct a special matrix $B(t) = [b_{i,j}(t)] \in \mathbb{C}^{n\times n}$, whose entries depend continuously on the parameter t in $[0, 1]$, such that

$$\begin{cases} b_{i,i}(t) = a_{i,i}, \ r_i(B(t)) = r_i(A), \text{ for all } 1 \leq i \leq n, \text{ and all } 0 \leq t \leq 1, \text{ and} \\ \mathcal{C}(B(t)) = \mathcal{C}(A) \text{ for all } 0 < t \leq 1. \end{cases} \tag{4.18}$$

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Kent State University
etna@mcs.kent.edu

To this end, write

$$B(t) := \left[\begin{array}{c|c} B_{1,1}(t) & B_{1,2}(t) \\ \hline A_{2,1} & A_{2,2} \end{array}\right], \tag{4.19}$$

i.e., the rows $p+1 \le \ell \le n$ of $B(t)$ are exactly those of A, and are independent of t. We note from (4.16) that

$$a_{1,2} \cdot a_{2,3} \cdots a_{p-1,p} \cdot a_{p,1} \neq 0, \tag{4.20}$$

and the rows of $B(t)$ are then defined, for all $t \in [0,1]$, by

$$\begin{cases} b_{i,i}(t) := a_{i,i} \text{ for all } 1 \le i \le p; \\ |b_{i,i+1}(t)| := (1-t)r_i(A) + t|a_{i,i+1}|, \text{ and } |b_{i,j}(t)| := t|a_{i,j}| \quad (j \neq i, i+1), \\ \qquad \text{for all } 1 \le i < p; \\ |b_{p,1}(t)| := (1-t)r_p(A) + t|a_{p,1}|, \text{ and } |b_{p,j}(t)| := t|a_{p,j}| \quad (\text{all } j \neq 1, p). \end{cases} \tag{4.21}$$

It is evident that the entries of $B(t)$ are all continuous in the variable t of $[0,1]$. Moreover, $B(t)$ and A have the same diagonal entries and the same row sums for all $0 \le t \le 1$, and, as $a_{i,j} \neq 0$ implies $b_{i,j}(t) \neq 0$ for all $0 < t \le 1$, then $B(t)$ and A have the same circuits in their directed graphs for all $0 < t \le 1$. As A is weakly irreducible by hypothesis, it follows that $B(t)$ is weakly irreducible for all $0 < t \le 1$. Also, from (4.3), $B(t) \in \Omega_{\mathcal{B}}(A)$ for all $0 < t \le 1$, and from (4.12), $B(0) \in \overline{\Omega}_{\mathcal{B}}(A)$. Hence, from (4.5),

$$\sigma(B(t)) \subseteq \mathcal{B}(A) \text{ for all } 0 < t \le 1. \tag{4.22}$$

But as $\mathcal{B}(A)$ is a closed set from (2.7), and as the eigenvalues of $B(t)$ are continuous functions of t, for $0 \le t \le 1$, we further have, for the limiting case $t = 0$, that

$$\sigma(B(0)) \subseteq \mathcal{B}(A), \tag{4.23}$$

where, from the definitions in (4.21),

$$B(0) = \left[\begin{array}{c|c} B_{1,1}(0) & O \\ \hline A_{2,1} & A_{2,2} \end{array}\right],$$

with

$$B_{1,1}(0) = \begin{bmatrix} a_{1,1} & r_1(A)e^{i\theta_1} & & & \\ & a_{2,2} & r_2(A)e^{i\theta_2} & & \\ & & \ddots & \ddots & \\ & & & a_{p-1,p-1} & r_{p-1}(A)e^{i\theta_{p-1}} \\ r_p(A)e^{i\theta_p} & & & & a_{p,p} \end{bmatrix}. \tag{4.24}$$

We note from (4.21) that the nondiagonal entries in the first p rows of $B(t)$ are defined only in terms of their moduli, which allows us to *fix* the arguments of certain nondiagonal nonzero entries of $B_{1,1}(0)$ through the factors $\{e^{i\theta_j}\}_{j=1}^p$ where the $\{\theta_j\}_{j=1}^p$ are all real. (These factors appear in $B_{1,1}(0)$ of (4.24).) The partitioned form of $B(0)$ gives us that

$$\sigma(B(0)) = \sigma(B_{1,1}(0)) \cup \sigma(A_{2,2}), \tag{4.25}$$

and, from the special cyclic-like form of $B_{1,1}(0)$ in (4.24), it is easily seen that each eigenvalue λ of $B_{1,1}(0)$ satisfies

$$\prod_{i=1}^{p} |\lambda - a_{i,i}| = \prod_{i=1}^{p} r_i(A), \tag{4.26}$$

for any real choices of $\{\theta_j\}_{j=1}^p$ in (4.24). But (4.26), when coupled with the definition in (3.4), immediately gives us that $\lambda \in \partial\mathcal{B}_\gamma(A)$, and, as all different choices of the real numbers $\{\theta_j\}_{j=1}^p$, in $B_{1,1}(0)$ of (4.24), give eigenvalues of $B_{1,1}(0)$ which cover the *entire boundary* of $\mathcal{B}_\gamma(A)$, we have

$$\bigcup_{\theta_1,\cdots,\theta_p real} \sigma(B_{1,1}(0)) = \partial\mathcal{B}_\gamma(A). \tag{4.27}$$

This can be used as follows. Let z be any boundary point of $\mathcal{B}(A)$ of (2.7). As $\mathcal{B}(A)$ is the union of a finite number of closed sets $\mathcal{B}_\gamma(A)$, this implies that there is a circuit γ of $\mathcal{C}(A)$ with $z \in \partial\mathcal{B}_\gamma(A)$, where $\mathcal{B}_\gamma(A)$ is defined in (3.4). As the result of (4.27) is valid for any γ of $\mathcal{C}(A)$, then each boundary point z of $\mathcal{B}(A)$ is an eigenvalue of some matrix in $\overline{\Omega}_{\mathcal{B}}(A)$ of (4.12), i.e.,

$$\partial\mathcal{B}(A) \subseteq \sigma(\overline{\Omega}_{\mathcal{B}}(A)), \tag{4.28}$$

which is the desired first inclusion of (4.14).

To investigate how the eigenvalues of $\overline{\hat{\Omega}}_{\mathcal{B}}(A)$ of (4.13) fill out $\mathcal{B}(A)$, we make a small change in the definition of the matrix $B(t)$ of (4.18) and (4.21). Let $\{\tau_i\}_{i=1}^p$ be any positive numbers such that

$$0 < \tau_i \le r_i(A) \quad (1 \le i \le p), \tag{4.29}$$

and let $\tilde{B}(t) = [\tilde{b}_{i,j}(t)] \in \mathbb{C}^{n\times n}$ have the same partitioned form as $B(t)$ of (4.19), but with (4.21) replaced by

$$\begin{cases} \tilde{b}_{i,i}(t) := a_{i,i} \text{ for all } 1 \le i \le p, \text{ and} \\ |\tilde{b}_{i,j}(t)| := \dfrac{\tau_i}{r_i(A)} |b_{i,j}(t)| \quad (j \ne i), \text{ for } 1 \le i \le p. \end{cases} \tag{4.30}$$

Then, $\tilde{B}(t)$ and A have the same diagonal entries, the row sums of $\tilde{B}(t)$ now satisfy $r_j(\tilde{B}(t)) = \tau_j$ for all $1 \le j \le p$, all $0 \le t \le 1$, and $\tilde{B}(t)$ and A have the same circuits for all $0 < t \le 1$. From (4.4), $\tilde{B}(t) \in \hat{\Omega}_{\mathcal{B}}(A)$ for all $0 < t \le 1$, and from (4.13), $\tilde{B}(0) \in \overline{\hat{\Omega}}_{\mathcal{B}}(A)$. In analogy with (4.23), we have

$$\tilde{B}(0) = \left[\begin{array}{c|c} \tilde{B}_{1,1}(0) & O \\ \hline A_{2,1} & A_{2,2} \end{array}\right],$$

with

$$\tilde{B}_{1,1}(0) = \begin{bmatrix} a_{1,1} & \tau_1 e^{i\theta_1} & & & \\ & a_{2,2} & \tau_2 e^{i\theta_2} & & \\ & & \ddots & \ddots & \\ & & & a_{p-1,p-1} & \tau_{p-1} e^{i\theta_{p-1}} \\ \tau_p e^{i\theta_p} & & & & a_{p,p} \end{bmatrix}, \tag{4.31}$$

where

$$\sigma(\tilde{B}(0)) = \sigma(\tilde{B}_{1,1}(0)) \cup \sigma(A_{2,2}). \tag{4.32}$$

It similarly follows that any eigenvalue λ of $\tilde{B}_{1,1}(0)$ in (4.31) now satisfies

$$\prod_{j=1}^{p} |\lambda - a_{i,i}| = \prod_{i=1}^{p} \tau_i, \tag{4.33}$$

for any choice of the real numbers $\{\theta_j\}_{j=1}^{p}$ in $\tilde{B}_{1,1}(0)$ of (4.31). Using the fact that $\{\tau_i\}_{i=1}^{p}$ are any numbers satisfying (4.29) and that $\{\theta_i\}_{i=1}^{p}$ are any real numbers, it follows from (3.4) and closure considerations that all the eigenvalues of $\tilde{B}_{1,1}(0)$ fill out $\mathcal{B}_\gamma(A)$, i.e.,

$$\bigcup_{\tau_i' s, \theta_j' s} \sigma(\tilde{B}_{1,1}(0)) = \mathcal{B}_\gamma(A). \tag{4.34}$$

As this holds for any $\gamma \in \mathcal{C}(A)$, where $\tilde{B}(0) \in \overline{\hat{\Omega}}_{\mathcal{B}}(A)$, then

$$\sigma(\overline{\hat{\Omega}}_{\mathcal{B}}(A)) = \mathcal{B}(A), \tag{4.35}$$

the desired final result of (4.14). $\Box$

5. A Comparison of Minimal Geršgorin Sets and Brualdi Lemniscate Sets. Given $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$ and $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T > \mathbf{0}$ in $\mathbb{R}^n$, then with $X := \operatorname{diag}[x_1, x_2, \cdots, x_n]$, we have that $X^{-1}AX = [\frac{a_{i,j}x_j}{x_i}]$, where, since A and $X^{-1}AX$ are similar matrices, $\sigma(A) = \sigma(X^{-1}AX)$. On setting

$$r_i^{\mathbf{x}}(A) := \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{|a_{i,j}|x_j}{x_i} \quad (\text{all } i \in N), \tag{5.1}$$

then Geršgorin Theorem, applied to $X^{-1}AX$, gives

$$\sigma(A) = \sigma(X^{-1}AX) \subseteq \bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - a_{i,i}| \leq r_i^{\mathbf{x}}(A)\} =: \mathcal{G}^{\mathbf{x}}(A). \tag{5.2}$$

As this eigenvalue inclusion holds for each $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T > \mathbf{0}$ in $\mathbb{R}^n$, we have

$$\sigma(A) \subseteq \bigcap_{\mathbf{x}>\mathbf{0}} \mathcal{G}^{\mathbf{x}}(A) =: \mathcal{G}^{\mathcal{R}}(A), \tag{5.3}$$

and $\mathcal{G}^{\mathcal{R}}(A)$ is called (cf. [5]) the **minimal Geršgorin set** for A, relative to the collection of all weighted row sums $\{r_i^{\mathbf{x}}(A)\}_{i=1}^n$. As $\mathcal{G}^{\mathbf{x}}(A)$ is, for each $\mathbf{x} > \mathbf{0}$, the union of n closed disks in the complex plane $\mathbb{C}$, then $\mathcal{G}^{\mathbf{x}}(A)$ is compact, as is $\mathcal{G}^{\mathcal{R}}(A)$.

Another natural question is how the Brualdi lemniscate set $\mathcal{B}(A)$ of (2.7), a compact set in $\mathbb{C}$, compares with the minimal Geršgorin set $\mathcal{G}^{\mathcal{R}}(A)$ of (5.3), for every $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$. The first apparent difference is that the Brualdi lemniscate set $\mathcal{B}(A)$ requires A to be weakly irreducible, whereas the minimal Geršgorin set $\mathcal{G}^{\mathcal{R}}(A)$ does not make this restriction. But, if we assume that $A = [a_{i,j}] \in \mathbb{C}^{n\times n}$ is weakly reducible, we can compare the associated sets in the complex plane. Given $A = [a_{i,j}] \in \mathbb{C}^{n\times n}, n \geq 2$, we consider the following set of matrices, associated with A:

$$\begin{aligned} \Delta(A) := \ & \{B = [b_{i,j}] \in \mathbb{C}^{n\times n} : b_{i,i} = a_{i,i} \text{ and} \\ & |b_{i,j}| \leq |a_{i,j}| \text{ for all } i \neq j; i,j \in N\}. \end{aligned} \tag{5.4}$$

Lemma 3. *For any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}, n \geq 2$, which is weakly irreducible, then* (cf. (4.13))

$$\Delta(A) \subseteq \overline{\hat{\Omega}}_{\mathcal{B}}(A). \tag{5.5}$$

Proof. We first note that as A is weakly irreducible, then $r_i(A) > 0$ for all $i \in N$. Consider any matrix $B = [b_{i,j}]$ in $\Delta(A)$, so that from (5.4),

$$r_i(B) \leq r_i(A) \quad \text{for all } i \in N. \tag{5.6}$$

Set $S_i(A) := \{j \in N : j \neq i \text{ and } |a_{i,j}| > 0\}$, for each $i \in N$. Then, $S_i(A) \neq \emptyset$ for any $i \in N$, since A is weakly irreducible. If there is a j in $S_i(A)$ for which $b_{i,j} = 0$, we note that

$$r_i(B) < r_i(A). \tag{5.7}$$

Then, for a fixed $\epsilon > 0$, replace this (i,j)-th entry of B by any number having modulus ϵ, and do the same for every k in $S_i(A)$ for which $b_{i,k} = 0$, leaving the remaining entries in this i-th row, of B, unchanged. On carrying out this procedure for all rows of the matrix B, a matrix $B(\epsilon)$, in $\mathbb{C}^{n\times n}$, is created, whose entries are continuous in the parameter ϵ, and for which the circuit set $\mathcal{C}(B(\epsilon))$ of $B(\epsilon)$ is identical with the circuit set $\mathcal{C}(A)$ of A, for each

$\epsilon > 0$. In addition, because of the strict inequality in (5.7) whenever $b_{i,j} = 0$ with $j \in S_i(A)$, it follows, for all $\epsilon > 0$ sufficiently small, that

(5.8) $$r_i(B(\epsilon)) \leq r_i(A) \quad \text{for all } i \in N.$$

Hence from (4.4), $B(\epsilon) \in \hat{\Omega}_{\mathcal{B}}(A)$ for all $\epsilon > 0$, sufficiently small. As such, we see from the definition in (4.13) that $B(0) = B \in \overline{\hat{\Omega}}_{\mathcal{B}}(A)$, and as this holds for any $B \in \Delta(A)$, the inclusion of (5.5) is valid. □

This brings us to the following new result which is both surprising and simple.

Theorem 4. *For any $A = [a_{i,j}] \in \mathbb{C}^{n\times n}, n \geq 2$, which is weakly irreducible, then*

(5.9) $$\mathcal{G}^{\mathcal{R}}(A) \subseteq \mathcal{B}(A).$$

Remark. The word "minimal" in the minimal Geršgorin set $\mathcal{G}^{\mathcal{R}}(A)$ seems to be appropriate!

Proof. It is known from [5] that $\mathcal{G}^{\mathcal{R}}(A)$ of (5.3) satisfies

(5.10) $$\mathcal{G}^{\mathcal{R}}(A) = \sigma(\Delta(A)),$$

and as $\Delta(A) \subseteq \overline{\hat{\Omega}}_{\mathcal{B}}(A)$ from (5.5) of Lemma 3, then $\sigma(\Delta(A)) \subseteq \sigma(\overline{\hat{\Omega}}_{\mathcal{B}}(A))$. But as $\sigma(\overline{\hat{\Omega}}_{\mathcal{B}}(A) = \mathcal{B}(A)$ from (4.14) of Theorem 2, then these inclusions give that

$$\mathcal{G}^{\mathcal{R}}(A) \subseteq \mathcal{B}(A),$$

the desired result of (5.9). □

6. An Example. To illustrate the above results, consider the matrix

(6.1) $$F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ \frac{1}{2} & i & \frac{1}{2} & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -i \end{bmatrix}.$$

Then, F is irreducible, with $\mathcal{C}(F) = (1,2) \cup (1,2,3,4)$, with row sums $r_i(F) = 1$ for all $1 \leq i \leq 4$. In this case, we have from (2.7) that $\mathcal{B}(F)$ consists of the union of the two closed sets

(6.2) $$\begin{cases} \mathcal{B}_{\gamma_1}(F) := \{z \in \mathbb{C} : |z-1| \cdot |z-i| \leq 1\} = K_{1,2}(A), \text{ and} \\ \\ \mathcal{B}_{\gamma_2}(F) := \{z \in \mathbb{C} : |z^4 - 1| \leq 1\}. \end{cases}$$

These sets are shown in Figure 2, where $\mathcal{B}_{\gamma_2}(F)$ has the shape of a four-leaf clover.

Next, any matrix h in $\hat{\Omega}_B(F)$ can be expressed from (4.4) as

(6.3) $$H = \begin{bmatrix} 1 & \tau_1 e^{i\theta_1} & 0 & 0 \\ \tau_2 s e^{i\theta_2} & i & \tau_3(1-s)e^{i\theta_3} & 0 \\ 0 & 0 & -1 & \tau_4 e^{i\theta_4} \\ \tau_5 e^{i\theta_5} & 0 & 0 & -i \end{bmatrix},$$

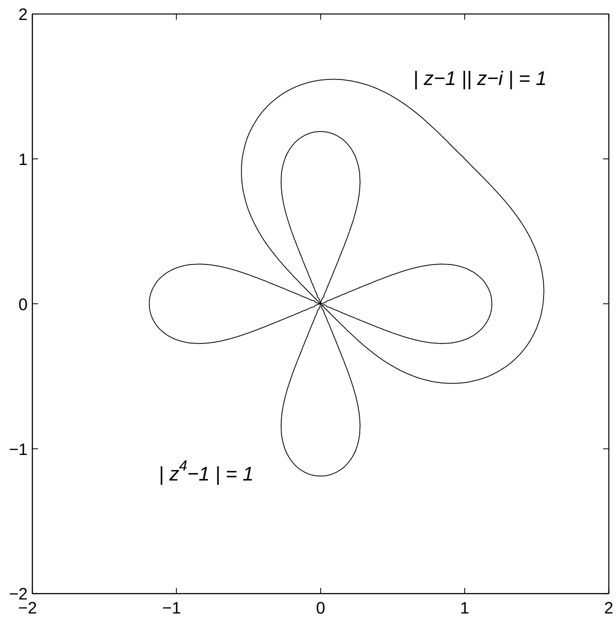

Figure 2

where $0 < \tau_i \leq 1$ and $0 \leq \theta_i \leq 2\pi$ for all $1 \leq i \leq 5$ and $0 < s < 1$. To show how the eigenvalues of H fill out $\mathcal{B}(F)$, we take random numbers $\{\tau_i\}_{i=1}^{5}$, random numbers $\{\theta_i\}_{i=1}^{5}$ from $[0, 2\pi)$, and a random number s from $(0, 1)$, and the eigenvalues of all these matrices, are plotted in Figure 3. Figure 3 indeed shows that these eigenvalues of F tend to fill out $\mathcal{B}(F)$.

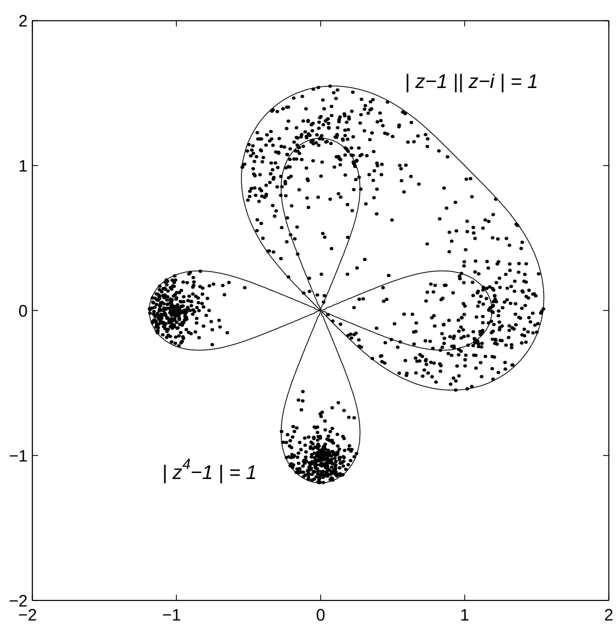

Figure 3

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Kent State University
etna@mcs.kent.edu

It is of interest to note the following **near paradox** arising from Theorem 2. As an example, F of (6.1) is irreducible, and it is a known result of Brualdi [3, Cor. 2.11] that a boundary point z of $\mathcal{B}(F)$ can be an eigenvalue of F only if z is a boundary point of *each* of the lemniscates $\mathcal{B}_{\gamma_1}(F)$ and $\mathcal{B}_{\gamma_2}(F)$ of (6.2). But from Figure 2, it is apparent that $z = 0$ is the *only* point for which $\partial\mathcal{B}_{\gamma_1}(F)$ and $\partial\mathcal{B}_{\gamma_2}(F)$ have a common point. Yet, (4.14) of Theorem 2 gives the nearly contradictory result that, for *each* point of $\partial\mathcal{B}(F)$, there is an arbitrarily close eigenvalue of some matrix in $\Omega_{\mathcal{B}}(F)$. The difference, of course, lies in the fact that Cor. 2.11 of [2] applies to the *fixed* matrix F, while the common data of (4.2) apply to *all* matrices in $\Omega_{\mathcal{B}}(F)$.

Next, we consider the minimal Geršgorin set, $\mathcal{G}^{\mathcal{R}}(F)$, for the matrix F of (6.1). The boundary of this set, $\partial\mathcal{G}^{\mathcal{R}}(F)$, can be verified to be

(6.4) $$\partial\mathcal{G}^{\mathcal{R}}(F) = \left\{ z \in \mathbb{C} : |z^4 - 1| = \frac{1}{2}(|z+1| \cdot |z+i|) + \frac{1}{2} \right\},$$

and $\partial\mathcal{G}^{\mathcal{R}}(F)$ is shown in Figure 4. Note that $\partial\mathcal{G}^{\mathcal{R}}(F)$ consists of **three** separate components.

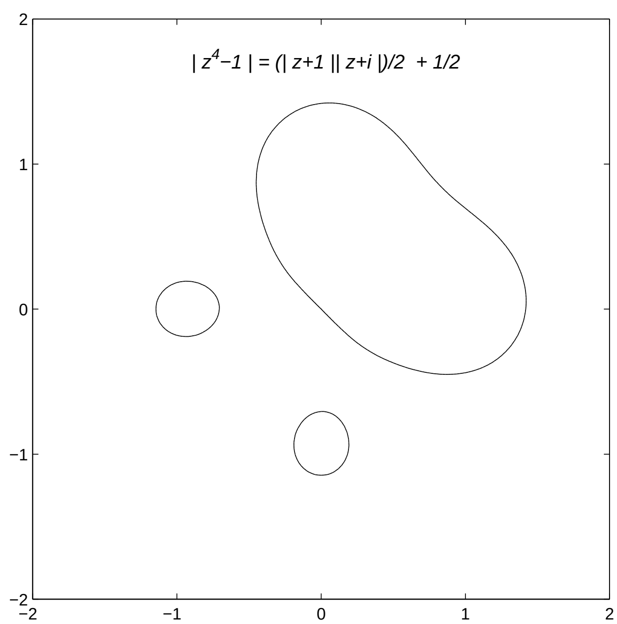

Figure 4

Next, we see from (5.4) that any matrix J in $\Delta(F)$ is of the form

(6.5) $$J = \begin{bmatrix} 1 & \tau_1 e^{i\theta_1} & 0 & 0 \\ \frac{\tau_2 e^{i\theta_2}}{2} & i & \frac{\tau_3 e^{i\theta_3}}{2} & 0 \\ 0 & 0 & -1 & \tau_4 e^{i\theta_4} \\ \tau_5 e^{i\theta_5} & 0 & 0 & -i \end{bmatrix},$$

where $0 \le \tau_i \le 1$ and $0 \le \theta_i \le 2\pi$, for all $1 \le i \le 5$. We similarly consider the eigenvalues of j of (6.5), where we take random choices of $\{s_i\}_{i=1}^{5}$ in $[0, 1]$, and random choices of

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Kent State University
etna@mcs.kent.edu

$\{\theta_i\}_{i=1}^5$ in $[0, 2\pi]$ in (6.5). These eigenvalues are plotted in Figure 5, which show again how they fill out $\mathcal{G}^{\mathcal{R}}(F)$. In Figure 6, we show both $\mathcal{B}(F)$ and $\mathcal{G}^{\mathcal{R}}(F)$, and we directly see that

$$\mathcal{G}^{\mathcal{R}}(F) \subsetneqq \mathcal{B}(F). \tag{6.6}$$

That (6.6) holds can be seen from the fact that each matrix J of (6.5) is necessarily a matrix H of (6.4), but not conversely.

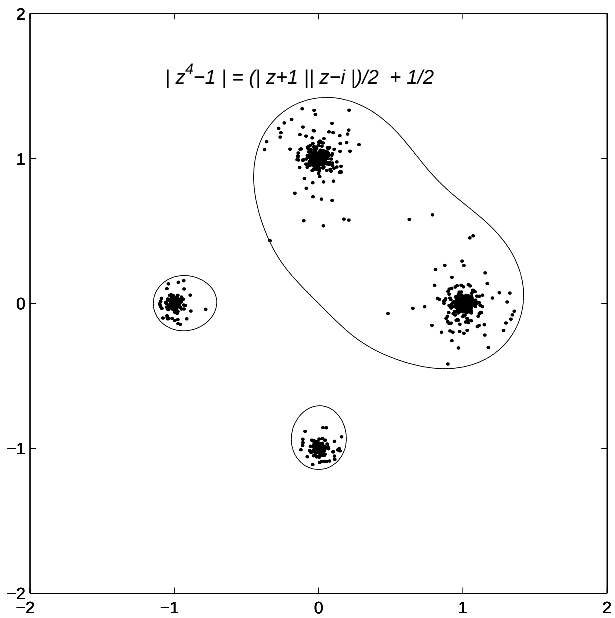

Figure 5

7. A Final Equality. The result, of (5.9) of Theorem 4, shows that the minimal Geršgorin set $\mathcal{G}^{\mathcal{R}}(A)$ is always a subset of the Brualdi lemniscate set. Adding to this from the inclusion of (3.1) and (1.6), we then have

$$\mathcal{G}^{\mathcal{R}}(A) \subseteq \mathcal{B}(A) \subseteq K(A) \subseteq \mathcal{G}(A), \tag{7.1}$$

for any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2$, which is weakly irreducible. But, the last three sets in (7.1) have *no* dependence on weighted row sums, while the first set certainly does from its definition in (5.3). To see the effect that weighted sums can have, consider any $\mathbf{x} = [x_1, x_2, \cdots, x_n] > \mathbf{0}$, and with $r_i^{\mathbf{x}}(A)$ of (5.1), we define (cf. (1.3)), in analogy with (5.2) and (5.3),

$$K_{i,j}^{\mathbf{x}}(A) := \{z \in \mathbb{C} : |z - a_{i,i}| \cdot |z - a_{j,j}| \leq r_i^{\mathbf{x}}(A) \cdot r_j^{\mathbf{x}}(A)\} \quad (1 \leq i, j \leq n; i \neq j), \tag{7.2}$$

and (cf. (1.4))

$$\mathcal{K}^{\mathbf{x}}(A) := \bigcup_{\substack{i,j=1 \\ i \neq j}}^{n} K_{i,j}^{\mathbf{x}}(A), \text{ and } \mathcal{K}^{\mathcal{R}}(A) := \bigcap_{\mathbf{x} > \mathbf{0}} \mathcal{K}^{\mathbf{x}}(A). \tag{7.3}$$

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Kent State University
etna@mcs.kent.edu

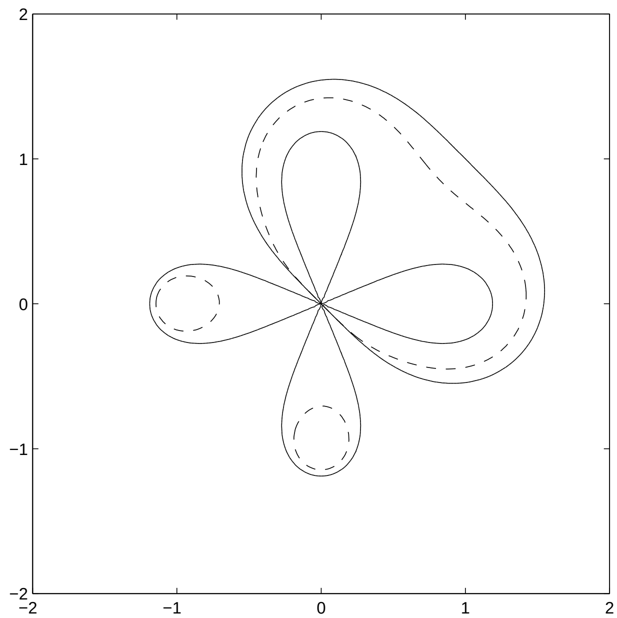

Figure 6

Similarly, for any circuit γ of $\Gamma(A)$, we similarly define (cf. (3.4) and (2.7))

$$\mathcal{B}_\gamma^{\mathbf{x}}(A) := \{z \in \mathbb{C} : \prod_{i\in\gamma} |z - a_{i,i}| \leq \prod_{i\in\gamma} r_i^{\mathbf{x}}(A)\}, \tag{7.4}$$

and

$$\mathcal{B}^{\mathbf{x}}(A) = \bigcup_{\gamma\in C(A)} \mathcal{B}_\gamma^{\mathbf{x}}(A), \text{ and } \mathcal{B}^{\mathcal{R}}(A) := \bigcap_{\mathbf{x}>\mathbf{0}} \mathcal{B}^{\mathbf{x}}(A). \tag{7.5}$$

As we see from (1.1) and (5.2), $r_i^{\mathbf{x}}(A) = r_i(X^{-1}AX)$. Hence, the last three inclusions of (7.1), applied to the matrix $X^{-1}AX$, become

$$\mathcal{B}^{\mathbf{x}}(A) \subseteq \mathcal{K}^{\mathbf{x}}(A) \subseteq \mathcal{G}^{\mathbf{x}}(A), \tag{7.6}$$

and as (7.6) holds for any $\mathbf{x} > \mathbf{0}$, then

$$\mathcal{B}^{\mathcal{R}}(A) \subseteq \mathcal{K}^{\mathcal{R}}(A) \subseteq \mathcal{G}^{\mathcal{R}}(A). \tag{7.7}$$

On the other hand, we have from (5.9) and (7.4) that

$$\mathcal{G}^{\mathcal{R}}(X^{-1}AX) \subseteq \mathcal{B}(X^{-1}AX) = \mathcal{B}^{\mathbf{x}}(A), \text{ for any } \mathbf{x} > \mathbf{0}.$$

But as is easily verified, $\mathcal{G}^{\mathcal{R}}(X^{-1}AX) = \mathcal{G}^{\mathcal{R}}(A)$ for any $\mathbf{x} > \mathbf{0}$, so that

$$\mathcal{G}^{\mathcal{R}}(A) \subseteq \mathcal{B}^{\mathbf{x}}(A) \text{ for any } \mathbf{x} > \mathbf{0}.$$

As this inclusion holds for all $\mathbf{x} > \mathbf{0}$, then with (7.5),

$$\mathcal{G}^{\mathcal{R}}(A) \subseteq \mathcal{B}^{\mathcal{R}}(A). \tag{7.8}$$

Thus, on combining (7.7) and (7.8), we immediately have the result of

Theorem 5. *For any $A = [a_{i,j}] \in \mathbb{C}^{n \times n}, n \geq 2$, which is weakly irreducible, then*

$$\mathcal{G}^{\mathcal{R}}(A) = \mathcal{K}^{\mathcal{R}}(A) = \mathcal{B}^{\mathcal{R}}(A). \tag{7.9}$$

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