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A TWO-LEVEL DISCRETIZATION METHOD FOR THE STATIONARY MHD EQUATIONS*

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Abstract. We describe and analyze a two-level finite-element method for discretizing the equations of stationary, viscous, incompressible magnetohydrodynamics (or MHD). These equations, which model the flow of electrically conducting fluids in the presence of electromagnetic fields, arise in plasma physics and liquid-metal technology as well as in geophysics and astronomy. We treat the equations under physically realistic ("nonideal") boundary conditions that account for the electromagnetic interaction of the fluid with the surrounding media.

The suggested algorithm involves solving a small, nonlinear problem on a coarse mesh and then one large, linear problem on a fine mesh. We prove well-posedness of the algorithm and optimal error estimates under a small-data assumption.

Key words. magnetohydrodynamics, Navier-Stokes equations, Maxwell's equations, variational methods, finite elements.

AMS subject classifications. 76W05, 65N30, 35Q30, 35Q35, 35Q60, 35A15, 65N12, 65N15.

1. Introduction. Under the assumptions of the magnetohydrodynamic (or MHD) approximation (see, for example, [13]), the flow of a viscous, incompressible, electrically conducting fluid, interacting with electromagnetic fields, is governed by the Navier-Stokes and pre-Maxwell equations, coupled via the Lorentz force and Ohm's law. These equations arise in plasma physics, geophysics, and astronomy as well as in connection with numerous engineering problems, such as controlled thermonuclear fusion, liquid-metal cooling of nuclear reactors, electromagnetic casting of metals, and MHD sea water propulsion.

We consider the stationary form of the complete, nonlinear system of MHD equations, in three space dimensions and under physically realistic ("nonideal") boundary and interface conditions that account for the electromagnetic interaction of the fluid with the outside world. This problem amounts to solving a coupled, nonlinear system of eight equations (involving two unknown three-dimensional vector fields and two unknown scalar fields) in a bounded domain of $\mathbb{R}^3$ in addition to an auxiliary, linear div-curl system in all of $\mathbb{R}^3$. In our approach, which is based on earlier work in [6], [7], and [8], the latter is reduced to the computation of certain singular integrals.

After finite-element discretization, the problem gives rise to a very large, nonlinear system of algebraic equations. If those are solved by means of a simple linearization and iteration scheme, the system matrix must be reassembled (or at least recalculated) at each step of the iteration, resulting in very high computational complexity. In the present work, we therefore propose a two-level finite-element discretization method that involves solving the full, nonlinear problem only on a rather coarse mesh, followed by the solution of just one large, linear system of equations on a much finer mesh.

We establish well-posedness of this algorithm in the case of unique solvability of the continuous problem (that is, under a small-data assumption) and prove that if the two meshsizes (H and h) are properly scaled ($H^2 \lesssim h \lesssim H$), then the resulting errors of approximation are of the same (optimal) order as those obtained by solving the full, nonlinear problem on the finer mesh.

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This two-level algorithm may be recursively applied to yield a multi-level method; in particular, our analysis implies convergence of a full multi-level Newton method, in which the meshsize is successively decreased by a factor of two. In order to minimize the number of levels and thus, the amount of work required to achieve the desired accuracy of approximation, one would choose the meshsize at each level to be proportional to the square of the meshsize at the previous level. With this scaling, however, the sheer size of the problem and the limitations of present-day computing hardware preclude the use of more than a very small number of levels. In light of these limitations and in order to simplify the exposition, we will focus our attention on the basic two-level algorithm and comment only briefly on its multi-level generalization (see Section 5).

The present paper extends earlier work of Layton et al. in [4] and [5], which is based on ideas proposed in [14] and [15]. Our scaling condition ($h \sim H^2$) is reminiscent of a similar condition in [9]. For additional background material, we refer to [1]–[3], [10], [12], and the references quoted in [4]–[8].

The remainder of the paper is organized as follows. After a brief description of the physical problem and its mathematical formulation (Section 2), we collect some preliminary results for the continuous problem and its finite-dimensional approximation (Sections 3 and 4); these results, the proofs of which may be found in [8], are needed for the subsequent analysis. In Section 5 we present our two-level discretization algorithm, prove its well-posedness (in the case of small data), and state an optimal error estimate. Section 6 is devoted to the proof of the error estimate, and some concluding remarks are given in Section 7.

2. The Problem. We are concerned with the stationary flow of a viscous, incompressible, electrically conducting fluid, confined to a region Ω (a bounded Lipschitz domain in $\mathbb{R}^3$), in the presence of stationary body forces, electric and magnetic fields, and electric currents. Assuming all external field sources (if any) to be known, the flow can be completely described in terms of the following unknown quantities: the fluid velocity $\mathbf{u}$ and pressure p, the current density $\mathbf{J}$ (in the fluid), the electric potential ϕ, and the magnetic field $\mathbf{B}$. The governing equations are the Navier-Stokes equations and Ohm's law,

$$-\eta\Delta\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathbf{J}\times\mathbf{B} = \mathbf{F} \tag{2.1}$$

and

$$\sigma^{-1}\mathbf{J} + \nabla\phi - \mathbf{u}\times\mathbf{B} = \mathbf{E}\,, \tag{2.2}$$

along with the divergence constraints,

$$\nabla\cdot\mathbf{u} = 0 \quad \text{and} \quad \nabla\cdot\mathbf{J} = 0\,, \tag{2.3}$$

reflecting the conservation of mass and charge. The viscosity η, density ρ, and conductivity σ of the fluid are positive parameters; $\mathbf{F}$ is a given body force, and $\mathbf{E}$ represents a given, externally generated electric field. (Physically, $\mathbf{E}$ should be assumed to be irrotational and could then be absorbed into the potential gradient; we allow an arbitrary field $\mathbf{E}$, for reasons of symmetry in the equations.)

The magnetic field $\mathbf{B}$ can be written as

$$\mathbf{B} = \mathbf{B}_0 + \mathcal{B}(\mathbf{J})\,, \tag{2.4}$$

where $\mathbf{B}_0$ comprises field components generated by known external sources (permanent magnets and/or electric currents flowing in circuits outside the fluid), while $\mathcal{B}(\mathbf{J})$ is induced by

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the unknown current $\mathbf{J}$ in the fluid. Under mild assumptions on $\mathbf{J}$, the Biot-Savart law implies that

$$\mathcal{B}(\mathbf{J})(x) = -\frac{\mu}{4\pi}\int_{\Omega}\frac{x-y}{|x-y|^3}\times\mathbf{J}(y)\,dy\,, \tag{2.5}$$

for $x \in \mathbb{R}^3$, where μ is the magnetic permeability. (For simplicity we assume the fluid, as well as any materials outside, to be nonmagnetic, so that μ is constant throughout space.)

Equations (2.1)–(2.3) need to be supplemented by suitable boundary conditions for $\mathbf{u}$ and $\mathbf{J}$ on Γ, the boundary of the region Ω occupied by the fluid; in the simplest case, $\mathbf{u} = 0$ and $\mathbf{J}\cdot\mathbf{n} = 0$, where $\mathbf{n}$ denotes the outward-pointing unit normal vector field on Γ. Here we allow the fluid to be mechanically driven through boundary forcing; this leads to a nonhomogeneous Dirichlet boundary condition,

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma,$$

where $\mathbf{g}$ must satisfy $\int_\Gamma \mathbf{g}\cdot\mathbf{n} = 0$ (since $\nabla\cdot\mathbf{u} = 0$ in Ω). We also allow electric current to enter and leave Ω through the boundary, that is, we prescribe the flux,

$$\mathbf{J}\cdot\mathbf{n} = j \quad \text{on } \Gamma,$$

where j must satisfy $\int_\Gamma j = 0$ (since $\nabla\cdot\mathbf{J} = 0$ in Ω). Obviously, if $j \neq 0$, then the current loop must be closed in the exterior of Ω, that is, we must have an external current distribution $\mathbf{J}_{\text{ext}}$ in $\mathbb{R}^3\setminus\overline{\Omega}$ with $\mathbf{J}_{\text{ext}}\cdot\mathbf{n} = j$ on Γ (and, of course, $\nabla\cdot\mathbf{J}_{\text{ext}} = 0$ in $\mathbb{R}^3\setminus\overline{\Omega}$). In fact, to arrive at a well-posed problem, we have to prescribe $\mathbf{J}_{\text{ext}}$ rather than just j, for otherwise, $\mathbf{B}_0$ (the magnetic field generated by sources outside the fluid) could not be directly determined. However, given $\mathbf{J}_{\text{ext}}$, we can write

$$\mathbf{B}_0(x) = \mathbf{B}_{\text{ext}}(x) - \frac{\mu}{4\pi}\int_{\mathbb{R}^3\setminus\overline{\Omega}}\frac{x-y}{|x-y|^3}\times\mathbf{J}_{\text{ext}}(y)\,dy\,, \tag{2.6}$$

for $x \in \mathbb{R}^3$, where $\mathbf{B}_{\text{ext}}$ comprises field components generated by external sources other than $\mathbf{J}_{\text{ext}}$ (if any). See [8] for further details.

We are now in a position to give a precise formulation of the problem.

PROBLEM 2.1. *Given positive parameters η, ρ, σ, and μ, and data*

$$\begin{aligned}
&\mathbf{F}\in\mathbf{H}^{-1}(\Omega),\ \mathbf{E}\in\mathbf{L}^2(\Omega),\\
&\mathbf{g}\in\mathbf{H}^{1/2}(\Gamma) \textit{ with } \textstyle\int_\Gamma \mathbf{g}\cdot\mathbf{n} = 0,\\
&\mathbf{J}_{\text{ext}}\in\mathbf{L}^2(\mathbb{R}^3\setminus\overline{\Omega}) \textit{ with } \nabla\cdot\mathbf{J}_{\text{ext}} = 0 \textit{ in } \mathbb{R}^3 \textit{ and } \textstyle\int_\Gamma \mathbf{J}_{\text{ext}}\cdot\mathbf{n} = 0,\\
&\mathbf{B}_{\text{ext}}\in\mathbf{W}^1(\mathbb{R}^3) \textit{ with } \nabla\cdot\mathbf{B}_{\text{ext}} = 0 \textit{ in } \mathbb{R}^3 \textit{ and } \nabla\times\mathbf{B}_{\text{ext}} = 0 \textit{ in } \Omega,
\end{aligned}$$

find

$$\begin{aligned}
&\mathbf{u}\in\mathbf{H}^1(\Omega) \textit{ with } \nabla\cdot\mathbf{u} = 0 \textit{ in } \Omega \textit{ and } \mathbf{u} = \mathbf{g} \textit{ on } \Gamma,\\
&\mathbf{J}\in\mathbf{L}^2(\Omega) \textit{ with } \nabla\cdot\mathbf{J} = 0 \textit{ in } \Omega \textit{ and } \mathbf{J}\cdot\mathbf{n} = \mathbf{J}_{\text{ext}}\cdot\mathbf{n} \textit{ on } \Gamma,\\
&p\in L^2(\Omega)/\mathbb{R} \textit{ and } \phi\in H^1(\Omega)/\mathbb{R},
\end{aligned}$$

such that Equations (2.1)–(2.3) are satisfied, with $\mathbf{B}$ given by (2.4), (2.5), and (2.6).

Here and in the sequel, L^2 and H^1 denote the usual Lebesgue and Sobolev spaces of square-integrable functions on the respective domains (that is, on Ω, $\mathbb{R}^3\setminus\overline{\Omega}$, or $\mathbb{R}^3$); $W^1(\mathbb{R}^3)$ is the completion of $H^1(\mathbb{R}^3)$ with respect to the norm $f \mapsto \|\nabla f\|_{L^2(\mathbb{R}^3)}$. We think of $H^{-1}(\Omega)$ as the norm dual of $H^1_0(\Omega)$, which is the subspace of $H^1(\Omega)$ consisting of the functions that vanish on Γ. Finally, $H^{1/2}(\Gamma)$ denotes the trace space of $H^1(\Omega)$, endowed with

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the usual infimum norm, and $H^{-1/2}(\Gamma)$ is the norm dual of $H^{1/2}(\Gamma)$. Throughout, bold-face type indicates a space of $\mathbb{R}^3$-valued functions (so that, for example, $\mathbf{L}^2(\Omega) = (L^2(\Omega))^3$).

For all of the following analysis, we assume a set of parameters η, ρ, σ, μ and a set of data $\mathbf{F}$, $\mathbf{E}$, $\mathbf{g}$, $\mathbf{J}_{\text{ext}}$, $\mathbf{B}_{\text{ext}}$ to be given as in Problem 2.1, and we set $j := \mathbf{J}_{\text{ext}} \cdot \mathbf{n}$.

3. Weak Formulation and Preliminary Results. We now give a weak formulation of Problem 2.1 and state some preliminary results, which will be needed in the subsequent analysis (see [8] for proofs and further details).

To begin with, let us define

$$\begin{array}{lll} \mathbf{Y}_1 := \mathbf{H}^1(\Omega), & \mathbf{Y}_2 := \mathbf{L}^2(\Omega), & \mathbf{Y} := \mathbf{Y}_1 \times \mathbf{Y}_2, \\ \mathbf{X}_1 := \mathbf{H}^1_0(\Omega), & \mathbf{X}_2 := \mathbf{L}^2(\Omega), & \mathbf{X} := \mathbf{X}_1 \times \mathbf{X}_2, \end{array}$$

and

$$M_1 := L^2(\Omega)/\mathbb{R}, \qquad M_2 := H^1(\Omega)/\mathbb{R}, \qquad M := M_1 \times M_2.$$

All these spaces are understood to be endowed with their natural Hilbert-space structures, inherited from $L^2(\Omega)$ and $H^1(\Omega)$.

Multiplying Equations (2.1)–(2.3) by appropriate test functions and integrating (by parts) over the domain Ω in the usual way, we obtain two variational equations of the form

$$\begin{aligned} a_0\big((\mathbf{u},\mathbf{J}),(\mathbf{v},\mathbf{K})\big) &+ a_1\big((\mathbf{u},\mathbf{J}),(\mathbf{u},\mathbf{J}),(\mathbf{v},\mathbf{K})\big) \\ &+ b\big((\mathbf{v},\mathbf{K}),(p,\phi)\big) = \int_\Omega \mathbf{F}\cdot\mathbf{v} + \int_\Omega \mathbf{E}\cdot\mathbf{K} \qquad \forall\,(\mathbf{v},\mathbf{K}) \in \mathbf{X} \end{aligned} \tag{3.1}$$

and

$$b\big((\mathbf{u},\mathbf{J}),(q,\psi)\big) = \int_\Gamma j\,\psi \qquad \forall\,(q,\psi) \in M\,, \tag{3.2}$$

where $a_0 : \mathbf{Y}\times\mathbf{Y} \to \mathbb{R}$ (a bilinear form), $a_1 : \mathbf{Y}\times\mathbf{Y}\times\mathbf{Y} \to \mathbb{R}$ (a trilinear form), and $b : \mathbf{Y}\times M \to \mathbb{R}$ (a bilinear form) are given by

$$\begin{aligned} &a_0\big((\mathbf{v}_1,\mathbf{K}_1),(\mathbf{v}_2,\mathbf{K}_2)\big) \\ &\quad:= \eta\int_\Omega (\nabla\mathbf{v}_1):(\nabla\mathbf{v}_2) + \sigma^{-1}\int_\Omega \mathbf{K}_1\cdot\mathbf{K}_2 \\ &\qquad\qquad + \int_\Omega \Big(\big(\mathbf{K}_2\times\mathbf{B}_0\big)\cdot\mathbf{v}_1 - \big(\mathbf{K}_1\times\mathbf{B}_0\big)\cdot\mathbf{v}_2\Big)\,, \\ &a_1\big((\mathbf{v}_1,\mathbf{K}_1),(\mathbf{v}_2,\mathbf{K}_2),(\mathbf{v}_3,\mathbf{K}_3)\big) \\ &\quad:= \frac{\rho}{2}\int_\Omega \Big(\big((\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\big)\cdot\mathbf{v}_3 - \big((\mathbf{v}_1\cdot\nabla)\mathbf{v}_3\big)\cdot\mathbf{v}_2\Big) \\ &\qquad\qquad + \int_\Omega \Big(\big(\mathbf{K}_3\times\mathcal{B}(\mathbf{K}_1)\big)\cdot\mathbf{v}_2 - \big(\mathbf{K}_2\times\mathcal{B}(\mathbf{K}_1)\big)\cdot\mathbf{v}_3\Big)\,, \end{aligned}$$

and

$$b\big((\mathbf{v},\mathbf{K}),(q,\psi)\big) := -\int_\Omega \Big(\nabla\cdot\mathbf{v} - \tfrac{1}{|\Omega|}\textstyle\int_\Omega \nabla\cdot\mathbf{v}\Big)q + \displaystyle\int_\Omega \mathbf{K}\cdot(\nabla\psi)\,.$$

We note that the term in a_1 that stems from the convective force in Equation (2.1) has been "skew-symmetrized." As a consequence, the form $a_1\big((\mathbf{v}_0,\mathbf{K}_0),(\cdot,\cdot),(\cdot,\cdot)\big)$ is skew-symmetric on $\mathbf{Y}\times\mathbf{Y}$, for every $(\mathbf{v}_0,\mathbf{K}_0) \in \mathbf{Y}$. Furthermore, the divergence in the first integral in b has been projected onto the space of L^2-functions with mean zero. Therefore, the form b is well defined on $\mathbf{Y} \times M$, independent of the choice of representatives for the quotient spaces $M_1 = L^2(\Omega)/\mathbb{R}$ and $M_2 = H^1(\Omega)/\mathbb{R}$. None of these changes affects the continuous problem, but they are useful for treating the discrete problem.

With the above definitions, Problem 2.1 is equivalent to the following.

PROBLEM 3.1. *Find $(\mathbf{u}, \mathbf{J}) \in \mathbf{Y}$ with $\mathbf{u}|_\Gamma = \mathbf{g}$ and $(p, \phi) \in M$ such that Equations (3.1) and (3.2) are satisfied.*

LEMMA 3.2.

(a) The forms a_0, a_1, and b are bounded on their respective domains, with norms denoted by $\|a_0\|$, $\|a_1\|$, and $\|b\|$, respectively.

(b) The form a_0 is positive definite on $\mathbf{X} \times \mathbf{X}$; that is, there exists a number $\alpha > 0$ such that

$$a_0\big((\mathbf{v}, \mathbf{K}), (\mathbf{v}, \mathbf{K})\big) \geq \alpha \, \|(\mathbf{v}, \mathbf{K})\|^2_{\mathbf{Y}} \qquad \forall \, (\mathbf{v}, \mathbf{K}) \in \mathbf{X} \, .$$

(c) The form b satisfies the Ladyzhenskaya-Babuska-Brezzi (LBB) or inf-sup condition on $\mathbf{X} \times M$; that is, there exists a number $\beta > 0$ such that

$$\inf_{(q,\psi)\in M} \sup_{(\mathbf{v},\mathbf{K})\in\mathbf{X}} \frac{b\big((\mathbf{v}, \mathbf{K}), (q, \psi)\big)}{\|(\mathbf{v}, \mathbf{K})\|_{\mathbf{Y}} \|(q, \psi)\|_M} \geq \beta \, .$$

THEOREM 3.3. *Problem 3.1 is well posed for small data. That is, if the data $\mathbf{F}$, $\mathbf{E}$, $\mathbf{g}$, $\mathbf{J}_{\text{ext}}$, and $\mathbf{B}_{\text{ext}}$ are sufficiently small in $\mathbf{H}^{-1}(\Omega)$, $\mathbf{L}^2(\Omega)$, $\mathbf{H}^{1/2}(\Gamma)$, $\mathbf{L}^2(\mathbb{R}^3 \setminus \overline{\Omega})$, and $\mathbf{W}^1(\mathbb{R}^3)$, respectively, then Problem 3.1 has a unique solution $(\mathbf{u}, \mathbf{J}, p, \phi)$. Moreover, the solution satisfies*

$$\|(\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|} \tag{3.3}$$

(*with α and $\|a_1\|$ as in Lemma 3.2).*

We refer the interested reader to [8] for a more detailed analysis of the well-posedness of Problem 3.1. Here we just note that the smallness assumptions on the data can be made quite explicit and must be interpreted relative to the parameters of the problem, η, ρ, σ, and μ. For example, given *any* set of data $(\mathbf{F}, \mathbf{E}, \mathbf{g}, \mathbf{J}_{\text{ext}}, \mathbf{B}_{\text{ext}})$, the necessary smallness assumptions are satisfied (and Problem 3.1 has a unique solution) provided that the viscosity η and electric resistivity σ^{-1} of the fluid are sufficiently large.

Another noteworthy fact is that the estimate (3.3) *implies* uniqueness. That is, if Problem 3.1 has a solution $(\mathbf{u}, \mathbf{J}, p, \phi)$ satisfying (3.3), then there are no other solutions.

4. Finite-Dimensional Approximation and Basic Error Estimates. Let B denote a Banach space and $(B^h)_{h\in I}$ a family of finite-dimensional subspaces of B, where I is a subset of the interval $(0, 1)$ having 0 as its only limit point. We say that $(B^h)_{h\in I}$ is a finite-dimensional approximation of B (or that B^h approximates B, for short) if for every $f \in B$, we have $E_B(h, f) \to 0$ as $h \to 0$, where

$$E_B(h, f) := \inf_{f^h\in B^h} \|f - f^h\|_B$$

denotes the error of best approximation of f by elements of B^h.

In all of the following, we assume that $(\mathbf{Y}^h_1)_{h\in I}$, $(\mathbf{Y}^h_2)_{h\in I}$, $(M^h_1)_{h\in I}$, and $(M^h_2)_{h\in I}$ are finite-dimensional approximations of the spaces $\mathbf{Y}_1 = \mathbf{H}^1(\Omega)$, $\mathbf{Y}_2 = \mathbf{L}^2(\Omega)$, $M_1 = L^2(\Omega)/\mathbb{R}$, and $M_2 = H^1(\Omega)/\mathbb{R}$, respectively. This implies, of course, that the product spaces $\mathbf{Y}^h := \mathbf{Y}^h_1 \times \mathbf{Y}^h_2$ and $M^h := M^h_1 \times M^h_2$ approximate $\mathbf{Y} = \mathbf{Y}_1 \times \mathbf{Y}_2$ and $M = M_1 \times M_2$,

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respectively. Recalling that $\mathbf{X}_1 = \mathbf{H}_0^1(\Omega)$, $\mathbf{X}_2 = \mathbf{Y}_2$, and $\mathbf{X} = \mathbf{X}_1 \times \mathbf{X}_2$, we also set $\mathbf{X}_1^h := \mathbf{Y}_1^h \cap \mathbf{X}_1$, $\mathbf{X}_2^h := \mathbf{Y}_2^h$, and $\mathbf{X}^h := \mathbf{X}_1^h \times \mathbf{X}_2^h$. Finally, we let $\mathbf{Y}_1^h|_\Gamma$ denote the trace space of $\mathbf{Y}_1^h$, that is, the subspace $\{\mathbf{v}^h|_\Gamma;\ \mathbf{v}^h \in \mathbf{Y}_1^h\}$ of $\mathbf{H}^{1/2}(\Gamma)$. Note that automatically, $(\mathbf{Y}_1^h|_\Gamma)_{h\in I}$ is a finite-dimensional approximation of $\mathbf{H}^{1/2}(\Gamma)$, the trace space of $\mathbf{Y}_1$. However, an extra assumption (see Assumption 4.3 below) is needed to guarantee that $(\mathbf{X}_1^h)_{h\in I}$ approximates $\mathbf{X}_1$.

We also choose a family $(\mathbf{g}^h)_{h\in I}$ of approximate boundary data $\mathbf{g}^h \in \mathbf{Y}_1^h|_\Gamma$ such that $\mathbf{g}^h \to \mathbf{g}$ in $\mathbf{H}^{1/2}(\Gamma)$ as $h \to 0$. We then consider a family of finite-dimensional approximations to Problem 3.1, as follows.

PROBLEM 4.1. *Find* $(\mathbf{u}^h, \mathbf{J}^h) \in \mathbf{Y}^h$ *with* $\mathbf{u}^h|_\Gamma = \mathbf{g}^h$ *and* $(p^h, \phi^h) \in M^h$ *such that*

$$\begin{aligned} &a_0\big((\mathbf{u}^h,\mathbf{J}^h),(\mathbf{v}^h,\mathbf{K}^h)\big) + a_1\big((\mathbf{u}^h,\mathbf{J}^h),(\mathbf{u}^h,\mathbf{J}^h),(\mathbf{v}^h,\mathbf{K}^h)\big) \\ &\quad + b\big((\mathbf{v}^h,\mathbf{K}^h),(p^h,\phi^h)\big) = \int_\Omega \mathbf{F}\cdot\mathbf{v}^h + \int_\Omega \mathbf{E}\cdot\mathbf{K}^h \qquad \forall\,(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{X}^h \end{aligned} \tag{4.1}$$

and

$$b\big((\mathbf{u}^h,\mathbf{J}^h),(q^h,\psi^h)\big) = \int_\Gamma j\,\psi^h \qquad \forall\,(q^h,\psi^h)\in M^h\,. \tag{4.2}$$

In order to prove the well-posedness (for small data) of Problem 4.1 and to establish optimal error estimates, we need two conditions on the finite-dimensional spaces involved. First, we assume that the form b satisfies the inf-sup condition on $\mathbf{X}^h \times M^h$, uniformly with respect to $h \in I$.

ASSUMPTION 4.2. *There exists a number* $\beta > 0$ *such that*

$$\inf_{(q^h,\psi^h)\in M^h}\ \sup_{(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{X}^h} \frac{b\big((\mathbf{v}^h,\mathbf{K}^h),(q^h,\psi^h)\big)}{\|(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}\|(q^h,\psi^h)\|_M} \ge \beta \qquad \forall\, h\in I\,.$$

Our second assumption is needed to deal with the nonhomogeneous essential boundary condition for the velocity field.

ASSUMPTION 4.3. *There exists a uniformly bounded family* $(\Pi^h)_{h\in I}$ *of linear projections* Π^h *from* $\mathbf{Y}_1$ *onto* $\mathbf{Y}_1^h$ *such that* $\Pi^h(\mathbf{X}_1) \subset \mathbf{X}_1$ *for all* $h \in I$.

REMARK 4.4.

(a) Uniform inf-sup conditions are standard in the theory of mixed variational problems. Numerous pairs of finite-element spaces satisfying such conditions have been devised and analyzed in the literature (see [8] and [10] for specific examples that are relevant in connection with Problem 4.1).

(b) Assumption 4.3 is less standard, but known to be satisfied for most of the commonly used finite-element spaces (see [11]). The crucial property that distinguishes the projections Π^h *in Assumption 4.3 from, say, the* orthogonal *projections of* $\mathbf{Y}_1$ *onto* $\mathbf{Y}_1^h$*, is that they preserve homogeneous Dirichlet boundary values. One immediate consequence of this property is that the spaces* $\mathbf{X}_1^h = \mathbf{Y}_1^h \cap \mathbf{X}_1$ *approximate the space* $\mathbf{X}_1$.

(c) Another consequence of Assumption 4.3 is the existence of a uniformly bounded family $(\Lambda^h)_{h\in I}$ *of linear lifting operators* $\Lambda^h : \mathbf{Y}_1^h|_\Gamma \to \mathbf{Y}_1^h$*. These lifting operators are needed to deal with the nonhomogeneous essential boundary condition for the velocity field; their uniform boundedness is crucial in deriving uniform (that is,* h*-independent) estimates for Problem 4.1.*

(d) As observed in Remark 2. above, Assumption 4.3 ensures that the spaces $\mathbf{X}^h \times M^h$ *approximate the space* $\mathbf{X}\times M$*. That being the case, Assumption 4.2 implies that*

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the inf-sup condition on $\mathbf{X} \times M$ *(see Lemma 3.2(c)) holds* with the same constant β *as in Assumption 4.2.*

In analogy to Theorem 3.3, we now obtain the well-posedness of Problem 4.1, under smallness assumptions that are independent of h.

THEOREM 4.5. *There exists a positive constant* c*, independent of* h*, such that if the data* $(\mathbf{F}, \mathbf{E}, \mathbf{g}, \mathbf{J}_{\text{ext}}, \mathbf{B}_{\text{ext}})$ *and* $(\mathbf{F}, \mathbf{E}, \mathbf{g}^h, \mathbf{J}_{\text{ext}}, \mathbf{B}_{\text{ext}})$ *have norms less than* c *in* $\mathbf{H}^{-1}(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}^{1/2}(\Gamma) \times \mathbf{L}^2(\mathbb{R}^3 \setminus \overline{\Omega}) \times \mathbf{W}^1(\mathbb{R}^3)$*, then Problems 3.1 and 4.1 have unique solutions* $(\mathbf{u}, \mathbf{J}, p, \phi)$ *and* $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$*, respectively. Moreover, these solutions satisfy*

$$\|(\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|} \quad \textit{and} \quad \|(\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|}$$

(*with* α *and* $\|a_1\|$ *as in Lemma 3.2).*

The following result is an optimal estimate for the error in approximating the solution of Problem 3.1, in the case of unique solvability, by a solution of Problem 4.1. (See the beginning of this section regarding notation.)

THEOREM 4.6. *Let* $\|a_0\|$, $\|a_1\|$, *and* $\|b\|$ *denote the norms of the forms* a_0, a_1, *and* b. *Choose constants* α *and* β *as in Lemma 3.2(b) and Assumption 4.2, and let* λ *be an upper bound for the norms of the lifting operators* Λ^h *of Remark 4.4(c).*

Suppose that $(\mathbf{u}, \mathbf{J}, p, \phi)$ *and* $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$ *are solutions of Problems 3.1 and 4.1, respectively. Set* $\nu := \|(\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}}$, $\nu^h := \|(\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}}$*, and assume that* $\nu < \frac{\alpha}{\|a_1\|}$*. Then we have*

$$\begin{aligned}
&\|(\mathbf{u}, \mathbf{J}) - (\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}} \\
&\qquad \leq \left(1 + \tfrac{\|a_0\| + (\nu + \nu^h)\|a_1\|}{\alpha - \nu\|a_1\|}\right)\left(1 + \tfrac{\|b\|}{\beta}\right) \\
&\qquad\qquad \left((1+\lambda) E_{\mathbf{Y}}(h, (\mathbf{u}, \mathbf{J})) + \lambda \|\mathbf{g} - \mathbf{g}^h\|_{\mathbf{H}^{1/2}(\Gamma)}\right) + \tfrac{\|b\|}{\alpha - \nu\|a_1\|} E_M(h, (p, \phi))
\end{aligned}$$

and

$$\begin{aligned}
&\|(p, \phi) - (p^h, \phi^h)\|_M \\
&\qquad \leq \tfrac{\|a_0\| + (\nu + \nu^h)\|a_1\|}{\beta} \|(\mathbf{u}, \mathbf{J}) - (\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}} + \left(1 + \tfrac{\|b\|}{\beta}\right) E_M(h, (p, \phi))\,.
\end{aligned}$$

COROLLARY 4.7. *Suppose that the data* $\mathbf{F}$, $\mathbf{E}$, $\mathbf{g}$, $\mathbf{J}_{\text{ext}}$, *and* $\mathbf{B}_{\text{ext}}$ *and the discretization parameter* h *are sufficiently small to guarantee the existence of unique solutions* $(\mathbf{u}, \mathbf{J}, p, \phi)$ *and* $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$ *of Problems 3.1 and 4.1, respectively, satisfying* $\|(\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|}$ *and* $\|(\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|}$ *(with* α *and* $\|a_1\|$ *as in Lemma 3.2).*

Then there exists a constant c*, independent of* h*, such that*

$$\|(\mathbf{u}, \mathbf{J}, p, \phi) - (\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)\|_{\mathbf{Y} \times M} \;\leq\; c\left(E_{\mathbf{Y} \times M}(h, (\mathbf{u}, \mathbf{J}, p, \phi)) + \|\mathbf{g} - \mathbf{g}^h\|_{\mathbf{H}^{1/2}(\Gamma)}\right).$$

In particular, $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h) \to (\mathbf{u}, \mathbf{J}, p, \phi)$ *in* $\mathbf{Y} \times M$*, as* $h \to 0$.

We note that it is possible (and numerically feasible) to choose the approximate boundary data $\mathbf{g}^h$ so that $\|\mathbf{g} - \mathbf{g}^h\|_{\mathbf{H}^{1/2}(\Gamma)}$ is of the same order as $E_{\mathbf{Y}_1}(h, \mathbf{u})$. See [8] for details.

5. Two-Level Algorithm. We now present a two-level algorithm for the approximation of solutions to Problem 3.1, in the case of unique solvability. As in Section 4, we choose finite-dimensional approximations $\mathbf{X}^h$, $\mathbf{Y}^h$, and M^h of the spaces $\mathbf{X}$, $\mathbf{Y}$, and M, satisfying Assumptions 4.2 and 4.3, and approximate boundary data $\mathbf{g}^h$ with $\mathbf{g}^h \to \mathbf{g}$ in $\mathbf{H}^{1/2}(\Gamma)$. The idea of the algorithm is the following: Instead of directly solving Problem 4.1 on a fine grid (with meshsize h), first solve Problem 4.1 on a coarse grid (with meshsize H) and then solve a

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suitable linearization of Problem 4.1 on the fine grid. We will show that with proper scaling of the meshsizes h and H, the resulting errors are of the same order as those obtained by solving the full, nonlinear problem on the fine grid. The algorithm may be applied recursively, so that optimal accuracy of approximation can be achieved by solving one small, nonlinear problem and then a sequence of linear problems on successively finer grids (we elaborate on this aspect at the end of this section).

ALGORITHM 5.1.

Step 1. *Solve the following nonlinear problem (on a coarse mesh).*

PROBLEM 5.2. *Find* $(\mathbf{u}^H, \mathbf{J}^H) \in \mathbf{Y}^H$ *with* $\mathbf{u}^H|_\Gamma = \mathbf{g}^H$ *and* $(p^H, \phi^H) \in M^H$ *such that*

$$\begin{aligned} a_0\big((\mathbf{u}^H, \mathbf{J}^H), (\mathbf{v}^H, \mathbf{K}^H)\big) &+ a_1\big((\mathbf{u}^H, \mathbf{J}^H), (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{v}^H, \mathbf{K}^H)\big) \\ &+ b\big((\mathbf{v}^H, \mathbf{K}^H), (p^H, \phi^H)\big) \\ &= \int_\Omega \mathbf{F}\cdot\mathbf{v}^H + \int_\Omega \mathbf{E}\cdot\mathbf{K}^H \qquad \forall\,(\mathbf{v}^H, \mathbf{K}^H) \in \mathbf{X}^H \end{aligned} \tag{5.1}$$

and

$$b\big((\mathbf{u}^H, \mathbf{J}^H), (q^H, \psi^H)\big) = \int_\Gamma j\,\psi^H \qquad \forall\,(q^H, \psi^H) \in M^H\,. \tag{5.2}$$

Step 2. *Solve the following linear problem (on a fine mesh).*

PROBLEM 5.3. *Find* $(\mathbf{u}^h, \mathbf{J}^h) \in \mathbf{X}^h$ *with* $\mathbf{u}^h|_\Gamma = \mathbf{g}^h$ *and* $(p^h, \phi^h) \in M^h$ *such that*

$$\begin{aligned} &a_0\big((\mathbf{u}^h, \mathbf{J}^h), (\mathbf{v}^h, \mathbf{K}^h)\big) + a_1\big((\mathbf{u}^h, \mathbf{J}^h), (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{v}^h, \mathbf{K}^h)\big) \\ &+ a_1\big((\mathbf{u}^H, \mathbf{J}^H), (\mathbf{u}^h, \mathbf{J}^h), (\mathbf{v}^h, \mathbf{K}^h)\big) - a_1\big((\mathbf{u}^H, \mathbf{J}^H), (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{v}^h, \mathbf{K}^h)\big) \\ &+ b\big((\mathbf{v}^h, \mathbf{K}^h), (p^h, \phi^h)\big) = \int_\Omega \mathbf{F}\cdot\mathbf{v}^h + \int_\Omega \mathbf{E}\cdot\mathbf{K}^h \qquad \forall\,(\mathbf{v}^h, \mathbf{K}^h) \in \mathbf{X}^h \end{aligned} \tag{5.3}$$

and

$$b\big((\mathbf{u}^h, \mathbf{J}^h), (q^h, \psi^h)\big) = \int_\Gamma j\,\psi^h \qquad \forall\,(q^h, \psi^h) \in M^h\,. \tag{5.4}$$

Note that Problem 5.3 is nothing but the Newton linearization of Problem 4.1 at the point $(\mathbf{u}^H, \mathbf{J}^H, p^H, \phi^H)$.

Under suitable smallness assumptions on the data, the well-posedness of Problem 5.2 is guaranteed by Theorem 4.5. As for Problem 5.3, we have the following.

LEMMA 5.4. *Suppose that* $(\mathbf{u}^H, \mathbf{J}^H) \in \mathbf{Y}^H$ *satisfies* $\nu^H := \|(\mathbf{u}^H, \mathbf{J}^H)\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|}$ *(with* α *and* $\|a_1\|$ *as in Lemma 3.2). Then Problem 5.3 has a unique solution* $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$.

Proof. Let $(\mathbf{u}^H, \mathbf{J}^H) \in \mathbf{Y}^H$ be given with $\nu^H := \|(\mathbf{u}^H, \mathbf{J}^H)\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|}$. Choosing a lifting $\mathbf{u}_0^h \in \mathbf{Y}_1^h$ for $\mathbf{g}^h$ and setting $\mathbf{u}^h = \mathbf{u}_0^h + \hat{\mathbf{u}}^h$ in (5.3) and (5.4), we obtain an equivalent pair of equations of the form

$$\begin{aligned} a^h\big((\hat{\mathbf{u}}^h, \mathbf{J}^h), (\mathbf{v}^h, \mathbf{K}^h)\big) + b^h\big((\mathbf{v}^h, \mathbf{K}^h), (p^h, \phi^h)\big) \\ = \ell_1^h(\mathbf{v}^h, \mathbf{K}^h) \qquad \forall\,(\mathbf{v}^h, \mathbf{K}^h) \in \mathbf{X}^h \end{aligned} \tag{5.5}$$

and

$$b^h\big((\hat{\mathbf{u}}^h, \mathbf{J}^h), (q^h, \psi^h)\big) = \ell_2^h(q^h, \psi^h) \qquad \forall\,(q^h, \psi^h) \in M^h\,, \tag{5.6}$$

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where a^h is a bilinear form on $\mathbf{X}^h \times \mathbf{X}^h$, defined by

$$\begin{aligned} a^h\big((\mathbf{v}_1^h, \mathbf{K}_1^h), (\mathbf{v}_2^h, \mathbf{K}_2^h)\big) &:= a_0\big((\mathbf{v}_1^h, \mathbf{K}_1^h), (\mathbf{v}_2^h, \mathbf{K}_2^h)\big) \\ &+ a_1\big((\mathbf{v}_1^h, \mathbf{K}_1^h), (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{v}_2^h, \mathbf{K}_2^h)\big) + a_1\big((\mathbf{u}^H, \mathbf{J}^H), (\mathbf{v}_1^h, \mathbf{K}_1^h), (\mathbf{v}_2^h, \mathbf{K}_2^h)\big)\,, \end{aligned}$$

while b^h is the restriction of b to $\mathbf{X}^h \times M^h$, and ℓ_1^h and ℓ_2^h are certain linear functionals on $\mathbf{X}^h$ and M^h, respectively.

The form a^h is positive definite; in fact,

$$a^h\big((\mathbf{v}^h, \mathbf{K}^h), (\mathbf{v}^h, \mathbf{K}^h)\big) \geq \big(\alpha - \nu^H \|a_1\|\big) \|(\mathbf{v}^h, \mathbf{K}^h)\|_{\mathbf{Y}}^2\,,$$

for all $(\mathbf{v}^h, \mathbf{K}^h) \in \mathbf{X}^h$, thanks to Lemma 3.2(b) and the skew-symmetry of the form a_1 with respect to its second and third arguments. This, along with the fact that b^h satisfies an inf-sup condition (Assumption 4.2), allows us to apply Corollary 4.1 in [1, Chapter I]. We infer the existence and uniqueness of a solution $(\hat{\mathbf{u}}^h, \mathbf{J}^h, p^h, \phi^h)$ of Equations (5.5) and (5.6) and, thereby, the well-posedness of Problem 5.3. $\square$

REMARK 5.5. *Under the assumptions of Lemma 5.4, let $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$ be the unique solution of Problem 5.3 (the linear problem) and let $(\tilde{\mathbf{u}}^h, \tilde{\mathbf{J}}^h, \tilde{p}^h, \tilde{\phi}^h)$ be a solution of Problem 4.1 (the corresponding nonlinear problem). It is not hard to show that*

$$\|(\tilde{\mathbf{u}}^h, \tilde{\mathbf{J}}^h) - (\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}} \leq \frac{\|a_1\|}{\alpha - \nu^H \|a_1\|} \|(\tilde{\mathbf{u}}^h, \tilde{\mathbf{J}}^h) - (\mathbf{u}^H, \mathbf{J}^H)\|_{\mathbf{Y}}^2$$

and

$$\|(\tilde{p}^h, \tilde{\phi}^h) - (p^h, \phi^h)\|_M \leq \frac{\|a_1\|}{\beta}\left(1 + \frac{\|a_0\| + 2\nu^H \|a_1\|}{\alpha - \nu^H \|a_1\|}\right) \|(\tilde{\mathbf{u}}^h, \tilde{\mathbf{J}}^h) - (\mathbf{u}^H, \mathbf{J}^H)\|_{\mathbf{Y}}^2\,,$$

with constants as in Theorem 4.6. This is essentially a special case of the usual quadratic error estimate for Newton's method.

We will now state an optimal error estimate for Algorithm 5.1. In principle, such an estimate could be obtained by combining our basic error estimate for Problem 4.1 (Theorem 4.6) with the Newton estimate in Remark 5.5. The following, however, is a slightly sharper result, which we will derive by directly comparing the solution of Problem 5.3 to the solution of the continuous problem, Problem 3.1.

THEOREM 5.6. *Let $\|a_0\|$, $\|a_1\|$, and $\|b\|$ denote the norms of the forms a_0, a_1, and b. Choose constants α and β as in Lemma 3.2(b) and Assumption 4.2, and let λ be an upper bound for the norms of the lifting operators Λ^h of Remark 4.4(c).*

In addition, suppose that $(\mathbf{u}^H, \mathbf{J}^H) \in \mathbf{Y}$ satisfies $\nu^H := \|(\mathbf{u}^H, \mathbf{J}^H)\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|}$ and that $(\mathbf{u}, \mathbf{J}, p, \phi)$ and $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$ are solutions of Problems 3.1 and 5.3, respectively. Then we have

$$\begin{aligned} &\|(\mathbf{u}, \mathbf{J}) - (\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}} \\ &\quad \leq \left(1 + \tfrac{\|a_0\| + 2\nu^H \|a_1\|}{\alpha - \nu^H \|a_1\|}\right)\left(1 + \tfrac{\|b\|}{\beta}\right) \\ &\qquad \Big((1+\lambda) E_{\mathbf{Y}}(h, (\mathbf{u}, \mathbf{J})) + \lambda \|\mathbf{g} - \mathbf{g}^h\|_{\mathbf{H}^{1/2}(\Gamma)}\Big) \\ &\qquad + \tfrac{\|b\|}{\alpha - \nu^H \|a_1\|} E_M(h, (p, \phi)) + \tfrac{\|a_1\|}{\alpha - \nu^H \|a_1\|} \|(\mathbf{u}, \mathbf{J}) - (\mathbf{u}^H, \mathbf{J}^H)\|_{\mathbf{Y}}^2 \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} &\|(p, \phi) - (p^h, \phi^h)\|_M \\ &\quad \leq \left(1 + \tfrac{\|b\|}{\beta}\right) E_M(h, (p, \phi)) + \tfrac{\|a_0\| + 2\nu^H \|a_1\|}{\beta} \|(\mathbf{u}, \mathbf{J}) - (\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}} \\ &\qquad + \tfrac{\|a_1\|}{\beta} \|(\mathbf{u}, \mathbf{J}) - (\mathbf{u}^H, \mathbf{J}^H)\|_{\mathbf{Y}}^2\,. \end{aligned} \tag{5.8}$$

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Before turning to the proof of Theorem 5.6, we state a simple corollary, discuss the question of optimal scaling of the meshsizes h and H, and comment on a possible multi-level generalization of Algorithm 5.1.

COROLLARY 5.7. *Suppose that the data* $\mathbf{F}$, $\mathbf{E}$, $\mathbf{g}$, $\mathbf{J}_{\text{ext}}$, *and* $\mathbf{B}_{\text{ext}}$ *and the discretization parameter* H *are sufficiently small to guarantee the existence of unique solutions* $(\mathbf{u}, \mathbf{J}, p, \phi)$ *and* $(\mathbf{u}^H, \mathbf{J}^H, p^H, \phi^H)$ *of Problems 3.1 and 5.2, respectively, satisfying* $\|(\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|}$ *and* $\|(\mathbf{u}^H, \mathbf{J}^H)\|_{\mathbf{Y}} < \frac{\alpha}{\|a_1\|}$ *(with* α *and* $\|a_1\|$ *as in Lemma 3.2), and let* $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$ *denote the unique solution of Problem 5.3.*

Then there exist constants c_1 *and* c_2, *independent of* h *and* H, *such that*

$$\begin{aligned}
&\|(\mathbf{u}, \mathbf{J}, p, \phi) - (\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)\|_{\mathbf{Y}\times M} \\
&\qquad \le c_1\Big(E_{\mathbf{Y}\times M}(h, (\mathbf{u}, \mathbf{J}, p, \phi)) + \|\mathbf{g} - \mathbf{g}^h\|_{\mathbf{H}^{1/2}(\Gamma)}\Big) \\
&\qquad\qquad + c_2\Big(E_{\mathbf{Y}\times M}(H, (\mathbf{u}, \mathbf{J}, p, \phi)) + \|\mathbf{g} - \mathbf{g}^H\|_{\mathbf{H}^{1/2}(\Gamma)}\Big)^2 .
\end{aligned}$$

Corollary 5.7 implies that with proper scaling of the meshsizes h and H, the two-level method (Algorithm 5.1) yields the same accuracy of approximation as that obtained by solving Problem 4.1 (the full, nonlinear system of equations) on the fine grid. For example, if $\|\mathbf{g} - \mathbf{g}^h\|_{\mathbf{H}^{1/2}(\Gamma)}$ and the errors of best approximation of $\mathbf{u}$, $\mathbf{J}$, p, and ϕ by elements of $\mathbf{Y}_1^h$, $\mathbf{Y}_2^h$, M_1^h, and M_2^h all behave as powers of h, then the scaling $h \sim H^2$ (or more generally, $H^2 \lesssim h \lesssim H$) guarantees optimal accuracy of Algorithm 5.1.

In view of Theorem 5.6, it is clear that the algorithm may be applied recursively, with a sequence of meshes satisfying $h_{l-1}^2 \lesssim h_l \lesssim h_{l-1}$. First the nonlinear problem is solved on the coarsest mesh (h_0); then a sequence of linear problems is solved on finer and finer meshes ($h_1, h_2, \ldots$). Here, the linear problem at level l is the Newton linearization of the nonlinear problem (at that level) about the approximate solution obtained at level $l-1$. Since we can choose $h_l = h_{l-1}/2$, our analysis implies in particular the convergence of a traditional, multi-level Newton method, provided that the initial mesh is fine enough.

The scaling $h_l \sim h_{l-1}^2$ would be optimal in the sense that it minimizes the number of levels and thus, the amount of work required to achieve the prescribed accuracy; but as was noted already in the introduction, with this scaling only a very small number of levels (two or three) is numerically feasible, in view of the formidable size of the problem.

6. Proof of Theorem 5.6. Let all assumptions of Theorem 5.6 be satisfied. Being solutions of Problems 3.1 and 5.3, respectively, $(\mathbf{u}, \mathbf{J}, p, \phi)$ and $(\mathbf{u}^h, \mathbf{J}^h, p^h, \phi^h)$ satisfy Equations (3.1) and (5.3). Using the same test function $(\mathbf{v}^h, \mathbf{K}^h)$ in both equations, subtracting one from the other, and rearranging terms, we obtain

$$\begin{aligned}
&a_0\big((\mathbf{u}, \mathbf{J}) - (\mathbf{u}^h, \mathbf{J}^h), (\mathbf{v}^h, \mathbf{K}^h)\big) \\
&\quad + a_1\big((\mathbf{u}, \mathbf{J}) - (\mathbf{u}^h, \mathbf{J}^h), (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{v}^h, \mathbf{K}^h)\big) \\
&\quad + a_1\big((\mathbf{u}^H, \mathbf{J}^H), (\mathbf{u}, \mathbf{J}) - (\mathbf{u}^h, \mathbf{J}^h), (\mathbf{v}^h, \mathbf{K}^h)\big) \\
&\quad + a_1\big((\mathbf{u}, \mathbf{J}) - (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{u}, \mathbf{J}) - (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{v}^h, \mathbf{K}^h)\big) \\
&\quad + b\big((\mathbf{v}^h, \mathbf{K}^h), (p, \phi) - (p^h, \phi^h)\big) = 0 .
\end{aligned} \tag{6.1}$$

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Now let $(\mathbf{w}^h, \mathbf{L}^h) \in \mathbf{Y}^h$, $(r^h, \chi^h) \in M^h$, and $(\mathbf{v}^h, \mathbf{K}^h) \in \mathbf{X}^h$. Then

$$
\begin{aligned}
(6.2)\qquad & a_0\big((\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h), (\mathbf{v}^h, \mathbf{K}^h)\big) \\
&\qquad + a_1\big((\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h), (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{v}^h, \mathbf{K}^h)\big) \\
&\qquad + a_1\big((\mathbf{u}^H, \mathbf{J}^H), (\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h), (\mathbf{v}^h, \mathbf{K}^h)\big) \\
&\qquad + a_1\big((\mathbf{u}, \mathbf{J}) - (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{u}, \mathbf{J}) - (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{v}^h, \mathbf{K}^h)\big) \\
&\qquad + b\big((\mathbf{v}^h, \mathbf{K}^h), (r^h, \chi^h) - (p^h, \phi^h)\big) \\
&\quad = a_0\big((\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}, \mathbf{J}), (\mathbf{v}^h, \mathbf{K}^h)\big) \\
&\qquad + a_1\big((\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}, \mathbf{J}), (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{v}^h, \mathbf{K}^h)\big) \\
&\qquad + a_1\big((\mathbf{u}^H, \mathbf{J}^H), (\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}, \mathbf{J}), (\mathbf{v}^h, \mathbf{K}^h)\big) \\
&\qquad + b\big((\mathbf{v}^h, \mathbf{K}^h), (r^h, \chi^h) - (p, \phi)\big)\,.
\end{aligned}
$$

(Note that the difference between the left-hand and right-hand sides of (6.2) equals the left-hand side of (6.1).)

Now suppose that $(\mathbf{w}^h, \mathbf{L}^h) \in \mathbf{Y}^h$ satisfies $\mathbf{w}^h|_\Gamma = \mathbf{g}^h$ and

$$
b\big((\mathbf{w}^h, \mathbf{L}^h), (q^h, \psi^h)\big) = \int_\Gamma j\,\psi^h \qquad \forall\,(q^h, \psi^h) \in M^h\,.
$$

Then, $(\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h) \in \mathbf{X}^h$ and

$$
b\big((\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h), (q^h, \psi^h)\big) = 0 \qquad \forall\,(q^h, \psi^h) \in M^h\,.
$$

Setting $(\mathbf{v}^h, \mathbf{K}^h) = (\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h)$ in (6.2) and recalling the skew-symmetry of the form a_1 with respect to its second and third arguments, we obtain

$$
\begin{aligned}
(6.3)\qquad & a_0\big((\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h), (\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h)\big) \\
&\qquad + a_1\big((\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h), (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h)\big) \\
&\quad = a_0\big((\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}, \mathbf{J}), (\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h)\big) \\
&\qquad + a_1\big((\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}, \mathbf{J}), (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h)\big) \\
&\qquad + a_1\big((\mathbf{u}^H, \mathbf{J}^H), (\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}, \mathbf{J}), (\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h)\big) \\
&\qquad + b\big((\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h), (r^h, \chi^h) - (p, \phi)\big) \\
&\qquad - a_1\big((\mathbf{u}, \mathbf{J}) - (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{u}, \mathbf{J}) - (\mathbf{u}^H, \mathbf{J}^H), (\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h)\big)\,,
\end{aligned}
$$

where $(r^h, \chi^h) \in M^h$ is arbitrary. The left-hand side of (6.3) is bounded from below by

$$
\big(\alpha - \nu^H \|a_1\|\big) \|(\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}}^2\,,
$$

while the right-hand side of (6.3) is bounded from above by

$$
\begin{aligned}
&\Big(\big(\|a_0\| + 2\nu^H\|a_1\|\big)\|(\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}} + \|b\|\,\|(r^h, \chi^h) - (p, \phi)\|_M \\
&\qquad + \|a_1\|\,\|(\mathbf{u}, \mathbf{J}) - (\mathbf{u}^H, \mathbf{J}^H)\|_{\mathbf{Y}}^2\Big)\|(\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}}\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|(\mathbf{u}, \mathbf{J}) - (\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}} &\le \|(\mathbf{u}, \mathbf{J}) - (\mathbf{w}^h, \mathbf{L}^h)\|_{\mathbf{Y}} + \|(\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}^h, \mathbf{J}^h)\|_{\mathbf{Y}} \\
&\le \Big(1 + \tfrac{\|a_0\| + 2\nu^H\|a_1\|}{\alpha - \nu^H\|a_1\|}\Big)\|(\mathbf{w}^h, \mathbf{L}^h) - (\mathbf{u}, \mathbf{J})\|_{\mathbf{Y}} \\
&\qquad + \tfrac{\|b\|}{\alpha - \nu^H\|a_1\|}\|(r^h, \chi^h) - (p, \phi)\|_M \\
&\qquad + \tfrac{\|a_1\|}{\alpha - \nu^H\|a_1\|}\|(\mathbf{u}, \mathbf{J}) - (\mathbf{u}^H, \mathbf{J}^H)\|_{\mathbf{Y}}^2\,.
\end{aligned}
$$

Taking infima with respect to $(\mathbf{w}^h, \mathbf{L}^h)$ and (r^h, χ^h), we get

$$\begin{aligned} \|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}} &\le \Big(1+\tfrac{\|a_0\|+2\nu^H\|a_1\|}{\alpha-\nu^H\|a_1\|}\Big)E^*_{\mathbf{Y}}(h,(\mathbf{u},\mathbf{J})) \\ &\quad + \tfrac{\|b\|}{\alpha-\nu^H\|a_1\|}E_M(h,(p,\phi)) + \tfrac{\|a_1\|}{\alpha-\nu^H\|a_1\|}\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^H,\mathbf{J}^H)\|^2_{\mathbf{Y}}\,, \end{aligned} \tag{6.4}$$

where

$$E^*_{\mathbf{Y}}(h,(\mathbf{u},\mathbf{J})) := \inf_{\substack{(\mathbf{w}^h,\mathbf{L}^h)\in\mathbf{Y}^h,\ \mathbf{w}^h|_\Gamma=\mathbf{g}^h \\ b((\mathbf{w}^h,\mathbf{L}^h),(q^h,\psi^h))=\int_\Gamma j\,\psi^h\ \forall (q^h,\psi^h)\in M^h}} \|(\mathbf{w}^h,\mathbf{L}^h)-(\mathbf{u},\mathbf{J})\|_{\mathbf{Y}}\,.$$

It was shown in [8] (see Steps 2 and 3 of the proof of Theorem 3.8, loc. cit.) that

$$E^*_{\mathbf{Y}}(h,(\mathbf{u},\mathbf{J})) \le \Big(1+\frac{\|b\|}{\beta}\Big)\Big((1+\lambda)E_{\mathbf{Y}}(h,(\mathbf{u},\mathbf{J})) + \lambda\|\mathbf{g}-\mathbf{g}^h\|_{\mathbf{H}^{1/2}(\Gamma)}\Big)\,.$$

Substituting this into (6.4), we arrive at the desired estimate (5.7).

Now let $(r^h,\chi^h)\in M^h$ and $(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{X}^h$. Recalling (6.1), we obtain

$$\begin{aligned} & b\big((\mathbf{v}^h,\mathbf{K}^h),(r^h,\chi^h)-(p^h,\phi^h)\big) \\ &\quad = b\big((\mathbf{v}^h,\mathbf{K}^h),(r^h,\chi^h)-(p,\phi)\big) + b\big((\mathbf{v}^h,\mathbf{K}^h),(p,\phi)-(p^h,\phi^h)\big) \\ &\quad = b\big((\mathbf{v}^h,\mathbf{K}^h),(r^h,\chi^h)-(p,\phi)\big) \\ &\qquad - a_0\big((\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h),(\mathbf{v}^h,\mathbf{K}^h)\big) \\ &\qquad - a_1\big((\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h),(\mathbf{u}^H,\mathbf{J}^H),(\mathbf{v}^h,\mathbf{K}^h)\big) \\ &\qquad - a_1\big((\mathbf{u}^H,\mathbf{J}^H),(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h),(\mathbf{v}^h,\mathbf{K}^h)\big) \\ &\qquad - a_1\big((\mathbf{u},\mathbf{J})-(\mathbf{u}^H,\mathbf{J}^H),(\mathbf{u},\mathbf{J})-(\mathbf{u}^H,\mathbf{J}^H),(\mathbf{v}^h,\mathbf{K}^h)\big) \\ &\quad \le \Big(\|b\|\,\|(r^h,\chi^h)-(p,\phi)\|_M \\ &\qquad + \big(\|a_0\|+2\nu^H\|a_1\|\big)\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}} \\ &\qquad + \|a_1\|\,\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^H,\mathbf{J}^H)\|^2_{\mathbf{Y}}\Big)\|(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}\,. \end{aligned}$$

It follows that

$$\begin{aligned} & \sup_{(\mathbf{v}^h,\mathbf{K}^h)\in\mathbf{X}^h} \frac{b\big((\mathbf{v}^h,\mathbf{K}^h),(r^h,\chi^h)-(p^h,\phi^h)\big)}{\|(\mathbf{v}^h,\mathbf{K}^h)\|_{\mathbf{Y}}} \\ &\quad \le \|b\|\,\|(r^h,\chi^h)-(p,\phi)\|_M + \big(\|a_0\|+2\nu^H\|a_1\|\big)\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}} \\ &\qquad + \|a_1\|\,\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^H,\mathbf{J}^H)\|^2_{\mathbf{Y}}\,. \end{aligned} \tag{6.5}$$

By virtue of Assumption 4.2, the left-hand side of (6.5) is bounded from below by $\beta\,\|(r^h,\chi^h)-(p^h,\phi^h)\|_M$, and we conclude that

$$\begin{aligned} \|(p,\phi)-(p^h,\phi^h)\|_M &\le \|(p,\phi)-(r^h,\chi^h)\|_M + \|(r^h,\chi^h)-(p^h,\phi^h)\|_M \\ &\le \Big(1+\tfrac{\|b\|}{\beta}\Big)\|(r^h,\chi^h)-(p,\phi)\|_M + \tfrac{\|a_0\|+2\nu^H\|a_1\|}{\beta}\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^h,\mathbf{J}^h)\|_{\mathbf{Y}} \\ &\quad + \tfrac{\|a_1\|}{\beta}\|(\mathbf{u},\mathbf{J})-(\mathbf{u}^H,\mathbf{J}^H)\|^2_{\mathbf{Y}}\,. \end{aligned}$$

The desired estimate (5.8) follows by taking the infimum with respect to (r^h,χ^h).□

7. Concluding Remarks. We described and analyzed a two-level finite-element discretization method for the stationary MHD equations in three space dimensions, under physically realistic boundary and interface conditions. The algorithm involves solving one small, nonlinear problem on a coarse mesh and one large, linear problem on a much finer mesh. We established the well-posedness of the algorithm under a small-data assumption and proved that if the two meshsizes (H and h) are properly scaled ($H^2 \lesssim h \lesssim H$), then the error of approximation is of the same (optimal) order as that obtained by solving the full, nonlinear problem on the finer mesh.

The algorithm may be applied recursively to yield a multi-level method. One implication is the convergence of a traditional multi-level Newton method (where the meshsize is successively decreased by a factor of two) provided that the initial mesh is fine enough. The convergence theory given by Shaidurov [12, Chapter 6.6] for the (significantly less complex) Navier-Stokes equations shares this main restriction.

If the meshsize at each level is chosen to be proportional to the square of the meshsize at the previous level (which is optimal in that it minimizes the number of levels and thus, the amount of work required to achieve the desired accuracy), then the sheer size of the problem and the limitations of present-day computing hardware preclude the use of more than a very small number of levels. However, already our basic two-level algorithm offers some distinct advantages over more traditional approaches. Specifically, since we have to solve only one large, linear system, we avoid the repeated finite-element assembly of large system matrices that would be necessary, for example, if the nonlinear problem was solved iteratively on a fine mesh. Even if the large, linear system itself is solved by a multi-level method, all intermediate linear systems can be computed without finite-element reassembly (see, for example, [2, Chapter 9]). By comparison, a traditional multi-level Newton method for the full, nonlinear problem would require finite-element reassembly at each new level since the point of linearization changes from level to level. An additional advantage of our two-level discretization over multi-level methods is its potentially greater geometric flexibility.

The computational efficiency and robustness of our algorithm will be the subject of future investigation.

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