A NEW GEOMETRIC ACCELERATION OF THE VON NEUMANN-HALPERIN PROJECTION METHOD

WILLIAMS LÓPEZ†

Abstract. We develop a geometrical acceleration scheme for the von Neumann-Halperin alternating projection method, when applied to the problem of finding the projection of a point onto the intersection of a finite number of closed subspaces of a Hilbert space. We study the convergence properties of the new scheme. We also present some encouraging preliminary numerical results to illustrate the performance of the new scheme when compared with a well-known geometrical acceleration scheme, and also with the original von Neumann-Halperin alternating projection method.

Key words. von Neumann-Halperin algorithm, alternating projection methods, orthogonal projections, acceleration schemes

AMS subject classifications. 52A20, 46C07, 65H10, 47J25

1. Introduction. Let $H$ be a Hilbert space with inner product $\langle ., . \rangle$. For a given $x_0 \in H$ and many closed subspaces $M_1, \ldots, M_p$ in $H$, we consider the best approximation problem: find the closest point to $x_0$ in $M = \cap_{i=1}^p M_i$, which can be stated as an optimization problem as follows:

$$\text{(1.1) minimize } \|x_0 - x\| \text{ subject to } x \in M,$$

where, for any $z \in H$, $\|z\|^2 = \langle z, z \rangle$. The unique solution $x^*$ of problem (1.1) is called the projection of $x_0$ onto $M$ and is denoted as $P_M(x_0)$.

In 1933, von Neumann solved problem (1.1) for the particular case of two closed subspaces.

THEOREM 1.1 (von Neumann [28]). If $M_1$ and $M_2$ are closed subspaces in $H$, then for each $x_0 \in H$,

$$\lim_{k \to \infty} (P_{M_2}P_{M_1})^k(x_0) = P_{M_1 \cap M_2}(x_0).$$

Figure 1.1 shows the geometric interpretation of Theorem 1.1. The extension to more than two subspaces was established in 1962 by Halperin.

THEOREM 1.2 (I. Halperin [23]). If $M_1, \ldots, M_p$ are closed subspaces in $H$, then for each $x_0 \in H$,

$$\lim_{k \to \infty} (P_{M_p}P_{M_{p-1}} \cdots P_{M_1})^k(x_0) = P_{\cap_{i=1}^p M_i}(x_0).$$

Theorem 1.2 suggests an algorithm, called the method of alternating projections (or MAP for short); see [11, 16], which can be described as follows: for any $x_0 \in H$, set

$$\text{(1.2) } x_0^k = x_{p}^{k-1}$$

$$x_i^k = P_{M_i}(x_{i-1}^k) \quad i = 1, 2, \ldots, p,$$

for $k \in Z^+$, with initial value $x_0^0 = x_0$. The MAP is closely related to Kaczmarz alternating projection method [25] for solving linear systems of equations.
ACCELERATION OF THE VON NEUMANN-HALPERIN PROJECTION METHOD

THEOREM 1.3. For any $i = 1, 2, \ldots, p$, the sequence $\{x^k\}$ generated by (1.2) converges to $P_M(x_0)$.

Proof. Let $i \in \{1, 2, \ldots, p\}$. Then $x^{k+1}_i = (P_{M_i} P_{M_{i-1}} \cdots P_{M_1})(x^k_p)$, for all $k \geq 0$, where $x^k_p = (P_{M_p} P_{M_{p-1}} \cdots P_{M_1})(x_0)$. From Theorem 1.2 we have that $x^k_p \to P_M(x_0)$ when $k \to \infty$ and since $P_{M_i}, P_{M_{i-1}}, \ldots, P_{M_1}$ are bounded linear operators, then

$$x^{k+1}_i = (P_{M_i} P_{M_{i-1}} \cdots P_{M_1})(x^k_p) \to (P_{M_i} P_{M_{i-1}} \cdots P_{M_1})(P_M(x_0)) = P_M(x_0)$$

when $k \to \infty$.

Applications of MAP in various areas of mathematics can be found in [16, 9, 10, 3, 7, 8, 22]. The MAP has an $r$-linear rate of convergence that can be very slow when the angles between the subspaces (in the sense of Friedrichs [20]) are small; see, e.g., [16]. Franchetti and Light [19] showed that the convergence in Theorem 1.1 may be arbitrarily slow if $M_1 + M_2$ is not closed (i.e., if the angle between $M_1$ and $M_2$ is zero). Consequently, several acceleration schemes have been proposed; see, e.g., [4, 21, 24, 12]. Motivated by a recent work of López and Raydan [26], in this work we present and analyze a geometrical acceleration scheme for MAP in (1.2), which solves problem (1.1). The new scheme involves finding points closest to a solution $(x^*, x^*)$ in the product space $H \times H$, where $x^*$ is the solution of (1.1).

The rest of this paper is organized as follows. In Section 2 we provide information about already existing acceleration schemes for MAP. In Section 2.1 we develop the new acceleration scheme and discuss its theoretical properties. In Section 3 we present encouraging preliminary numerical results.

2. Accelerations for MAP. The sequences generated by the method of alternating projections often converge very slowly. Such a slow convergence is observed, e.g., if $M_1$ and $M_2$ are two closed subspaces with angles close to 0 (see, e.g., [19]). Since the method of alternating projections has many practical applications, any acceleration technique seems to be important.

Several acceleration schemes with a geometrical flavor have been proposed to improve the performance of MAP; see, e.g., De Pierro and Iusem [13], Dos Santos [14], Gearhart and Koshy [21], Bauschke et al. [4], Gubin, Polyak and Raik [22], Martínez [27], Appleby and Smolarski [1] and Censor [6]. In Section 2.1, we describe a geometrically appealing acceleration scheme following the presentation in [21]. Some other different acceleration ideas have also been developed (see, e.g., Echebest et al. [17, 18] and Scolnik et al. [29, 30]) based on the so-called projected aggregation method (PAM). Other specialized acceleration scheme
ideas have been developed by Hernández-Ramos et. al. [24] based on the use of the conjugate
gradient method for minimizing a related convex quadratic map. Usually vector extrapolation
algorithms are used for accelerating the convergence of MAP; see, e.g., [3, 8, 7].

Dykstra’s algorithm [15, 5] solves problem (1.1) when the involved sets are closed and
convex (not necessarily closed subspaces). Dykstra’s algorithm can be viewed as a natural
extension to convex sets of von Neumann-Halperin’s MAP for subspaces in a Hilbert space.
The sequences generated by Dykstra’s algorithm often converge very slowly. Such a slow
convergence is observed, e.g., if \( M_1 \) is a hyperplane and \( M_2 \) is a ball which is tangent to \( M_1 \); see, e.g., [2, Example 5.3]. An acceleration scheme has been developed for Dykstra’s algorithm
[26]. In Section 2.1 we develop a new acceleration scheme for von Neumann-Halperin’s MAP
on closed subspaces, which is related to the scheme given in [26].

2.1. A new acceleration scheme. We now discuss an acceleration scheme for MAP
to solve problem (1.1). For that we need to consider an auxiliary sequence in the product
space \( H \times H \) that will be denoted as \( H^2 \). If \( H \) is a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \),
we will use the inner product in \( H^2 \) defined by \( \langle (x, y), (w, z) \rangle = \langle x, w \rangle + \langle y, z \rangle \), for all
\( (x, y), (w, z) \in H^2 \) and norm \( \| (x, y) \|^2 = \| x \|^2 + \| y \|^2 \), for all \( (x, y) \in H^2 \).
Thus \( H^2 \) is a Hilbert space. For each \( k \in \mathbb{Z}^+ \), let us define
\[
\hat{x}_k = (x^k_{p-1}, x^k_p) \in H^2, \tag{2.1}
\]
where \( x^k_{p-1} \) and \( x^k_p \) are defined in (1.2). Let us also denote
\[ \hat{x}^* = (P_M (x_0), P_M (x_0)) \in H^2. \]
It follows that \( \{ \hat{x}_k \} \) is a sequence in \( H^2 \) that converges to \( \hat{x}^* \). Our idea to accelerate the
convergence of MAP consists in building another sequence in \( H^2 \) which converges faster to \( \hat{x}^* \) than \( \{ \hat{x}_k \} \). For that we need to define a suitable subspace \( \Pi \) of \( H^2 \) that contains \( \hat{x}^* \), as follows:
\[ \Pi = \{ (x, x) : x \in H \}. \]

Clearly, \( \Pi \) is a closed subspace in \( H^2 \) and \( \hat{x}^* \in \Pi \). Let us denote by \( P_\Pi (\hat{x}) \) the orthogonal
projection of \( \hat{x} \in H^2 \) onto \( \Pi \). For each \( k \in \mathbb{Z}^+ \), let us consider \( P_\Pi (\hat{x}_k) \in \Pi \), where \( \{ \hat{x}_k \} \) is given by (2.1).

**Theorem 2.1.** For all \( k \geq 1 \), \( \| P_\Pi (\hat{x}_k) - \hat{x}^* \| \leq \| \hat{x}_k - \hat{x}^* \| \).

**Proof.** For all \( k \geq 1 \), since \( \hat{x}_k - P_\Pi (\hat{x}_k) \) is orthogonal to \( \Pi \), and since \( P_\Pi (\hat{x}_k) \) and \( \hat{x}^* \)
belong to \( \Pi \), then \( \hat{x}_k - P_\Pi (\hat{x}_k) \) is orthogonal to \( P_\Pi (\hat{x}_k) - \hat{x}^* \). Hence
\[
\| \hat{x}_k - \hat{x}^* \|^2 = \| \hat{x}_k - P_\Pi (\hat{x}_k) + P_\Pi (\hat{x}_k) - \hat{x}^* \|^2 = \| \hat{x}_k - P_\Pi (\hat{x}_k) \|^2 + \| P_\Pi (\hat{x}_k) - \hat{x}^* \|^2.
\]
Consequently, \( \| \hat{x}_k - \hat{x}^* \|^2 \geq \| P_\Pi (\hat{x}_k) - \hat{x}^* \|^2 \), thus \( \| P_\Pi (\hat{x}_k) \|^2 - \| \hat{x}_k - \hat{x}^* \|^2 \leq \| \hat{x}_k - \hat{x}^* \|^2 \).

The sequence \( \{ P_\Pi (\hat{x}_k) \} \) will play a key role in the development of the acceleration
scheme for MAP.

**Remark 2.2.** If \( (x, y) \in H^2 \), then (see, e.g., [26])
\[
P_\Pi (x, y) = 1/2(x + y, x + y). \tag{2.2}
\]
Notice that computing the projection onto \( \Pi \) only requires us to compute the average of the
two involved vectors, i.e., it requires a very inexpensive calculation.

Let \( \{ \hat{x}_k \} \) be the sequence defined by (2.1). For any \( k \geq 1 \), with \( P_\Pi (\hat{x}_{k+1}) \neq P_\Pi (\hat{x}_k) \),
let \( \delta_k \in \Pi \) be the projection of \( \hat{x}^* \) onto the line that goes through \( P_\Pi (\hat{x}_{k+1}) \) and \( P_\Pi (\hat{x}_k) \) (see
Therefore

\[ \hat{o}_k = P_\Pi(\hat{x}_k) + \alpha_k(P_\Pi(\hat{x}_{k+1}) - P_\Pi(\hat{x}_k)). \]

Therefore

\[ (P_\Pi(\hat{x}_k) - \hat{x}^* + \alpha_k(P_\Pi(\hat{x}_{k+1}) - P_\Pi(\hat{x}_k)), P_\Pi(\hat{x}_{k+1}) - P_\Pi(\hat{x}_k)) = 0. \]

Solving for \( \alpha_k \), from (2.3), gives

\[ \alpha_k = \frac{\langle P_\Pi(\hat{x}_k) - \hat{x}^*, P_\Pi(\hat{x}_k) - P_\Pi(\hat{x}_{k+1}) \rangle}{\| P_\Pi(\hat{x}_{k+1}) - P_\Pi(\hat{x}_k) \|^2}. \]

In the following result we will give a formula for \( \alpha_k \) in (2.4) that does not require knowledge of \( \hat{x}^* \).

**Theorem 2.3.** Let \{\( \hat{x}_k \)\} be defined by (2.1). Then \( \langle \hat{x}^*, P_\Pi(\hat{x}_k) - P_\Pi(\hat{x}_{k+1}) \rangle = 0 \) for all \( k \geq 1 \).

**Proof.** Let \( k \geq 1 \). Since \( \hat{x}^* \in \Pi \) and the projection \( P_\Pi \) is self-adjoint, we obtain

\[ \langle \hat{x}^*, P_\Pi(\hat{x}_k) - P_\Pi(\hat{x}_{k+1}) \rangle = \langle \hat{x}^*, \hat{x}_k - \hat{x}_{k+1} \rangle = \langle (x^*, x^*), (x^*_{p-1} - x^*_{p+1}, x^*_p - x^*_{p+1}) \rangle = \langle x^*, x^*_{p-1} - x^*_{p+1} \rangle + \langle x^*, x^*_p - x^*_{p+1} \rangle. \]

On the other hand, since \( x^* \in M = \cap_{i=1}^p M_i \) and the projections \( P_{M_i} \) are self-adjoint, we have

\[ \langle x^*, x^*_p - x^*_{p+1} \rangle = \langle x^*, x^*_p \rangle - \langle x^*, x^*_{p+1} \rangle = \langle x^*, x^*_p \rangle - \langle x^*, (P_{M_p} P_{M_{p-1}} \ldots P_{M_1})(x^*_p) \rangle = \langle x^*, x^*_p \rangle - \langle x^*, x^*_p \rangle = 0. \]

Similarly, we also obtain \( \langle x^*, x^*_{p-1} - x^*_{p+1} \rangle = 0. \) \( \square \)

**Corollary 2.4.** Let \( \alpha_k \), for some \( k \), be given by (2.4). Then

\[ \alpha_k = \frac{\langle P_\Pi(\hat{x}_k), P_\Pi(\hat{x}_k) - P_\Pi(\hat{x}_{k+1}) \rangle}{\| P_\Pi(\hat{x}_{k+1}) - P_\Pi(\hat{x}_k) \|^2}. \]

**Proof.** This is a consequence of Theorem 2.3 and (2.4). \( \square \)
Let us define the sequence \( \{ \hat{o}_k \} \) in \( \Pi \), for \( k \geq 1 \), as follow

\[
\hat{o}_k = \begin{cases} 
  P_{\Pi}(\hat{x}_{k+1}) & \text{if } \|P_{\Pi}(\hat{x}_{k+1}) - P_{\Pi}(\hat{x}_k)\| = 0, \\
  P_{\Pi}(\hat{x}_k) + \alpha_k(P_{\Pi}(\hat{x}_{k+1}) - P_{\Pi}(\hat{x}_k)) & \text{if } \|P_{\Pi}(\hat{x}_{k+1}) - P_{\Pi}(\hat{x}_k)\| \neq 0,
\end{cases}
\]

where \( \alpha_k \) is given by (2.5).

**Theorem 2.5.** Let \( \{ \hat{o}_k \} \) be given by (2.6). Then \( \|\hat{o}_k - \hat{x}^*\| \leq \|\hat{x}_{k+1} - \hat{x}^*\| \) for all \( k \geq 1 \).

**Proof.** Let \( k \geq 1 \). If \( \|P_{\Pi}(\hat{x}_{k+1}) - P_{\Pi}(\hat{x}_k)\| = 0 \), then from Theorem 2.1 and (2.6) it follows that \( \|\hat{o}_k - \hat{x}^*\| = \|P_{\Pi}(\hat{x}_{k+1}) - \hat{x}^*\| \leq \|\hat{x}_{k+1} - \hat{x}^*\| \). On the other hand, if \( \|P_{\Pi}(\hat{x}_{k+1}) - P_{\Pi}(\hat{x}_k)\| \neq 0 \), then \( \hat{o}_k \) is chosen to minimize the distance to \( \hat{x}^* \) along the line connecting \( P_{\Pi}(\hat{x}_{k+1}) \) and \( P_{\Pi}(\hat{x}_k) \). Therefore \( \|\hat{o}_k - \hat{x}^*\| \leq \|P_{\Pi}(\hat{x}_{k+1}) - \hat{x}^*\| \) and, from Theorem 2.1, it follows that \( \|\hat{o}_k - \hat{x}^*\| \leq \|\hat{x}_{k+1} - \hat{x}^*\| \). \( \square \)

**Remark 2.6.** For the particular case of closed subspaces, the sequence \( \{ \hat{o}_k \} \) defined by (2.6) accelerates the convergence of the sequence \( \{ \hat{x}_k \} \) defined by (2.1).

**Remark 2.7.** Since the sequence \( \{ \hat{o}_k \} \) defined by (2.6) belongs to \( \Pi \), for any \( k \geq 1 \), there exists a unique \( o_k \in H \) such that

\[
\hat{o}_k = (o_k, o_k) \in \Pi.
\]
Theorem 2.8. Let \( \{o_k\} \) the sequence defined by (2.7). Then, for each \( k \geq 1 \),
\[
\|o_k - P_M(x_0)\| \leq \max\{\|x_{p-1}^{k+1} - P_M(x_0)\|, \|x_p^{k+1} - P_M(x_0)\|\}.
\]

Proof. Let \( k \geq 1 \). From Theorem 2.5, we have that
\[
\sqrt{2}\|o_k - P_M(x_0)\| = \sqrt{\|o_k - P_M(x_0)\|^2 + \|o_k - P_M(x_0)\|^2} = \|o_k - \hat{x}^*\|
\]
\[
\leq \|\hat{x}_{k+1} - \hat{x}^*\|
\]
\[
= \sqrt{\|x_{p-1}^{k+1} - P_M(x_0)\|^2 + \|x_p^{k+1} - P_M(x_0)\|^2}
\]
\[
\leq \sqrt{2} \max\{\|x_{p-1}^{k+1} - P_M(x_0)\|, \|x_p^{k+1} - P_M(x_0)\|\}
\]
and the result is established.

Corollary 2.9. The sequence \( \{o_k\} \) defined by (2.7) converges to \( P_M(x_0) \).

Proof. This is a consequence of Theorem 2.8 and Theorem 1.3.

Remark 2.10. Observe that the sequence \( \{\hat{o}_k\} \) in \( H^2 \) accelerates the convergence of the sequence \( \{\hat{x}_k\} \) in \( H^2 \) (see Theorem 2.5). However, Theorem 2.8 does not guarantee acceleration when the sequence \( \{o_k\} \) in \( H \) is compared to the original MAP sequence \( \{x_p^k\} \) in \( H \), since a “maximal” is involved. Nevertheless, based on numerical experimentation, the sequence \( \{o_k\} \) almost always accelerates the convergence of the original MAP sequence \( \{x_p^k\} \).

We will now describe explicitly the sequence \( \{o_k\} \) defined by (2.7). If \( \|P_M(\hat{x}_{k+1}) - P_M(\hat{x}_k)\| = 0 \), then from (2.1) and (2.2) we have that
\[
\hat{o}_k = P_M(\hat{x}_{k+1}) = P_M(x_{p-1}^{k+1}, x_p^{k+1}) = 1/2 (x_{p-1}^{k+1} + x_p^{k+1}, x_p^{k+1} + x_{p-1}^{k+1}) = (o_k, o_k) \in \Pi.
\]
In this case \( o_k = 1/2 (x_{p-1}^{k+1} + x_p^{k+1}) \in H \). If, on the other hand, \( \|P_M(\hat{x}_{k+1}) - P_M(\hat{x}_k)\| \neq 0 \), then
\[
\hat{o}_k = P_M(\hat{x}_k) + o_k(\Pi(\hat{x}_{k+1}) - P_M(\hat{x}_k)) = (o_k, o_k) \in \Pi,
\]
where
\[
o_k = 1/2 (x_p^k + o_k (x_{p-1}^{k+1} - x_p^k) + x_p^{k+1} + o_k (x_{p-1}^{k+1} - x_p^{k+1})) \in H.
\]

From Corollary 2.9 it follows that
\[
o_k = 1/2 (x_p^k + o_k (x_{p-1}^{k+1} - x_p^k) + x_p^{k+1} + o_k (x_{p-1}^{k+1} - x_p^{k+1})) \rightarrow P_M(x_0)
\]
when \( k \rightarrow \infty \). Since \( P_{M_p} \) is a bounded linear operator (where \( P_{M_p} \) is the projection operator onto \( M_p \)),
\[
P_{M_p} \left[ 1/2 (x_p^k + o_k (x_{p-1}^{k+1} - x_p^k) + x_p^{k+1} + o_k (x_{p-1}^{k+1} - x_p^{k+1})) \right] \rightarrow P_{M_p}(P_M(x_0))
\]
when \( k \rightarrow \infty \). Consequently
\[
1/2 \left[ x_p^k + o_k (x_{p-1}^{k+1} - x_p^k) + x_p^{k+1} + o_k (x_{p-1}^{k+1} - x_p^{k+1}) \right] \rightarrow P_M(x_0)
\]
when \( k \to \infty \). Therefore

\[
x^k_p + \alpha_k (x^k_{p-1} - x^k_p) \to P_M(x_0)
\]

when \( k \to \infty \). Equation (2.8) suggests how to define an accelerated sequence in \( H \) and a specialized algorithm for which convergence to the solution \( P_M(x_0) \) will be later established. Nevertheless, to achieve this it is convenient to write \( \alpha_k \) given by (2.5) as a function of the original MAP iterations in \( H \).

**Lemma 2.11.** Let \( \alpha_k \), for some \( k \), be given by (2.5). Then

\[
\alpha_k = \frac{\langle x^k_{p-1} + x^k_p, x^k_{p-1} - x^k_{p-1} \rangle + \langle x^k_p, x^k_{p-1} - x^k_p \rangle}{\|x^k_{p-1} + x^k_{p-1} - x^k_p\|^2}.
\]

**Proof.** Let \( k \geq 1 \). The numerator in (2.5) can be written as

\[
\langle P_{\Pi}(\tilde{x}_k), P_{\Pi}(\tilde{x}_{k+1}) \rangle = \langle P_{\Pi}(\tilde{x}_k), \tilde{x}_k - \tilde{x}_{k+1} \rangle = \frac{1}{2} \langle x^k_{p-1} + x^k_{p}, x^k_{p-1} + x^k_{p} \rangle - \frac{1}{2} \langle x^k_{p-1} + x^k_{p}, x^k_{p-1} + x^k_{p} \rangle = \frac{1}{2} (\langle x^k_{p-1} + x^k_{p}, x^k_{p-1} - x^k_{p} \rangle + \langle x^k_{p-1} + x^k_{p}, x^k_{p} - x^k_{p} \rangle).
\]

On the other hand

\[
P_{\Pi}(\tilde{x}_{k+1}) - P_{\Pi}(\tilde{x}_k) = \frac{1}{2} (x^k_{p-1} + x^k_{p} + x^k_{p-1} + x^k_{p}) - \frac{1}{2} (x^k_{p-1} + x^k_{p}, x^k_{p-1} + x^k_{p}) = \frac{1}{2} (\langle x^k_{p-1} + x^k_{p}, x^k_{p-1} - x^k_{p} \rangle + \langle x^k_{p-1} + x^k_{p}, x^k_{p} - x^k_{p} \rangle).
\]

Therefore the denominator in (2.5) can be written as

\[
\|P_{\Pi}(\tilde{x}_{k+1}) - P_{\Pi}(\tilde{x}_k)\|^2 = \frac{1}{4} \left( 2 \|x^k_{p-1} + x^k_{p} - x^k_{p-1}\|^2 \right) = \frac{1}{2} \|x^k_{p-1} + x^k_{p} - x^k_{p}\|^2,
\]

and the result holds. \( \square \)

Let \( \{x^k_i\} \) be the MAP’s iterates given by (1.2). Let us now define a new sequence \( \{\alpha'_k\} \) in \( H \), for \( k \geq 1 \), as follows

\[
\alpha'_k = \begin{cases} 
    x^k_{p-1} & \text{if } \|x^k_{p-1} + x^k_p - x^k_{p-1}\| = 0, \\
    x^k_p + \alpha_k (x^k_{p-1} + x^k_p) & \text{if } \|x^k_{p-1} + x^k_p - x^k_{p-1}\| \neq 0,
\end{cases}
\]

where \( \alpha_k \) is given by (2.9).

**Theorem 2.12.** The sequence \( \{\alpha'_k\} \) defined by (2.10) converges to \( P_M(x_0) \).

**Proof.** This is a consequence of Corollary 2.9, equation (2.8), and Theorem 1.3. \( \square \)

We are now ready to present our acceleration scheme for MAP.

**Algorithm 1.** Let \( M_i, i = 1, \ldots, p \), be \( p \) closed subspaces in \( H \). Given \( x_0 \in H \); set \( k = 1 \).

1. set \( x^0_p = x_0 \)
2. \( x^k_p = (P_{M_{p-1}} \cdots P_{M_1})(x^k_{p-1}) \)
3. for \( k = 1, 2, \ldots \) do
   1. \( x^{k+1}_{p-1} = (P_{M_{p-1}} \cdots P_{M_1})(x^k_p) \)
   2. \( x^{k+1}_p = P_{M_p}(x^{k+1}_{p-1}) \)


if \( \|x_{p+1}^{k+1} + x_{p+1}^{k} - x_{p}^{k-1} - x_{p}^{k}\| \neq 0 \) then

compute \( \alpha_k \) using (2.9), and set \( d_k' = x_p^k + \alpha_k(x_{p}^{k+1} - x_{p}^{k}) \)

else

set \( d_k' = x_p^{k+1} \)

end if

end for

In the following, we will compare our scheme, given in Algorithm 1, with the acceleration scheme given by Gearhart and Koshy [21]. Therefore, we will give a brief explanation of the Gearhart-Koshy scheme. Let us denote by \( x_0 \) the given starting point and by \( Q \) the composition of the projection operators, i.e, \( Q = P_{M_1}P_{M_2} \cdots P_{M_p} \), where \( P_{M_i} \) is the projection operator onto \( M_i \) for all \( i \). Let \( x_k \) be the \( k \) th iterate, and let \( Qx_k \) be the next iterate after applying a sweep of MAP. The idea is to search along the line through the points \( x_k \) and \( Qx_k \) to obtain the point closest to the solution \( x^* = \bigcap_{i=1}^p M_i(x_0) \). Let us represent any point on this line as

\[ x^k(t) = tQx_k + (1-t)x_k = x_k + t(Qx_k - x_k), \]

for some real number \( t \). Let \( t_k \) be the value of \( t \) for which this point is the closest to \( x^* \). Then,

\[ \langle x^k(t_k) - x^*, x_k - Qx_k \rangle = 0. \]  

Now, since \( x^* \in \bigcap_{i=1}^p M_i \) and the projections \( P_{M_i} \) are self-adjoint, it follows

\[ \langle x^*, Qx_k \rangle = \langle P_{M_1}P_{M_2} \cdots P_{M_p}x^*, x_k \rangle = \langle x^*, x_k \rangle. \]

Consequently, \( \langle x^*, x_k - Qx_k \rangle = 0 \), and so \( x^* \) can be eliminated from (2.11) to obtain

\[ \langle x^k(t_k), x_k - Qx_k \rangle = 0. \]

Solving for \( t_k \) gives

\[ t_k = \frac{\langle x_k, x_k - Qx_k \rangle}{\|x_k - Qx_k\|^2}. \]

To summarize, we have the following steps.

**Algorithm 2.** (Gearhart-Koshy) Let \( M_i, i = 1, \ldots, p \), be \( p \) closed subspaces in \( H \). Given \( x_0 \in H \).

for \( k = 1, 2, \ldots \) do

\[ Q_k = Qx_k \]

\[ t_k = \frac{\langle x_k, x_k - Qx_k \rangle}{\|x_k - Qx_k\|^2} \]

\[ x_{k+1} = t_kQ_k + (1-t_k)x_k \]

end for

3. Numerical experiments. We compare our acceleration scheme, given in Algorithm 1, with the acceleration scheme given by Gearhart and Koshy (Algorithm 2), and with the original MAP with no acceleration given by (1.2). All computations were performed in MATLAB. For all experiments we know the exact solution \( x^* \), and we stop each algorithm when the absolute error (the distance from the \( k \)-th iterate to \( x^* \)) is less than or equal to \( 10^{-6} \).
For our first experiment, we consider the following three subspaces of the space of square real matrices $\mathbb{R}^{3\times 3}$, with the Frobenius norm $\|A\|_F^2 = \langle A, A \rangle = \text{trace}(A^T A)$:

- $M_1 = \{ A \in \mathbb{R}^{3\times 3} : A^T = A \}$,
- $M_2 = \{ A \in \mathbb{R}^{3\times 3} : a_{i,j+1} = 0, \ i = 1, 2, \ j = i + 1, \ldots, 3 \}$,
- $M_3 = \{ A \in \mathbb{R}^{3\times 3} : a_{1,1} = a_{1,3} = a_{3,1} = a_{3,3} \}$.

If $A = (a_{i,j}) \in \mathbb{R}^{3\times 3}$, then $P_{M_i}(A) = (A^T + A)/2$,

$$P_{M_i}(A) = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad \text{and} \quad P_{M_3}(A) = \begin{bmatrix} P & A_{12} & P \\ A_{21} & A_{22} & A_{23} \\ P & A_{32} & P \end{bmatrix},$$

where $P = (A_{11} + A_{13} + A_{31} + A_{33})/4$.

We choose

$$A_0 = \begin{bmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 70 & 80 & 90 \end{bmatrix}$$

as the initial given point. Then

$$P_{M_1 \cap M_2 \cap M_3}(A_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

since

$$A \in \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{bmatrix} : x \in \mathbb{R} \right\} = M_1 \cap M_2 \cap M_3,$$

$$\langle A, A_0 - P_{M_1 \cap M_2 \cap M_3}(A_0) \rangle = \text{trace} \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 10 & 20 & 30 \\ 40 & 0 & 60 \\ 70 & 80 & 90 \end{bmatrix} \right) = 0.$$
FIG. 3.1. Acceleration for three subspaces in $\mathbb{R}^{3 \times 3}$.

FIG. 3.2. Acceleration for solving $Ax = 0$, where $A \in \mathbb{R}^{5 \times 5}$.

Acknowledgements. I thank the three anonymous referees for constructive remarks and additional references.

REFERENCES


FIG. 3.3. Acceleration for solving $Ax = 0$, where $A \in \mathbb{R}^{10 \times 10}$.  


ACCELERATION OF THE VON NEUMANN-HALPERIN PROJECTION METHOD


