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THE NUMBER OF ZEROS OF UNILATERAL POLYNOMIALS OVER COQUATERNIONS AND RELATED ALGEBRAS*

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Abstract. We have proved that unilateral polynomials over the nondivision algebras in $\mathbb{R}^4$ have at most $n(2n-1)$ zeros, when the polynomial has degree n. Moreover, we have created an algorithm for finding all zeros of polynomials over these algebras using a real polynomial of degree $2n$, called *companion polynomial*. The algebras in question are coquaternions, $\mathbb{H}_{\rm coq}$, nectarines, $\mathbb{H}_{\rm nec}$, and conectarines, $\mathbb{H}_{\rm con}$. Besides the isolated and hyperbolic zeros we introduce a new type of zeros, the *unexpected* zeros. There is a formal algorithm, and there are numerical examples. In a tutorial section on similarity we show how to find the similarity transformation of two algebra elements to be known as similar, where a singular value decomposition of a certain real 4×4 matrix related to the two similar elements has to be applied. We show that there is a strong indication that an algorithm by Serôdio, Pereira, and Vitória [Comput. Math. Appl., 42 (2001), pp. 1229–1237], designed for finding zeros of quaternionic polynomials, is also valid in the nondivision algebras in $\mathbb{R}^4$ and it produces—though with another technique—the same zeros as those proposed in this paper.

Key words. number of zeros of polynomials over nondivision algebras in $\mathbb{R}^4$, number of zeros of polynomials over coquaternions, number of zeros of polynomials over nectarines, number of zeros of polynomials over conectarines, unexpected zeros, computation of all zeros of polynomials over nondivision algebras in $\mathbb{R}^4$

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1. Introduction.

We present an algorithm for finding all zeros of unilateral polynomials of degree n in one of the algebras: $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$. Beyond the *isolated* and the *hyperbolic* zeros, there will be a new type of zero which we will call an *unexpected zero*. As an essential result we determine the maximal number of zeros of unilateral polynomials of degree n over any of the noncommutative algebras in $\mathbb{R}^4$, in particular in *coquaternions* ($\mathbb{H}_{\rm coq}$), *nectarines* ($\mathbb{H}_{\rm nec}$), and *conectarines* ($\mathbb{H}_{\rm con}$). These algebras are also nondivision algebras, which means that there are noninvertible algebra elements different from the zero element. The notation $\mathbb{H}$ will be reserved—in honor of Hamilton—for the field of quaternions; see [8]. In order to support our result we have developed an algorithm for finding all zeros of unilateral polynomials over the mentioned algebras. The explicit names of these algebras were introduced by Cockle [1, 2] and Schmeikal [23] for the last two algebras. The letter $\mathcal{A}$ will denote one of these three algebras. For algebras in general, see Garling [5]. Algebras in $\mathbb{R}^N$ are in many cases called *geometric algebras* [6]. If we write nondivision algebra(s), we always mean the noncommutative nondivision algebras $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$.

For finding all zeros and their number of unilateral polynomials of degree n over quaternions $\mathbb{H}$, see Janovská and Opfer [14] and also Serôdio, Pereira, and Vitória [24] and De Leo, Ducati, and Leonardi [3]. The main ingredient in [14] for finding zeros of a quaternionic polynomial of degree n is a real polynomial of degree $2n$, which is called *companion polynomial* by the authors of [14], and is denoted by q. At the end of this paper, we will explain why the name *companion polynomial* is reasonable. In order to distinguish the zeros of the given polynomial p from the solutions of $q(z)=0$ we called these solutions *roots* of q. We found that the number of zeros of quaternionic polynomials p cannot exceed the degree, which coincides with a result published in 1965 by Gordon and Motzkin [7]. Since zeros may fill a whole similarity class, the count of zeros must be per similarity class which contains a zero.

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The importance of the notion of *similarity* was already discovered in 1936 by Wolf [25]. For nondivision algebras it is reasonable also to introduce the notion of *quasi similarity*. At a later occasion, we will point out the few cases in which quasi similarity and similarity differ.

In another paper by Janovská and Opfer [11], we have introduced an algorithm for finding zeros of unilateral coquaternionic polynomials also by the means of the real companion polynomial q. The pairs of conjugate complex roots of q gave rise to a zero of p, however, for the real roots of q we did not find a connection to the zeros of p. Any conclusion on how many zeros may exist in $\mathcal{A}$ could not be established. In [11] we extended the search for zeros by employing Newton's method where a special technique described by Lauterbach and Opfer in [17] was used. Other attempts to find zeros by Newton's method can be found in [4, 16].

In this paper, we fill this gap and show how to find all zeros of unilateral polynomials over $\mathcal{A}$ again by employing the companion polynomial, which also allows the conclusion that there are maximally $\binom{2n}{2} = n(2n-1)$ zeros of a polynomial of degree n, for instance, quadratic polynomials in $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$ may have up to 6 and cubic polynomials up to 15 zeros. It will be shown that the essential gist is not to consider the individual real roots of the companion polynomial, but to consider all *pairs* of real roots. A positive minimum number of zeros does not exist in $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$, since it was shown in [11] that there are polynomials without zeros. This is in some analogy with the fact that there are matrices in these algebras which have no eigenvalues [10]. The algorithm for finding all zeros, which implies the above upper bound, will be presented in the sequel.

2. Definitions and elementary properties. The polynomials considered here will have the form

$$p(z) = \sum_{j=0}^{n} a_j z^j, \qquad a_j, z \in \mathcal{A},\ a_n, a_0 \text{ invertible.} \tag{2.1}$$

Let $\mathcal{A}$ be one of the three algebras $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$, and if a specific algebra is chosen, we say that p is a polynomial *over* $\mathcal{A}$. The algebra of quaternions $\mathbb{H}$ is not included in this investigation since there are already publications with algorithms for finding all zeros of unilateral polynomials with quaternionic coefficients; see [3, 14, 24].

We denote algebra elements from $\mathcal{A}$ in the simple form $a = (a_1, a_2, a_3, a_4)$. The four units in any $\mathbb{R}^4$ algebra will be denoted by $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ so that one can also use the representation

$$a = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}, \qquad a_j \in \mathbb{R},\ j = 1, 2, 3, 4.$$

For completeness we present the multiplication rules for $\mathbb{H}_{\rm coq}$, $\mathbb{H}_{\rm nec}$, $\mathbb{H}_{\rm con}$ in Table 2.1.

TABLE 2.1
The three multiplication tables for $\mathbb{H}_{\rm coq}$, $\mathbb{H}_{\rm nec}$, $\mathbb{H}_{\rm con}$.

$\mathbb{H}_{\rm coq}$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	1	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	1

$\mathbb{H}_{\rm nec}$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	1	$\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	-1	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$-\mathbf{j}$	$-\mathbf{i}$	1

$\mathbb{H}_{\rm con}$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	1	$\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	1	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$-\mathbf{j}$	$\mathbf{i}$	-1

DEFINITION 2.1. *Let G be any noncommutative algebra. The* center *of G, denoted by $\mathcal{C}_G$, is the subset of G whose elements commute with all elements of G.*

LEMMA 2.2. *The center of all algebras in $\mathcal{A}$ is*

$$\mathcal{C}_{\mathcal{A}} = \mathbb{R},$$

where $\mathbb{R}$ is identified with algebra elements of the form $(a, 0, 0, 0) \in \mathcal{A},\ a \in \mathbb{R}$.

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Proof. It is clear that $\mathbb{R}$ belongs to the center. Let $a \notin \mathbb{R}$. Then the assumption $ab = ba$ for all $b \in \mathcal{A}$ leads to a contradiction. ∎

LEMMA 2.3. *Let us denote the four units in $\mathcal{A}$ by*

$$\mathit{unit}_1 = 1, \quad \mathit{unit}_2 = \mathbf{i}, \quad \mathit{unit}_3 = \mathbf{j}, \quad \mathit{unit}_4 = \mathbf{k}.$$

Then the product $\mathit{unit}_r \mathit{unit}_s$ is real if and only if $r = s,\ 1 \le r, s \le 4$.

Proof. In all three Tables 2.1 only the diagonal elements are real. ∎

We denote the first component a_1 of $a = (a_1, a_2, a_3, a_4)$ by $a_1 = \Re(a)$ and call a_1 the *real part* of a in all algebras considered here. The multiplication rules and Lemma 2.3 imply

$$\Re(ab) = \Re(ba) \qquad \text{for all } a, b \in \mathcal{A}. \tag{2.2}$$

DEFINITION 2.4. *Let $a = (a_1, a_2, a_3, a_4) \in \mathcal{A}$. We define the* conjugate *of a, denoted either by $\overline{a}$ or by $\operatorname{conj}(a)$, by*

$$\overline{a} = \operatorname{conj}(a) = (a_1, -a_2, -a_3, -a_4). \tag{2.3}$$

For the product $a\overline{a}$ we use the notation

$$\operatorname{abs}_2(a) = a\overline{a}. \tag{2.4}$$

The importance of these two notions is expressed in the following lemma.

LEMMA 2.5. *Let $a, b \in \mathcal{A}$. Then*

1. $\overline{ab} = \overline{b}\,\overline{a}, \quad a + \overline{a} = 2\Re(a)$,
2. $\operatorname{abs}_2(a) = a\overline{a} = \overline{a}a \in \mathbb{R}, \quad \operatorname{abs}_2(\overline{a}) = \operatorname{abs}_2(a)$,
3. *a is invertible if and only if $\operatorname{abs}_2(a) \neq 0$.*
4. *Let $\operatorname{abs}_2(a) \neq 0$. Then*

$$a^{-1} = \frac{\overline{a}}{\operatorname{abs}_2(a)}\,.$$

5. *The function $\operatorname{abs}_2 : \mathcal{A} \to \mathbb{R}$ defined in (2.4) is multiplicative, which means*

$$\operatorname{abs}_2(ab) = \operatorname{abs}_2(ba) = \operatorname{abs}_2(a)\operatorname{abs}_2(b). \tag{2.5}$$

For invertible a, (2.5) implies

$$1 = \operatorname{abs}_2(aa^{-1}) = \operatorname{abs}_2(a)\operatorname{abs}_2(a^{-1}).$$

6. $$\operatorname{abs}_2(a) = \begin{cases} a_1^2 + a_2^2 + a_3^2 + a_4^2 & \text{for } a \in \mathbb{H}, \\ a_1^2 + a_2^2 - a_3^2 - a_4^2 & \text{for } a \in \mathbb{H}_{\text{coq}}, \\ a_1^2 - a_2^2 + a_3^2 - a_4^2 & \text{for } a \in \mathbb{H}_{\text{nec}}, \\ a_1^2 - a_2^2 - a_3^2 + a_4^2 & \text{for } a \in \mathbb{H}_{\text{con}}. \end{cases}$$

Proof. See [11]. ∎

Since similarity is an important concept in our investigation we will repeat the essential features.

2.1. Similarity and quasi similarity.

We start with the principal definition.

DEFINITION 2.6. *Let $a, b \in \mathcal{A}$. Then a, b are said to be* similar*, denoted by $a \sim b$, if there is an invertible $h \in \mathcal{A}$ such that*

$$h^{-1}ah = b. \tag{2.6}$$

We note that similarity is an equivalence relation. We call the transformation $a \to h^{-1}ah$ a *similarity transformation* of a. We have a very simple lemma.

LEMMA 2.7. *Let $a, b \in \mathcal{A}$.*

1. *Let a, b both be real. Then a, b are similar if and only if a, b are identical.*
2. *Let a or b be real but not both. Then a, b are not similar.*

Proof. Real elements commute with all algebra elements. The defining equation (2.6) under assumption 1 implies $a = b$. In the second case it also implies $a = b$. Because one of the two elements a, b is not real and the other is real, the equation $a = b$ can never be valid in case 2. ∎

THEOREM 2.8. *Let $a, b \in \mathcal{A}$ be similar. Then,*

$$\Re(a) = \Re(b), \quad \mathrm{abs}_2(a) = \mathrm{abs}_2(b). \tag{2.7}$$

Proof. We put $b = h^{-1}ah$ and apply (2.2): $\Re(h^{-1}ah) = \Re(hh^{-1}a) = \Re(a) = \Re(b)$. We apply (2.5) and (2.6): $\mathrm{abs}_2(h^{-1}ah) = \mathrm{abs}_2(h^{-1})\mathrm{abs}_2(h)\mathrm{abs}_2(a) = \mathrm{abs}_2(a) = \mathrm{abs}_2(b)$. ∎

The main question is now whether (2.7) implies similarity. Here we refer to [11, Lemma 4.3]. This lemma says:

$$\text{Let } a, b \in \mathbb{H}_{\mathrm{coq}} \backslash \mathbb{R} \text{ and let (2.7) be valid. Then } a \sim b. \tag{2.8}$$

The proof is by matrix arguments, and it would also apply to $\mathbb{H}_{\mathrm{nec}}$ and to $\mathbb{H}_{\mathrm{con}}$ instead of $\mathbb{H}_{\mathrm{coq}}$. However, for quaternions $\mathbb{H}$, (2.7) is a necessary and sufficient condition for similarity without any restriction. See also [21] for coquaternions, where the condition $a, b \notin \mathbb{R}$ is omitted.

THEOREM 2.9. *Let $a, b \in \mathbb{H}_{\mathrm{coq}} \backslash \mathbb{R}$ and $a \sim b$. Then a similarity transformation, expressed by an invertible $h \in \mathbb{H}_{\mathrm{coq}}$ can be found by computing the kernel (= null space) of the homogeneous, singular matrix equation*

$$\mathbf{Mh} = \mathbf{0},$$

where $\mathbf{M}$ is the real 4×4 matrix equivalent to Sylvester's equation

$$ah - hb = 0, \quad h \text{ invertible}.$$

See [12, 15]. The kernel of $\mathbf{M}$ can be computed by applying a singular value decomposition to the matrix $\mathbf{M}$. Details can be deduced from Example 2.11.

Let $a \in \mathcal{A}$ and $\mathrm{abs}_2(a) - (\Re(a))^2 = 0$. In such a situation it is sometimes desirable that $(\Re(a), 0, 0, 0)$ and a are in the same similarity class. This can be achieved by slightly changing the definition of similarity to the condition which is given in (2.7).

DEFINITION 2.10. *Let $a, b \in \mathcal{A}$. The two elements a, b are said to be* quasi similar*, abbreviated as $a \overset{q}{\sim} b$, if (2.7) is valid. The quasi similarity classes will be denoted by $[a]_q$.*

Quasi similarity is also an equivalence relation. It is clear that similarity implies quasi similarity and that

$$[a] \subset [a]_q \quad \text{for all } a \in \mathcal{A}.$$

See [11] for more details.

EXAMPLE 2.11. Let $a = (1, 5, 4, 3) \in \mathbb{H}_{\mathrm{coq}}$ and $b = (1, 1, 1, 0) \in \mathbb{H}_{\mathrm{coq}}$. According to (2.8), these elements are similar in $\mathbb{H}_{\mathrm{coq}}$ and they are both quasi similar to 1 because of $\Re(a) = \Re(b) = \mathrm{abs}_2(a) = \mathrm{abs}_2(b) = 1$. We will furnish a direct proof of $a \sim b$ by finding the corresponding similarity transformation explicitly by using Theorem 2.9. For $\mathbf{M}$ we find in this case (see [12])

$$\mathbf{M} = \begin{bmatrix} 0 & -4 & 3 & 3 \\ 4 & 0 & 3 & -5 \\ 3 & 3 & 0 & -6 \\ 3 & -5 & 6 & 0 \end{bmatrix}. \tag{2.9}$$

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This matrix $\mathbf{M}$ has rank 2 and, therefore, the corresponding kernel has dimension 2 which implies that the kernel contains invertible elements. In order to find the corresponding similarity transformation, we apply a singular value decomposition (abbreviated svd) to $\mathbf{M}$ and obtain $\mathtt{svd}(\mathbf{M}) = [U, S, V]$ (using MATLAB notation) where U, S, V are again 4×4 matrices. More details can be found in the classical reference by Horn and Johnson [9, p. 414]. The last two columns of V contain two linearly independent vectors spanning the kernel. This result can be found in standard textbooks; see, e.g., [19, p. 311]. These two vectors are here

$$[h_1, h_2] = \begin{bmatrix} -0.682852186027397 & 0.454741481629430 \\ 0.338967359220833 & 0.635870976388783 \\ 0.623898892364393 & 0.302521739509271 \\ -0.171942413403282 & 0.545306229009106 \end{bmatrix}. \tag{2.10}$$

As elements of $\mathbb{H}_{\mathrm{coq}}$, h_1, h_2 are invertible. Now we make the following numerical checks:

$$\begin{aligned} h_1^{-1} a h_1 &= (1.000000000000000, 1.000000000000003, \\ &\qquad\qquad 1.000000000000003, -0.000000000000004), \\ h_2^{-1} a h_2 &= (1.000000000000001, 1.000000000000000, \\ &\qquad\qquad 1.000000000000004, -0.000000000000001), \\ h_1 b h_1^{-1} &= (1.000000000000000, 5.000000000000009, \\ &\qquad\qquad 4.000000000000004, \quad 3.000000000000008), \\ h_2 b h_2^{-1} &= (1.000000000000000, 5.000000000000002, \\ &\qquad\qquad 4.000000000000004, \quad 2.999999999999999), \end{aligned}$$

and the check is affirmative, a and b are indeed similar in $\mathbb{H}_{\mathrm{coq}}$ within computer precision. The computations were carried out in MATLAB with about 15 significant decimal digits.

3. Finding zeros from similarity classes. We will treat the following problem: Given a polynomial p over $\mathcal{A}$ and a quasi similarity class $[z]_q \subset \mathcal{A}$, which is known to contain a zero $z_0 \in [z]_q$ of p. How to find the zero? The main idea is to write the polynomial p in a formally linear form. For this purpose, we use the identity

$$z^2 = -\mathrm{abs}_2(z) + 2\Re(z)z,$$

which is valid in $\mathcal{A}$ and in $\mathbb{H}$. It implies

$$z^k = \alpha_k + \beta_k z, \qquad \alpha_k, \beta_k \in \mathbb{R}, \tag{3.1}$$

$$\alpha_0 = 1, \qquad \beta_0 = 0, \tag{3.2}$$

$$\alpha_{k+1} = -\mathrm{abs}_2(z)\beta_k, \qquad \beta_{k+1} = \alpha_k + 2\Re(z)\beta_k \qquad k = 0, 1, \ldots, n-1. \tag{3.3}$$

This means that for a given $z \in \mathcal{A}$, the representation (3.1) is easily computable. For a first application of (3.1) in $\mathbb{H}$, see [20]. See also Horn and Johnson, [9, p. 87], for a matrix equivalent of (3.1). If we restrict our attention to one quasi similarity class $[z]_q$, then the coefficients $\alpha_k, \beta_k, k \geq 0$, are constant on this class. This follows from (2.7). If we apply (3.1) to all powers in the polynomial p, then we obtain

$$p(z) = \sum_{k=0}^{n} a_k z^k = \sum_{k=0}^{n} a_k(\alpha_k + \beta_k z) = \sum_{k=0}^{n} \alpha_k a_k + \left(\sum_{k=0}^{n} \beta_k a_k \right) z =: A + Bz, \tag{3.4}$$

and $A, B \in \mathcal{A}$ are constant on the quasi similarity class $[z]_q$. Though we have written A, B without arguments, they depend on z. More precisely, A, B depend on $\operatorname{abs}_2(z)$ and on $\Re(z)$ but not fully on z.

THEOREM 3.1. *In the representation $p(z) = A + Bz$, let B be invertible on the given class $[z]_q$, and let $[z]_q$ contain a zero z_0 of p. Then*

$$z_0 = -B^{-1}A$$

is the only zero of p in $[z]_q$. If $A = B = 0$, then all elements in $[z]_q$ are zeros of p.

Proof. From (3.4) it follows that $p(z_0) = 0$. Let there be two distinct zeros, $z_0, z_1 \in [z]_q$. Then, $p(z_0) = A + Bz_0 = 0$ and $p(z_1) = A + Bz_1 = 0$, which implies $B(z_0 - z_1) = 0$. If B is invertible, then $z_0 = z_1$ would follow, a contradiction. Thus, B is noninvertible if there are two distinct zeros $z_1, z_2 \in [z]_q$. The last part is obvious. □

THEOREM 3.2. *Let $B \neq 0$, but let B be noninvertible on the given class $[z]_q$, and let $[z]_q$ contain a zero of p. Assume that there is a real constant γ such that*

$$A + \gamma B = 0.$$

Then, for all real α, the quantity

$$z_0 = \alpha\overline{B} + \gamma \tag{3.5}$$

is a zero of p, provided that $z_0 \in [z]_q$.

Proof. We have

$$p(z_0) = A + Bz_0 = -\gamma B + B\big(\alpha\overline{B} + \gamma\big) = -\gamma B + \alpha B\overline{B} + \gamma B = 0.$$

The quasi similarity $z_0 \overset{q}{\sim} z$ has to be checked separately and will restrict the possible values of α. □

LEMMA 3.3. *Let in Theorem 3.2 $\Re(B) \neq 0$. Then there is at most one α which defines a zero z_0 which is contained in the quasi similarity class $[z]_q$.*

Proof. Since the real part is fixed in the whole quasi similarity class $[z]_q$, the equation $\Re(z_0) = \Re(\alpha\overline{B} + \gamma) = \alpha\Re(B) + \gamma$ allows several real parts for varying α. This is a contradiction. □

DEFINITION 3.4. *Zeros z_0 of p with the property that there is no other zero in $[z_0]_q$ are called* isolated. *Zeros z_0 with the property that all elements in $[z_0]_q$ are zeros are called* hyperbolic. *See [11, p. 139]. Zeros z_0 which are computed by formula (3.5) are called* unexpected *zeros.*

It should be noted that the similarity classes $[z]$ either contain infinitely many elements in case $[z]$ does not contain real elements or $[z]$ consist of a single element, which is possible only if $z \in \mathbb{R}$. However, in $\mathcal{A}$ there are no quasi similarity classes which contain only one element.

Examples related to the Theorems 3.1, 3.2 will be presented later.

4. The companion polynomial and its roots. Conjugation plays an important role in the following definition.

DEFINITION 4.1. *Let p be a polynomial of degree n of the form defined in (2.1). The real polynomial q of degree $2n$ defined by*

$$q(z) = \sum_{j,k=0}^{n} \overline{a_j} a_k z^{j+k} = \sum_{\ell=0}^{2n} b_\ell z^\ell, \qquad b_\ell = \sum_{j=\max(0,\ell-n)}^{\min(\ell,n)} \overline{a_j} a_{\ell-j} \in \mathbb{R}, \qquad 0 \le \ell \le 2n, \tag{4.1}$$

is called the companion polynomial *of p.*

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In [11, Lemma 6.2] it is shown that the coefficients b_ℓ defined in (4.1) are real. We will reserve the word *zeros* for solutions z of $p(z) = 0$ and will use the word *roots* for the solutions of $q(z) = 0$. Since q has even degree $2n$ and the coefficients b_ℓ of q are all real, there is an even number $2n_1$ of complex roots and an even number $2n_2$ of real roots such that $2(n_1 + n_2) = 2n$ where $n_1 = 0$ or $n_2 = 0$ is possible. And it is clear that the complex roots always appear in complex conjugate pairs. There is one important property of q, which will be used on certain occasions, namely

$$q(z) = p(z)\overline{p(z)} \quad \text{for all } z \in \mathbb{R}. \tag{4.2}$$

It follows that a real zero of p will be a real double root of q since the reals commute with all algebra elements. Note that $p(z)\overline{p(z)} = 0$ does in general not imply $p(z) = 0$. In another paper [11], we have called a z with $p(z)\overline{p(z)} = 0$ a *singular point* of p. The companion polynomial q, though not with that name, was already introduced 1941 by Niven [18]. In a later paper (2004) it was called *basic polynomial* by Pogorui and Shapiro [20].

THEOREM 4.2. *Let q be the companion polynomial of p, and let it have at least one pair of complex conjugate roots $c = u \pm v\mathbf{i}$, where $u, v \in \mathbb{R}$, $v > 0$. Define*

$$s := \begin{cases} u + v\mathbf{i} & \textit{for } \mathcal{A} = \mathbb{H}_{\rm coq}, \\ u + v\mathbf{j} & \textit{for } \mathcal{A} = \mathbb{H}_{\rm nec}, \\ u + v\mathbf{k} & \textit{for } \mathcal{A} = \mathbb{H}_{\rm con}. \end{cases} \tag{4.3}$$

Then in $[s]_q$ there may be a zero s_0 of p which can be found by applying one of the Theorems 3.1 or 3.2. In all cases $\operatorname{abs}_2(s) = u^2 + v^2$.

Proof. For $\mathbb{H}$ the proof is given in [14]. In [11] it is shown for $\mathcal{A} = \mathbb{H}_{\rm coq}$ that under the given conditions, $[s]_q$ contains a zero of p. The remaining part follows from Theorem 3.1. The proof given in [11] can easily be extended to the remaining two algebras $\mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$. $\square$

Since there are at most $n_1 \le n$ pairs of complex conjugate roots, p may have at most n_1 zeros derived from complex zeros of q. For $\mathbb{H}$, the paper [14] contains a complete description on how to find all (maximally n) zeros of p over $\mathbb{H}$. An extension to two-sided polynomials over $\mathbb{H}$ was given in [13].

THEOREM 4.3. *Let the companion polynomial q of p have at least one pair of real roots, r_1, r_2, and assume (without loss of generality) that $r_1 \ge r_2$. Define*

$$u := \frac{1}{2}(r_1 + r_2), \quad v := \frac{1}{2}(r_1 - r_2), \tag{4.4}$$

$$s := \begin{cases} u + v\mathbf{j} \textit{ or } u + v\mathbf{k} & \textit{for } \mathcal{A} = \mathbb{H}_{\rm coq}, \\ u + v\mathbf{i} \textit{ or } u + v\mathbf{k} & \textit{for } \mathcal{A} = \mathbb{H}_{\rm nec}, \\ u + v\mathbf{i} \textit{ or } u + v\mathbf{j} & \textit{for } \mathcal{A} = \mathbb{H}_{\rm con}. \end{cases} \tag{4.5}$$

Then in $[s]_q$ there may be a zero s_0 of p which can be found by applying one of the Theorems 3.1 or 3.2. If $r_1 = r_2$, then $s = u = r_1$ is a real zero of p and possibly also an unexpected zero of p. See Example 4.9. In all cases $\operatorname{abs}_2(s) = u^2 - v^2$.

Proof. Let $p(z) = A + Bz$; see (3.4). The proof has to be made under three assumptions: i: B is invertible, ii: $A = B = 0$, iii: $B \ne 0$ and B is not invertible. Assume $r_1 > r_2$. Though we have written A, B without an argument, both A and B depend on s and both are constant on $[s]_q$. We will show that s and s_0 are quasi similar which means that (2.7), mentioned in Theorem 2.8, is valid for s, s_0. The real part of s is $\Re(s) = u$, and $\operatorname{abs}_2(s) = u^2 - v^2 = (u + v)(u - v) = r_1 r_2$ in all three algebras $\mathcal{A}$. We have to show that

$$\Re(s_0) = u, \quad \operatorname{abs}_2(s_0) = u^2 - v^2 = r_1 r_2. \tag{4.6}$$

Let us assume that B is invertible. We have

$$\begin{aligned}\Re(s_0) &= -\frac{1}{2}(B^{-1}A+\overline{B^{-1}A}) = \frac{-1}{2\,\mathrm{abs}_2(B)}(\overline{B}A+\overline{A}B),\\ \mathrm{abs}_2(s_0) &= (-B^{-1}A)(\overline{-B^{-1}A}) = \frac{\mathrm{abs}_2(A)}{\mathrm{abs}_2(B)}.\end{aligned}$$

We note that the powers s^k, $k \geq 0$, of s have the form

$$s^k = \begin{cases} u_k + v_k\mathbf{j} & \text{for } \mathcal{A} = \mathbb{H}_{\mathrm{coq}}, \\ u_k + v_k\mathbf{i} & \text{for } \mathcal{A} = \mathbb{H}_{\mathrm{nec}} \text{ and for } \mathcal{A} = \mathbb{H}_{\mathrm{con}}, \end{cases} \tag{4.7}$$

where in all algebras

$$u_k = \frac{r_1^k + r_2^k}{2}, \quad v_k = \frac{r_1^k - r_2^k}{2}, \qquad k = 0, 1, \ldots$$

This can be shown by induction using $\mathbf{j}^2 = 1$ in $\mathbb{H}_{\mathrm{coq}}$ and $\mathbf{i}^2 = 1$ in $\mathbb{H}_{\mathrm{nec}}$ and in $\mathbb{H}_{\mathrm{con}}$. If we compare s^k from (4.7) with s^k from (3.1) we obtain in $\mathcal{A}$

$$\alpha_k = u_k - u\frac{v_k}{v}, \quad \beta_k = \frac{v_k}{v}.$$

From here,

$$\begin{aligned} A &= \sum_{k=0}^n \alpha_k a_k = \sum_{k=0}^n \left(u_k - u\frac{v_k}{v}\right) a_k = \sum_{k=0}^n u_k a_k - \frac{u}{v}\sum_{k=0}^n v_k a_k \\ &= \frac{1}{2}(p(r_1)+p(r_2)) - \frac{u}{2v}\,(p(r_1)-p(r_2)) \\ &= \frac{-r_2}{r_1-r_2}p(r_1) + \frac{r_1}{r_1-r_2}p(r_2), \\ B &= \sum_{k=0}^n \beta_k a_k = \frac{1}{v}\sum_{k=0}^n v_k a_k = \frac{1}{r_1-r_2}\,(p(r_1)-p(r_2))\,. \end{aligned}$$

These formulas imply, by using (4.2) frequently,

$$\begin{aligned} \mathrm{abs}_2(A) &= A\overline{A} \\ &= \left(\frac{-r_2}{r_1-r_2}p(r_1) + \frac{r_1}{r_1-r_2}p(r_2)\right)\left(\frac{-r_2}{r_1-r_2}\overline{p(r_1)} + \frac{r_1}{r_1-r_2}\overline{p(r_2)}\right) \\ &= \left(\frac{-r_2}{r_1-r_2}\right)^2 p(r_1)\overline{p(r_1)} + \left(\frac{r_1}{r_1-r_2}\right)^2 p(r_2)\overline{p(r_2)} \\ &\quad + \left(\frac{-r_2}{r_1-r_2}\right)\left(\frac{r_1}{r_1-r_2}\right)p(r_1)\overline{p(r_2)} + \left(\frac{r_1}{r_1-r_2}\right)\left(\frac{-r_2}{r_1-r_2}\right)p(r_2)\overline{p(r_1)} \\ &= -2\left(\frac{r_1r_2}{(r_1-r_2)^2}\right)\Re\left(p(r_1)\overline{p(r_2)}\right). \\ \mathrm{abs}_2(B) &= B\overline{B} \\ &= \left(\frac{1}{r_1-r_2}\right)^2\left(p(r_1)\overline{p(r_1)} + p(r_2)\overline{p(r_2)} - 2\Re\big(p(r_1)\overline{p(r_2)}\,\big)\right) \\ &= -2\left(\frac{1}{r_1-r_2}\right)^2\Re\left(p(r_1)\overline{p(r_2)}\right), \end{aligned}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

$$\begin{aligned}\overline{B}A + \overline{A}B &= \frac{1}{r_1 - r_2}\left(\overline{p(r_1)} - \overline{p(r_2)}\right)\left(\frac{-r_2}{r_1 - r_2}p(r_1) + \frac{r_1}{r_1 - r_2}p(r_2)\right)\\ &\quad + \frac{1}{r_1 - r_2}\left(\frac{-r_2}{r_1 - r_2}\overline{p(r_1)} + \frac{r_1}{r_1 - r_2}\overline{p(r_2)}\right)(p(r_1) - p(r_2))\\ &= 2\frac{r_1 + r_2}{(r_1 - r_2)^2}\Re\left(\overline{p(r_1)}p(r_2)\right).\end{aligned}$$

Finally, also using (2.2),

$$\frac{\operatorname{abs}_2(A)}{\operatorname{abs}_2(B)} = r_1 r_2, \quad -\frac{\overline{B}A + \overline{A}B}{2\operatorname{abs}_2(B)} = u,$$

which coincides with (4.6). The last part of the theorem is obvious. ∎

The motivation for the use of the formulas (4.4), (4.5) is taken from [10, Table 5], where under certain conditions the two eigenvalues of a 2×2 matrix are the sum and the difference of two real numbers.

DEFINITION 4.4. *Let a pair of real roots r_1, r_2 or a pair of conjugate complex roots $u \pm v\mathbf{i}$ of the companion polynomial q have the property that it defines a zero s_0 of the given polynomial p by applying one of the Theorems 4.2, 4.3. Then we say that the pair of roots of q* generates *a zero s_0 of p.*

THEOREM 4.5. *Let p be a polynomial of degree n as defined in (2.1) over $\mathcal{A}$. Then the companion polynomial q of p generates at most $n(2n-1)$ zeros of p.*

Proof. Let the companion polynomial q (defined in (4.1)) have only real roots such that their number is $2n$. Then the number of real pairs is $\binom{2n}{2} = 2n(2n-1)/2 = n(2n-1)$ and according to Theorem 4.3, each pair may generate a zero of p. In Example 7.1 below with a polynomial of degree $n = 3$ we will show that the upper bound $2(2n-1) = 15$ of zeros will be attained. Another example of a polynomial of degree 4 with the maximum number of zeros $n(2n-1) = 28$ is presented in Example 7.4. ∎

We can more precisely estimate the number of zeros of p if the companion polynomial has $2n_1$ (nonreal) complex roots and $2n_2$ real roots.

THEOREM 4.6. *Let p be a polynomial of degree n over $\mathcal{A}$, and let the roots of the companion polynomial q be $r_1, r_2, \ldots, r_{2n}$. Assume that the first $2n_1$ roots are (nonreal) complex and that the remaining $2n_2 := 2n - 2n_1$ roots are real. Then the number of zeros of p is*

$$\#\{z : p(z) = 0\} \le n_1 + \binom{2n - 2n_1}{2} = n_1 + (n - n_1)(2n - 2n_1 - 1),$$

where all quasi similar zeros are counted as one zero. The maximum, $n(2n-1)$, is attained for $n_1 = 0$ when there are no complex roots of q and when all real roots of q are pairwise distinct.

Proof. The result follows from formulas (4.3) and (4.5) in Theorems 4.2, 4.3. In (4.3) there are at most n_1 complex roots with positive imaginary part, and in (4.5) there are at most $\binom{2n-2n_1}{2}$ real pairs. ∎

EXAMPLE 4.7. We start with an extremely simple example. Let

$$p(z) = d - z, \quad z, d \in \mathcal{A}.$$

Then the companion polynomial is

$$q(z) = z^2 - 2\Re(d)z + \operatorname{abs}_2(d),$$

and the zeros of q are $\Re(d) \pm \sqrt{(\Re(d))^2 - \mathrm{abs}_2(d)}$. The formally linear form of p is

$$p(z) = A + Bz, \qquad A = d, B = -1,$$

and A, B do not depend on z, and B is always invertible. Thus, independent of q, we have

$$z_0 = -AB^{-1} = -d(-1)^{-1} = d,$$

and z_0 is an isolated zero of p.

EXAMPLE 4.8. Let

$$\begin{aligned} p(z) &= (z-1)(z-2) = z^2 - 3z + 2 \\ &= \big(-\mathrm{abs}_2(z) + 2\big) + \big(2\Re(z) - 3\big)z =: A(z) + B(z)z \end{aligned}$$

be a polynomial over $\mathcal{A}$. The companion polynomial q is in all algebras $\mathcal{A}$ and in $\mathbb{H}$

$$q(z) = z^4 - 6z^3 + 13z^2 - 12z + 4,$$

and the zeros of q are in all cases $1, 1, 2, 2$. For $\mathbb{H}$ this implies that 1 and 2 are real zeros of p, and there are no other zeros of p in $\mathbb{H}$. Let $\mathcal{A} = \mathbb{H}_{\mathrm{coq}}$. There are three distinct real pairs $(1, 1)$, $(1, 2)$, $(2, 2)$. We apply Theorem 4.3 and find for the first and the last pair $s = 1, s = 2$, respectively, and $A(1) = 1, A(2) = -2, B(1) = -1, B(2) = 1$. This implies

$$s_0 = -B(1)^{-1}A(1) = 1, \quad s_0 = -B(2)^{-1}A(2) = 2,$$

and both zeros s_0 of p are isolated. However, in $\mathbb{H}_{\mathrm{coq}}$ the pair $(1, 2)$ defines, using the same theorem, $s = \frac{1}{2}(3 + \mathbf{j})$ or $s = \frac{1}{2}(3 + \mathbf{k})$, and in both cases we have $A = B = 0$ and the above mentioned (similar) zeros are hyperbolic zeros. Thus, p over $\mathbb{H}_{\mathrm{coq}}$ has 3 zeros. The same is valid in $\mathbb{H}_{\mathrm{nec}}$ and in $\mathbb{H}_{\mathrm{con}}$ if we apply Theorem 4.3 correspondingly.

We will furnish an example which shows by an application of Theorem 3.2 the existence of unexpected zeros.

EXAMPLE 4.9. For $a \in \mathcal{A}$ but $a \notin \mathbb{R}$ and $(\Re(a))^2 - \mathrm{abs}_2(a) = 0$, we define

$$\begin{aligned} p(z) &:= z^2 - 2az + a^2 \\ &= (-\mathrm{abs}_2(z) + a^2) + 2(\Re(z) - a)z =: A(\mathrm{abs}_2(z)) + B(\Re(z))z. \end{aligned}$$

It is easy to see that $p(a) = 0$. In this case the companion polynomial is

$$q(z) = (z - \Re(a))^4.$$

It defines only one pair of real roots $\big(\Re(a), \Re(a)\big)$, and the evaluation of A and B at $s = \Re(a)$ yields

$$A = -\mathrm{abs}_2(\Re(a)) + a^2 = -(\Re(a))^2 + a^2, \quad B = 2(\Re(a) - a). \tag{4.8}$$

LEMMA 4.10. *In Example 4.9 we have*

$$B \neq 0, \quad \Re(B) = 0, \quad B \textit{ noninvertible}, \quad A = -\Re(a)B.$$

Proof. The first two properties follow from the last part of (4.8). For the third one we have

$$\begin{aligned} B\overline{B} &= 4(\Re(a) - a)(\Re(a) - \overline{a}) = 4\big((\Re(a))^2 - \Re(a)(a + \overline{a}) + a\overline{a}\big) \\ &= 4(a\overline{a} - (\Re(a))^2) = 0. \end{aligned}$$

Thus, B is noninvertible. Finally,

$$-\Re(a)B = -2(\Re(a))^2+2\Re(a)a = -(\Re(a))^2-\mathrm{abs}_2(a)+2\Re(a)a = -(\Re(a))^2+a^2 = A. \qquad \square$$

This lemma implies that Theorem 3.2 is applicable, which shows that for all real α

$$z_0 = \alpha\overline{B} + \Re(a) = 2\alpha(\Re(a) - \overline{a}) + \Re(a)$$

is a zero of p. Let $a = (a_1, a_2, a_3, a_4)$. Then the zeros have the form

$$z_0 = (a_1, \alpha a_2, \alpha a_3, \alpha a_4) \qquad \text{for all } \alpha \in \mathbb{R}.$$

Thus, they are all quasi similar to $a_1 = \Re(a)$ and are unexpected zeros. However, not all elements quasi similar to $\Re(a)$ belong to that quasi similarity class. The unexpected zeros consist of an infinite subset of $[\Re(a)]_q$ but do not exhaust this set.

It should be noted that the similarity classes $[z]$ either contain infinitely many elements, in which case $[z]$ does not contain real elements, or $[z]$ consist of a single element, which is possible only if $z \in \mathbb{R}$. However, in $\mathcal{A}$ there are no quasi similarity classes which contain only one element.

5. All zeros of p are generated by roots of q. We will show that all zeros of p are generated by roots of q.

THEOREM 5.1. *Let p have a zero s_0 where the similarity class $[s_0]$ contains an element of the form*

$$\begin{aligned} s &:= u + v\mathbf{i}, && \textit{if } p \textit{ is a polynomial over } \mathbb{H}_{\mathrm{coq}}, \\ s &:= u + v\mathbf{j}, && \textit{if } p \textit{ is a polynomial over } \mathbb{H}_{\mathrm{nec}}, \\ s &:= u + v\mathbf{k}, && \textit{if } p \textit{ is a polynomial over } \mathbb{H}_{\mathrm{con}}, \qquad u, v \in \mathbb{R}, v > 0 \textit{ in all cases.} \end{aligned}$$

Then there exists a (nonreal) complex s such that $q(s) = 0$ and s generates s_0.

Proof. The result follows from Theorem 6.10 for coquaternions in [11], which can be adapted to the other two algebras $\mathbb{H}_{\mathrm{nec}}, \mathbb{H}_{\mathrm{con}}$. $\square$

In the paper [11, p. 146], we have written with respect to the algebra of coquaternions: “The previous theorem tells us that we can find all zeros of p employing the companion polynomial provided that the zero has a complex number in its equivalence class. ... but all others cannot be found." This is now not true anymore. We are able to find all zeros by employing the companion polynomial, and the gap is closed by Theorem 5.2.

THEOREM 5.2. *Let p have a zero s_0 where the similarity class $[s_0]$ contains an element of the form (see also (4.5))*

$$\begin{aligned} s &:= u + v\mathbf{j}, && \textit{if } p \textit{ is a polynomial over } \mathbb{H}_{\mathrm{coq}}, \\ s &:= u + v\mathbf{i}, && \textit{if } p \textit{ is a polynomial over } \mathbb{H}_{\mathrm{nec}} \textit{ or over } \mathbb{H}_{\mathrm{con}}, \qquad u, v \in \mathbb{R}, v > 0 \textit{ in all cases.} \end{aligned}$$

Then there exists a pair of real, distinct roots r_1, r_2 of q which generates s_0.

Proof. In all three algebras it is easy to retrieve s from s_0 uniquely. And the equations $(r_1 + r_2)/2 = u$, $(r_1 - r_2)/2 = v$ have the unique solution $r_1 = u + v, r_2 = u - v$. For the further proof we will use the identity for the companion polynomial q taken from [11, Formula (6.3)], which reads

$$q(z) = \mathrm{abs}_2(A) + 2\Re(\overline{B}A)z + \mathrm{abs}_2(B)z^2.$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

For the coefficients of this real quadratic equation we insert the results from the proof of Theorem 4.3. Then the standard solutions of $q(z) = 0$ are $z = r_1$ and $z = r_2$. Thus, the real pair r_1, r_2 generates s_0. □

COROLLARY 5.3. *All unilateral polynomials of degree n over $\mathbb{H}_{\text{coq}}, \mathbb{H}_{\text{nec}}, \mathbb{H}_{\text{con}}$ have at most $n(2n-1)$ zeros, which means that there are at most $n(2n-1)$ similarity classes which contain zeros.*

Proof. In Theorem 4.5 we have shown that a polynomial may have $n(2n-1)$ zeros. In Theorems 5.1, 5.2 we have shown that this number cannot be exceeded. □

6. An algorithm to find all zeros of polynomials over $\mathcal{A}$. In order to find all zeros of a given polynomial p over $\mathcal{A}$, we follow the steps of Algorithm 6.1.

ALGORITHM 6.1. Algorithm for finding all zeros of a polynomial p over $\mathcal{A}$ defined in (2.1) by means of the companion polynomial.

1. Let $a_0, a_1, \ldots, a_n$ be the coefficients of the polynomial p over $\mathcal{A}$. Assume that a_n is invertible.
2. Define an empty list of zeros of p.
3. Compute the real coefficients $c_0, c_1, \ldots, c_{2n}$ of the companion polynomial q by formula (4.1).
4. Compute all $2n$ real and complex roots of q by a standard routine.

   `for` all complex roots $u + v\mathbf{i}$ with $v > 0$ of q `do`
5. Define the algebra element *root* $= (u, v, 0, 0)$ if $\mathcal{A} = \mathbb{H}_{\text{coq}}$.
6. Define the algebra element *root* $= (u, 0, v, 0)$ if $\mathcal{A} = \mathbb{H}_{\text{nec}}$.
7. Define the algebra element *root* $= (u, 0, 0, v)$ if $\mathcal{A} = \mathbb{H}_{\text{con}}$.
8. Compute A, B at *root* by using formula (3.4).
9. Apply Theorems 3.1, 3.2.
10. If the result is a zero s_0, then add s_0 to the list of zeros of p. Also note the type of zero (isolated, hyperbolic, unexpected).

    `end for`

    `for` all real pairs r_1, r_2 of the roots of q `do` [do not distinguish between (r_1, r_2) and (r_2, r_1)]
11. Define $u = (r_1 + r_2)/2$, $v = \text{abs}\big((r_1 - r_2)/2\big)$.
12. Define the algebra element *root* $= (u, 0, v, 0)$ if $\mathcal{A} = \mathbb{H}_{\text{coq}}$.
13. Define the algebra element *root* $= (u, v, 0, 0)$ if $\mathcal{A} = \mathbb{H}_{\text{nec}}$.
14. Define the algebra element *root* $= (u, v, 0, 0)$ if $\mathcal{A} = \mathbb{H}_{\text{con}}$.
15. Repeat all steps from 8. to 10.

    `end for`

The result of this algorithm is a list of zeros of p, where the number of entries may vary from 0 to $n(2n-1)$. The list may contain multiple entries.

REMARK 6.2. In order to produce an executable program from Algorithm 6.1, the following computational steps must be possible:

1. Adding and multiplying algebra elements
2. Finding the companion polynomial
3. Finding roots of real polynomials
4. Finding A, B
5. Finding inverses of algebra elements
6. Evaluating a polynomial over $\mathcal{A}$ at an algebra element in $\mathcal{A}$

All these steps can be easily accomplished if one makes use of the *overloading technique* offered by MATLAB.

ETNA
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7. Examples. We present a few numerical examples.

EXAMPLE 7.1. For the coquaternionic polynomial p of degree 3 defined by the coefficients

$$a_3 = (2, 2, -1, 0), \quad a_2 = (-1, 0, -5, -1), \quad a_1 = (-4, -5, 1, 1), \quad a_0 = (2, -2, 2, 3),$$

we have presented 8 zeros in [11, Table 10.1]. In this case the companion polynomial q of degree six is defined by the coefficients $b_6 = 7, -14, -59, 24, 61, -6, b_0 = -5$, and it has exactly 6 pairwise distinct real zeros. With this paper we know that there must be 15 zeros of p. In Table 7.1 we present the missing 7 zeros. The other examples [11, Examples 10.2–10.4] have the correct number of zeros.

TABLE 7.1
The seven missing zeros of Example 10.1 in [11].

$1.410018698387151,$	$40.927688450784920,$	$-26.484628029183256,$	-31.296139541593462
$2.078329585493254,$	$35.227789879357942,$	$-23.037052468108019,$	-26.708143691872522
$1.780207170581877,$	$-3.512185413662750,$	$3.899454035433289,$	1.136051036343325
$-0.820915616403146,$	$-0.132277571822474,$	$0.994132668916126,$	-0.607528109039788
$-1.119038031314515,$	$-0.708374333154589,$	$0.481092542977948,$	-1.004459188532533
$-0.331689112894335,$	$70.975467125897083,$	$-43.119928985136582,$	-56.379387168520203
$-0.629811527805284,$	$0.558924803050916,$	$-0.631505659586322,$	-0.225026759123903

If we measure the error of the zeros z of Table 7.1 by

$$e := \frac{||p(z)||}{||z||},$$

where $||\cdot||$ is the Euclidean norm in $\mathbb{R}^4$, then in all cases $e \le 10^{-10}$.

EXAMPLE 7.2. We consider a slightly altered polynomial in the coquaternions $\mathbb{H}_{\mathrm{coq}}$, namely $\pi(z) = p(z)(z-1)$, where p is the polynomial defined in Example 7.1. We expect the same zeros of p as before with the additional zero $z = 1$. However, the companion polynomial q has degree 8 and could define maximally 28 zeros. The coefficients of q are $b_8 = 7, -28, -24, 128, -46, -104, 68, 4, b_0 = -5$. The computed roots r_j,$1 \le j \le 8$, of q are all real and the double root 1 is listed as r_5 and r_6. For the cases $(r_5, r_j), (r_6, r_j), j > 6$, we found $A + B = 0, A \ne 0,\ B \ne 0, B$ not invertible, and $\Re(B) \ne 0$ such that Lemma 3.3 applies and indicates that these pairs do not define a zero of p. The pair $(5, 6)$ defines the zero 1, and all other pairs define the zeros known from Example 7.1. Altogether there are 16 zeros as expected.

EXAMPLE 7.3. The above polynomial p in algebra $\mathbb{H}_{\mathrm{nec}}$ defines a companion polynomial q with two real roots and two pairs of complex conjugate roots. Thus, p has three zeros. The polynomial $\pi(z) = p(z)(z-1)$ in $\mathbb{H}_{\mathrm{nec}}$, where p is defined in Example 7.1 and is of degree 4, has 4 zeros.

EXAMPLE 7.4. The same polynomial p in $\mathbb{H}_{\mathrm{con}}$ defines a companion polynomial q with 4 real roots and one pair of complex conjugate roots. Thus, p has 7 zeros. The polynomial $\pi(z) = p(z)(z-1)$, again in $\mathbb{H}_{\mathrm{con}}$, where p is defined in Example 7.1 and is of degree 4, has 8 zeros.

We will end this section with a rare species of a polynomial p, namely one which has degree $n = 4$ where the corresponding companion polynomial q has only real roots.

EXAMPLE 7.5. Let $\mathcal{A} = \mathbb{H}_{\mathrm{coq}}$ and

$$p(z) = (1, 1, -2, 0)z^4 + (4, 2, 0, 3)z^3 + (-4, 0, 2, 4)z^2 + (-4, -2, -4, 0)z + (3, 2, 1, -3).$$

The polynomial is not monic, but the highest coefficient is invertible in $\mathbb{H}_{\mathrm{coq}}$, and one could divide all coefficients by the highest coefficient. However, this does not effect the zeros of p. The companion polynomial is

$$q(x) = -2x^8 + 12x^7 + 11x^6 - 84x^5 - 30x^4 + 98x^3 - 24x + 3,$$

and the 8 roots of q (by `roots` of MATLAB) are

$$\begin{aligned} r_1 &= -2.111197229212920, & r_5 &= 0.584463189626376,\\ r_2 &= -1.264898455994286, & r_6 &= 0.767284309708206,\\ r_3 &= -0.580911934832195, & r_7 &= 2.860897723805642,\\ r_4 &= 0.134343403046716, & r_8 &= 5.610018993852465. \end{aligned}$$

Applying Theorem 4.3 we obtain 28 possible similarity classes defined by

$$u_{j,k} = 0.5(r_k + r_j) + 0.5(r_k - r_j)\mathbf{j} \quad \text{or} \quad 0.5(r_k + r_j) + 0.5(r_k - r_j)\mathbf{k},$$
$$j = 1, 2, \ldots 8, k = j+1, j+2, \ldots, 8,$$

which may contain zeros. Checking these similarity classes by Theorem 3.1, we find that the formally linear polynomial form $p(z) = A + Bz$ has in all cases the property that B is invertible which means that all 28 pairs $(r_j, r_k), j < k$, define a zero of p and that all these zeros belong to a similarity class of the form $[a + \sqrt{-b}\,\mathbf{j}]$, $b < 0$. The zero of p corresponding to the pair (r_1, r_2) is

$$x_1 = (-1.688047842603601, -0.168989609556503, 0.405751318682548, 0.207313190398666),$$

and the zero corresponding to the pair (r_7, r_8) is

$$x_{28} = (4.235458358828954, -7.292058894146280, 6.971671162937881, 2.541523372096755).$$

This example shows that the maximal number of zeros, $n(2n-1) = 28$, is attained.

For $n > 4$ we were not able to find coquaternionic polynomials of degree n where the corresponding companion polynomial q of degree $2n$ had $2n$ real roots. This implies the following problem: Given $n > 4$, can we find a coquaternionic polynomial of degree n with the maximal number of $\binom{2n}{2}$ zeros?

If one is interested in polynomials over commutative algebras, then one should consult [11, 22].

8. A relation to an algorithm by Serôdio, Pereira, and Vitória. The algorithm published in [24] is tailored to the quaternionic case. It is based on the companion matrix of a monic polynomial over $\mathbb{H}$,

$$p(z) := z^n + a_{n-1}z^{n-1} + \cdots + a_0, \quad z, a_j \in \mathbb{H},\ j = 0, 1, \ldots, n-1,\ a_0 \neq 0 \tag{8.1}$$

and the quaternionic companion matrix is

$$\mathbf{C} := \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ & \ddots & \ddots & & \\ 0 & 0 & \ddots & 0 & -a_{n-2} \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{bmatrix}.$$

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Now, the $2n$ real or complex eigenvalues of $\mathbf{C}$ are determined by transferring $\mathbf{C}$ to a real $2n \times 2n$ matrix. These eigenvalues are used to apply Niven's algorithm resulting in the zeros of p. For details, see [24].

The results of our observations are put into the following conjecture.

CONJECTURE 8.1. *Let p be the polynomial defined in (8.1) over $\mathbb{H}$ or over $\mathcal{A}$. Then the $2n$ real or complex eigenvalues of $\mathbf{C}$ coincide with the $2n$ real or complex roots of the corresponding companion polynomial of p.*

This conjecture is based on many numerical experiments, and for degree $n \leq 3$ we can prove it. This means that the algorithm [24] can also be extended to algebras in $\mathcal{A}$, and the algorithms [11, 14] and [24] produce the same zeros, provided that Conjecture 8.1 is true.

This connection between the companion matrix and the companion polynomial also justifies the name companion polynomial.

9. Epilogue. The algebra elements considered here have an isomorphic image in 2×2 matrices. Details are given in [11] including the form of the matrices. This means that the whole paper could have been based on matrix equations. The details would be different, but the main results would be the same. As authors we had to make a decision and our decision favored algebra elements.

In [11] one also finds that the three algebras $\mathcal{A}$ are isomorphic. Here one also has to make a decision. If a certain problem has to be solved in two distinct, but isomorphic algebras, say $\mathcal{A}_1$ and $\mathcal{A}_2$, then one can apply the isomorphism rules to reduce the problem in Algebra $\mathcal{A}_2$ to algebra $\mathcal{A}_1$ and solve it there. The solution then has to be be retransformed to algebra $\mathcal{A}_2$. Another technique, which we prefer, is the adaption of the algebraic rules to the corresponding algebras. In our computer program we could adapt the algebraic rules to the rules for one specific algebra by just setting one integer variable to the corresponding algebra number.

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