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QUASI-OPTIMAL CONVERGENCE RATES FOR ADAPTIVE BOUNDARY ELEMENT METHODS WITH DATA APPROXIMATION. PART II: HYPER-SINGULAR INTEGRAL EQUATION*

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Abstract. We analyze an adaptive boundary element method with fixed-order piecewise polynomials for the hyper-singular integral equation of the Laplace-Neumann problem in 2D and 3D which incorporates the approximation of the given Neumann data into the overall adaptive scheme. The adaptivity is driven by some residual-error estimator plus data oscillation terms. We prove convergence with quasi-optimal rates. Numerical experiments underline the theoretical results.

Key words. boundary element method, hyper-singular integral equation, a posteriori error estimate, adaptive algorithm, convergence, optimality

AMS subject classifications. 65N30, 65N15, 65N38

1. Introduction and outline. Data approximation is an important subject in numerical computations, and reliable numerical algorithms have to properly account for it. The present work proves quasi-optimal convergence rates for an adaptive boundary element method (ABEM) that includes data errors. While the earlier work [13] was concerned with weakly-singular integral equations, the present work considers the hyper-singular integral equation

$$Wg = (1/2 - K')\phi \qquad \text{on } \Gamma := \partial\Omega \tag{1.1}$$

for given boundary data ϕ and a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with polygonal respectively polyhedral boundary $\partial\Omega$; see Section 2 for the precise statement of the integral operators W and K' involved. In the spirit of, e.g., [10, 24], we prove convergence and quasi-optimality of some standard adaptive algorithms of the type

$$\boxed{\texttt{solve}} \longrightarrow \boxed{\texttt{estimate}} \longrightarrow \boxed{\texttt{mark}} \longrightarrow \boxed{\texttt{refine}} \tag{1.2}$$

which are steered by the weighted-residual estimator from [7] plus data oscillation terms. The proposed algorithm employs the L^2-orthogonal projection to replace the given data ϕ in the Galerkin scheme by some discrete data $\Pi_\ell\phi$. The benefit of such an approach is that the implementation of (1.2) has to deal with discrete integral operators only. Since reliable quadratures for these (with polynomial ansatz and test functions) are well-understood, see e.g. [22], such an approach is superior to the data-dependent integration of $K'\phi$, where possible singularities of ϕ as well as the singular kernel of the boundary integral operator K' have to be treated simultaneously. We note that an overall goal of ABEM research is to incorporate the

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quadrature error control into the adaptive algorithm. While this is a long-term goal and requires substantial research on its own, the present work provides a first step into this direction.

Preliminary convergence and quasi-optimality results for *lowest-order ABEM* have independently been achieved in [14, 15]. While [14] is concerned with the weakly-singular integral equation for the Laplacian on polygonal/polyhedral boundaries, the work [15] treats general weakly-singular and hyper-singular integral equations on smooth boundaries. In either work, the heart of the matter are novel inverse estimates for the integral operators involved. These have recently been generalized to arbitrary polynomials on polygonal/polyhedral boundaries in [4]. Besides this, certain properties of the error estimator are required that have to be analyzed beyond the particular lowest-order case. For the weakly-singular integral equation, this has been done in [13], while the analysis for the hyper-singular integral equation is the topic of the present work.

The contribution of this work may thus be summarized as follows: Unlike [15], we address the important question of data approximation in the adaptive algorithm considered and prove convergence and quasi-optimality in this case. Owing to the method of proof, the analysis of the error estimator in [15] is restricted to lowest-order ABEM, i.e., globally continuous and piecewise affine ansatz and test functions. Instead, the analysis given here covers continuous piecewise polynomials of arbitrary but fixed order. Finally, the overall presentation aims to give a deeper insight into which *basic properties* of the error estimator are really mandatory to prove optimal convergence for ABEM. A further qualitative improvement over, e.g., [10, 14, 15, 24] is that our analysis avoids the use of a lower bound (the so-called efficiency) and hence relaxes the dependencies of optimal marking parameters.

Outline. The work and its main results are organized as follows: Section 2 fixes the functional analytic framework of the (stabilized) hyper-singular integral equation (1.1) and its Galerkin discretization by piecewise polynomials. In Section 3 we introduce and analyze the weighted-residual error estimator. Moreover, we analyze the overall a posteriori error control in case of data approximation of ϕ by discontinuous piecewise polynomials in terms of data oscillation terms (Theorem 3.8). Section 4 gives a precise statement of the adaptive loop (1.2) and proves linear convergence of the overall error estimator with respect to the iteration step ℓ (Corollary 4.3). In Section 5, we prove that the adaptive algorithm leads asymptotically to optimal convergence rates (Theorem 5.1). Conclusions are drawn in Section 6, and possible extensions of the analysis to indirect integral formulations and screen problems are discussed. Numerical experiments in 2D conclude the work in Section 7.

Throughout the work, the symbol $\lesssim$ abbreviates '$\leq$ up to a multiplicative constant', and $\simeq$ means that both estimates $\lesssim$ and $\gtrsim$ hold.

2. Preliminaries. This section gives a brief overview of the functional analytic setting of the hyper-singular integral equation. For more details, we refer to the monographs [17, 21]; a comprehensive discussion of the boundary element spaces employed in the present work can be found in the monograph [22].

Throughout, we assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$, with connected and polygonal/polyhedral boundary $\Gamma = \partial\Omega$. With the fundamental solution of the Laplacian

$$
\begin{aligned}
G(x,y) &= -\frac{1}{2\pi}\,\log|x-y| && \text{for } d=2,\\
G(x,y) &= \frac{1}{4\pi}\,\frac{1}{|x-y|} && \text{for } d=3,
\end{aligned}
$$

the hyper-singular integral equation (1.1) involves the hyper-singular integral operator W as well as the adjoint of the double-layer integral operator K. These operators are formally

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defined by

$$Wv(x) := -\partial_{n_x} \int_\Gamma \partial_{n_y} G(x,y)\, v(y)\, d\Gamma(y), \tag{2.1}$$

$$Kv(x) := \int_\Gamma \partial_{n_y} G(x,y)\, v(y)\, d\Gamma(y), \tag{2.2}$$

where n_z denotes the outer unit normal vector at $z \in \Gamma$ and ∂_{n_z} is the normal derivative.

2.1. Sobolev spaces. Let $L^2(\Gamma)$ and $H^1(\Gamma)$ denote the usual Lebesgue and Sobolev spaces on $\Gamma = \partial\Omega$. The norm on $H^1(\Gamma)$ reads

$$\|u\|^2_{H^1(\Gamma)} = \|u\|^2_{L^2(\Gamma)} + \|\nabla u\|^2_{L^2(\Gamma)},$$

where $\nabla(\cdot)$ denotes the arc-length derivative or the surface gradient for $d = 2, 3$. Sobolev spaces of fractional order $0 < s < 1$ are defined by interpolation

$$H^s(\Gamma) = [L^2(\Gamma); H^1(\Gamma)]_s,$$

where $[\cdot;\cdot]_s$ denotes interpolation by the K-method. For abbreviation, let $H^0(\Gamma) := L^2(\Gamma)$. The Sobolev spaces $H^{-s}(\Gamma)$ for $0 < s \le 1$ are defined by duality

$$H^{-s}(\Gamma) = H^s(\Gamma)^*,$$

where duality is understood with respect to the extended $L^2(\Gamma)$-scalar product $\langle \cdot\,,\,\cdot\rangle$. We note that $H^{1/2}(\Gamma)$ is equivalently characterized as the trace space of $H^1(\Omega)$.

2.2. Hyper-singular integral operator. The hyper-singular integral operator W from equation (2.1) is a well-defined linear and continuous operator

$$W : H^s(\Gamma) \to H^{s-1}(\Gamma) \qquad \text{for all } 0 \le s \le 1.$$

Moreover, W is symmetric and positive semi-definite on $H^{1/2}(\Gamma)$, i.e.,

$$\langle Wv\,,\,w\rangle = \langle Ww\,,\,v\rangle \quad \text{and} \quad \langle Wv\,,\,v\rangle \ge 0, \qquad \text{for all } v, w \in H^{1/2}(\Gamma).$$

Since Γ is connected, the kernel of W is one-dimensional and spanned by the constant functions. The bilinear form

$$\langle\!\langle v\,,\,w\rangle\!\rangle_W := \langle Wv\,,\,w\rangle$$

is a scalar product on $H^{1/2}_\star(\Gamma) := \big\{v \in H^{1/2}(\Gamma) \,:\, \langle 1\,,\,v\rangle = 0\big\}$. Therefore,

$$\langle\!\langle v\,,\,w\rangle\!\rangle_{W+S} := \langle Wv\,,\,w\rangle + \langle 1\,,\,v\rangle\langle 1\,,\,w\rangle, \qquad \text{for } v, w \in H^{1/2}(\Gamma),$$

defines a scalar product on $H^{1/2}(\Gamma)$. According to the Rellich compactness theorem, the induced norm $|\!|\!|v|\!|\!|^2_{W+S} = \langle\!\langle v\,,\,v\rangle\!\rangle_{W+S}$ is an equivalent norm on $H^{1/2}(\Gamma)$.

2.3. Neumann problem. The double-layer integral operator K from (2.2) is a well-defined linear and continuous operator

$$K :\, H^s(\Gamma) \to H^s(\Gamma) \qquad \text{for all } 0 \le s \le 1.$$

Moreover, its adjoint is a well-defined linear and continuous operator

$$K' :\, H^{-s}(\Gamma) \to H^{-s}(\Gamma) \qquad \text{for all } 0 \le s \le 1.$$

For the particular right-hand side $f = (1/2 - K')\phi$ in (1.1), the hyper-singular integral equation is an equivalent formulation of the Neumann problem

$$(2.3)\qquad \begin{aligned} -\Delta P &= 0 \qquad \text{in } \Omega,\\ \partial_n P &= \phi \qquad \text{on } \Gamma, \end{aligned}$$

in the following sense: the hyper-singular integral equation (1.1) admits a unique solution $u \in H^{1/2}_\star(\Gamma)$. If uniqueness of the potential $P \in H^1(\Omega)$ from (2.3) is enforced by $\int_\Gamma P d\Gamma = 0$, then $u = P|_\Gamma$, i.e., u is the trace of P on Γ.

We note that existence and uniqueness of the solution $u \in H^{1/2}_\star(\Gamma)$ of (1.1) follow from the Lax-Milgram lemma: as $\langle Wv\,,\,1\rangle = 0$ and $\langle(1/2-K')\psi\,,\,1\rangle = 0$ for all $v \in H^{1/2}(\Gamma)$ and $\psi \in H^{-1/2}(\Gamma)$, the hyper-singular integral equation is equivalently recast into the variational formulation

$$\langle\!\langle u\,,\,v\rangle\!\rangle_{W+S} = \langle(1/2 - K')\phi\,,\,v\rangle, \qquad \text{for all } v \in H^{1/2}(\Gamma).$$

According to the Lax-Milgram lemma, the latter equation admits a unique solution, and $|\Gamma|\,\langle u\,,\,1\rangle = \langle\!\langle Wu\,,\,1\rangle\!\rangle_{W+S} = \langle(1/2 - K')\phi\,,\,1\rangle = 0$ proves $u \in H^{1/2}_\star(\Gamma)$.

2.4. Admissible triangulations. For $d = 2$, we suppose that $\mathcal{T}_\star$ is a partition of Γ into finitely many compact affine line segments. For $d = 3$, we suppose that $\mathcal{T}_\star$ is a triangulation of Γ into finitely many compact and flat surface triangles which is regular in the sense of Ciarlet, i.e., the intersection of two elements $T, T' \in \mathcal{T}_\star$ with $T \neq T'$ is either empty, or a common node, or a common edge. In these cases, we say that $\mathcal{T}_\star$ is an *admissible triangulation* of Γ. Throughout, we assume that all triangulations are admissible.

Let $|\cdot|$ denote the surface measure, i.e., $|T| = \mathrm{diam}(T)$ for an affine line segment and $d = 2$. With an admissible triangulation $\mathcal{T}_\star$, we associate its local mesh-width

$$h_\star \in L^\infty(\Gamma), \quad h_\star|_T := h_\star(T) := |T|^{1/(d-1)}, \qquad \text{for all } T \in \mathcal{T}_\star.$$

Throughout, quantities associated with a given triangulation $\mathcal{T}_\star$ have the same index, e.g., $h_\star$ for the associated mesh-width function or $G_\star$ for the corresponding Galerkin solution (see below).

For $d = 2$, we say that $\mathcal{T}_\star$ is γ-shape regular if

$$\frac{\mathrm{diam}(T)}{\mathrm{diam}(T')} \le \gamma \qquad \text{for all neighboring elements } T, T' \in \mathcal{T}_\star.$$

For $d = 3$, $\mathcal{T}_\star$ is called γ-shape regular if

$$\frac{\mathrm{diam}(T)}{|T|^{1/2}} \le \gamma \qquad \text{for all elements } T \in \mathcal{T}_\star.$$

Since essentially all estimates will depend on γ-shape regular triangulations, this will implicitly be assumed throughout.

2.5. Discrete spaces. Let $T_{\mathrm{ref}} = [0,1]$ for $d = 2$ and $T_{\mathrm{ref}} = \mathrm{conv}\{(0,0),(1,0),(0,1)\}$ for $d = 3$ denote the reference simplices in $\mathbb{R}^{d-1}$. By assumption, each element $T \in \mathcal{T}_\star$ is the image of T_{ref} under an affine bijection $\gamma_T : T_{\mathrm{ref}} \to T$. Let $\mathcal{P}^p(T_{\mathrm{ref}})$ denote the space of all polynomials of degree $\le p$ on the reference element. We then define spaces of $\mathcal{T}_\star$-piecewise polynomials by

$$(2.4)\qquad \mathcal{P}^p(\mathcal{T}_\star) := \big\{V_\star : \Gamma \to \mathbb{R} \,:\, \forall T \in \mathcal{T}_\star \quad V_\star \circ \gamma_T \in \mathcal{P}^p(T_{\mathrm{ref}})\big\}$$

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and

$$\mathcal{S}^{p+1}(\mathcal{T}_\star) := \mathcal{P}^{p+1}(\mathcal{T}_\star) \cap C(\Gamma). \tag{2.5}$$

We note that $\mathcal{P}^p(\mathcal{T}_\star) \subset L^2(\Gamma) \subset H^{-1/2}(\Gamma)$ and $\mathcal{S}^{p+1}(\mathcal{T}_\star) \subset H^1(\Gamma) \subset H^{1/2}(\Gamma)$ for all $p \in \mathbb{N}_0$.

3. A posteriori error estimation. In this section, we recall the weighted-residual error estimator from [7]. We give a new proof for its reliability and derive the properties needed for the later convergence and quasi-optimality analysis. Unlike [15], where similar results have been derived for the first time, our analysis covers arbitrary polynomials of fixed degree $p \geq 0$ and is essentially independent of the mesh-refinement strategy used. In addition, we incorporate the approximation of the given right-hand side data by piecewise polynomials into the overall a posteriori error estimation. The benefit is that the later implementation has to deal with discrete integral operators only so that the question of *reliable quadrature* is much simplified.

3.1. Mesh-refinement. For admissible triangulations $\mathcal{T}_\ell$, $\mathcal{T}_\star$ of Γ, we write $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ and say that $\mathcal{T}_\star$ is an arbitrary refinement of $\mathcal{T}_\ell$, if

$$T = \bigcup \{T' \in \mathcal{T}_\star \,:\, T' \subseteq T\}, \qquad \text{for all } T \in \mathcal{T}_\ell,$$

and

$$|T'| \leq |T|/2, \quad \text{for all } T \in \mathcal{T}_\ell \backslash \mathcal{T}_\star \qquad \text{and all} \qquad T' \in \mathcal{T}_\star \quad \text{with } T' \subseteq T,$$

i.e., each element $T \in \mathcal{T}_\ell$ is the union of its successors, and refinement ensures that the surface area is at least halved. Note that $\mathcal{T}_\ell \backslash \mathcal{T}_\star$ denotes the set of refined elements, while $\mathcal{T}_\star \backslash \mathcal{T}_\ell$ then consists of their successors. The assumptions posed imply that $\bigcup(\mathcal{T}_\ell \backslash \mathcal{T}_\star) = \bigcup(\mathcal{T}_\star \backslash \mathcal{T}_\ell)$ with the pointwise estimates $h_\star \leq 2^{-1/(d-1)}\, h_\ell$ on $\bigcup(\mathcal{T}_\ell \backslash \mathcal{T}_\star)$ and $h_\star = h_\ell$ on $\bigcup(\mathcal{T}_\ell \cap \mathcal{T}_\star)$.

3.2. Auxiliary results. This subsection recalls and states some facts which are used for the a posteriori error analysis. First, we shall need certain inverse estimates.

LEMMA 3.1. *Let $\mathcal{T}_\star$ be an admissible triangulation of Γ. For all $\psi \in L^2(\Gamma)$ and $v \in H^1(\Gamma)$, it holds that*

$$C_{\rm inv}^{-1} \|h_\star^{1/2}(1/2 - K')\psi\|_{L^2(\Gamma)} \leq \|\psi\|_{H^{-1/2}(\Gamma)} + \|h_\star^{1/2}\psi\|_{L^2(\Gamma)} \tag{3.1}$$

as well as

$$C_{\rm inv}^{-1} \|h_\star^{1/2} W v\|_{L^2(\Gamma)} \leq \|v\|_{H^{1/2}(\Gamma)} + \|h_\star^{1/2}\nabla v\|_{L^2(\Gamma)}. \tag{3.2}$$

In particular, for all $\Psi_\star \in \mathcal{P}^p(\mathcal{T}_\star)$ and $V_\star \in \mathcal{S}^{p+1}(\mathcal{T}_\star)$, it holds that

$$\|h_\star^{1/2}\Psi_\star\|_{L^2(\Gamma)} + \|h_\star^{1/2}(1/2 - K')\Psi_\star\|_{L^2(\Gamma)} \leq C_{\rm inv}\|\Psi_\star\|_{H^{-1/2}(\Gamma)} \tag{3.3}$$

as well as

$$\|h_\star^{1/2}\nabla V_\star\|_{L^2(\Gamma)} + \|h_\star^{1/2} W V_\star\|_{L^2(\Gamma)} \leq C_{\rm inv}\|V_\star\|_{H^{1/2}(\Gamma)}. \tag{3.4}$$

The constant $C_{\rm inv} > 0$ depends only on the γ-shape regularity of $\mathcal{T}_\star$, the boundary Γ, and the polynomial degree $p \geq 0$.

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Proof. The estimates (3.1)–(3.2) are proved in [4, Theorem 1], while those in (3.3)–(3.4) follow directly by employing the inverse estimates from [16, Theorem 3.6] for $\|\cdot\|_{H^{-1/2}(\Gamma)} \gtrsim \|h_\star^{1/2}(\cdot)\|_{L^2(\Gamma)}$ and for $\|\cdot\|_{H^{1/2}(\Gamma)} \gtrsim \|h_\star^{1/2}\nabla(\cdot)\|_{L^2(\Gamma)}$ from [9, Cor. 3.2] for $d=2$ and from [3, Proposition 5] for $d=3$. ∎

Second, we rely on the Scott-Zhang projection [23] for quasi-interpolation in $H^{1/2}(\Gamma)$. While the original work from [23] is concerned with the integer-order Sobolev space $H^1(\Omega)$ on Lipschitz domains, the approach is generalized in [3] to fractional-order Sobolev spaces $H^s(\Gamma)$ and $\widetilde{H}^s(\Gamma)$ for $0 \le s \le 1$ on boundaries. Since the precise construction will matter below, we briefly sketch it: fix a set $\mathcal{N}_\star$ of Lagrange nodes for the space $\mathcal{S}^{p+1}(\mathcal{T}_\star)$. For each $z \in \mathcal{N}_\star$, choose an *arbitrary* element $T_z \in \mathcal{T}_\star$ with $z \in T_z$. Let $\phi_z \in \mathcal{S}^{p+1}(\mathcal{T}_\star)$ denote the Lagrange basis function associated with z, and let $\phi_z^\star \in \mathcal{P}^{p+1}(T_z)$ be the L^2-dual basis function, i.e., $\int_{T_z} \phi_z^\star \phi_{z'}\,dx = \delta_{zz'}$, for all $z' \in \mathcal{N}_\star$, with Kronecker's delta $\delta_{zz'}$. Then, the Scott-Zhang projection $J_\star$ defined by

$$J_\star v := \sum_{z\in\mathcal{N}_\star} \left(\int_{T_z} \phi_z^\star\, v\,dx\right)\phi_z \tag{3.5}$$

has the following properties.

LEMMA 3.2. *$J_\star : L^2(\Gamma) \to \mathcal{S}^{p+1}(\mathcal{T}_\star)$ is a well-defined linear projection, i.e.,*

$$J_\star V_\star = V_\star, \qquad \text{for all } V_\star \in \mathcal{S}^{p+1}(\mathcal{T}_\star). \tag{3.6}$$

In particular,

$$(J_\star v)(z) = v(z), \qquad \text{for all } z \in \mathcal{N}_\star \text{ and } v \in L^2(\Gamma) \text{ with } v|_{T_z} \in \mathcal{P}^{p+1}(T_z), \tag{3.7}$$

where T_z is the element chosen for z in (3.5). Moreover, $J_\star$ is stable in $H^s(\Gamma)$ for all $0 \le s \le 1$, i.e.,

$$\|J_\star v\|_{H^s(\Gamma)} \le C_{\rm sz}\,\|v\|_{H^s(\Gamma)}, \qquad \text{for all } v \in H^s(\Gamma), \tag{3.8}$$

and has a local first-order approximation property

$$\|h_\star^{-s}(1-J_\star)v\|_{L^2(\Gamma)} \le C_{\rm sz}\,\|v\|_{H^s(\Gamma)}, \qquad \text{for all } v \in H^s(\Gamma). \tag{3.9}$$

The constant $C_{\rm sz} > 0$ depends only on Γ, $0 \le s \le 1$, the polynomial degree $p \in \mathbb{N}_0$, and the γ-shape regularity of $\mathcal{T}_\star$.

Proof. Well-posedness and the projection property (3.6)–(3.7) as well as L^2-stability (i.e., (3.8) for $s = 0$) follow by construction; see also [3, Lemma 3]. For $T \in \mathcal{T}_\star$, let

$$\omega_\star(T) := \bigcup\{T' \in \mathcal{T}_\star \,:\, T \cap T' \neq \emptyset\}$$

denote the patch of T. Following the original arguments in [23], it is noted in [3] that

$$\operatorname{diam}(T)^{-1}\,\|(1-J_\star)v\|_{L^2(T)} + \|\nabla J_\star v\|_{L^2(T)} \lesssim \|\nabla v\|_{L^2(\omega_\star(T))}$$

for all $v \in H^1(\Gamma)$ and $T \in \mathcal{T}_\star$. By γ-shape regularity, this implies the global estimate

$$\|h_\star^{-1}(1-J_\star)v\|_{L^2(\Gamma)} + \|\nabla J_\star v\|_{L^2(\Gamma)} \lesssim \|\nabla v\|_{L^2(\Gamma)} \le \|v\|_{H^1(\Gamma)}, \qquad \text{for all } v \in H^1(\Gamma).$$

Together with the L^2-stability (3.8) for $s=0$, one thus obtains H^1-stability (3.8) for $s=1$ as well as the approximation estimate (3.9) for $s=0$ and $s=1$. The general case $0<s<1$ in (3.8)–(3.9) therefore follows by interpolation. ∎

Bootstrapping [8, Theorem 4.1] by use of idempotency of projections, we obtain the following result, which will allow us to control the error incurred by approximating the Neumann data.

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LEMMA 3.3. *Suppose that $\pi_\star : L^2(\Gamma) \to X_\star$ is the L^2-orthogonal projection onto a closed subspace $X_\star \subseteq L^2(\Gamma)$ with $\mathcal{P}^0(\mathcal{T}_\star) \subseteq X_\star$. Then, it holds that*

$$\|(1-\pi_\star)\psi\|_{H^{-s}(\Gamma)} \leq C_{\rm apx} \|h_\star^s (1-\pi_\star)\psi\|_{L^2(\Gamma)}, \qquad \text{for all } \psi \in L^2(\Gamma).$$

The constant $C_{\rm apx} > 0$ depends only on Γ and $0 \leq s \leq 1$.

REMARK 3.4. The present work is concerned with the h-version of the BEM, i.e., where the polynomial degree p is fixed. Our primary reason for fixing p is that the mechanism for error reduction of the residual error estimator (see Section 3.3 below) is based on element refinement and not suited for tracking the effect of increasing the (local) polynomial degree. Since we believe that an explicit tracking of the degree would just render the notation more cumbersome, we have opted to suppress it. Nevertheless, we now give some pointers as to how the polynomial degree enters the above Lemmas 3.1–3.3. For a variable polynomial degree distribution $\mathbf{p}$, the key notion is that of a γ-shape regular one: associate with each element T a degree $p_T \in \mathbb{N}_0$, and collect these degrees into the degree vector $\mathbf{p}$. One denotes by $p_\mathcal{T}$ the piecewise constant function $p_\mathcal{T}|_T = p_T$. The spaces $\mathcal{P}^{\mathbf{p}+1}(\mathcal{T})$ and $\mathcal{S}^{\mathbf{p}+1}(\mathcal{T})$ are then defined analogously to (2.4)–(2.5). The concept of γ-shape regularity of the degree distribution requires

$$\gamma^{-1}(p_T+1) \leq p_{T'}+1 \leq \gamma\,(p_T+1), \qquad \text{for all } T, T' \in \mathcal{T} \text{ with } T \cap T' \neq \emptyset.$$

In this setting, the bounds (3.3)–(3.4) of Lemma 3.1 remain valid if $h_\star$ is replaced with $h_\star/(p_\star+1)^2$ as shown in [4]. Lemma 3.2 generalizes to this setting in a weaker form: in [18], an approximation operator $J_\star$ is provided, which is based on local averaging, with the stability bound (3.8) and the approximation property (3.9) (with $h_\star$ replaced with $h_\star/(p_\star+1)$); however, the projection properties (3.6)–(3.7) are not directly available from the literature. We note that this lack would impact some arguments of the present paper. Finally, the statement of Lemma 3.3 generalizes to the hp-setting when $h_\star$ is replaced with $h_\star/(p_\star+1)$.

3.3. Weighted-residual error estimator. For given $f \in L^2(\Gamma)$, let $u \in H^{1/2}(\Gamma)$ and $U_\star \in \mathcal{S}^{p+1}(\mathcal{T}_\star)$ be the unique solutions to

$$\langle\!\langle u\,,\,v\rangle\!\rangle_{W+S} = \langle f\,,\,v\rangle, \qquad \text{for all } v \in H^{1/2}(\Gamma) \tag{3.10}$$

and

$$\langle\!\langle U_\star\,,\,V_\star\rangle\!\rangle_{W+S} = \langle f\,,\,V_\star\rangle, \qquad \text{for all } V_\star \in \mathcal{S}^{p+1}(\mathcal{T}_\star). \tag{3.11}$$

We employ the weighted-residual error estimator [7] with local contributions

$$\eta_\star(T) := \|h_\star^{1/2}(f - WU_\star)\|_{L^2(T)}, \tag{3.12a}$$

and let

$$\eta_\star := \eta_\star(\mathcal{T}_\star) \qquad \text{with} \quad \eta_\star(\mathcal{E}_\star)^2 := \sum_{T\in\mathcal{E}_\star} \eta_\star(T)^2, \qquad \text{for all } \mathcal{E}_\star \subseteq \mathcal{T}_\star. \tag{3.12b}$$

Note that $\eta_\star(\mathcal{E}_\star) = \|h_\star^{1/2}(f - WU_\star)\|_{L^2(\bigcup \mathcal{E}_\star)}$. The following proposition collects the *basic properties* of $\eta_\star$. The discrete reliability (3.15) has first been proved in [15]. For technical reasons, the proof in [15] relied on the use of newest vertex bisection and is restricted to the lowest-order case $p = 0$ since it uses the norm localization techniques from [11, 12]. Our proof refines the arguments from [7], where reliability (3.17) is proved, but the constant $C_{\rm dlr}$

depends on the shape of all possible node patches. Using the Scott-Zhang projection from Lemma 3.2, we see that $C_{\rm dlr}$, in fact, depends only on the γ-shape regularity. A further qualitative improvement over [15] is that our discrete reliability estimate (3.15) involves only the refined elements $\mathcal{T}_\ell\backslash\mathcal{T}_\star$, while the original result of [15] is based on the refined elements plus one additional layer of non-refined elements.

PROPOSITION 3.5. *Let $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$, and let U_ℓ and $U_\star$ be the corresponding Galerkin solutions from* (3.11). *Then, the weighted-residual error estimator η_ℓ from* (3.12) *satisfies the following properties* (i)–(iii).

(i). *Stability on non-refined elements: there exists a constant $C_{\rm stab} > 0$ such that*

$$|\eta_\star(\mathcal{T}_\star \cap \mathcal{T}_\ell) - \eta_\ell(\mathcal{T}_\ell \cap \mathcal{T}_\star)| \le C_{\rm stab}\,\|U_\star - U_\ell\|_{H^{1/2}(\Gamma)}. \tag{3.13}$$

(ii). *Reduction on refined elements: there exist constants $0 < q_{\rm red} < 1$ and $C_{\rm red} > 0$ such that*

$$\eta_\star^2(\mathcal{T}_\star \setminus \mathcal{T}_\ell) \le q_{\rm red}\,\eta_\ell^2(\mathcal{T}_\ell \setminus \mathcal{T}_\star) + C_{\rm red}\|U_\star - U_\ell\|^2_{H^{1/2}(\Gamma)}. \tag{3.14}$$

(iii). *Discrete reliability: there exists a constant $C_{\rm dlr} > 0$ such that*

$$\|U_\star - U_\ell\|_{H^{1/2}(\Gamma)} \le C_{\rm dlr}\eta_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\star). \tag{3.15}$$

The constants $C_{\rm stab}, C_{\rm red}, C_{\rm dlr} > 0$ and $0 < q_{\rm red} < 1$ depend only on Γ, the γ-shape regularity of $\mathcal{T}_\ell$ and $\mathcal{T}_\star$, and the polynomial degree $p \in \mathbb{N}_0$.

Proof. (i) Proof of stability, (3.13). The reverse triangle inequality and $h_\star = h_\ell$ on the non-refined region $\bigcup(\mathcal{T}_\star \cap \mathcal{T}_\ell)$ prove

$$\begin{aligned}|\eta_\star(\mathcal{T}_\star \cap \mathcal{T}_\ell) - \eta_\ell(\mathcal{T}_\star \cap \mathcal{T}_\ell)| &\le \|h_\star^{1/2}(f - WU_\star) - h_\ell^{1/2}(f - WU_\ell)\|_{L^2(\bigcup(\mathcal{T}_\star\cap\mathcal{T}_\ell))}\\ &\le \|h_\star^{1/2}W(U_\star - U_\ell)\|_{L^2(\Gamma)}.\end{aligned}$$

By use of the inverse estimate (3.4), we conclude the proof with $C_{\rm stab} = C_{\rm inv}$.

(ii) Proof of reduction, (3.14). Recall that $h_\star \le q h_\ell$ with $q = 2^{-1/(d-1)} < 1$ in the refined region $\bigcup(\mathcal{T}_\star\backslash\mathcal{T}_\ell) = \bigcup(\mathcal{T}_\ell\backslash\mathcal{T}_\star)$. Together with the triangle inequality and the inverse estimate (3.4), this yields

$$\begin{aligned}\eta_\star(\mathcal{T}_\star\backslash\mathcal{T}_\ell) &= \|h_\star^{1/2}(f - WU_\star)\|_{L^2(\bigcup(\mathcal{T}_\star\backslash\mathcal{T}_\ell))}\\ &\le \|h_\star^{1/2}(f - WU_\ell)\|_{L^2(\bigcup(\mathcal{T}_\star\backslash\mathcal{T}_\ell))} + \|h_\star^{1/2}W(U_\star - U_\ell)\|_{L^2(\bigcup(\mathcal{T}_\star\backslash\mathcal{T}_\ell))}\\ &\le q^{1/2}\|h_\ell^{1/2}(f - WU_\ell)\|_{L^2(\bigcup(\mathcal{T}_\star\backslash\mathcal{T}_\ell))} + C_{\rm inv}\,\|U_\star - U_\ell\|_{H^{1/2}(\Gamma)}\\ &= q^{1/2}\,\eta_\ell(\mathcal{T}_\ell\backslash\mathcal{T}_\star) + C_{\rm inv}\,\|U_\star - U_\ell\|_{H^{1/2}(\Gamma)}.\end{aligned}$$

Young's inequality $(a+b)^2 \le (1+\delta)a^2 + (1+\delta^{-1})b^2$, for all $a, b \in \mathbb{R}$ and $\delta > 0$, concludes the proof of (3.14) with $q_{\rm red} = (1+\delta)q$ and $C_{\rm red} = (1+\delta^{-1})C_{\rm inv}^2$ if $\delta > 0$ is chosen sufficiently small.

(iii) Proof of discrete reliability, (3.15). We employ norm equivalence and Galerkin orthogonality to see

$$\|U_\star - U_\ell\|^2_{H^{1/2}(\Gamma)} \simeq |\!|\!|U_\star - U_\ell|\!|\!|^2_{W+S} = \langle\!\langle U_\star - U_\ell\,,\,(1 - J_\ell)(U_\star - U_\ell)\rangle\!\rangle_{W+S}$$

with J_ℓ being the Scott-Zhang projection onto $\mathcal{S}^{p+1}(\mathcal{T}_\ell)$ from Lemma 3.2. The Galerkin orthogonality together with $W1 = 0$ implies $|\Gamma|\,\langle U_\star - U_\ell\,,\,1\rangle = \langle\!\langle U_\star - U_\ell\,,\,1\rangle\!\rangle_{W+S} = 0$. This yields

$$\begin{aligned}\|U_\star - U_\ell\|^2_{H^{1/2}(\Gamma)} &\simeq \langle\!\langle U_\star - U_\ell\,,\,(1 - J_\ell)(U_\star - U_\ell)\rangle\!\rangle_{W+S}\\ &= \langle f - WU_\ell\,,\,(1 - J_\ell)(U_\star - U_\ell)\rangle.\end{aligned}$$

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Since the results of Lemma 3.2 do not depend on the precise choice of the elements $T_z \in \mathcal{T}_\ell$ associated with the Lagrangian nodes $z \in \mathcal{N}_\ell$ of $\mathcal{S}^{p+1}(\mathcal{T}_\ell)$, we may suppose that $T_z \in \mathcal{T}_\ell \cap \mathcal{T}_\star$ if $z \in \mathcal{N}_\ell \cap \bigcup(\mathcal{T}_\ell \cap \mathcal{T}_\star)$. According to the projection property (3.7), this implies

$$(1 - J_\ell)(U_\star - U_\ell) = 0 \quad \text{in the non-refined region } \bigcup(\mathcal{T}_\ell \cap \mathcal{T}_\star).$$

The Cauchy-Schwarz inequality and the approximation property (3.9) thus yield

$$\begin{aligned}\langle f - WU_\ell\,,\,(1 - J_\ell)(U_\star - U_\ell)\rangle &\\ &\le \|h_\ell^{1/2}(f - WU_\ell)\|_{L^2(\bigcup(\mathcal{T}_\ell\setminus\mathcal{T}_\star))}\|h_\ell^{-1/2}(1 - J_\ell)(U_\star - U_\ell)\|_{L^2(\Gamma)}\\ &\lesssim \|h_\ell^{1/2}(f - WU_\ell)\|_{L^2(\bigcup(\mathcal{T}_\ell\setminus\mathcal{T}_\star))}\|U_\star - U_\ell\|_{H^{1/2}(\Gamma)}\\ &= \eta_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_\star)\,\|U_\star - U_\ell\|_{H^{1/2}(\Gamma)}.\end{aligned}$$

Combining the last three estimates, we conclude the proof. ∎

The properties (i)–(iii) of Proposition 3.5 are called *basic properties* as they provide the essential mathematical ingredients to prove linear convergence (Section 4) and quasi-optimal convergence rates (Section 5) for adaptive algorithms. In fact, the following properties (iv)–(vi) are derived from algebraic postprocessing of (i)–(iii).

COROLLARY 3.6. *Let $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ with the corresponding Galerkin solutions U_ℓ and $U_\star$ from* (3.11)*. Then, the weighted-residual error estimator η_ℓ from* (3.12) *satisfies the following properties* (iv)–(vi).

(iv). *Quasi-monotonicity: there exists a constant $C_{\rm mon} > 0$ such that*

$$\eta_\star \le C_{\rm mon}\,\eta_\ell. \tag{3.16}$$

(v). *Reliability: with the constant $C_{\rm dlr}$ from discrete reliability* (3.15)*, we have*

$$\|u - U_\ell\|_{H^{1/2}(\Gamma)} \le C_{\rm dlr}\,\eta_\ell. \tag{3.17}$$

(vi). *Estimator reduction: the following implication is valid:*

$$\theta\,\eta_\ell^2 \le \eta_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_\star)^2 \quad\Longrightarrow\quad \eta_\star^2 \le q_{\rm est}\,\eta_\ell^2 + C_{\rm est}\,\|U_\star - U_\ell\|_{H^{1/2}(\Gamma)}^2. \tag{3.18}$$

The constants $C_{\rm est} > 0$ and $0 < q_{\rm est} < 1$ depend only on $0 < \theta \le 1$ and $C_{\rm stab}$, $C_{\rm red}$, $q_{\rm red}$, while $C_{\rm mon}$ depends only on $C_{\rm dlr}$, $C_{\rm stab}$, $C_{\rm red}$.

Proof. (iv) Proof of quasi-monotonicity, (3.16). Stability (3.13) together with the reduction property (3.14) shows that

$$\eta_\star^2 \lesssim \eta_\ell^2 + \|U_\star - U_\ell\|_{H^{1/2}(\Gamma)}^2.$$

With discrete reliability (3.15), this implies that

$$\eta_\star^2 \lesssim (1 + C_{\rm dlr})\eta_\ell^2$$

and concludes the proof of (iv).

(v) Proof of reliability, (3.17). Since the stabilized hyper-singular integral operator is $H^{1/2}$-elliptic, each Galerkin solution $U_\star \in \mathcal{S}^{p+1}(\mathcal{T}_\star)$ satisfies the Céa-type quasi-optimality

$$\|u - U_\star\|_{H^{1/2}(\Gamma)} \lesssim \min_{V_\star\in\mathcal{S}^{p+1}(\mathcal{T}_\star)} \|u - V_\star\|_{H^{1/2}(\Gamma)}.$$

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For given $\varepsilon > 0$, standard density results imply

$$\|u - U_\star\|_{H^{1/2}(\Gamma)} \leq \varepsilon$$

provided that the *global* mesh-width $\|h_\star\|_{L^\infty(\Gamma)} \ll 1$ is sufficiently small. Consequently, for given $\varepsilon > 0$ and $\mathcal{T}_\ell$, there exists a refinement $\mathcal{T}_\star$ of $\mathcal{T}_\ell$ with $\|u - U_\star\|_{H^{1/2}(\Gamma)} \leq \varepsilon$. The triangle inequality and discrete reliability (3.15) thus yield

$$\|u - U_\ell\|_{H^{1/2}(\Gamma)} \leq \varepsilon + \|U_\star - U_\ell\|_{H^{1/2}(\Gamma)} \leq \varepsilon + C_{\rm dlr}\, \eta_\ell.$$

The left-hand side as well as the right-hand side of this estimate are independent of $\mathcal{T}_\star$. Hence, passing to the limit $\varepsilon \to 0$ concludes the proof of (v).

(vi) Proof of estimator reduction, (3.18). We exploit stability (3.13) and reduction (3.14). For $\delta > 0$, Young's inequality gives

$$\begin{aligned}\eta_\star^2 &= \eta_\star^2(\mathcal{T}_\ell \cap \mathcal{T}_\star) + \eta_\star^2(\mathcal{T}_\star \setminus \mathcal{T}_\ell)\\ &\leq (1+\delta)\eta_\ell^2(\mathcal{T}_\ell \cap \mathcal{T}_\star) + q_{\rm red}\, \eta_\ell^2(\mathcal{T}_\ell \setminus \mathcal{T}_\star) + \big((1+\delta^{-1})C_{\rm stab}^2 + C_{\rm red}\big)\, \|U_\star - U_\ell\|_{H^{1/2}(\Gamma)}^2.\end{aligned}$$

The assumption $\theta\eta_\ell^2 \leq \eta_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\star)^2$ then implies

$$\begin{aligned}(1+\delta)\eta_\ell^2(\mathcal{T}_\ell \cap \mathcal{T}_\star) + q_{\rm red}\, \eta_\ell^2(\mathcal{T}_\ell \setminus \mathcal{T}_\star) &\leq (1+\delta)\eta_\ell^2 - (1+\delta-q_{\rm red})\eta_\ell^2(\mathcal{T}_\ell \setminus \mathcal{T}_\star)\\ &\leq \big(1+\delta-\theta(1+\delta-q_{\rm red})\big)\eta_\ell^2.\end{aligned}$$

For sufficiently small $\delta > 0$, the combination of the last two estimates proves (3.18) with $q_{\rm est} = 1+\delta-\theta(1+\delta-q_{\rm red}) < 1$ and $C_{\rm est} = (1+\delta^{-1})C_{\rm stab}^2 + C_{\rm red}$. □

3.4. Control of the data approximation error. For an admissible triangulation $\mathcal{T}_\star$, we consider the L^2-orthogonal projection $\Pi_\star$ onto $\mathcal{P}^p(\mathcal{T}_\star)$, which for $\phi \in L^2(\Gamma)$ is given elementwise as the unique solution of

$$\int_T (1-\Pi_\star)\phi\, \Psi_\star\, dx = 0, \qquad \text{for all } T \in \mathcal{T}_\star \text{ and all } \Psi_\star \in \mathcal{P}^p(\mathcal{T}_\star).$$

Let

$$\text{(3.19a)} \qquad {\rm osc}_\star(T) := \|h_\star^{1/2}(1-\Pi_\star)\phi\|_{L^2(T)},$$

and let

$$\text{(3.19b)} \qquad {\rm osc}_\star := {\rm osc}_\star(\mathcal{T}_\star) \qquad \text{with} \quad {\rm osc}_\star(\mathcal{E}_\star)^2 := \sum_{T\in\mathcal{E}_\star} {\rm osc}_\star(T)^2, \qquad \text{for all } \mathcal{E}_\star \subseteq \mathcal{T}_\star.$$

In analogy to Proposition 3.5, the following proposition collects the basic properties of ${\rm osc}_\star$.

PROPOSITION 3.7. *Let* $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$. *Then, the data oscillation terms* ${\rm osc}_\ell$ *from* (3.19) *satisfies the following properties* (i)–(iii).

(i). *Stability on non-refined elements: it holds that*

$$\text{(3.20)} \qquad {\rm osc}_\star(\mathcal{T}_\star \cap \mathcal{T}_\ell) = {\rm osc}_\ell(\mathcal{T}_\star \cap \mathcal{T}_\ell).$$

(ii). *Reduction on refined elements: with* $q_{\rm osc} := 2^{-1/(d-1)} < 1$, *it holds that*

$$\text{(3.21)} \qquad {\rm osc}_\star^2(\mathcal{T}_\star \setminus \mathcal{T}_\ell) \leq q_{\rm osc}\, {\rm osc}_\ell^2(\mathcal{T}_\ell \setminus \mathcal{T}_\star).$$

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(iii). *Discrete reliability: there exists a constant* $C_{\rm osc} > 0$ *such that*

$$\|(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)} \le C_{\rm osc}\, {\rm osc}_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\star). \tag{3.22}$$

The constant $C_{\rm osc} > 0$ *depends only on* Γ.

Proof. Note that Π_ℓ is the $\mathcal{T}_\ell$-elementwise best approximation. First, this reveals the identity $\Pi_\ell\phi = \Pi_\star\phi$ in the non-refined region $\bigcup(\mathcal{T}_\ell \cap \mathcal{T}_\star)$ and hence proves (3.20). Recall that $h_\star \le q\, h_\ell$ with $q = 2^{-1/(d-1)}$ in the refined region $\bigcup(\mathcal{T}_\ell \backslash \mathcal{T}_\star) = \bigcup(\mathcal{T}_\star \backslash \mathcal{T}_\ell)$. This yields

$$\begin{aligned}{\rm osc}_\star(\mathcal{T}_\star \backslash \mathcal{T}_\ell) = \|h_\star^{1/2}(1-\Pi_\star)\phi\|_{L^2(\bigcup(\mathcal{T}_\star\backslash\mathcal{T}_\ell))} &\le q^{1/2}\, \|h_\ell^{1/2}(1-\Pi_\ell)\phi\|_{L^2(\bigcup(\mathcal{T}_\star\backslash\mathcal{T}_\ell))}\\ &= q^{1/2}\, {\rm osc}_\ell(\mathcal{T}_\ell\backslash\mathcal{T}_\star)\end{aligned}$$

and proves (3.21). To see (3.22), note that orthogonal projections satisfy elementwise

$$\Pi_\star(1-\Pi_\ell) = \Pi_\star - \Pi_\ell = (1-\Pi_\ell)\Pi_\star.$$

With Lemma 3.3 and $\Pi_\ell\phi = \Pi_\star\phi$ in $\bigcup(\mathcal{T}_\ell \cap \mathcal{T}_\star)$, we infer that

$$\begin{aligned}\|(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)} &= \|(1-\Pi_\ell)\Pi_\star\phi\|_{H^{-1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(1-\Pi_\ell)\Pi_\star\phi\|_{L^2(\Gamma)}\\ &= \|h_\ell^{1/2}(\Pi_\star - \Pi_\ell)\phi\|_{L^2(\bigcup(\mathcal{T}_\ell\backslash\mathcal{T}_\star))} = \|h_\ell^{1/2}\Pi_\star(1-\Pi_\ell)\phi\|_{L^2(\bigcup(\mathcal{T}_\ell\backslash\mathcal{T}_\star))}\\ &\le \|h_\ell^{1/2}(1-\Pi_\ell)\phi\|_{L^2(\bigcup(\mathcal{T}_\ell\backslash\mathcal{T}_\star))} = {\rm osc}_\ell(\mathcal{T}_\ell\backslash\mathcal{T}_\star).\end{aligned}$$

For the previous estimate, we have used that $h_\ell \in \mathcal{P}^0(\mathcal{T}_\ell) \subseteq \mathcal{P}^0(\mathcal{T}_\star)$ and that $\Pi_\star$ is the $\mathcal{T}_\star$-piecewise L^2-orthogonal projection. This proves (3.22) with $C_{\rm osc} = C_{\rm apx}$. □

3.5. Overall a posteriori error estimator. For given Neumann data $\phi \in L^2(\Gamma)$, let $g \in H^{1/2}(\Gamma)$ and $G_\star \in \mathcal{S}^{p+1}(\mathcal{T}_\star)$ be the unique solutions to

$$\langle\!\langle g\,,\, v\rangle\!\rangle_{W+S} = \langle(1/2-K')\phi\,,\, v\rangle, \qquad \text{for all } v \in H^{1/2}(\Gamma), \tag{3.23}$$

$$\langle\!\langle G_\star\,,\, V_\star\rangle\!\rangle_{W+S} = \langle(1/2-K')\Pi_\star\phi\,,\, V_\star\rangle, \qquad \text{for all } V_\star \in \mathcal{S}^{p+1}(\mathcal{T}_\star). \tag{3.24}$$

For the a posteriori error control, we define the local contributions

$$\rho_\star(T) := \|h_\star^{1/2}((1/2-K')\Pi_\star\phi - WG_\star)\|_{L^2(T)} + \|h_\star^{1/2}(1-\Pi_\star)\phi\|_{L^2(T)} \tag{3.25a}$$

and let

$$\rho_\star := \rho_\star(\mathcal{T}_\star), \qquad \text{with } \rho_\star(\mathcal{E}_\star)^2 := \sum_{T\in\mathcal{E}_\star} \rho_\star(T)^2 \quad \text{for any } \mathcal{E}_\star \subseteq \mathcal{T}_\star, \tag{3.25b}$$

that is, we consider the sum of the weighted-residual error estimator plus data oscillation terms. Compared to $\eta_\star$ from Section 3.3, the difference is that now the right-hand side f changes in each step of the adaptive loop, i.e., $f = (1/2-K')\Pi_\ell\phi$. The following theorem collects the properties of $\rho_\star$. As for the weighted-residual error estimator $\eta_\star$, an algebraic postprocessing of the *basic properties* (i)–(iii) reveals further properties (iv)–(vi) of $\rho_\star$ required for the convergence and quasi-optimality analysis below.

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THEOREM 3.8. *Let $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ with the corresponding Galerkin solutions G_ℓ and $G_\star$ from* (3.24)*. Then, the overall error estimator ρ_ℓ from* (3.25) *satisfies the following properties* (i)–(iii).

(i). *Stability on non-refined elements: there exists a constant $C_{\rm stab} > 0$ such that*

$$C_{\rm stab}^{-1}\,|\rho_\star(\mathcal{T}_\star\cap\mathcal{T}_\ell) - \rho_\ell(\mathcal{T}_\ell\cap\mathcal{T}_\star)| \leq \|G_\star - G_\ell\|_{H^{1/2}(\Gamma)} + \|(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)}. \tag{3.26}$$

(ii). *Reduction on refined elements: there exist constants $0 < q_{\rm red} < 1$ and $C_{\rm red} > 0$ such that*

$$\rho_\star^2(\mathcal{T}_\star\setminus\mathcal{T}_\ell) \leq q_{\rm red}\,\rho_\ell^2(\mathcal{T}_\ell\setminus\mathcal{T}_\star) + C_{\rm red}\left(\|G_\star - G_\ell\|_{H^{1/2}(\Gamma)}^2 + \|(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)}^2\right). \tag{3.27}$$

(iii). *Discrete reliability: there exists a constant $C_{\rm dlr} > 0$ such that*

$$\|G_\star - G_\ell\|_{H^{1/2}(\Gamma)} + \|(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)} \leq C_{\rm dlr}\rho_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_\star). \tag{3.28}$$

The constants $C_{\rm dlr}, C_{\rm stab}, C_{\rm red} > 0$ and $0 < q_{\rm red} < 1$ depend only on Γ, the γ-shape regularity of $\mathcal{T}_\ell$ and $\mathcal{T}_\star$, and the polynomial degree $p \in \mathbb{N}_0$. Moreover, these basic properties imply the following properties (iv)–(vi).

(iv). *Quasi-monotonicity: there exists a constant $C_{\rm mon} > 0$ such that*

$$\rho_\star \leq C_{\rm mon}\,\rho_\ell. \tag{3.29}$$

(v). *Reliability: with the constant $C_{\rm dlr}$ from discrete reliability* (3.28)*, it holds that*

$$\|g - G_\ell\|_{H^{1/2}(\Gamma)} \leq C_{\rm dlr}\,\rho_\ell.$$

(vi). *Estimator reduction: the Dörfler criterion $\theta\,\rho_\ell^2 \leq \rho_\ell(\mathcal{T}_\ell\backslash\mathcal{T}_\star)^2$ implies that*

$$\rho_\star^2 \leq q_{\rm est}\,\rho_\ell^2 + C_{\rm est}\left(\|G_\star - G_\ell\|_{H^{1/2}(\Gamma)}^2 + \|(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)}^2\right). \tag{3.30}$$

The constants $C_{\rm est} > 0$ and $0 < q_{\rm est} \leq 1$ depend only on $0 < \theta \leq 1$ and $C_{\rm stab}$, $C_{\rm red}$, $q_{\rm red}$, while $C_{\rm mon}$ depends only on $C_{\rm dlr}, C_{\rm stab}, C_{\rm red}$.

Proof. We first prove stability, (3.26). Recall that $h_\star = h_\ell$ on $\bigcup(\mathcal{T}_\ell\cap\mathcal{T}_\star)$. The reverse triangle inequality and stability (3.20) of the data oscillation prove

$$\begin{aligned}
&|\rho_\star(\mathcal{T}_\star\cap\mathcal{T}_\ell) - \rho_\ell(\mathcal{T}_\star\cap\mathcal{T}_\ell)|\\
&\quad\leq \|h_\star^{1/2}((1/2-K')\Pi_\star\phi - WG_\star) - h_\ell^{1/2}((1/2-K')\Pi_\ell\phi - WG_\ell)\|_{L^2(\bigcup(\mathcal{T}_\star\cap\mathcal{T}_\ell))}\\
&\quad\leq \|h_\star^{1/2}(1/2-K')(\Pi_\star - \Pi_\ell)\phi\|_{L^2(\Gamma)} + \|h_\star^{1/2}W(G_\star - G_\ell)\|_{L^2(\Gamma)}\\
&\quad\lesssim \|(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)} + \|G_\star - G_\ell\|_{H^{1/2}(\Gamma)},
\end{aligned}$$

where we have used the inverse estimates (3.3)–(3.4).

Second, we prove the discrete reliability, (3.28). To that end, let $G_{\ell,\star} \in \mathcal{S}^{p+1}(\mathcal{T}_\star)$ denote the unique Galerkin solution of

$$\langle\!\langle G_{\ell,\star}\,,\,V_\star\rangle\!\rangle_{W+S} = \langle(1/2-K')\Pi_\ell\phi\,,\,V_\star\rangle, \qquad \text{for all } V_\star \in \mathcal{S}^{p+1}(\mathcal{T}_\star).$$

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Note that $G_{\ell,\star} \in \mathcal{S}^{p+1}(\mathcal{T}_\star)$ and $G_\ell \in \mathcal{S}^{p+1}(\mathcal{T}_\ell)$ are Galerkin solutions for the same right-hand side $f = (1/2 - K')\Pi_\ell\phi$. Therefore, the discrete reliability (3.15) of the weighted-residual error estimator yields

$$\|G_{\ell,\star} - G_\ell\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}((1/2 - K')\Pi_\ell\phi - WG_\ell\|_{L^2(\bigcup(\mathcal{T}_\ell\setminus\mathcal{T}_\star))} \le \rho_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_\star). \tag{3.31}$$

Moreover, the stability of the Galerkin formulations and the adjoint double-layer potential yield

$$\begin{aligned} \|G_\star - G_{\ell,\star}\|_{H^{1/2}(\Gamma)} &\lesssim \|(1/2 - K')\Pi_\star\phi - (1/2 - K')\Pi_\ell\phi\|_{H^{-1/2}(\Gamma)} \\ &\lesssim \|(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)}. \end{aligned} \tag{3.32}$$

Therefore, the discrete reliability (3.22) of the data oscillations gives

$$\|G_\star - G_{\ell,\star}\|_{H^{1/2}(\Gamma)} + \|(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)} \lesssim \mathrm{osc}_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_\star) \le \rho_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_\star).$$

The combination of (3.31)–(3.32) proves (3.28).

The reduction property (3.27) is proved analogously. Arguing along the lines of Corollary 3.6, one sees that the basic properties (i)–(iii) already imply the further properties (iv)–(vi). Details are left to the reader. □

4. Linear convergence of adaptive BEM.

We consider the following adaptive mesh-refining algorithm.

ALGORITHM 4.1.

INPUT: *initial mesh* $\mathcal{T}_0$, *parameter* $0 < \theta \le 1$, *and set* $\ell := 0$.

(i). *Compute approximate data* $\Pi_\ell\phi$ *using the* L^2*-projection* $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^p(\mathcal{T}_\ell)$.

(ii). *Compute the Galerkin solution* $G_\ell \in \mathcal{S}^{p+1}(\mathcal{T}_\ell)$ *of* (3.24).

(iii). *Compute refinement indicators* $\rho_\ell(T)$ *from* (3.25) *for all* $T \in \mathcal{T}_\ell$.

(iv). *Determine a set of marked elements* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ *which satisfies the Dörfler marking criterion*

$$\theta\rho_\ell^2 \le \rho_\ell(\mathcal{M}_\ell)^2. \tag{4.1}$$

(v). *Refine at least the marked elements to obtain* $\mathcal{T}_{\ell+1} \in \mathtt{refine}(\mathcal{T}_\ell)$, *i.e.,* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}$.

(vi). *Increment* $\ell \leftarrow \ell + 1$ *and goto* (i).

OUTPUT: *the sequences of the error estimators* $(\rho_\ell)_{\ell\in\mathbb{N}}$ *and Galerkin solutions* $(G_\ell)_{\ell\in\mathbb{N}}$.

For the mesh-refinement in step (v), we suppose that the assumptions of Section 3.1 hold true, i.e., marked elements are refined into at least two sons of (at most) half area. Note that we do not impose any minimality condition on the set of marked elements $\mathcal{M}_\ell$ in step (iv) so that, formally, $\mathcal{M}_\ell = \mathcal{T}_\ell$ would also be a valid choice.

THEOREM 4.2. *Let* $\mathcal{T}_\star \in \mathtt{refine}(\mathcal{T}_\ell)$ *with the corresponding Galerkin solutions* $G_\ell \in \mathcal{S}^{p+1}(\mathcal{T}_\ell)$ *and* $G_\star \in \mathcal{S}^{p+1}(\mathcal{T}_\star)$ *of* (3.24). *Let* g_ℓ, $g_\star \in H^{1/2}(\Gamma)$ *denote the unique solutions to*

$$\begin{aligned} \langle\!\langle g_\ell\,,\,v\rangle\!\rangle_{W+S} &= \langle(1/2 - K')\Pi_\ell\phi\,,\,v\rangle, && \text{for all } v \in H^{1/2}(\Gamma), \\ \langle\!\langle g_\star\,,\,v\rangle\!\rangle_{W+S} &= \langle(1/2 - K')\Pi_\star\phi\,,\,v\rangle, && \text{for all } v \in H^{1/2}(\Gamma). \end{aligned}$$

Suppose that the set of refined elements satisfies the Dörfler marking

$$\theta\,\rho_\ell^2 \le \rho_\ell(\mathcal{T}_\ell\setminus\mathcal{T}_\star)^2$$

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for some $0 < \theta \leq 1$. *Then, there exist constants* α, $\beta > 0$ *and* $0 < \kappa < 1$ *such that the quasi-errors*

$$\Delta_\ell := \|g_\ell - G_\ell\|_{W+S}^2 + \alpha\rho_\ell^2 + \beta\mathrm{osc}_\ell^2,$$
$$\Delta_\star := \|g_\star - G_\star\|_{W+S}^2 + \alpha\rho_\star^2 + \beta\mathrm{osc}_\star^2$$

satisfy the contraction property

$$\Delta_\star \leq \kappa\,\Delta_\ell. \tag{4.2}$$

Moreover, it holds that

$$\alpha\rho_\ell^2 \leq \Delta_\ell \leq (C_{\mathrm{dlr}}^2 + \alpha + \beta)\,\rho_\ell. \tag{4.3}$$

The constants α, $\beta > 0$, *and* $0 < \kappa < 1$ *depend only on* C_{dlr}, C_{est}, q_{est} *as well as on* Γ. *Indirectly, they hence also depend on* $0 < \theta \leq 1$, *the* γ*-shape regularity of* $\mathcal{T}_\ell$ *and* $\mathcal{T}_\star$, *and the polynomial degree* $p \in \mathbb{N}_0$.

COROLLARY 4.3. *Suppose that all meshes* $\mathcal{T}_\ell$ *generated by Algorithm 4.1 are* γ*-shape regular. Then, Algorithm 4.1 guarantees* R*-linear convergence of the error estimator sequence*

$$\rho_{\ell+k}^2 \leq C_1\,\kappa^k\,\rho_\ell^2, \qquad \text{for all } k, \ell \in \mathbb{N}_0, \tag{4.4}$$

with $C_1 = \alpha^{-1}(C_{\mathrm{dlr}}^2 + \alpha + \beta)$ *and the constants* α, $\beta > 0$, *and* $0 < \kappa < 1$ *from Theorem 4.2.*

Proof. Theorem 4.2 applies with $\mathcal{T}_\star = \mathcal{T}_{\ell+1}$ since $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}$ satisfies the Dörfler marking (4.1). By induction, the contraction estimate (4.2) proves $\Delta_{\ell+k} \leq \kappa^k\,\Delta_\ell$ for all $k, \ell \in \mathbb{N}_0$. Together with the equivalence (4.3), this concludes the proof. □

Proof of Theorem 4.2. First, we recall the norm equivalence $\|\cdot\|_{W+S} \simeq \|\cdot\|_{H^{1/2}(\Gamma)}$, namely,

$$C_2^{-1}\,\|v\|_{H^{1/2}(\Gamma)} \leq \|v\|_{W+S} \leq C_2\,\|v\|_{H^{1/2}(\Gamma)}, \qquad \text{for all } v \in H^{1/2}(\Gamma), \tag{4.5}$$

where $C_2 > 0$ depends only on Γ.

Second, recall stability of the continuous formulation in the sense that

$$\|g_\star - g_\ell\|_{H^{1/2}(\Gamma)} \lesssim \|(1/2 - K')(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)} \lesssim \|(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)}.$$

Together with the norm equivalence (4.5) and discrete reliability (3.22) of the data oscillation, this leads to

$$C_3^{-1}\,\|g_\star - g_\ell\|_{W+S}^2 \leq \|(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)}^2 \leq C_{\mathrm{osc}}^2\,\mathrm{osc}_\ell(\mathcal{T}_\ell \backslash \mathcal{T}_\star)^2, \tag{4.6}$$

where $C_3 > 0$ depends only on Γ.

Third, the Galerkin orthogonality implies

$$\langle\!\langle g_\star - G_\star\,,\,V_\star\rangle\!\rangle_{W+S} = 0, \qquad \text{for all } V_\star \in \mathcal{S}^{p+1}(\mathcal{T}_\star).$$

This yields the Pythagorean identity

$$\|g_\star - G_\ell\|_{W+S}^2 = \|g_\star - G_\star\|_{W+S}^2 + \|G_\star - G_\ell\|_{W+S}^2.$$

For $\varepsilon > 0$, Young's inequality gives

$$\|g_\star - G_\ell\|_{W+S}^2 \leq (1+\varepsilon)\,\|g_\ell - G_\ell\|_{W+S}^2 + (1+\varepsilon^{-1})\,\|g_\star - g_\ell\|_{W+S}^2.$$

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Combining the last two observations with the stability estimate (4.6), we see that

$$\begin{aligned}\|\!|g_\star - G_\star\|\!|^2_{W+S} &\le (1+\varepsilon)\,\|\!|g_\ell - G_\ell\|\!|^2_{W+S} - \|\!|G_\star - G_\ell\|\!|^2_{W+S} \\ &\quad + (1+\varepsilon^{-1})C_3C_{\rm osc}^2\,{\rm osc}_\ell(\mathcal{T}_\ell\backslash\mathcal{T}_\star)^2.\end{aligned} \tag{4.7}$$

Fourth, we define $C_4 = (1+\varepsilon^{-1})C_3C_{\rm osc}^2 + \alpha C_{\rm est}C_{\rm osc}^2$ and combine the estimator reduction (3.30) with (4.7) to see

$$\begin{aligned}\Delta_\star &\le (1+\varepsilon)\,\|\!|g_\ell - G_\ell\|\!|^2_{W+S} - \|\!|G_\star - G_\ell\|\!|^2_{W+S} + (1+\varepsilon^{-1})C_3C_{\rm osc}^2\,{\rm osc}_\ell(\mathcal{T}_\ell\backslash\mathcal{T}_\star)^2 \\ &\qquad + \alpha\, q_{\rm est}\rho_\ell^2 + \alpha C_{\rm est}\left(\|G_\star - G_\ell\|^2_{H^{1/2}(\Gamma)} + \|(\Pi_\star - \Pi_\ell)\phi\|^2_{H^{-1/2}(\Gamma)}\right) + \beta\,{\rm osc}_\star^2 \\ &\le (1+\varepsilon)\,\|\!|g_\ell - G_\ell\|\!|^2_{W+S} + \alpha\, q_{\rm est}\rho_\ell^2 + C_4\,{\rm osc}_\ell(\mathcal{T}_\ell\backslash\mathcal{T}_\star)^2 + \beta\,{\rm osc}_\star^2 \\ &\qquad + (\alpha C_{\rm est}C_2^2 - 1)\,\|\!|G_\star - G_\ell\|\!|^2_{W+S}.\end{aligned} \tag{4.8}$$

With the choice $\alpha := C_{\rm est}^{-1}C_2^{-2}$, the last term vanishes.

Fifth, note that $h_\star \le q\,h_\ell$ with $q = 2^{-1/(d-1)}$ on $\bigcup(\mathcal{T}_\ell\backslash\mathcal{T}_\star)$, while $h_\star = h_\ell$ on $\bigcup(\mathcal{T}_\ell\cap\mathcal{T}_\star)$. With the characteristic function $\chi_{\bigcup(\mathcal{T}_\ell\backslash\mathcal{T}_\star)}$, this implies the pointwise estimate $(1-q)h_\ell\chi_{\bigcup(\mathcal{T}_\ell\backslash\mathcal{T}_\star)} \le h_\ell - h_\star$, and hence the best approximation property of $\Pi_\star$ yields

$$\begin{aligned}(1-q)\,{\rm osc}_\ell(\mathcal{T}_\ell\backslash\mathcal{T}_\star)^2 &= (1-q)\,\|h_\ell^{1/2}(1-\Pi_\ell)\phi\|^2_{L^2(\bigcup(\mathcal{T}_\ell\backslash\mathcal{T}_\star))} \\ &\le \|h_\ell^{1/2}(1-\Pi_\ell)\phi\|^2_{L^2(\Gamma)} - \|h_\star^{1/2}(1-\Pi_\ell)\phi\|^2_{L^2(\Gamma)} \\ &\le {\rm osc}_\ell^2 - {\rm osc}_\star^2.\end{aligned}$$

With the above choice of α and $\beta := C_4(1-q)^{-1}$, the estimate (4.8) becomes

$$\Delta_\star \le (1+\varepsilon)\,\|\!|g_\ell - G_\ell\|\!|^2_{W+S} + \alpha\, q_{\rm est}\rho_\ell^2 + \beta\,{\rm osc}_\ell^2. \tag{4.9}$$

Sixth, since g_ℓ and G_ℓ are determined by the same right-hand side $f = (1/2 - K')\Pi_\ell\phi$, the reliability estimate (3.17) of the weighted-residual error estimator yields

$$C_{\rm dlr}^{-1}\,\|g_\ell - G_\ell\|_{H^{1/2}(\Gamma)} \le \|h_\ell^{1/2}(1/2 - K')\Pi_\ell\phi - WG_\ell\|_{L^2(\Gamma)} \le \rho_\ell.$$

Together with the norm equivalence $\|\!|\cdot\|\!|_{W+S} \simeq \|\cdot\|_{H^{1/2}(\Gamma)}$, this leads to

$$C_5\,\|\!|g_\ell - G_\ell\|\!|^2_{W+S} \le \rho_\ell^2,$$

where $C_5 > 0$ depends only on $C_{\rm dlr}$ and Γ. Moreover, it holds that ${\rm osc}_\ell^2 \le \rho_\ell^2$. For arbitrary $\delta > 0$, we obtain from (4.9)

$$\Delta_\star \le (1+\varepsilon-\alpha\delta C_5)\,\|\!|g_\ell - G_\ell\|\!|^2_{W+S} + \alpha\,(q_{\rm est}+2\delta)\,\rho_\ell^2 + (\beta - \alpha\delta)\,{\rm osc}_\ell^2 \le \kappa\,\Delta_\ell,$$

where

$$\kappa := \max\big\{1+\varepsilon-\alpha\delta C_5\,,\, q_{\rm est}+2\delta\,,\,(\beta-\alpha\delta)/\beta\big\}.$$

The choice $\delta < (1-q_{\rm est})/2$ and $\varepsilon < \alpha\delta C_5$ finally concludes that $0 < \kappa < 1$. ∎

5. Quasi-optimal convergence rates. In this section, we prove that the usual implementation of Algorithm 4.1 leads to quasi-optimal convergence behavior in the following sense: suppose that adaptive mesh-refinement can provide a decay $\mathcal{O}(N^{-s})$ of the error estimator $\rho_\star$ with respect to the number N of elements and some algebraic convergence rate $s > 0$ if the *optimal meshes* are chosen (which do not have to be nested). Theorem 5.1 below then proves that the sequence of estimators ρ_ℓ generated by Algorithm 4.1 will also decay asymptotically with rate s.

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5.1. Additional assumptions. While all the previous results hold without any further assumptions on the mesh-refinement, the following assumptions are necessary for the optimality result of Theorem 5.1, where we further specify steps (iv)–(v) of Algorithm 4.1.

A1. We suppose that the set of marked elements $\mathcal{M}_\ell$ which satisfies (4.1) has minimal cardinality.

We note that the set $\mathcal{M}_\ell$ might not be unique in general and that its computation usually relies on sorting the refinement indicators $\rho_\ell(T)$. For the mesh-refinement in step (v), we suppose the following.

A2. For $d = 2$, the bisection algorithm from [2] is used. For $d = 3$, we use 2D newest vertex bisection; see, e.g., [19] and the references therein.

A3. In either case, we suppose that $\mathcal{T}_{\ell+1} = \mathtt{refine}(\mathcal{T}_\ell; \mathcal{M}_\ell)$ is the coarsest admissible refinement of $\mathcal{T}_\ell$ such that all marked elements $T \in \mathcal{M}_\ell$ have been bisected.

First, the choice of these mesh-refinement strategies guarantees that the meshes $\mathcal{T}_\ell$ generated by Algorithm 4.1 are uniformly γ-shape regular, where $\gamma > 0$ depends only on the initial mesh $\mathcal{T}_0$.

Second, it has first been observed in [6] for 2D newest vertex bisection that the number $\#\mathcal{T}_\ell$ of elements in $\mathcal{T}_\ell$ can be controlled by the number of marked elements, i.e.,

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C_{\mathrm{mesh}} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j, \tag{5.1}$$

where $C_{\mathrm{mesh}} > 0$ depends only on $\mathcal{T}_0$. While [6] requires an additional assumption on $\mathcal{T}_0$, this assumption has recently been removed in [19] so that the initial triangulation $\mathcal{T}_0$ is in fact an arbitrary admissible triangulation. For $d = 2$, the estimate (5.1) is proved in [2] for a bisection based refinement, where additional bisection of non-marked elements are required to ensure uniform γ-shape regularity. In either case, the proof of (5.1) naturally relies on assumption A3.

Finally, for two admissible triangulations $\mathcal{T}_\ell$ and $\mathcal{T}_\star$, let

$$\mathcal{T}_\ell \oplus \mathcal{T}_\star \in \mathtt{refine}(\mathcal{T}_\ell) \cap \mathtt{refine}(\mathcal{T}_\star)$$

be the coarsest admissible refinement of both $\mathcal{T}_\ell$ and $\mathcal{T}_\star$. Then, $\mathcal{T}_\ell \oplus \mathcal{T}_\star$ is in fact the overlay, and it holds that

$$\#(\mathcal{T}_\ell \oplus \mathcal{T}_\star) \le \#\mathcal{T}_\ell + \#\mathcal{T}_\star - \#\mathcal{T}_0; \tag{5.2}$$

see [24] for $d = 3$ and 2D newest vertex bisection and [2] for $d = 2$.

Overall, we note that the estimates (5.1)–(5.2) are required for the arguments of the proof and strongly tailored to the mesh-refinement strategy chosen in A2.

5.2. Optimality result. To quantify the convergence rate of Algorithm 4.1, we introduce for each $s > 0$ the quasi-norm

$$\|(g, \phi)\|_{\mathbb{A}_s} := \sup_{N \in \mathbb{N}_0} \inf_{\substack{\mathcal{T}_\star \in \mathtt{refine}(\mathcal{T}_0) \\ \#\mathcal{T}_\star - \#\mathcal{T}_0 \le N}} (N+1)^s \, \rho_\star.$$

Note that $\|(g, \phi)\|_{\mathbb{A}_s} < \infty$ for some $s > 0$ implies that a convergence rate

$$\rho_\star \lesssim (\#\mathcal{T}_\star - \#\mathcal{T}_0)^{-s}$$

could be achieved if the optimal meshes $\mathcal{T}_\star$ are chosen. The following theorem states that each possible rate $s > 0$ will be recovered by Algorithm 4.1, i.e., the meshes generated are asymptotically optimal.

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THEOREM 5.1. *Let $0 < \theta < \theta_{\rm opt} := (1 + C_{\rm stab}^2 C_{\rm dlr}^2)^{-1}$. Then, the adaptively generated meshes of Algorithm 4.1 satisfy*

$$c_{\rm opt}\|(g,\phi)\|_{\mathbb{A}_s} \le \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1)^s \rho_\ell \le C_{\rm opt}\|(g,\phi)\|_{\mathbb{A}_s}. \tag{5.3}$$

The constant $c_{\rm opt} > 0$ depends only on $d = 2, 3$, whereas the constant $C_{\rm opt} > 0$ depends on θ, $C_{\rm mesh}$, $C_{\rm dlr}$, $C_{\rm red}$, $C_{\rm stab}$, $q_{\rm red}$ as well as the polynomial degree p, s, and the γ-shape regularity of $\mathcal{T}_0$.

We note that in contrast to the FEM literature, e.g., [10, 24], and the first results on ABEM [14, 15], the upper bound $\theta_{\rm opt}$ on optimal marking parameters is independent of any lower bound for the error (the so-called *efficiency* of the estimator). The proof needs some preparations. The following result shows that Dörfler marking (4.1) is not only sufficient (4.4) but even necessary to obtain linear convergence of the error estimator.

LEMMA 5.2. *For any $0<\theta<\theta_{\rm opt}$, there exists $0 < \kappa_0 < 1$ such that all $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ satisfy the implication*

$$\rho_\star^2 \le \kappa_0 \rho_\ell^2 \qquad \Longrightarrow \qquad \theta\rho_\ell^2 \le \rho_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\star)^2. \tag{5.4}$$

The constant κ_0 depends only on θ, $C_{\rm stab}$, and $C_{\rm dlr}$.

Proof. The stability result (3.26) and Young's inequality show for $\delta > 0$ that

$$\begin{aligned}\rho_\ell^2 &= \rho_\ell(\mathcal{T}_\ell \cap \mathcal{T}_\star)^2 + \rho_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\star)^2\\ &\le (1+\delta)\rho_\star(\mathcal{T}_\star \cap \mathcal{T}_\ell)^2 + \rho_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\star)^2\\ &\qquad + (1+\delta^{-1})C_{\rm stab}^2\big(\|G_\star - G_\ell\|_{H^{1/2}(\Gamma)} + \|(\Pi_\star - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)}\big)^2.\end{aligned}$$

The assumption $\rho_\star^2 \le \kappa_0\rho_\ell^2$ together with discrete reliability (3.28) imply

$$\rho_\ell^2 \le (1+\delta)\kappa_0\rho_\ell^2 + (1 + (1+\delta^{-1})C_{\rm stab}^2 C_{\rm dlr}^2)\rho_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\star)^2,$$

and hence

$$\theta\rho_\ell^2 \le \rho_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_\star)^2 \qquad \text{for all } \ 0 \le \theta < \theta(\kappa_0) := \sup_{\delta>0} \frac{1-(1+\delta)\kappa_0}{1+(1+\delta^{-1})C_{\rm stab}^2 C_{\rm dlr}^2}.$$

For each $\theta < \theta_{\rm opt}$, there exist $\delta, \kappa_0 > 0$ such that

$$\theta < \frac{1-(1+\delta)\kappa_0}{1+(1+\delta^{-1})C_{\rm stab}^2 C_{\rm dlr}^2} < \theta_{\rm opt},$$

and hence $\theta < \theta(\kappa_0)$. This concludes the proof. ∎

The definition of the quasi-norm $\|\cdot\|_{\mathbb{A}_s}$ allows one to find *optimal* meshes with a number of elements comparable to the adaptively generated meshes. This is stated in the following lemma.

LEMMA 5.3. *Let $0 < \kappa < 1$ and let $s > 0$ be such that $\|(g,\phi)\|_{\mathbb{A}_s} < \infty$. Then, for all admissible meshes $\mathcal{T}_\ell$, there exists a refinement $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ with*

$$\rho_\star^2 \le \kappa\rho_\ell^2 \qquad \text{and} \qquad \#\mathcal{T}_\star - \#\mathcal{T}_\ell + 1 \le C_6\|(g,\phi)\|_{\mathbb{A}_s}^{1/s}\rho_\ell^{-1/s}. \tag{5.5}$$

The constant $C_6 > 0$ depends only on $C_{\rm mon}$, κ, and $s > 0$.

Proof. Arguing as in [10, 24], the definition of $\|\cdot\|_{\mathbb{A}_s}$ provides for each sufficiently small $\varepsilon > 0$ a mesh $\mathcal{T}_\varepsilon \in \texttt{refine}(\mathcal{T}_0)$, which satisfies

$$\#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 + 1 \lesssim \|(g,\phi)\|_{\mathbb{A}_s}^{1/s}\varepsilon^{-1/s} \qquad \text{and} \qquad \rho_\varepsilon \le \varepsilon.$$

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For $\varepsilon := C_{\rm mon}^{-1}\kappa^{1/2}\rho_\ell$, define $\mathcal{T}_\star := \mathcal{T}_\ell \oplus \mathcal{T}_\varepsilon$, and verify with (5.2)

$$\#\mathcal{T}_\star - \#\mathcal{T}_\ell + 1 \le \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 + 1 \lesssim \|(g,\phi)\|_{\mathbb{A}_s}^{1/s}\rho_\ell^{-1/s}.$$

Since $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\varepsilon)$ and by the choice of ε, quasi-monotonicity (3.29) shows that

$$\rho_\star^2 \le C_{\rm mon}^2\rho_\varepsilon^2 \le \kappa\rho_\ell^2.$$

This concludes the proof. □

Proof of Theorem 5.1. Choose $\kappa > 0$ sufficiently small such that the implication (5.4) holds true. Given $\mathcal{T}_\ell$, Lemma 5.3 provides a mesh $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ with (5.5). Therefore, Lemma 5.2 implies that $\mathcal{T}_\ell \setminus \mathcal{T}_\star$ satisfies the Dörfler marking (4.1). Since A1 states that $\mathcal{M}_\ell$ is a set of minimal cardinality which satisfies Dörfler marking, there holds

$$\#\mathcal{M}_\ell + 1 \le \#(\mathcal{T}_\ell \setminus \mathcal{T}_\star) + 1 \le \#\mathcal{T}_\star - \#\mathcal{T}_\ell + 1 \lesssim \|(g,\phi)\|_{\mathbb{A}_s}^{1/s}\rho_\ell^{-1/s}.$$

This and the mesh-closure estimate (5.1) imply

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1 \lesssim \sum_{j=0}^{\ell-1}(\#\mathcal{M}_j + 1) \lesssim \|(g,\phi)\|_{\mathbb{A}_s}^{1/s}\sum_{j=0}^{\ell-1}\rho_j^{-1/s}.$$

The R-linear convergence of Corollary 4.3 together with the convergence of the geometric series show that

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1 \lesssim \|(g,\phi)\|_{\mathbb{A}_s}^{1/s}\rho_\ell^{-1/s}C_1^{-1/s}\sum_{j=0}^{\ell-1}\kappa^{(\ell-j)/s} \lesssim \|(g,\phi)\|_{\mathbb{A}_s}^{1/s}\rho_\ell^{-1/s}.$$

This implies the upper bound in (5.3). The lower bound in (5.3) follows from elementary arguments and the fact that each refined element is split into at most two sons for $d = 2$ and into at most four sons for $d = 3$. This concludes the proof. □

6. Conclusions and remarks.

6.1. Conclusions on convergence results. In contrast to the FEM, the right-hand sides in the BEM typically involve boundary integral operators, which cannot be evaluated exactly in practice. Thus, the analysis of the data error is mandatory. To compute the right-hand side term $(1/2 - K')\phi$ numerically in our model problem (1.1), we follow the earlier work [5] and replace the exact data ϕ by its L^2-projection Φ_ℓ onto discontinuous piecewise polynomials. This approach thus decouples the problem of integrating the singular kernel of the integral operator K' from integrating the possibly singular data ϕ to compute $K'\phi$. On their own, both problems are well understood. Moreover, in 2D (see [20]) one can even find analytic formulas to compute the term $K'\Phi_\ell$ exactly, while there exist black-box quadrature algorithms to compute $K'\Phi_\ell$ in 3D; see, e.g., [22].

Based on the weighted-residual error estimator from [7], we introduced an overall error estimator which controls both, the discretization error as well as the data approximation error (Theorem 3.8). For the resulting adaptive algorithm, linear convergence (Corollary 4.3) even with quasi-optimal rates (Theorem 5.1) is shown. Throughout, the analysis applies to the Galerkin BEM based on piecewise polynomials of arbitrary but fixed maximal order $p \ge 1$.

We note that linear convergence (Corollary 4.3) as well as the optimality result of Theorem 5.1 also hold if data approximation is avoided, i.e., $\Pi_\star$ is taken as the identity in (3.24) and hence $\mathrm{osc}_\star = 0$ throughout. Therefore, this work generalizes [15] from the lowest order case $p = 0$ to general $p \ge 0$.

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6.2. Extension to indirect BEM. Linear convergence (Corollary 4.3) as well as the optimality result of Theorem 5.1 also hold for the indirect BEM (3.10) with the Galerkin discretization (3.11), where the analysis is even simpler. The necessary properties of the error estimator are provided by Proposition 3.5 and Corollary 3.6.

Moreover, the analysis can easily be adapted to the indirect BEM with data approximation, where (3.11) becomes

$$\langle\!\langle U_\star\,,\,V_\star\rangle\!\rangle_{W+S} = \langle \Pi_\star f\,,\,V_\star\rangle, \qquad \text{for all } V_\star \in \mathcal{S}^{p+1}(\mathcal{T}_\star).$$

Details follow by simplifying the proof of Theorem 3.8, while the proofs of linear convergence and optimal convergence rates hold accordingly.

6.3. Extension to screen problems. For a connected and relatively open screen $\Gamma \subsetneqq \partial\Omega$, let $\widetilde{H}^1(\Gamma)$ denote the space of all $H^1(\partial\Omega)$-functions which are supported on $\overline{\Gamma}$. One defines

$$\widetilde{H}^s(\Gamma) = [L^2(\Gamma); \widetilde{H}^1(\Gamma)]_s \qquad \text{and} \qquad H^s(\Gamma) = [L^2(\Gamma); H^1(\Gamma)]_s$$

by interpolation for $0 < s < 1$ and the corresponding dual spaces

$$\widetilde{H}^{-s}(\Gamma) = H^s(\Gamma)' \qquad \text{and} \qquad H^{-s}(\Gamma) = \widetilde{H}^s(\Gamma)'$$

with respect to the extended $L^2(\Gamma)$-scalar product $\langle\cdot\,,\,\cdot\rangle$. Then, $W : \widetilde{H}^s(\Gamma) \to H^{s-1}(\Gamma)$ is a well-defined linear and continuous operator. Moreover, W is symmetric and elliptic on $\widetilde{H}^{1/2}(\Gamma)$. For a given right-hand side $f \in H^{-1/2}(\Gamma)$, the hyper-singular integral equation $Wu = f$ thus fits into the setting of the Lax-Milgram lemma, and (unlike the case $\Gamma = \partial\Omega$) the stabilization can be omitted.

The analysis of the weighted-residual error estimator in Proposition 3.5 holds verbatim. On a technical side, one requires that the inverse estimates of Lemma 3.1 remain valid, which is, in fact, the case. We refer to the references [3, 4, 16] also given above. Moreover, our analysis requires an appropriate Scott-Zhang projection in $\widetilde{H}^{1/2}(\Gamma)$, which is defined and analyzed in [3], and Lemma 3.2 transfers to this case as well. Overall, also linear convergence (Corollary 4.3) and optimality (Theorem 5.1) remain valid.

7. Numerical experiments. This section reports on some numerical experiments in 2D with first-order $\mathcal{S}^1(\mathcal{T}_\star)$ and second-order $\mathcal{S}^2(\mathcal{T}_\star)$ boundary elements. All experiments are conducted in MATLAB by means of the library `HILBERT` [1].

7.1. Direct BEM for the 2D Neumann problem. We consider the hyper-singular integral equation (3.23) on the boundary of the Z-shaped domain sketched in Figure 7.1. The Neumann data $\phi \in H^{-1/2}(\Gamma)$ are given by the normal derivative of the potential

$$P(x, y) = r^{4/7}\cos(4/7\varphi),$$

where (r, φ) denote the polar coordinates of $(x, y) \in \mathbb{R}^2$, i.e., $(x, y) = r(\cos(\varphi), \sin(\varphi))$. Up to an additive constant, the exact solution $g \in H^{1/2}_\star(\Gamma)$ is the trace $P|_\Gamma$ of the potential P. We solve the perturbed discrete system (3.24) in each step of the adaptive algorithm from Section 4. Moreover, Algorithm 4.1 is steered by the local error indicators $\rho_\ell(T)$ from (3.25). We note that the energy norm $\|\!|g|\!\|_{W+S}$ of the exact solution is unknown. Therefore, we employ [3, Lemma 6] for $s = 1/2$ and estimate the energy error by

$$\|\!|g - G_\ell|\!\|_{W+S} \lesssim \|h_\ell^{1/2}\nabla(g - G_\ell)\|_{L^2(\Gamma)} + \mathrm{osc}_\ell =: \mathrm{err}_\ell. \tag{7.1}$$

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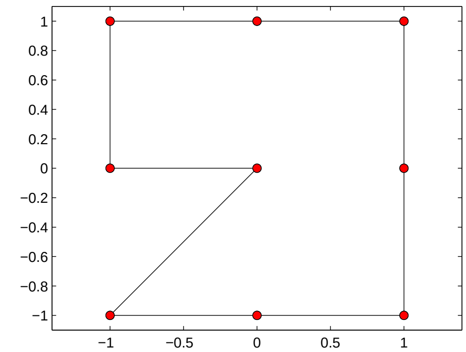

FIG. 7.1. *Z-shaped domain Ω with boundary $\Gamma = \partial\Omega$ and initial triangulation of Γ into 9 boundary elements for the numerical experiment of Section 7.1.*

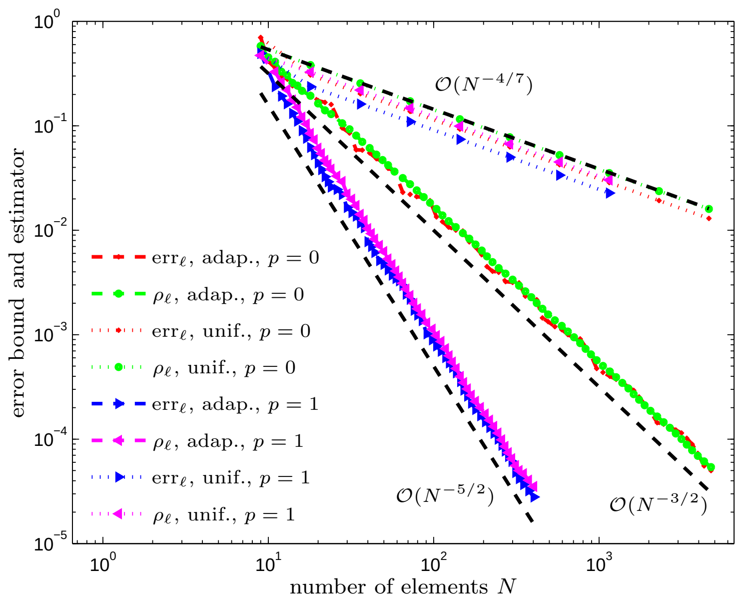

FIG. 7.2. *Hyper-singular integral equation of Section 7.1 on the Z-shaped domain, sketched in Figure 7.1. Uniform mesh-refinement leads to a suboptimal convergence $\mathcal{O}(N^{-4/7})$ for both $p = 0$ and $p = 1$, whereas the adaptive strategy recovers the optimal convergence $\mathcal{O}(N^{-(3/2+p)})$.*

In Figure 7.2, we compare adaptive ($\theta = 0.25$) versus uniform ($\theta = 1$) mesh-refinement for $p = 0, 1$, i.e., for an approximation of u by polynomials of degree 1 or 2. Since the exact solution satisfies $g \in H^{1/2+4/7-\varepsilon}(\Gamma)$, for all $\varepsilon > 0$, theory predicts the convergence order $|||g - G_\ell|||_{W+S} = \mathcal{O}(N_\ell^{-s})$ with $s = 4/7$ for uniform mesh-refinement. This is confirmed by our numerical results from Figure 7.2 for both $p = 0$ and $p = 1$, whereas the adaptive strategies for $p = 0, 1$ recover the optimal orders $s = 3/2 + p$. Moreover, we observe a good coincidence of $\rho_\ell \simeq \mathrm{err}_\ell$. We note, however, that both ρ_ℓ as well as err_ℓ are only proved to provide upper bounds of $|||g - G_\ell|||_W$, while lower bounds as well as the relation $\rho_\ell \simeq \mathrm{err}_\ell$ remain mathematically an open issue. This leaves an interesting question for future research.

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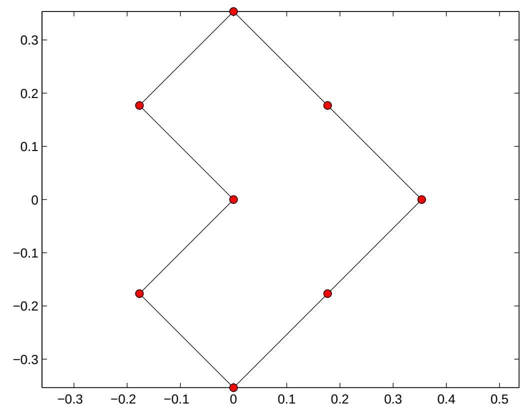

FIG. 7.3. *L-shaped domain* Ω *with boundary* $\Gamma = \partial\Omega$ *and initial triangulation of* Γ *into* 8 *boundary elements for the numerical experiment of Section 7.2.*

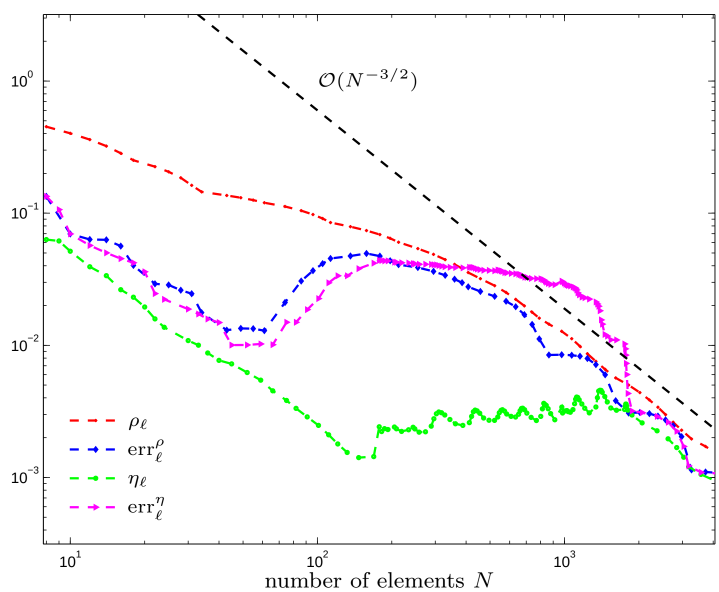

FIG. 7.4. *Hyper-singular integral equation of Section 7.2 on the L-shaped domain, sketched in Figure 7.3. Comparison of* ρ_ℓ*-adaptive mesh-refinement vs.* η_ℓ*-adaptive mesh-refinement, where* $\rho_\ell^2 = \eta_\ell^2 + \mathrm{osc}_\ell^2$*, to examine the influence of the data oscillations. One observes that* η_ℓ*-adaptivity leads to a higher number of adaptive steps than* ρ_ℓ*-adaptivity.*

7.2. 2D problem with oscillatory data. Let $\Gamma = \partial\Omega$ be the boundary of the L-shaped domain $\Omega \subset \mathbb{R}^2$ sketched in Figure 7.3. We consider the hyper-singular integral equation (3.23). The Neumann data $\phi \in H^{-1/2}(\Gamma)$ are given by the normal derivative of the potential

$$P(x,y) = r^{2/3}\cos(2/3\varphi) + \beta\big(\sin(\alpha x')\sinh(\alpha y') + \sin(\alpha y')\sinh(\alpha x')\big),$$

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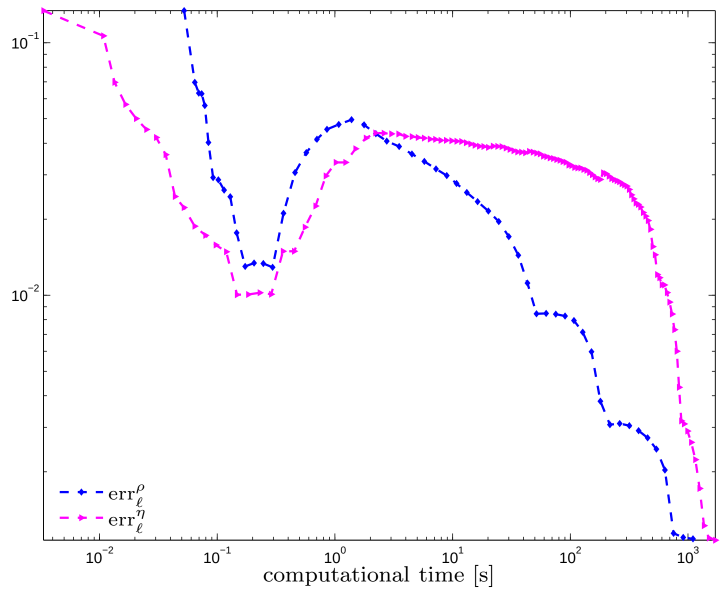

FIG. 7.5. *Hyper-singular integral equation of Section 7.2 on the L-shaped domain, sketched in Figure 7.3. Comparison of ρ_ℓ-adaptive mesh-refinement vs. η_ℓ-adaptive mesh-refinement with respect to the computational time, where $\rho_\ell^2 = \eta_\ell^2 + \mathrm{osc}_\ell^2$, to examine the influence of the data oscillations. Due to the higher number of adaptive steps, η_ℓ-adaptivity performs worse than ρ_ℓ-adaptivity.*

where (r, φ) denote the polar coordinates of $(x, y) \in \mathbb{R}^2$, $\alpha = 240\pi$, $\beta = 10^{-3}\sinh(\alpha)$, and

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\frac{3\pi}{4} & -\sin\frac{3\pi}{4} \\ \sin\frac{3\pi}{4} & \cos\frac{3\pi}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Hence, the data consists of a term which is singular at the reentrant corner $(x, y) = (0, 0)$ and an oscillatory perturbation term. Up to an additive constant, the exact solution $g \in H^{1/2}(\Gamma)$ is given as the trace $P|_\Gamma$ of the potential P.

To examine the influence of the data oscillation on the adaptive mesh-refinement, we steer Algorithm 4.1 first with the local contributions $\rho_\ell(T)^2 = \eta_\ell(T)^2 + \mathrm{osc}_\ell(T)^2$ and then with $\eta_\ell(T)^2$ only. We note that ρ_ℓ is a mathematically guaranteed upper bound for the error $|||u - U_\ell|||_W$, while η_ℓ is not. In both cases, we use $\theta = 1/4$ and lowest-order discretizations $p = 0$, i.e., affine elements to approximate u. As in the previous Section 7.1, the energy error is not known exactly, and we use (7.1) to compute an upper error bound. Figure 7.4 shows the outcome of ρ_ℓ-adaptive versus η_ℓ-adaptive mesh-refinement. Besides ρ_ℓ and η_ℓ, we plot the corresponding reliable error bounds (denoted by err_ℓ^ρ and err_ℓ^η). Asymptotically, both methods appear to converge with optimal order $\mathcal{O}(N^{-3/2})$. However, we observe that Algorithm 4.1 needs significantly more steps for η_ℓ-adaptive mesh-refinement than for ρ_ℓ-adaptive mesh-refinement. A similar observation can be made with respect to Figure 7.5, where we plot the errors err_ℓ^ρ respectively err_ℓ^η versus the computational time. Altogether, it is thus preferable to use ρ_ℓ instead of only η_ℓ.

7.3. 2D slit problem. Let $\Gamma := (-1, 1) \times \{0\} \subseteq \mathbb{R}^2$. We consider the hyper-singular integral equation

$$\langle\langle g\,,\,v\rangle\rangle_W = \langle \phi\,,\,v\rangle, \qquad \text{for all } v \in \widetilde{H}^{1/2}(\Gamma),$$

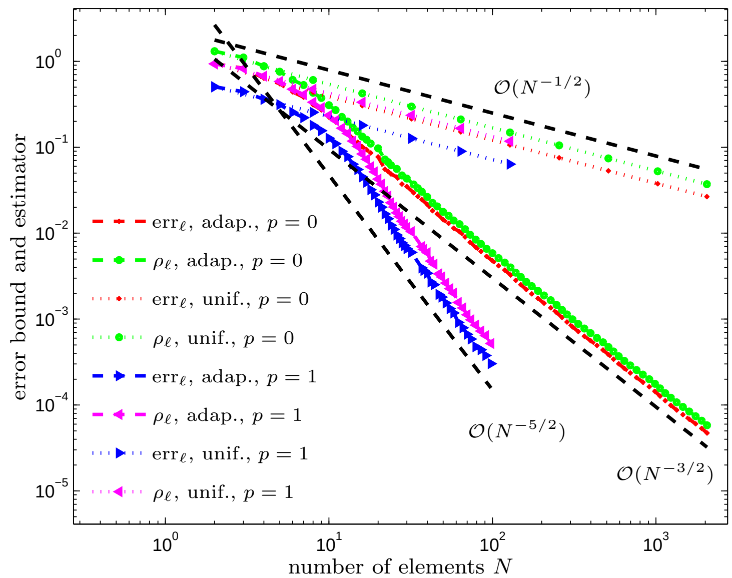

FIG. 7.6. *Hyper-singular integral equation of Section 7.3 on the slit domain* $\Gamma := (-1, 1) \times \{0\}$. *Uniform mesh-refinement leads to suboptimal convergence* $\mathcal{O}(N^{-1/2})$ *for both* $p = 0$ *and* $p = 1$*, whereas the adaptive strategy recovers the optimal convergence* $\mathcal{O}(N^{-(3/2+p)})$.

with the right-hand side $\phi = 1$ and the exact solution $g(x, 0) = 2\sqrt{1 - x^2}$. Since $\Pi_\ell \phi = \phi$, the discrete formulation reads

$$\langle\!\langle G_\ell \,,\, V_\ell \rangle\!\rangle_W = \langle \Pi_\ell \phi \,,\, V_\ell \rangle = \langle \phi \,,\, V_\ell \rangle, \qquad \text{for all } V_\ell \in \mathcal{S}_0^{p+1}(\mathcal{T}_\ell),$$

where $\mathcal{S}_0^{p+1}(\mathcal{T}_\ell) = \mathcal{S}^{p+1}(\mathcal{T}_\ell) \cap \widetilde{H}^{1/2}(\Gamma)$. Moreover, the oscillation term vanishes in (3.25). Thus, the local error indicators, which are used to steer Algorithm 4.1, read

$$\rho_\ell(T) = \| h_\ell^{1/2} (W G_\ell - \phi) \|_{L^2(T)}, \qquad \text{for all } T \in \mathcal{T}_\ell.$$

The Galerkin orthogonality allows one to compute the error in the energy norm by

$$|\!|\!| g - G_\ell |\!|\!|_W^2 = |\!|\!| g |\!|\!|_W^2 - |\!|\!| G_\ell |\!|\!|_W^2 = \pi - |\!|\!| G_\ell |\!|\!|_W^2 =: \mathrm{err}_\ell^2.$$

We stress that $g \in \widetilde{H}^{1/2}(\Gamma) \cap H^{1-\varepsilon}(\Gamma)$, for all $\varepsilon > 0$, but $g \notin H^1(\Gamma)$. Theory predicts a convergence rate $\mathrm{err}_\ell = \mathcal{O}(N_\ell^{-s})$ with $s = 1/2$ for uniform mesh-refinement. This is confirmed by the numerical experiments for both $p = 0, 1$; see Figure 7.6. In contrast to that, the adaptive strategy from Algorithm 4.1 with $\theta = 0.25$ regains the optimal order of convergence $s = 3/2 + p$ in either case $p = 0, 1$.

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