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NON–STANDARD ORTHOGONALITY FOR MEIXNER POLYNOMIALS*

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Abstract. In this work, we obtain a non–standard orthogonality property for Meixner polynomials $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$, with $0 < \mu < 1$ and $\gamma \in \mathbb{R}$, that is, we show that they are orthogonal with respect to some discrete inner product involving difference operators. The non–standard orthogonality can be used to recover properties of these Meixner polynomials, e. g., linear relations for the classical Meixner polynomials.

Key words. Meixner polynomials, inner product involving difference operators, non–standard orthogonality.

AMS subject classifications. 33C45.

1. Introduction. Let γ and μ be real numbers such that $\gamma > 0$ and $0 < \mu < 1$. It is well known that classical monic Meixner polynomials $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$ can be defined by their explicit representation in terms of the hypergeometric function ${}_2\mathrm{F}_1$ (see [8], section 2.7, p. 49 and [2], p. 42):

$$M_n^{(\gamma,\mu)}(x) = (\gamma)_n \left(\frac{\mu}{\mu-1}\right)^n \;{}_2\mathrm{F}_1\left(\begin{array}{c} -n,-x \\ \gamma \end{array} \middle| 1-\frac{1}{\mu}\right), \tag{1.1}$$

where $x \in [0,+\infty)$, ${}_2\mathrm{F}_1\left(\begin{array}{c} -n,-x \\ \gamma \end{array} \middle| 1-\frac{1}{\mu}\right) = \sum_{k=0}^{+\infty} \frac{(-n)_k(-x)_k}{(\gamma)_k k!}\left(1-\frac{1}{\mu}\right)^k$ and $(\gamma)_n$ denotes the usual Pochhammer symbol

$$(b)_0 = 1, \quad (b)_n = b(b+1)\dots(b+n-1), \quad b\in\mathbb{R}, \quad \forall n\geq 1.$$

Simplifying expression (1.1), we get

$$M_n^{(\gamma,\mu)}(x) = \left(\frac{\mu}{\mu-1}\right)^n \sum_{k=0}^{n} \binom{n}{k} (\gamma+k)_{n-k}(x-k+1)_k \left(1-\frac{1}{\mu}\right)^k, n\geq 0. \tag{1.2}$$

We must notice that expression (1.2) is valid for every value of the real parameter γ and, in this way, it can be used to define Meixner polynomials for all $\gamma\in\mathbb{R}$.

From the explicit representation (1.2), we can deduce that Meixner polynomials $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$ satisfy, for every real value of γ, the three-term recurrence relation

$$\begin{aligned} &M_{-1}^{(\gamma,\mu)}(x) = 0, \quad M_0^{(\gamma,\mu)}(x) = 1, \\ &xM_n^{(\gamma,\mu)}(x) = M_{n+1}^{(\gamma,\mu)}(x) + \beta_n^{(\gamma,\mu)} M_n^{(\gamma,\mu)}(x) + \gamma_n^{(\gamma,\mu)} M_{n-1}^{(\gamma,\mu)}(x), \quad n\geq 0, \end{aligned} \tag{1.3}$$

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where

$$\beta_n^{(\gamma,\mu)} = \frac{n(1+\mu)+\mu\gamma}{1-\mu}, \quad \gamma_n^{(\gamma,\mu)} = \frac{n\mu(n-1+\gamma)}{(\mu-1)^2}. \tag{1.4}$$

Whenever $\gamma \neq 0, -1, -2, \ldots$, we have $\gamma_n^{(\gamma,\mu)} \neq 0$, for all $n \geq 1$, and Favard's theorem (see Chihara [5], p. 21) ensures the orthogonality of the sequence $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$ with respect to some quasi–definite linear functional. If $\gamma > 0$ the functional is positive definite and the polynomials are orthogonal with respect to the weight function $\rho^{(\gamma,\mu)}(x) = \dfrac{\mu^x\Gamma(\gamma+x)}{\Gamma(\gamma)\Gamma(x+1)}$ on the interval $[0,+\infty)$. For $\gamma = 0, -1, -2, \ldots$, from expression (1.4), we deduce that the coefficient $\gamma_n^{(\gamma,\mu)}$ vanishes for some value of n. So, in this case, we can not deduce orthogonality results from Favard's theorem.

The main aim of this paper is to obtain orthogonality properties for the sequence of Meixner polynomials $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$, with $\gamma \in \mathbb{R}$ and $0 < \mu < 1$. In fact, we are going to show that they are orthogonal with respect to a discrete inner product involving difference operators.

Similar results for different families of classical polynomials, but in the continuous case, have been obtained by several authors. For instance, K. H. Kwon and L. L. Littlejohn, in [6], established the orthogonality of the generalized Laguerre polynomials $\{L_n^{(-k)}\}_{n\geq 0}, \quad k \geq 1$, with respect to a Sobolev inner product of the form:

$$\langle f, g\rangle = (\, f(0), f'(0), \ldots, f^{(k-1)}(0)\,)\,\mathbf{A}\begin{pmatrix} g(0) \\ g'(0) \\ \vdots \\ g^{(k-1)}(0)\end{pmatrix} + \int_0^{+\infty} f^{(k)}(x)g^{(k)}(x)e^{-x}dx,$$

being $\mathbf{A}$ a symmetric $k \times k$ real matrix. In [7], the same authors showed that the Jacobi polynomials $\{P_n^{(-1,-1)}\}_{n\geq 0}$, are orthogonal with respect to the inner product

$$(f,g)_1 = d_1 f(1)g(1) + d_2 f(-1)g(-1) + \int_{-1}^{1} f'(x)g'(x)dx,$$

where d_1 and d_2 are real numbers.

Later, in [9], T. E. Pérez and M. A. Piñar gave an unified approach to the orthogonality of the generalized Laguerre polynomials $\{L_n^{(\alpha)}\}_{n\geq 0}$, for any real value of the parameter α, by proving their orthogonality with respect to a Sobolev non–diagonal inner product, whereas, in [10], they have shown how to use this orthogonality to obtain different properties of the generalized Laguerre polynomials.

M. Alfaro, M.L. Rezola, T.E. Pérez and M.A. Piñar, in [1], have studied sequences of polynomials which are orthogonal with respect to a Sobolev bilinear form defined by

$$\mathcal{B}_S^{(N)}(f,g) = (\, f(c), f'(c), \ldots, f^{(N-1)}(c)\,)\,\mathbf{A}\begin{pmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(N-1)}(c)\end{pmatrix} + \langle u, f^{(N)}g^{(N)}\rangle, \tag{1.5}$$

where u is a quasi–definite linear functional, $c \in \mathbb{R}$, N is a positive integer, and $\mathbf{A}$ is a symmetric $N \times N$ real matrix such that each of its principal submatrices is regular. In particular, they deduced that Jacobi polynomials $\{P_n^{(-N,\beta)}\}_{n\geq 0}$, for $\beta+N$ not a negative integer, are orthogonal with respect to (1.5), for u the Jacobi functional Jacobi corresponding to the weight function $\rho^{(0,\beta+N)}(x) = (1+x)^{\beta+N}$ and $c = 1$.

In a recent paper [3], M. Álvarez de Morales, T.E. Pérez and M.A. Piñar have studied the sequence of the monic Gegenbauer polynomials $\{C_n^{(-N+\frac{1}{2})}\}_{n\geq 0}$, for $N \geq 1$ a positive integer. They have shown that this sequence is orthogonal with respect to a Sobolev inner product of the form

$$(f,g)_S^{(2N)} = \tag{1.6}$$
$$= (\, F(1)|F(-1)\,)\,\mathbf{A}\,(\, G(1)|G(-1)\,)^T + \int_{-1}^{1} f^{(2N)}(x) g^{(2N)}(x)(1-x^2)^N dx,$$

where

$$(\, F(1)|F(-1)\,) = (\, f(1), f'(1), \ldots, f^{(N-1)}(1), f(-1), f'(-1), \ldots, f^{(N-1)}(-1)\,)\,,$$

$\mathbf{A} = \mathcal{Q}^{-1}\mathbf{D}(\mathcal{Q}^{-1})^T$, $\mathcal{Q}$ is a regular matrix whose elements are the consecutives derivatives of the Gegenbauer polynomials evaluated at the points 1 and -1, and $\mathbf{D}$ is an arbitrary diagonal positive definite matrix.

The structure of the paper is as follows. In Section 2, from the explicit representation of monic Meixner polynomials, $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$, for $\gamma \in \mathbb{R}$, we deduce some of their usual properties, namely the three-term recurrence relation, the difference property, the second order difference equation, etc. In Section 3, we define an inner product involving difference operators of the form

$$(f,g)_S^{(K,\gamma+K)} = \sum_{x=0}^{+\infty} F(x)\mathbf{\Lambda}^{(K)} G(x)^T \rho^{(\gamma+K,\mu)}(x), \quad x \in [0,+\infty), \tag{1.7}$$

where $K \geq 0$ is a non negative integer,

$$F(x) = (\, f(x), \quad \Delta f(x), \quad \ldots, \quad \Delta^K f(x)\,)\,,$$

Δ and ∇ are, respectively, the forward and backward difference operators defined by

$$\Delta f(x) = f(x+1) - f(x), \quad \nabla f(x) = f(x) - f(x-1),$$

$\rho^{(\gamma+K,\mu)}$ denotes the weight function associated with the classical Meixner polynomials $\{M_n^{(\gamma+K,\mu)}\}_{n\geq 0}$, and $\mathbf{\Lambda}^{(K)}$ is a real symmetric and positive definite $(K+1)\times(K+1)$ matrix. We show that the sequences of polynomials, $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$, for $\gamma \in \mathbb{R}$ and $0 < \mu < 1$, is a sequence of monic orthogonal polynomials (MOPS) with respect to the inner product $(.,.)_S^{(K,\gamma+K)}$, where $K \geq \max\{0, [-\gamma+1]\}$.

Section 4 of the paper is devoted to the study of a difference operator, $\mathcal{F}^{(K)}$, which is defined on the space of the real polynomials, $\mathbb{P}$, and is symmetric with respect to the inner product (1.7). From the expression of this operator, we can establish several relations between Meixner polynomials $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$, and classical Meixner polynomials $\{M_n^{(\gamma+K,\mu)}\}_{n\geq 0}$. Finally, in Section 5, we study the sequence of Meixner polynomials $\{M_n^{(-N,\mu)}\}_{n\geq 0}$, $N = 0, 1, 2, \ldots$.

2. The Meixner polynomials. Let γ and μ be real numbers such that $\gamma > 0$ and $0 < \mu < 1$. The explicit representation of the n-th classical monic Meixner polynomials is given by

$$M_n^{(\gamma,\mu)}(x) = \left(\frac{\mu}{\mu-1}\right)^n \sum_{k=0}^{n} \binom{n}{k} (\gamma+k)_{n-k}(x-k+1)_k \left(1-\frac{1}{\mu}\right)^k, n \geq 0. \tag{2.1}$$

Notice that for every real value of the parameter γ, expression (2.1) defines a monic polynomial of exact degree n. In this way, for $\gamma \in \mathbb{R}$, we can define a family of monic polynomials $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$, which is a basis of the linear space of real polynomials, $\mathbb{P}$. These polynomials will be called *generalized Meixner polynomials*.

Very simple and straightforward manipulations of the explicit representation show that the main algebraic properties of the classical Meixner polynomials remains for the generalized Meixner polynomials.

PROPOSITION 2.1. *Let γ be an arbitrary real number and $0 < \mu < 1$. Then, the generalized Meixner polynomials $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$ satisfy the following properties:*

i) Three-term recurrence relation

$$\begin{aligned} &M_{-1}^{(\gamma,\mu)}(x) = 0, \quad M_0^{(\gamma,\mu)}(x) = 1, \\ &xM_n^{(\gamma,\mu)}(x) = M_{n+1}^{(\gamma,\mu)}(x) + \beta_n^{(\gamma,\mu)} M_n^{(\gamma,\mu)}(x) + \gamma_n^{(\gamma,\mu)} M_{n-1}^{(\gamma,\mu)}(x), \quad n \geq 0, \end{aligned} \tag{2.2}$$

where

$$\beta_n^{(\gamma,\mu)} = \frac{n(1+\mu)+\mu\gamma}{1-\mu}, \quad \gamma_n^{(\gamma,\mu)} = \frac{n\mu(n-1+\gamma)}{(\mu-1)^2}. \tag{2.3}$$

ii) For any integer k, $0 \leq k \leq n$, we have

$$ii.1) \quad \Delta^k M_n^{(\gamma,\mu)}(x) = (n-k+1)_k M_{n-k}^{(\gamma+k,\mu)}(x), \tag{2.4}$$

$$ii.2) \quad \nabla^k M_n^{(\gamma,\mu)}(x) = (n-k+1)_k M_{n-k}^{(\gamma+k,\mu)}(x-k). \tag{2.5}$$

iii) Structure relations

$$iii.1) \quad \left(\frac{x+\gamma}{n}\right) \Delta M_n^{(\gamma,\mu)}(x) = M_n^{(\gamma,\mu)}(x) + \left(\frac{\gamma+n-1}{1-\mu}\right) M_{n-1}^{(\gamma,\mu)}(x), \tag{2.6}$$

$$iii.2) \quad \frac{x}{n}\nabla M_n^{(\gamma,\mu)}(x) = M_n^{(\gamma,\mu)}(x) + \left(\frac{\mu}{\mu-1}\right)(1-\gamma-n) M_{n-1}^{(\gamma,\mu)}(x). \tag{2.7}$$

iv) Second order difference equation

$$x\Delta\nabla y + [(\mu-1)x + \mu\gamma]\Delta y + (1-\mu)ny = 0, \tag{2.8}$$

where $y = M_n^{(\gamma,\mu)}(x)$.
v) Δ*-representation*

$$M_n^{(\gamma,\mu)}(x) = \frac{1}{n+1}\Delta M_{n+1}^{(\gamma,\mu)}(x) + \frac{\mu}{1-\mu}\Delta M_n^{(\gamma,\mu)}(x). \tag{2.9}$$

vi) ∇*-representation*

$$M_n^{(\gamma,\mu)}(x) = \frac{1}{n+1}\nabla M_{n+1}^{(\gamma,\mu)}(x) + \frac{1}{1-\mu}\nabla M_n^{(\gamma,\mu)}(x). \tag{2.10}$$

PROPOSITION 2.2. *For the same conditions of Proposition 2.1, we have*

i) $$M_n^{(\gamma,\mu)}(x) = \sum_{i=0}^{k}\binom{k}{i}\left(\frac{\mu}{\mu-1}\right)^i \Delta^i M_n^{(\gamma-k,\mu)}(x) =$$

$$= \left(I + \frac{\mu}{\mu-1}\Delta\right)^k M_n^{(\gamma-k,\mu)}(x). \tag{2.11}$$

ii) $$M_n^{(\gamma,\mu)}(x) = \sum_{i=0}^{k}\binom{k}{i}\left(\frac{1}{\mu-1}\right)^i \nabla^i M_n{}^{(\gamma-k,\mu)}(x+k) =$$

$$= \left(I + \frac{1}{\mu-1}\nabla\right)^k M_n^{(\gamma-k,\mu)}(x+k). \tag{2.12}$$

Proof. i) The case $k=0$ is trivial. Using expression (2.9) and property (2.4), we obtain

$$\left(I + \frac{\mu}{\mu-1}\Delta\right) M_n^{(\gamma-1,\mu)}(x) = M_n^{(\gamma-1,\mu)}(x) + \frac{\mu}{\mu-1}\Delta M_n^{(\gamma-1,\mu)}(x) =$$
$$= \frac{1}{n+1}\Delta M_{n+1}^{(\gamma-1,\mu)}(x) = M_n^{(\gamma,\mu)}(x).$$

Finally, the result follows from the identity

$$\left(I + \frac{\mu}{\mu-1}\Delta\right)^n f(x) = \sum_{j=0}^{n}\binom{n}{j}\left(\frac{\mu}{\mu-1}\right)^j \Delta^j f(x). \tag{2.13}$$

ii) It is analogous to i) but now using the property

$$\left(I + \frac{1}{\mu-1}\nabla\right)^n f(x) = \sum_{j=0}^{n}\binom{n}{j}\left(\frac{1}{\mu-1}\right)^j \nabla^j f(x). \tag{2.14}$$

□

3. Non-standard orthogonality. Let $K \geq 0$ be an integer. Let us define a lower triangular $(K+1)\times(K+1)$ matrix $\mathbf{L}(K) =$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \binom{K}{1}\left(\frac{\mu}{\mu-1}\right) & 1 & 0 & \dots & 0 \\ \binom{K}{2}\left(\frac{\mu}{\mu-1}\right)^2 & \binom{K-1}{1}\left(\frac{\mu}{\mu-1}\right) & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{K}{K}\left(\frac{\mu}{\mu-1}\right)^K & \binom{K-1}{K-1}\left(\frac{\mu}{\mu-1}\right)^{K-1} & \binom{K-2}{K-2}\left(\frac{\mu}{\mu-1}\right)^{K-2} & \dots & 1 \end{pmatrix}$$

From this, we can define a real symmetric matrix $\mathbf{\Lambda}^{(K)}$ by means of

$$\mathbf{\Lambda}^{(K)} := \mathbf{L}(K)\mathbf{L}(K)^T. \tag{3.1}$$

If we denote $\mathbf{\Lambda}^{(K)} = (\,\lambda_{m,k}\,)_{m,k=0}^{K}$, then

$$\lambda_{m,k} = \sum_{p=0}^{\min\{m,k\}} (-1)^{m+k}\binom{K-p}{m-p}\binom{K-p}{k-p}\left(\frac{\mu}{1-\mu}\right)^{m+k-2p}, \quad 0 \leq m,k \leq K.$$

Obviously, $\mathbf{\Lambda}^{(K)}$ is positive definite since expression (3.1) constitutes the Cholesky factorization for $\mathbf{\Lambda}^{(K)}$, (see [12], p. 174), and $\det\left(\mathbf{\Lambda}^{(K)}\right) = 1$.

Letting $K \geq 0$ be an integer and $\gamma > -K$ a real number, we define an inner product involving the difference operators by means of the expression

$$(f,g)_{\Delta}^{(K,\gamma)} = \sum_{x=0}^{+\infty} F(x)\mathbf{\Lambda}^{(K)}G(x)^T\rho^{(\gamma+K,\mu)}(x), \quad x \in [0,+\infty), \tag{3.2}$$

where $F(x)$ are $G(x)$ are two vectors, defined by

$$F(x) = (\,f(x), \quad \Delta f(x), \quad \dots, \quad \Delta^K f(x)\,)\,,$$
$$G(x) = (\,g(x), \quad \Delta g(x), \quad \dots, \quad \Delta^K g(x)\,)\,,$$

and

$$\rho^{(\gamma+K,\mu)}(x) = \frac{\mu^x\Gamma(\gamma+K+x)}{\Gamma(\gamma+K)\Gamma(x+1)},$$

is the Meixner weight function.

Since $\gamma + K > 0$, the series (3.2) converges and, as consequence of the positive definite character of the symmetric matrix $\mathbf{\Lambda}^{(K)}$, we conclude that $(.,.)_{\Delta}^{(K,\gamma)}$ is an inner product. By analogy with the Sobolev inner products, the inner product (3.2) will be called Δ–Sobolev inner product.

REMARK 1. In the case $K = 0$ and, therefore, $\gamma > 0$, the inner product (3.2) is the standard inner product associated to the classical weight function $\rho^{(\gamma,\mu)}$, i. e.,

$$(f,g) = (f,g)_\Delta^{(0,\gamma)} = \sum_{x=0}^{+\infty} f(x)g(x)\rho^{(\gamma,\mu)}(x), \quad x \in [0,+\infty).$$

REMARK 2. Substituting the explicit expression for the elements of $\mathbf{\Lambda}^{(K)}$ in (3.2), we obtain

$$(f,g)_\Delta^{(K,\gamma)} = \sum_{x=0}^{+\infty} \sum_{m,k=0}^{K} \lambda_{m,k} \Delta^m \left(f(x)\right) \Delta^k \left(g(x)\right) \rho^{(\gamma+K,\mu)}(x). \tag{3.3}$$

From now on, we denote by ρ the Meixner classical weight function $\rho^{(\gamma+K,\mu)}$.

From the explicit expression of the matrix $\mathbf{L}(K)$, it is possible to obtain a representation of the inner product (3.2) in terms of the forward difference operator Δ.

PROPOSITION 3.1. *Let γ and μ be real numbers such that $0 < \mu < 1$, and let $K \geq 0$ be an integer with $\gamma + K > 0$. Then, for two arbitrary polynomials f and g, the inner product (3.2) can be written in the form:*

$$\begin{aligned}(f,g)_\Delta^{(K,\gamma)} &= \\ &= \sum_{x=0}^{+\infty} \sum_{j=0}^{K} \left(I + \frac{\mu}{\mu-1}\Delta\right)^{K-j} \Delta^j f(x) \left(I + \frac{\mu}{\mu-1}\Delta\right)^{K-j} \Delta^j g(x)\rho(x).\end{aligned} \tag{3.4}$$

Proof. Using expression (2.13), the product $F(x)\mathbf{L}(K)$ transforms into

$$\left(\, f(x), \quad \Delta f(x), \quad \dots, \quad \Delta^K f(x) \,\right) \mathbf{L}(K) =$$

$$= \left(\left(I + \frac{\mu}{\mu-1}\Delta\right)^K f(x), \quad \left(I + \frac{\mu}{\mu-1}\Delta\right)^{K-1} \Delta f(x), \quad \dots, \quad \Delta^K f(x) \right).$$

□

In the following Proposition, we establish a recurrent expression for the inner product defined in (3.2).

PROPOSITION 3.2. *In the above conditions, the inner product (3.2) can be written in the following recurrent form*

$$(f,g)_\Delta^{(K,\gamma)} = \left(\left(I + \frac{\mu}{\mu-1}\Delta\right) f, \left(I + \frac{\mu}{\mu-1}\Delta\right) g \right)_\Delta^{(K-1,\gamma)} + \sum_{x=0}^{+\infty} \Delta^K f(x) \Delta^K g(x) \rho(x). \tag{3.5}$$

Proof. From (3.4), we have

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$$(f,g)_{\Delta}^{(K,\gamma)} = \sum_{x=0}^{+\infty}\sum_{j=0}^{K}\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-j}\Delta^j f(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-j}\Delta^j g(x)\rho(x) =$$

$$= \sum_{x=0}^{+\infty}\sum_{j=0}^{K-1}\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-j}\Delta^j f(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-j}\Delta^j g(x)\rho(x) +$$

$$+ \sum_{x=0}^{+\infty}\Delta^K f(x)\Delta^K g(x)\rho(x) =$$

$$= \sum_{x=0}^{+\infty}\sum_{j=0}^{K-1}\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-j-1}\Delta^j\left(\left(I+\frac{\mu}{\mu-1}\Delta\right)f(x)\right)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-j-1}\cdot$$

$$\Delta^j\left(\left(I+\frac{\mu}{\mu-1}\Delta\right)g(x)\right)\rho(x) + \sum_{x=0}^{+\infty}\Delta^K f(x)\Delta^K g(x)\rho(x) =$$

$$= \left(\left(I+\frac{\mu}{\mu-1}\Delta\right)f, \left(I+\frac{\mu}{\mu-1}\Delta\right)g\right)_{\Delta}^{(K-1,\gamma)} + \sum_{x=0}^{+\infty}\Delta^K f(x)\Delta^K g(x)\rho(x).$$

□

In the following theorem we establish the orthogonality of generalized Meixner polynomials $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$, for $\gamma\in\mathbb{R}$ and $0<\mu<1$, with respect to the inner product (3.4).

THEOREM 3.3. *Let γ and μ be real numbers such that $0<\mu<1$. The sequence of generalized Meixner polynomials $\{M_n^{(\gamma,\mu)}\}_{n\geq 0}$ is a MOPS with respect to the Δ–Sobolev inner product $(.,.)_{\Delta}^{(K,\gamma)}$, where $K\geq\max\{0,[-\gamma+1]\}$.*

Proof. We compute the inner product of two generalized Meixner polynomials $M_n^{(\gamma,\mu)}$ and $M_m^{(\gamma,\mu)}$. From relations (2.4) and (2.11), we get

$$\left(M_n^{(\gamma,\mu)}, M_m^{(\gamma,\mu)}\right)_{\Delta}^{(K,\gamma)} = \sum_{x=0}^{+\infty}\sum_{j=0}^{K}\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-j}\Delta^j M_n^{(\gamma,\mu)}(x)$$

$$\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-j}\Delta^j M_m^{(\gamma,\mu)}(x)\rho(x) =$$

$$= \sum_{x=0}^{+\infty}\sum_{j=0}^{K}(n-j+1)_j(m-j+1)_j\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-j}M_{n-j}^{(\gamma+j,\mu)}(x)$$

$$\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-j}M_{m-j}^{(\gamma+j,\mu)}(x)\rho(x) =$$

$$= \sum_{x=0}^{+\infty}\sum_{j=0}^{K}(n-j+1)_j(m-j+1)_j M_{n-j}^{(\gamma+K,\mu)}(x)M_{m-j}^{(\gamma+K,\mu)}(x)\rho(x),$$

where we assume $M_i^{(\gamma+K,\mu)}=0$, for $i<0$. The result follows from the orthogonality of classical Meixner polynomials $\{M_i^{(\gamma+K,\mu)}\}_{i\geq 0}$ with respect to the weight function ρ. □

REMARK 1. Using the **same** matrix $\mathbf{\Lambda}^{(K)}$, we can obtain an alternative version of the above result for the inner product given by

$$(f,g)_{\nabla}^{(K,\gamma)} = \sum_{x=0}^{+\infty} \tilde{F}(x)\mathbf{\Lambda}^{(K)}\tilde{G}(x)^T \rho^{(\gamma+K,\mu)}(x), \tag{3.6}$$

where $\tilde{F}(x)$ and $\tilde{G}(x)$ are two vectors defined by

$$\begin{aligned} \tilde{F}(x) &= (\, f(x), \quad \nabla f(x+1), \quad \ldots, \quad \nabla^K f(x+K)\,)\,, \\ \tilde{G}(x) &= (\, g(x), \quad \nabla g(x+1), \quad \ldots, \quad \nabla^K g(x+K)\,)\,. \end{aligned}$$

REMARK 2. In the case of the operator ∇ and using a **different** matrix, $\hat{\mathbf{\Lambda}}^{(K)}$, we can consider the inner product

$$(f,g)_{\nabla}^{(K,\gamma)} = \sum_{x=0}^{+\infty} \hat{F}(x)\hat{\mathbf{\Lambda}}^{(K)}\hat{G}(x)^T \rho^{(\gamma+K,\mu)}(x-K), \tag{3.7}$$

where $\hat{F}(x)$ and $\hat{G}(x)$ are the vectors

$$\begin{aligned} \hat{F}(x) &= (\, f(x), \quad \nabla f(x), \quad \ldots, \quad \nabla^K f(x)\,)\,, \\ \hat{G}(x) &= (\, g(x), \quad \nabla g(x), \quad \ldots, \quad \nabla^K g(x)\,)\,, \end{aligned}$$

and $\hat{\mathbf{\Lambda}}^{(K)}$ is a real symmetric and positive definite matrix whose elements are defined by

$$\hat{\lambda}_{i,j} = \sum_{p=0}^{\min\{i,j\}} (-1)^{i+j} \binom{K-p}{i-p}\binom{K-p}{j-p}\left(\frac{1}{1-\mu}\right)^{i+j-2p}, \quad 0 \le i,j \le K.$$

4. The difference operator $\mathcal{F}^{(K)}$. In this section, we will define a difference operator $\mathcal{F}^{(K)}$, defined on the linear space of real polynomials $\mathbb{P}$, symmetric with respect to the Δ–Sobolev inner product (3.4). Using this operator we will deduce the existence of several relations involving the sequence of generalized Meixner polynomials $\{M_n^{(\gamma,\mu)}\}_{n\ge 0}$, and the sequence of classical Meixner polynomials $\{M_n^{(\gamma+K,\mu)}\}_{n\ge 0}$.

We define the difference operator $\mathcal{F}^{(K)}$ by means of

$$\mathcal{F}^{(K)} := \frac{\Phi(x;K)}{\rho(x)} \sum_{m,k=0}^{K} (-1)^m \lambda_{m,k} \nabla^m \left(\rho(x)\Delta^k\right), \tag{4.1}$$

where $\Phi(x;K) = \mu^K (x+\gamma)_K$ and $\rho(x) = \dfrac{\mu^x \Gamma(\gamma+K+x)}{\Gamma(\gamma+K)\Gamma(x+1)}$, with $x \in [0,+\infty)$, γ, $\mu \in \mathbb{R}$, such that $0<\mu<1$, and $K \ge 0$ an integer.

Expanding the expression of $\mathcal{F}^{(K)}$, we can write (4.1) in the form:

$$\mathcal{F}^{(K)} = \frac{\Phi(x;K)}{\rho(x)} \left(I, -\nabla, \ldots, (-\nabla)^K\right) \mathbf{\Lambda}^{(K)} \begin{pmatrix} \rho(x)I \\ \rho(x)\Delta \\ \vdots \\ \rho(x)\Delta^K \end{pmatrix}. \tag{4.2}$$

If we now substitute the elements of the matrix $\mathbf{\Lambda}^{(K)}$ and use relations (2.13) and (2.14) in (4.2), we obtain a simple expression for the operator $\mathcal{F}^{(K)}$:

$$\mathcal{F}^{(K)} = \frac{\Phi(x;K)}{\rho(x)} \sum_{j=0}^{K} (-\nabla)^j \left(I + \frac{\mu}{1-\mu}\nabla\right)^{K-j} \rho(x) \left(I + \frac{\mu}{\mu-1}\Delta\right)^{K-j} \Delta^j, \tag{4.3}$$

Expression (4.3) can be written in recurrent form, as we show in the following Proposition.

PROPOSITION 4.1. *Let $K \geq 0$ be a given integer and let γ and μ be real numbers such that $0 < \mu < 1$. Then,*

$$\begin{aligned} \mathcal{F}^{(0)} &= I, \\ \mathcal{F}^{(K)} &= [(1-\mu)x - \mu\gamma]\, \mathcal{F}^{(K-1)}\Delta - x\nabla\left(\mathcal{F}^{(K-1)}\Delta\right) + \\ &\quad + \frac{\Phi(x;K)}{\rho(x)} \left(I + \frac{\mu}{1-\mu}\nabla\right)^{K} \rho(x) \left(I + \frac{\mu}{\mu-1}\Delta\right)^{K}, \quad K \geq 1. \end{aligned} \tag{4.4}$$

Proof. For $K = 0$, we get $\mathcal{F}^{(0)} = I$ from (4.3). Now, we deduce the recurrence expression. From (4.3), we have

$$\mathcal{F}^{(K)} = \mathcal{F}_1^{(K)} + \mathcal{F}_2^{(K)},$$

where

$$\begin{aligned} \mathcal{F}_1^{(K)} &= \frac{\Phi(x;K)}{\rho(x)} \sum_{j=1}^{K} (-\nabla)^j \left(I + \frac{\mu}{1-\mu}\nabla\right)^{K-j} \rho(x) \left(I + \frac{\mu}{\mu-1}\Delta\right)^{K-j} \Delta^j, \\ \mathcal{F}_2^{(K)} &= \frac{\Phi(x;K)}{\rho(x)} \left(I + \frac{\mu}{1-\mu}\nabla\right)^{K} \rho(x) \left(I + \frac{\mu}{\mu-1}\Delta\right)^{K}. \end{aligned}$$

Then,

$$\begin{aligned} &\mathcal{F}_1^{(K)} = \\ &= -\frac{\Phi(x;K)}{\rho(x)}\nabla \left(\sum_{j=0}^{K-1} (-\nabla)^j \left(I + \frac{\mu}{1-\mu}\nabla\right)^{K-j-1} \rho(x) \left(I + \frac{\mu}{\mu-1}\Delta\right)^{K-j-1} \Delta^{j+1} \right) \\ &= -\nabla \left(\frac{\Phi(x+1;K)}{\rho(x+1)} \sum_{j=0}^{K-1} (-\nabla)^j \left(I + \frac{\mu}{1-\mu}\nabla\right)^{K-j-1} \rho(x). \right. \\ &\quad \left. \left(I + \frac{\mu}{\mu-1}\Delta\right)^{K-j-1} \Delta^{j+1} \right) + \nabla\left(\frac{\Phi(x+1;K)}{\rho(x+1)}\right) \\ &\quad \sum_{j=0}^{K-1} (-\nabla)^j \left(I + \frac{\mu}{1-\mu}\nabla\right)^{K-j-1} \rho(x) \left(I + \frac{\mu}{\mu-1}\Delta\right)^{K-j-1} \Delta^{j+1} = \end{aligned}$$

$$= -\nabla\left((x+1)\mathcal{F}^{(K-1)}\Delta\right) + \Delta\left(\frac{\Phi(x;K)}{\rho(x)}\right)$$
$$\sum_{j=0}^{K-1}(-\nabla)^j\left(I+\frac{\mu}{1-\mu}\nabla\right)^{K-j-1}\rho(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-j-1}\Delta^{j+1} =$$
$$= -\nabla\left((x+1)\mathcal{F}^{(K-1)}\Delta\right) + \left[\frac{\Phi(x+1;K)}{\rho(x+1)} - \frac{\Phi(x;K)}{\rho(x)}\right]$$
$$\sum_{j=0}^{K-1}(-\nabla)^j\left(I+\frac{\mu}{1-\mu}\nabla\right)^{K-j-1}\rho(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-j-1}\Delta^{j+1} =$$
$$= -\nabla\left((x+1)\mathcal{F}^{(K-1)}\Delta\right) + [x+1-\mu(x+\gamma)]\,\mathcal{F}^{(K-1)}\Delta =$$
$$= [(1-\mu)x-\mu\gamma]\,\mathcal{F}^{(K-1)}\Delta - x\nabla\left(\mathcal{F}^{(K-1)}\Delta\right).$$

□

PROPOSITION 4.2. *We have*

$$\mathcal{F}^{(K)}x^n = F(n,K)x^n + \dots, \quad n \geq K, \tag{4.5}$$

where

$$F(n,K) = \sum_{i=0}^{K}(1-\mu)^i\frac{n!}{(n-i)!}\left(\frac{\mu}{1-\mu}\right)^{K-i}(\gamma+n)_{K-i} > 0,$$

denotes the leading coefficient of the polynomial $\mathcal{F}^{(K)}x^n$, for γ, $\mu \in \mathbb{R}$, $0 < \mu < 1$, and $K \geq 0$ an integer.

Proof. We will prove the result by induction. From Proposition 4.1, we get

$$\mathcal{F}^{(K)} = \mathcal{F}_1^{(K)} + \mathcal{F}_2^{(K)},$$

where

$$\begin{aligned}\mathcal{F}_1^{(K)} &= [(1-\mu)x-\mu\gamma]\,\mathcal{F}^{(K-1)}\Delta - x\nabla\left(\mathcal{F}^{(K-1)}\Delta\right),\\ \mathcal{F}_2^{(K)} &= \frac{\Phi(x;K)}{\rho(x)}\left(I+\frac{\mu}{1-\mu}\nabla\right)^K\rho(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^K.\end{aligned} \tag{4.6}$$

If $K=0$, then since $\mathcal{F}^{(0)} = I$, the result is trivial. For $K=1$, we have

$$\mathcal{F}^{(1)} = \frac{(1-\gamma)\mu}{\mu-1}I - \frac{\mu}{(\mu-1)^2}\left([(\mu-1)x+\mu(\gamma-1)]\,\nabla + \mu(x-1+\gamma)\Delta\nabla\right);$$

thus, $\mathcal{F}^{(1)}$ preserves the degree.

We assume that the result is true for $K-1$. Then, by induction, the operator $\mathcal{F}_1^{(K)}$ preserves the degree of the polynomials. Therefore, we have only to show that the operator $\mathcal{F}_2^{(K)}$ preserves it.

In the case $K=0$, the result is trivial since $\mathcal{F}_2^{(0)}$ is the identity operator. For $K=1$, we deduce that

$$\begin{aligned}
&\mathcal{F}_2^{(1)} = \frac{\Phi(x;1)}{\rho(x)}\left(I+\frac{\mu}{1-\mu}\nabla\right)\rho(x)\left(I+\frac{\mu}{\mu-1}\Delta\right) = \\
&= \mu\frac{(x+\gamma)}{\rho(x)}\left[\rho(x)I+\frac{\mu}{\mu-1}\rho(x)\Delta+\frac{\mu}{1-\mu}\nabla(\rho(x)I)-\left(\frac{\mu}{1-\mu}\right)^2\nabla(\rho(x)\Delta)\right] = \\
&= \left(\frac{\mu\gamma}{1-\mu}\right)I-\left(\frac{\mu}{\mu-1}\right)^2\gamma\Delta+\frac{\mu}{1-\mu}x\nabla-\left(\frac{\mu}{1-\mu}\right)^2 x\Delta\nabla,
\end{aligned}$$

so the operator $\mathcal{F}_2^{(1)}$ preserves the degree. We assume that the result is true for $K-1$ and we are going to prove it for K. We have

$$\begin{aligned}
&\mathcal{F}_2^{(K)} = \frac{\Phi(x;K)}{\rho(x)}\left(I+\frac{\mu}{1-\mu}\nabla\right)^K\rho(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^K = \\
&= \frac{\Phi(x;K)}{\rho(x)}\left(I+\frac{\mu}{1-\mu}\nabla\right)^{K-1}\rho(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-1}\left(I+\frac{\mu}{\mu-1}\Delta\right) \\
&+ \frac{\mu}{1-\mu}\frac{\Phi(x;K)}{\rho(x)}\nabla\left(\left(I+\frac{\mu}{1-\ mu}\nabla\right)^{K-1}\rho(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-1}\left(I+\frac{\mu}{\mu-1}\Delta\right)\right) \\
&= \mu(x+\gamma)\mathcal{F}_2^{(K-1)}\left(I+\frac{\mu}{\mu-1}\Delta\right)+\frac{\mu}{1-\mu}\nabla\left(\frac{\Phi(x+1;K)}{\rho(x+1)}\left(I+\frac{\mu}{1-\mu}\nabla\right)^{K-1}\right. \\
&\qquad \left.\rho(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-1}\left(I+\frac{\mu}{\mu-1}\Delta\right)\right)+\frac{\mu}{\mu-1}\nabla\left(\frac{\Phi(x+1;K)}{\rho(x+1)}\right) \\
&\qquad \left(I+\frac{\mu}{1-\mu}\nabla\right)^{K-1}\rho(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-1}\left(I+\frac{\mu}{\mu-1}\Delta\right) = \\
&= \mu(x+\gamma)\mathcal{F}_2^{(K-1)}\left(I+\frac{\mu}{\mu-1}\Delta\right)+\frac{\mu}{1-\mu}\nabla\left((x+1)\mathcal{F}_2^{(K-1)}\left(I+\frac{\mu}{\mu-1}\Delta\right)\right) \\
&+ \frac{\mu}{\mu-1}\Delta\left(\frac{\Phi(x;K)}{\rho(x)}\right)\left(I+\frac{\mu}{1-\mu}\nabla\right)^{K-1}\rho(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{K-1}\left(I+\frac{\mu}{\mu-1}\Delta\right) \\
&= \mu(x+\gamma)\mathcal{F}_2^{(K-1)}\left(I+\frac{\mu}{\mu-1}\Delta\right)+\frac{\mu}{1-\mu}\nabla\left((x+1)\mathcal{F}_2^{(K-1)}\left(I+\frac{\mu}{\mu-1}\Delta\right)\right) \\
&+ \frac{\mu}{\mu-1}\left[x+1-\mu(x+\gamma)\right]\mathcal{F}_2^{(K-1)}\left(I+\frac{\mu}{\mu-1}\Delta\right) = \\
&= \left[\frac{\mu}{\mu-1}x+\frac{1}{1-\mu}\mu(x+\gamma)\right]\mathcal{F}_2^{(K-1)}\left(I+\frac{\mu}{\mu-1}\Delta\right) \\
&+ \frac{\mu}{1-\mu}x\nabla\left(\mathcal{F}_2^{(K-1)}\left(I+\frac{\mu}{\mu-1}\Delta\right)\right) = \\
&= \frac{\mu\gamma}{(1-\mu)}\mathcal{F}_2^{(K-1)}\left(I+\frac{\mu}{\mu-1}\Delta\right)+\frac{\mu}{1-\mu}x\nabla\left(\mathcal{F}_2^{(K-1)}\left(I+\frac{\mu}{\mu-1}\Delta\right)\right).
\end{aligned}$$

Finally, identifying the leading coefficients and using a recurrence reasoning, we deduce that

$$F(n,K) = \sum_{i=0}^{K} (1-\mu)^i \frac{n!}{(n-i)!} \left(\frac{\mu}{1-\mu}\right)^{K-i} (\gamma+n)_{K-i} > 0.$$

□

Next, we are going to show that the difference operator $\mathcal{F}^{(K)}$ is symmetric with respect to the inner product (3.2). First, we need the following lemma.

LEMMA 4.3. *Let f be an arbitrary polynomial of the space $\mathbb{P}$ and let $n \geq 0$ be an integer. Then,*

$$\sum_{x=0}^{+\infty} \Delta^n f(x)\rho(x) = (-1)^n \sum_{x=0}^{+\infty} f(x)\nabla^n \rho(x), \tag{4.7}$$

where $\rho(x) = \dfrac{\mu^x \Gamma(\gamma+K+x)}{\Gamma(\gamma+K)\Gamma(x+1)}$, with $x \in [0,+\infty)$, γ, $\mu \in \mathbb{R}$, with $0 < \mu < 1$, and $K \geq 0$ an integer.

Proof. For $n=0$ the result is trivial. If $n=1$, then using the relations $\Delta\left(f(x)g(x)\right) = \Delta f(x)g(x) + f(x+1)\Delta g(x)$, $\Delta f(x) = \nabla f(x+1)$ and $\rho(-1) \equiv 0$, we get

$$\begin{aligned} \sum_{x=0}^{+\infty} \Delta f(x)\rho(x) &= \sum_{x=0}^{+\infty} \left[\Delta\left(f(x)\rho(x-1)\right) - f(x)\Delta\rho(x-1)\right] = \\ &= -\sum_{x=0}^{+\infty} f(x)\Delta\rho(x-1) = -\sum_{x=0}^{+\infty} f(x)\nabla\rho(x). \end{aligned}$$

Now, we assume that the result is true for $n-1$ and we prove it for n.

$$\begin{aligned} \sum_{x=0}^{+\infty} \Delta^n f(x)\rho(x) &= \sum_{x=0}^{+\infty} \Delta\left(\Delta^{n-1} f(x)\right)\rho(x) = \\ &= -\sum_{x=0}^{+\infty} \Delta^{n-1} f(x)\nabla\rho(x) = -(-1)^{n-1} \sum_{x=0}^{+\infty} f(x)\nabla^{n-1}\left(\nabla\rho(x)\right). \end{aligned}$$

□

From expression (4.1), we can obtain a representation of the Δ–Sobolev inner product in terms of the inner product associated to the weight function ρ.

PROPOSITION 4.4. *Let f and g be two real polynomials of $\mathbb{P}$. Then,*

$$(\Phi(x;K)f, g)_{\Delta}^{(K,\gamma)} = \sum_{x=0}^{+\infty} f(x)\mathcal{F}^{(K)} g(x)\rho(x),$$

where $\Phi(x;K) = \mu^K (x+\gamma)_K$ and $\rho(x) = \dfrac{\mu^x \Gamma(\gamma+K+x)}{\Gamma(\gamma+K)\Gamma(x+1)}$, with $x \in [0,+\infty)$, γ, $\mu \in \mathbb{R}$, such that $0 < \mu < 1$, and $K \geq 0$ an integer.

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Proof. Using expression (4.1) and relation (4.7) in the definition of the inner product (3.3), we get

$$
\begin{aligned}
(\Phi(x;K)f,g)_{\Delta}^{(K,\gamma)} &= \sum_{x=0}^{+\infty}\sum_{m,k=0}^{K}\lambda_{m,k}\Delta^m\left(\Phi(x;K)f(x)\right)\Delta^k\left(g(x)\right)\rho(x) = \\
&= \sum_{x=0}^{+\infty}\sum_{m,k=0}^{K}(-1)^m\lambda_{m,k}\Phi(x;K)f(x)\nabla^m\left(\rho(x)\Delta^k\left(g(x)\right)\right) = \\
&= \sum_{x=0}^{+\infty}f(x)\sum_{m,k=0}^{K}(-1)^m\lambda_{m,k}\frac{\Phi(x;K)}{\rho(x)}\nabla^m\left(\rho(x)\Delta^k\left(g(x)\right)\right)\rho(x) = \\
&= \sum_{x=0}^{+\infty}f(x)\mathcal{F}^{(K)}g(x)\rho(x).
\end{aligned}
$$

□

THEOREM 4.5. *The difference operator $\mathcal{F}^{(K)}$ is symmetric with respect to the inner product (3.3), that is,*

$$
\left(\mathcal{F}^{(K)}f,g\right)_{\Delta}^{(K,\gamma)} = \left(f,\mathcal{F}^{(K)}g\right)_{\Delta}^{(K,\gamma)}.
$$

Proof. From expression (4.1), Proposition 4.4, relation (4.7) and the definition of the inner product (3.3), we conclude that

$$
\begin{aligned}
\left(\mathcal{F}^{(K)}f,g\right)_{\Delta}^{(K,\gamma)} &= \sum_{m,k=0}^{K}(-1)^m\lambda_{m,k}\left(\frac{\phi(x;K)}{\rho(x)}\nabla^m\left(\rho(x)\Delta^k f(x)\right),g(x)\right)_{\Delta}^{(K,\gamma)} = \\
&= \sum_{x=0}^{+\infty}\sum_{m,k=0}^{K}(-1)^m\lambda_{m,k}\nabla^m\left(\rho(x)\Delta^k f(x)\right)\mathcal{F}^{(K)}g(x) = \\
&= \sum_{x=0}^{+\infty}\sum_{m,k=0}^{K}\lambda_{m,k}\Delta^k f(x)\Delta^m\left(\mathcal{F}^{(K)}g(x)\right)\rho(x) = \\
&= \left(f,\mathcal{F}^{(K)}g\right)_{\Delta}^{(K,\gamma)}.
\end{aligned}
$$

□

PROPOSITION 4.6. *Let γ and μ be real numbers such that $0<\mu<1$, and let $K\geq 0$ be an integer. For every value of the integer n such that $n\geq K$, we have*

$$
\mathcal{F}^{(K)}M_n^{(\gamma,\mu)}(x) = F(n,K)M_n^{(\gamma,\mu)}(x). \tag{4.8}
$$

Proof. Writing the polynomial $\mathcal{F}^{(K)}M_n^{(\gamma,\mu)}$ in terms of generalized Meixner polynomials $\{M_i^{(\gamma,\mu)}\}_{i\geq 0}$, we get

$$
\mathcal{F}^{(K)}M_n^{(\gamma,\mu)}(x) = \sum_{i=0}^{n}\gamma_{n,i}M_i^{(\gamma,\mu)}(x),
$$

where the coefficients $\gamma_{n,i}$ are given by

$$\gamma_{n,i} = \frac{\left(\mathcal{F}^{(K)} M_n^{(\gamma,\mu)}, M_i^{(\gamma,\mu)}\right)_\Delta^{(K,\gamma)}}{\left(M_i^{(\gamma,\mu)}, M_i^{(\gamma,\mu)}\right)_\Delta^{(K,\gamma)}} = \frac{\left(M_n^{(\gamma,\mu)}, \mathcal{F}^{(K)} M_i^{(\gamma,\mu)}\right)_\Delta^{(K,\gamma)}}{\tilde{k}_i}.$$

Here, the symmetry of the operator $\mathcal{F}^{(K)}$ has been used. From Proposition 4.2 (see (4.5)), and the orthogonality of generalized Meixner polynomial $M_n^{(\gamma,\mu)}$ with respect to the Δ–Sobolev inner product, we deduce that $\gamma_{n,i} = 0$, for $0 \le i \le n-1$. $\square$

The following Proposition establishes several relations between generalized Meixner polynomials $\{M_n^{(\gamma,\mu)}\}_{n\ge 0}$ and classical Meixner polynomials $\{M_n^{(\gamma+K,\mu)}\}_{n\ge 0}$.

PROPOSITION 4.7. *Let γ and μ be real numbers such that $0 < \mu < 1$, and let $K \ge 0$ be an integer. The following relations hold:*

$$\text{(4.9)} \qquad \text{i)} \qquad \mu^K (x+\gamma)_K M_n^{(\gamma+K,\mu)}(x) = \sum_{i=n}^{n+K} \alpha_{n,i} M_i^{(\gamma,\mu)}(x), \quad n \ge 0,$$

where $\alpha_{n,n+K} = \mu^K, \quad \alpha_{n,n} = F(n,K)\dfrac{k_n}{\tilde{k}_n}$*;*

$$\text{(4.10)} \qquad \text{ii)} \qquad \mathcal{F}^{(K)} M_n^{(\gamma,\mu)}(x) = \sum_{i=n-K}^{n} \beta_{n,i} M_i^{(\gamma+K,\mu)}(x), \quad n \ge K,$$

where $\beta_{n,n} = F(n,K), \quad \beta_{n,n-K} = \mu^K \dfrac{\tilde{k}_n}{k_{n-K}}$.

Proof.

i) Expanding the polynomial $\mu^K (x+\gamma)_K M_n^{(\gamma+K,\mu)}(x)$ in terms of the generalized Meixner polynomials $\{M_i^{(\gamma,\mu)}\}_{i\ge 0}$, we obtain

$$\mu^K (x+\gamma)_K M_n^{(\gamma+K,\mu)}(x) = \sum_{i=0}^{n+K} \alpha_{n,i} M_i^{(\gamma,\mu)}(x),$$

where the coefficients $\alpha_{n,i}$ are

$$\alpha_{n,i} = \frac{\left(\mu^K (x+\gamma)_K M_n^{(\gamma+K,\mu)}, M_i^{(\gamma,\mu)}\right)_\Delta^{(K,\gamma)}}{\left(M_i^{(\gamma,\mu)}, M_i^{(\gamma,\mu)}\right)_\Delta^{(K,\gamma)}} = \frac{\displaystyle\sum_{x=0}^{+\infty} M_n^{(\gamma+K,\mu)}(x) \mathcal{F}^{(K)} M_i^{(\gamma,\mu)}(x) \rho(x)}{\tilde{k}_i}.$$

Now, using the orthogonality of classical polynomial $M_n^{(\gamma+K,\mu)}$, we deduce that $\alpha_{n,i} = 0$, for $0 \le i \le n-1$.

ii) Writing the polynomial $\mathcal{F}^{(K)} M_n^{(\gamma,\mu)}$ as a linear combination of the classical polynomials $\{M_i^{(\gamma+K,\mu)}\}_{i\ge 0}$, we get

$$\mathcal{F}^{(K)} M_n^{(\gamma,\mu)}(x) = \sum_{i=0}^{n} \beta_{n,i} M_i^{(\gamma+K,\mu)}(x).$$

The coefficients can been computed again, from Proposition 4.4, and so, we get

$$\beta_{n,i} = \frac{\displaystyle\sum_{x=0}^{+\infty} M_i^{(\gamma+K,\mu)}(x)\mathcal{F}^{(K)} M_n^{(\gamma,\mu)}(x)\rho(x)}{\displaystyle\sum_{x=0}^{+\infty} M_i^{(\gamma+K,\mu)}(x) M_i^{(\gamma+K,\mu)}(x)\rho(x)} = \frac{\left(\mu^K (x+\gamma)_K M_i^{(\gamma+K,\mu)}, M_n^{(\gamma,\mu)}\right)_{\Delta}^{(K,\gamma)}}{k_i}.$$

Finally, from the orthogonality of generalized polynomial $M_n^{(\gamma,\mu)}$, we conclude that $\beta_{n,i} = 0$, for $0 \le i \le n - K - 1$. ☐

The following Proposition, concerning the zeros of generalized polynomial $M_n^{(\gamma,\mu)}$, is a simple consequence of the orthogonality.

PROPOSITION 4.8. *Let γ and μ be real numbers such that $0 < \mu < 1$. For every $n > K = \max\{0, [-\gamma + 1]\}$, the generalized polynomial $M_n^{(\gamma,\mu)}$ has at least $(n - K)$ real zeros of odd multiplicity contained in the interval $[0, +\infty)$.*

Proof. Using Proposition 4.4, relation (4.8), and the orthogonality of generalized Meixner polynomial $M_n^{(\gamma,\mu)}$ with respect to $(.,.)_{\Delta}^{(K,K+\alpha)}$, we have

$$\begin{aligned}\left(\mu^K (x+\gamma)_K, M_n^{(\gamma,\mu)}\right)_{\Delta}^{(K,K+\alpha)} &= \sum_{x=0}^{+\infty} \mathcal{F}^{(K)} M_n^{(\gamma,\mu)} \rho(x) = \\ &= F(n,K) \sum_{x=0}^{+\infty} M_n^{(\gamma,\mu)} \rho(x) = 0,\end{aligned}$$

and then the polynomial $M_n^{(\gamma,\mu)}$ changes its sign in the interval $[0, +\infty)$.

Let $x_1, x_2, \ldots, x_r$ be the real and positive zeros of odd multiplicity of the polynomial $M_n^{(\gamma,\mu)}$, and denote by $q(x)$ the polynomial

$$q(x) = \prod_{i=1}^{r} (x - x_i).$$

Then,

$$\begin{aligned}\left(\mu^K (x+\gamma)_K q(x), M_n^{(\gamma,\mu)}\right)_{\Delta}^{(K,K+\alpha)} &= \sum_{x=0}^{+\infty} q(x)\mathcal{F}^{(K)} M_n^{(\gamma,\mu)} \rho(x) = \\ &= F(n,K) \sum_{x=0}^{+\infty} q(x) M_n^{(\gamma,\mu)} \rho(x) \neq 0,\end{aligned}$$

since $q(x)M_n^{(\gamma,\mu)} \ge 0, \quad \forall x \in [0, +\infty)$. So, from the orthogonality of generalized Meixner polynomial $M_n^{(\gamma,\mu)}$ with respect to $(.,.)_{\Delta}^{(K,K+\alpha)}$, we deduce that $r \ge n - K$. ☐

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5. The Meixner polynomials $\{M_n^{(-N,\mu)}\}_{n\geq 0}$. This section is devoted to the study of the generalized Meixner polynomials in the special case when the parameter $\gamma = -N$, for $N = 0, 1, 2, \ldots$. The special characteristics of monic Meixner polynomials $\{M_n^{(-N,\mu)}\}_{n\geq 0}$, with $0 < \mu < 1$, and the Δ–Sobolev inner product defined in (3.2), allow us to deduce properties for these polynomials.

Meixner polynomials satisfy the properties given in Propositions 2.1 and 2.2, respectively. Moreover, from the explicit representation of these polynomials, we can deduce some new properties, as it is shown in the following Proposition.

PROPOSITION 5.1. *Let $N \geq 0$ be an integer. Then, the monic Meixner polynomials $\{M_n^{(-N,\mu)}\}_{n\geq 0}$, with $0 < \mu < 1$, satisfy the following properties:*

i) $M_n^{(-N,\mu)}(0) = (-N)_n \left(\dfrac{\mu}{\mu-1}\right)^n, \quad n \geq 0;$

ii) $M_n^{(-N,\mu)}(0) = 0, \quad n \geq N+1;$

iii) $\Delta^k M_n^{(-N,\mu)}(0) = (n-k+1)_k(-N+k)_{n-k} \left(\dfrac{\mu}{\mu-1}\right)^{n-k}, \quad n \geq k;$

iv) $\Delta^k M_n^{(-N,\mu)}(0) = 0, \quad 0 \leq k \leq N, \quad n \geq N+1;$

v) $M_{N+1}^{(-N,\mu)}(x) = (x-N)_{N+1}.$

Proof.

i) We need only to replace $\gamma = -N$ and $x = 0$ in the explicit representation (2.1).

ii) If $n \geq N+1$, then $(-N)_n = 0$, and replacing it in i), the result follows.

iii) From relations i) and (2.4), for $n \geq 0$, we deduce that

$$\Delta^k M_n^{(-N,\mu)}(0) = (n-k+1)_k M_{n-k}^{(-N+k,\mu)}(0) = (n-k+1)_k(-N+k)_{n-k} \left(\frac{\mu}{\mu-1}\right)^{n-k}.$$

iv) If we consider $0 \leq k \leq N$ and $n \geq N+1$, then the Pochhammer symbol $(-N+k)_{n-k}$ vanishes, and using iii), we conclude $\Delta^k M_n^{(-N,\mu)}(0) = 0$.

v) Taking $n = N+1$ in (2.1), with $\gamma = -N$, we obtain

$$\begin{aligned} M_{N+1}^{(-N,\mu)}(x) &= \left(\frac{\mu}{\mu-1}\right)^{N+1} \sum_{k=0}^{N+1} \binom{N+1}{k} (-N+k)_{N+1-k}(x-k+1)_k \left(1-\frac{1}{\mu}\right)^k \\ &= \left(\frac{\mu}{\mu-1}\right)^{N+1} (x-(N+1)+1)_{N+1} \left(1-\frac{1}{\mu}\right)^{N+1} = (x-N)_{N+1}. \end{aligned}$$

□

PROPOSITION 5.2. *For every $n \geq N+1$, the Meixner polynomial $M_n^{(-N,\mu)}$ satisfies the relation*

$$M_n^{(-N,\mu)}(x) = (x-N)_{N+1} M_{n-N-1}^{(N+2,\mu)}(x-N-1). \tag{5.1}$$

Proof. The proof uses induction again. For $n = N+1$ in (5.1), we get

$$M_{N+1}^{(-N,\mu)}(x) = (x-N)_{N+1} M_0^{(N+2,\mu)}(x-N-1).$$

If $n = N+2$, from the recurrence relation (2.2) for the polynomials $M_{N+1}^{(-N,\mu)}$, using $\gamma_{N+1}^{(-N,\mu)} = 0$, $\beta_{N+1}^{(-N,\mu)} = \dfrac{N+1+\mu}{1-\mu} = N+1+\beta_0^{(N+2,\mu)}$, and the property v) of Proposition 5.1, we get

$$M_{N+2}^{(-N,\mu)}(x) = \left(x-N-1-\beta_0^{(N+2,\mu)}\right) M_{N+1}^{(-N,\mu)}(x) =$$
$$= (x-N)_{N+1}\left(x-N-1-\beta_0^{(N+2,\mu)}\right) = (x-N)_{N+1} M_1^{(N+2,\mu)}(x-N-1).$$

We will consider now the case $n \geq N+2$. Writing the three-term recurrence relation for the classical polynomials $M_{n-N-1}^{(N+2,\mu)}(x-N-1)$ and using

$$\beta_n^{(-N,\mu)} = N+1+\beta_{n-N-1}^{(N+2,\mu)}, \quad \gamma_n^{(-N,\mu)} = \gamma_{n-N-1}^{(N+2,\mu)},$$

expression (2.2) gives

$$M_{n-N}^{(N+2,\mu)}(x-N-1) = \left(x-N-1-\beta_{n-N-1}^{(N+2,\mu)}\right) M_{n-N-1}^{(N+2,\mu)}(x-N-1)-$$
$$-\gamma_{n-N-1}^{(N+2,\mu)} M_{n-N-2}^{(N+2,\mu)}(x-N-1) =$$
$$= \left(x-\beta_n^{(-N,\mu)}\right) M_{n-N-1}^{(N+2,\mu)}(x-N-1) - \gamma_n^{(-N,\mu)} M_{n-N-2}^{(N+2,\mu)}(x-N-1). \tag{5.2}$$

Now, using an inductive reasoning, from (5.2) we deduce that the three-term recurrence relation for the polynomials $(x-N)_{N+1} M_{n-N-1}^{(N+2,\mu)}(x-N-1)$ coincides with the one for the polynomials $M_n^{(-N,\mu)}$, and then the result follows. □

REMARK. From relation (5.1), we deduce that, for every value of $n \geq N+1$, the Meixner polynomial $M_n^{(-N,\mu)}$ has $N+1$ real zeros at the form $x = 0, 1, 2, \ldots, N$, as well as $n-N-1$ real zeros of odd multiplicity contained in the interval $[0, +\infty)$.

Finally, we will prove a difference relation between the Meixner polynomial $M_n^{(-N,\mu)}$, with $n \geq N+1$, and the classical Meixner polynomial $M_{n-N-1}^{(N+2,\mu)}$.

PROPOSITION 5.3. *For every* $n \geq N+1$, *we have*

$$\left(I + \frac{\mu}{\mu-1}\Delta\right)^{N+1} \nabla^{N+1} M_n^{(-N,\mu)}(x) = (n-N)_{N+1} M_{n-N-1}^{(N+2,\mu)}(x-N-1). \tag{5.3}$$

Proof. Using difference property (2.5), for $k = N+1$ and $\gamma = -N$, and using relation (2.11), we get

$$\left(I + \frac{\mu}{\mu-1}\Delta\right)^{N+1} \nabla^{N+1} M_n^{(-N,\mu)}(x) = \left(I + \frac{\mu}{\mu-1}\Delta\right)^{N+1} .$$
$$(n-N)_{N+1} M_{n-N-1}^{(1,\mu)}(x-N-1) = (n-N)_{N+1} M_{n-N-1}^{(N+2,\mu)}(x-N-1).$$

□

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5.1. Δ–Sobolev inner product. Let $N \geq 0$ be a given integer, and consider the sequence of Meixner polynomials $\{M_n^{(-N,\mu)}\}_{n\geq 0}$, with $0 < \mu < 1$. We know from Section 3, that this sequence is orthogonal with respect to the inner product defined in (3.4), for $K \geq \max\{0, [-\gamma+1]\} = N+1$. When $K = N+1$, the Δ–Sobolev inner product (3.4) reads

$$(f,g)_{\Delta}^{(N+1,-N)} =$$
$$= \sum_{x=0}^{+\infty}\sum_{j=0}^{N+1}\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N+1-j}\Delta^j f(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N+1-j}\Delta^j g(x)\mu^x. \tag{5.4}$$

Our purpose, in this particular case, is to obtain a simpler expression for the inner product (5.4).

LEMMA 5.4. *Let f be an arbitrary polynomial and $0 < \mu < 1$ a real number. We have*

$$\mu^x\left(I+\frac{\mu}{\mu-1}\Delta\right)^n f(x) = \frac{1}{(\mu-1)^n}\Delta^n\left(\mu^x f(x)\right), \quad n \geq 0.$$

Proof. Let use induction again. If $n = 0$, then the result is trivial. For $n = 1$, using the relations $\Delta(fg) = \Delta(f)g + f\Delta(g) + \Delta(f)\Delta(g)$ and $\Delta(\mu^x) = (\mu-1)\mu^x$, we get

$$\begin{aligned}
\frac{1}{\mu-1}\Delta\left(\mu^x f(x)\right) &= \frac{1}{\mu-1}\left[\mu^x\Delta(f(x)) + (\mu-1)\mu^x f(x) + (\mu-1)\mu^x\Delta(f(x))\right] = \\
&= \frac{\mu^x}{\mu-1}\left[(\mu-1)f(x) + \mu\Delta(f(x))\right] = \\
&= \mu^x\left(I+\frac{\mu}{\mu-1}\Delta\right)f(x).
\end{aligned}$$

Next, we assume that the result is true for $n-1$ and, we will prove it for n. From the induction hypothesis and the proof in the case $n = 1$, we deduce

$$\begin{aligned}
&\mu^x\left(I+\frac{\mu}{\mu-1}\Delta\right)^n f(x) = \mu^x\left(I+\frac{\mu}{\mu-1}\Delta\right)^{n-1}\left[\left(I+\frac{\mu}{\mu-1}\Delta\right)f(x)\right] = \\
&= \frac{1}{(\mu-1)^{n-1}}\Delta^{n-1}\left(\mu^x\left(I+\frac{\mu}{\mu-1}\Delta\right)f(x)\right) = \frac{1}{(\mu-1)^n}\Delta^n\left(\mu^x f(x)\right).
\end{aligned}$$

□

LEMMA 5.5. *With the above conditions,*

$$\mu^x\left(I+\frac{1}{\mu-1}\nabla\right)^n f(x) = \left(\frac{\mu}{\mu-1}\right)^n\nabla^n\left(\mu^x f(x)\right), \quad n \geq 0.$$

Proof. The proof is analogous to the proof of Lemma 5.4. □

LEMMA 5.6. *Let $n \geq 0$ be an integer, $0 < \mu < 1$ and let f and g be two real polynomials. We have*

$$\mu^x\left(I+\frac{\mu}{\mu-1}\Delta\right)f\left(I+\frac{\mu}{\mu-1}\Delta\right)g = \frac{1}{\mu-1}\left[\Delta\left(\mu^x fg\right) + \frac{\mu}{\mu-1}\mu^x\Delta f\Delta g\right].$$

Proof. We have

$$\mu^x\left(I+\frac{\mu}{\mu-1}\Delta\right)f\left(I+\frac{\mu}{\mu-1}\Delta\right)g=\mu^x\left[fg+\frac{\mu}{\mu-1}\left[\Delta(fg)-\Delta f\Delta g\right]+\right.$$
$$\left.+\left(\frac{\mu}{\mu-1}\right)^2\Delta f\Delta g\right]=\mu^x\left(I+\frac{\mu}{\mu-1}\Delta\right)fg+\frac{\mu}{(\mu-1)^2}\mu^x\Delta f\Delta g=$$
$$=\frac{1}{\mu-1}\Delta\left(\mu^x fg\right)+\frac{\mu}{(\mu-1)^2}\mu^x\Delta f\Delta g=\frac{1}{\mu-1}\left[\Delta\left(\mu^x fg\right)+\frac{\mu}{\mu-1}\mu^x\Delta f\Delta g\right].$$

□

LEMMA 5.7. *Let $N\geq 0$ be a given integer, $0<\mu<1$, and let f and g be two polynomials. Then the inner product (5.4) can be written in the form:*

$$(f,g)_{\Delta}^{(N+1,-N)}=\frac{1}{1-\mu}\left[(f,g)_D^{(N)}+\frac{\mu}{\mu-1}(\Delta f,\Delta g)_{\Delta}^{(N,-N+1)}\right]$$
$$+\sum_{x=0}^{+\infty}\Delta^{N+1}f(x)\Delta^{N+1}g(x)\mu^x,$$

where

$$(f,g)_D^{(N)}=\left(\,f(0),\Delta f(0),\ldots,\Delta^N f(0)\,\right)\boldsymbol{\Lambda}^{(N)}\begin{pmatrix}g(0)\\ \Delta g(0)\\ \ldots\\ \Delta^N g(0)\end{pmatrix},$$

and $\boldsymbol{\Lambda}^{(N)}$ is defined in (3.1).

Proof. Applying Lemma 5.6 to the inner product (5.4), we get

$$(f,g)_{\Delta}^{(N-1,-N)}=$$
$$=\sum_{x=0}^{+\infty}\sum_{j=0}^{N+1}\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N+1-j}\Delta^j f(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N+1-j}\Delta^j g(x)\mu^x$$
$$=\sum_{x=0}^{+\infty}\sum_{j=0}^{N}\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N+1-j}\Delta^j f(x)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N+1-j}\Delta^j g(x)\mu^x$$
$$+\sum_{x=0}^{+\infty}\Delta^{N+1}f(x)\Delta^{N+1}g(x)\mu^x$$
$$=\sum_{x=0}^{+\infty}\sum_{j=0}^{N}\left(I+\frac{\mu}{\mu-1}\Delta\right)\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N-j}\Delta^j f\left(I+\frac{\mu}{\mu-1}\Delta\right)$$
$$\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N-j}\Delta^j g\mu^x+\sum_{x=0}^{+\infty}\Delta^{N+1}f(x)\Delta^{N+1}g(x)\mu^x$$
$$=\frac{1}{1-\mu}\sum_{x=0}^{+\infty}\Delta\left(\sum_{j=0}^{N}\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N-j}\Delta^j f\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N-j}\Delta^j g\mu^x\right)$$

$$+\frac{\mu}{(\mu-1)^2}\sum_{x=0}^{+\infty}\sum_{j=0}^{N}\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N-j}\Delta^{j+1}f\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N-j}\Delta^{j+1}g\mu^x$$
$$+\sum_{x=0}^{+\infty}\Delta^{N+1}f(x)\Delta^{N+1}g(x)\mu^x$$
$$=\frac{1}{1-\mu}\sum_{j=0}^{N}\left|\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N-j}\Delta^{j}f\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N-j}\Delta^{j}g\mu^x\right|_{x=0}^{x=+\infty}$$
$$+\frac{\mu}{(\mu-1)^2}(\Delta f,\Delta g)_{\Delta}^{(N,-N+1)}+\sum_{x=0}^{+\infty}\Delta^{N+1}f(x)\Delta^{N+1}g(x)\mu^x$$
$$=\frac{1}{1-\mu}\left[(f,g)_D^{(N)}+\frac{\mu}{\mu-1}(\Delta f,\Delta g)_{\Delta}^{(N,-N+1)}\right]+\sum_{x=0}^{+\infty}\Delta^{N+1}f(x)\Delta^{N+1}g(x)\mu^x.$$

□

PROPOSITION 5.8. *With the above conditions, the inner product (5.4) can be expressed as:*

$$(f,g)_{\Delta}^{(N+1,-N)}=\frac{1}{1-\mu}\sum_{i=0}^{N}\left(\frac{\mu}{(\mu-1)^2}\right)^{i}(\Delta^i f,\Delta^i g)_D^{(N-i)}$$
$$+\left(\sum_{i=0}^{N+1}\left(\frac{\mu}{(\mu-1)^2}\right)^{i}\right)\sum_{x=0}^{+\infty}\Delta^{N+1}f(x)\Delta^{N+1}g(x)\mu^x. \tag{5.5}$$

Proof. Iterating in Lemma 5.7, the inner product (5.4) reads

$$(f,g)_{\Delta}^{(N+1,-N)}$$
$$=\frac{1}{1-\mu}\left[(f,g)_D^{(N)}+\frac{\mu}{\mu-1}(\Delta f,\Delta g)_{\Delta}^{(N,-N+1)}\right]+\sum_{x=0}^{+\infty}\Delta^{N+1}f(x)\Delta^{N+1}g(x)\mu^x$$
$$=\frac{1}{1-\mu}(f,g)_D^{(N)}$$
$$+\frac{\mu}{(\mu-1)^2}\left[\frac{1}{1-\mu}\left[(\Delta f,\Delta g)_D^{(N-1)}+\frac{\mu}{\mu-1}(\Delta^2 f,\Delta^2 g)_{\Delta}^{(N-1,-N+2)}\right]\right.$$
$$\left.+\sum_{x=0}^{+\infty}\Delta^{N+1}f(x)\Delta^{N+1}g(x)\mu^x\right]+\sum_{x=0}^{+\infty}\Delta^{N+1}f(x)\Delta^{N+1}g(x)\mu^x$$
$$=\frac{1}{1-\mu}\left[(f,g)_D^{(N)}+\frac{\mu}{(\mu-1)^2}(\Delta f,\Delta g)_D^{(N-1)}\right]+\left(\frac{\mu}{(\mu-1)^2}\right)^2$$
$$(\Delta^2 f,\Delta^2 g)_{\Delta}^{(N-1,-N+2)}+\left(1+\frac{\mu}{(\mu-1)^2}\right)\sum_{x=0}^{+\infty}\Delta^{N+1}f(x)\Delta^{N+1}g(x)\mu^x$$
$$=\ldots=\frac{1}{1-\mu}\sum_{i=0}^{N}\left(\frac{\mu}{(\mu-1)^2}\right)^{i}(\Delta^i f,\Delta^i g)_D^{(N-i)}$$

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$$+\left(\sum_{i=0}^{N+1}\left(\frac{\mu}{(\mu-1)^2}\right)^i\right)\sum_{x=0}^{+\infty}\Delta^{N+1}f(x)\Delta^{N+1}g(x)\mu^x.$$

□

REMARK. It is well known that the Meixner polynomials $\{M_n^{(-N,\mu)}\}$, with $0<\mu<1$, for $n=0,1,\dots,N$, are the Kravchuk polynomials $\{K_n^p(x,N+1)\}$, where the parameter p is given by the relation

$$\frac{1}{p}=1-\frac{1}{\mu}.$$

In this form, the previous result allows us to give properties of Δ–Sobolev orthogonality for the Kravchuk polynomials, since the term corresponding to the infinite sum in the Δ–Sobolev product (5.5), vanishes for all polynomial of degree less or equal to N.

5.2. The operator $\mathcal{F}^{(N+1)}$. In the particular case $\gamma=-N$, it is possible to find a compact expression for the operator $\mathcal{F}^{(N+1)}$ defined in (4.3).

Let us remember that when $K=N+1$ and $\gamma=-N$, we have

$$\rho(x)=\mu^x,\quad \Phi(x;N+1)=\mu^{N+1}(x-N)_{N+1},\quad 0<\mu<1.$$

So, if we use Lemmas 5.4 and 5.5, expression (4.3) can be written in the form

$$\begin{aligned}
&\mathcal{F}^{(N+1)}\\
&=\frac{\mu^{N+1}(x-N)_{N+1}}{\mu^x}\sum_{j=0}^{N+1}(-\nabla)^j\left(I+\frac{\mu}{1-\mu}\nabla\right)^{N+1-j}\mu^x\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N+1-j}\Delta^j\\
&=\frac{\mu^{N+1}(x-N)_{N+1}}{\mu^x}\sum_{j=0}^{N+1}(-1)^j\left(I+\frac{\mu}{1-\mu}\nabla\right)^{N+1-j}\\
&\quad\nabla^j\left(\mu^x\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N+1-j}\Delta^j\right)\\
&=\frac{\mu^{N+1}(x-N)_{N+1}}{\mu^x}\sum_{j=0}^{N+1}(-1)^j\left(I+\frac{\mu}{1-\mu}\nabla\right)^{N+1-j}\\
&\quad\Delta^j\left(\mu^{x-j}\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N+1-j}\nabla^j\right)\\
&=\frac{\mu^{N+1}(x-N)_{N+1}}{\mu^x}\sum_{j=0}^{N+1}\left(\frac{1-\mu}{\mu}\right)^j\left(I+\frac{\mu}{1-\mu}\nabla\right)^{N+1-j}\mu^x\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N+1}\nabla^j\\
&=\frac{\mu^{N+1}(x-N)_{N+1}}{\mu^x}\sum_{j=0}^{N+1}\left(\frac{1-\mu}{\mu}\right)^j\sum_{i=0}^{N+1-j}\binom{N+1-j}{i}\left(\frac{\mu}{1-\mu}\right)^i\\
&\quad\nabla^i\left(\mu^x\left(I+\frac{\mu}{\mu-1}\Delta\right)^{N+1}\nabla^j\right)
\end{aligned}$$

$$= \frac{\mu^{N+1}(x-N)_{N+1}}{\mu^x} \sum_{j=0}^{N+1} \left(\frac{1-\mu}{\mu}\right)^j \sum_{i=0}^{N+1-j} \binom{N+1-j}{i} (-1)^i \mu^x \left(I + \frac{1}{\mu-1}\nabla\right)^i \left(I + \frac{\mu}{\mu-1}\Delta\right)^{N+1} \nabla^j$$

$$= \mu^{N+1}(x-N)_{N+1} \left(I + \frac{\mu}{\mu-1}\Delta\right)^{N+1} \sum_{j=0}^{N+1} \left(\frac{1-\mu}{\mu}\right)^j \left(\frac{1}{1-\mu}\right)^{N+1-j} \nabla^{N+1-j}\nabla^j$$

$$= \left(\sum_{j=0}^{N+1} \left(\frac{\mu}{1-\mu}\right)^{N+1-j} (1-\mu)^j\right) (x-N)_{N+1} \left(I + \frac{\mu}{\mu-1}\Delta\right)^{N+1} \nabla^{N+1}.$$

In this way, the difference operator $\mathcal{F}^{(N+1)}$ could be expressed as

$$\mathcal{F}^{(N+1)} = m(\mu,N)(x-N)_{N+1} \left(I + \frac{\mu}{\mu-1}\Delta\right)^{N+1} \nabla^{N+1}, \tag{5.6}$$

where

$$m(\mu,N) = (1-\mu)^{N+1} \sum_{i=0}^{N+1} \left(\frac{\mu}{(\mu-1)^2}\right)^i.$$

REMARK. The operator $\mathcal{F}^{(N+1)}$ vanishes for every polynomial of degree less or equal to N.

From relation (4.5), we know that the operator $\mathcal{F}^{(N+1)}$ preserves the degree of the polynomials. In fact, we have

$$\mathcal{F}^{(N+1)} x^n = F(n, N+1) x^n + \dots, \quad n \ge N+1, \tag{5.7}$$

where $F(n, N+1)$ denotes the leading coefficient of the polynomial $\mathcal{F}^{(N+1)}x^n$, and

$$F(n, N+1) = m(\mu, N) \frac{n!}{(n-N-1)!}.$$

From Theorem 4.5, we deduce that the operator $\mathcal{F}^{(N+1)}$ is symmetric with respect to the inner product defined in (5.5). And from Proposition 4.4, we can obtain a representation of the Δ–Sobolev inner product in terms of the inner product associated to the weight function μ^x.

As a consequence of the relation (5.7) and the symmetric character of the operator $\mathcal{F}^{(N+1)}$, we deduce that the generalized Meixner polynomials $\{M_n^{(-N,\mu)}\}_{n\ge N+1}$, with $0 < \mu < 1$, are the eigenfunctions of the operator $\mathcal{F}^{(N+1)}$, as we state in the following Proposition.

PROPOSITION 5.9. *Let $N \ge 0$ be a given integer and $0 < \mu < 1$. Then, for every $n \ge N+1$, we have*

$$\mathcal{F}^{(N+1)} M_n^{(-N,\mu)}(x) = m(\mu, N) \frac{n!}{(n-N-1)!} M_n^{(-N,\mu)}(x). \tag{5.8}$$

Proof. Using the relation (4.8) for $K = N+1$ and Theorem 4.5, and identifying the leading coefficients, the result follows. □

From Proposition 4.4, we can deduce several relations between the generalized Meixner polynomials and the classical Meixner polynomials.

PROPOSITION 5.10. *Let $N \geq 0$ be a given integer and $0 < \mu < 1$. Then,*

$$\text{i)} \quad (x-N)_{N+1} M_n^{(1,\mu)}(x) = M_{n+N+1}^{(-N,\mu)}(x) + \sum_{i=n}^{n+N} \alpha_{n,i} M_i^{(-N,\mu)}(x), \quad n \geq 0; \tag{5.9}$$

ii) *For $n \geq N+1$, we get*

$$\begin{aligned} &\mathcal{F}^{(N+1)} M_n^{(-N,\mu)}(x) = \\ &\qquad = m(\mu, N) \frac{n!}{(n-N-1)!} M_n^{(1,\mu)}(x) + \sum_{i=n-N-1}^{n-1} \beta_{n,i} M_i^{(1,\mu)}(x). \end{aligned} \tag{5.10}$$

From the property of Δ–Sobolev orthogonality for the generalized Meixner polynomials we can obtain certain properties of these polynomials. In particular, using expression (5.6) of the difference operator $\mathcal{F}^{(N+1)}$, we can recover the properties (5.1) and (5.3).

PROPOSITION 5.11. *Let $n \geq N+1$ be an integer. Then, we have*

$$\text{i)} \quad M_n^{(-N,\mu)}(x) = (x-N)_{N+1} M_{n-N-1}^{(N+2,\mu)}(x-N-1);$$

$$\text{ii)} \quad \left(I + \frac{\mu}{\mu-1}\Delta\right)^{N+1} \nabla^{N+1} M_n^{(-N,\mu)}(x) = \frac{n!}{(n-N-1)!} M_{n-N-1}^{(N+2,\mu)}(x-N-1).$$

Proof. i) Writing the polynomial $(x-N)_{N+1} M_{n-N-1}^{(N+2,\mu)}(x-N-1)$ as a linear combination of the generalized Meixner polynomials $\{M_i^{(-N,\mu)}\}_{i\geq 0}$:

$$(x-N)_{N+1} M_{n-N-1}^{(N+2,\mu)}(x-N-1) = \sum_{i=0}^{n} a_{n,i} M_i^{(-N,\mu)}(x-N-1).$$

Using Proposition 4.4 and the expression of the operator $\mathcal{F}^{(N+1)}$, the coefficients $a_{n,i}$ are

$$a_{n,i} = \frac{\left((x-N)_{N+1} M_{n-N-1}^{(N+2,\mu)}(x-N-1), M_i^{(-N,\mu)}(x-N-1)\right)_\Delta^{(N+1,-N)}}{\left(M_i^{(-N,\mu)}(x-N-1), M_i^{(-N,\mu)}(x-N-1)\right)_\Delta^{(N+1,-N)}}$$

$$= \frac{\dfrac{1}{\mu^{N+1}} \displaystyle\sum_{x=N+1}^{+\infty} M_{n-N-1}^{(N+2,\mu)}(x-N-1)\mathcal{F}^{(N+1)} M_i^{(-N,\mu)}(x-N-1)\mu^x}{\tilde{k}_i}$$

$$= \frac{\dfrac{1}{\mu^{N+1}} m(\mu,N) \displaystyle\sum_{x=N+1}^{+\infty} M_{n-N-1}^{(N+2,\mu)}(x-N-1)(x-N)_{N+1}}{\tilde{k}_i}$$

$$+ \frac{\left(I + \dfrac{\mu}{\mu-1}\Delta\right)^{N+1} \nabla^{N+1} M_i^{(-N,\mu)}(x-N-1)\mu^{x-N-1}}{\tilde{k}_i},$$

and, applying the orthogonality for the classical Meixner polynomial $M_{n-N-1}^{(N+2,\mu)}(x-N-1)$ with respect to $\dfrac{(x-N)_{N+1}\mu^{x-N-1}}{(N+1)!}$, we obtain that $a_{n,i} = 0$, for $0 \le i < n$.

ii) From i) and using relation (5.8) as well as expression (5.6), we deduce that

$$m(\mu,N)\frac{n!}{(n-N-1)!}(x-N)_{N+1} M_{n-N-1}^{(N+2,\mu)}(x-N-1) =$$
$$= m(\mu,N)\frac{n!}{(n-N-1)!} M_n^{(-N,\mu)}(x) = \mathcal{F}^{(N+1)} M_n^{(-N,\mu)}(x) =$$
$$= m(\mu,N)(x-N)_{N+1}\left(I + \frac{\mu}{\mu-1}\Delta\right)^{N+1} \nabla^{N+1} M_n^{(-N,\mu)}(x).$$

Finally, we arrive at the result simplifying the factor $m(\mu,N)(x-N)_{N+1}$. □

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