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ON THE COMPUTATION OF THE DISTANCE TO QUADRATIC MATRIX POLYNOMIALS THAT ARE SINGULAR AT SOME POINTS ON THE UNIT CIRCLE*

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Dedicated to Lothar Reichel on the occasion of his 60th birthday

Abstract. For a quadratic matrix polynomial, the distance to the set of quadratic matrix polynomials which have singularities on the unit circle is computed using a bisection-based algorithm. The success of the algorithm depends on the eigenvalue method used within the bisection to detect the eigenvalues near the unit circle. To this end, the QZ algorithm along with the Laub trick is employed to compute the anti-triangular Schur form of a matrix resulting from a palindromic reduction of the quadratic matrix polynomial. It is shown that despite rounding errors, the Laub trick followed, if necessary, by a simple refinement procedure makes the results reliable for the intended purpose. Several numerical illustrations are reported.

Key words. distance to instability, quadratic matrix polynomial, palindromic pencil, QZ algorithm, Laub trick

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1. Introduction.

Robust stability of dynamic systems is often measured by the distance to instability, or stability radius, which is equal to the norm of the smallest perturbation under which the perturbed system loses its stability. For a continuous-time system $dx/dt = Ax$ with a square complex matrix A, the distance to instability (see [11, 12, 13, 26]) is given by

$$d_c(A) = \min_{\omega \in \mathbb{R}} \sigma_{\min}(i\omega I - A),$$

where $i = \sqrt{-1}$, I is the identity matrix, and $\sigma_{\min}$ denotes the smallest singular value of a matrix. R. Byers [5] and other authors (see, e.g., [1, 2, 3, 4, 21]) have exploited the remarkable fact that σ is a singular value of $i\omega I - A$ for some $\omega \in \mathbb{R}$ if and only if $i\omega$ is an eigenvalue of the Hamiltonian matrix

$$H(\sigma) = \begin{bmatrix} A & -\sigma I \\ \sigma I & -A^* \end{bmatrix},$$

where A^* denotes the conjugate transpose of A. This means that the imaginary eigenvalues of $H(\sigma)$ determine the σ-level set of the multivalued function

$$i\mathbb{R} \ni i\omega \mapsto \text{singular spectrum of } (i\omega I - A).$$

Note that $H(\sigma)$ has no eigenvalues on the imaginary axis if and only if $|\sigma| < d_c$.

When investigating the discrete-time stability of systems $x_{k+1} = Ax_k$, the distance to instability is determined by

$$d_d(A) = \min_{\omega \in \mathbb{R}} \sigma_{\min}(e^{i\omega} I - A),$$

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and the eigenvalues on the unit circle $e^{i\mathbb{R}} = \{\lambda \in \mathbb{C}\colon |\lambda| = 1\}$ of the linear symplectic matrix pencil

$$\lambda\mathcal{B}(\sigma) - \mathcal{A}(\sigma) = \lambda \begin{bmatrix} I & \sigma I \\ 0 & A^* \end{bmatrix} - \begin{bmatrix} A & 0 \\ \sigma I & I \end{bmatrix}$$

determine the σ-level set of the multivalued function

$$e^{i\mathbb{R}} \ni e^{i\omega} \mapsto \text{singular spectrum of } (e^{i\omega} I - A).$$

More general formulations of the level set approach described above can be found in [4, 8, 13]. The common feature of various variants of the level set approach is a reformulation of the initial problem to one that requires the decision whether a matrix or a matrix pencil has an eigenvalue on the imaginary axis or the unit circle. It is important to find out how this decision can be made reliable in spite of inaccuracies in the computed eigenvalues caused by roundoff errors. In the paper [5], it has been demonstrated that such a reliability can be achieved by the bisection method of [5] coupled with the structure-preserving methods such as those discussed in [14, 15, 22, 23, 24].

Below we deal with the distance to instability for the second-order discrete-time system

$$A_0 x_k + A_1 x_{k+1} + A_2 x_{k+2} = 0, \tag{1.1}$$

where $A_0, A_1, A_2 \in \mathbb{C}^{m\times m}$. When the system (1.1) is stable, that is, when all eigenvalues of the quadratic polynomial

$$Q(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 \tag{1.2}$$

are located in the open unit disk, the distance to instability (also called the complex stability radius) is given by

$$\begin{aligned} d := d(Q) = \min \Big\{ \|\Delta\|_2 \,\Big|\, &\exists \lambda \in \mathbb{C} \text{ such that} \\ &\det(A_0 + \Delta + \lambda A_1 + \lambda^2 A_2) = 0 \text{ and } |\lambda| \geq 1 \Big\}. \end{aligned} \tag{1.3}$$

Formula (1.3) gives the size of the smallest perturbation of the coefficients A_0, A_1, and A_2 that places an eigenvalue of the perturbed polynomial $Q(\lambda)$ on the unit circle. It corresponds to the distance of the matrix polynomial (1.2) to the set of unstable quadratic matrix polynomials. For matrices and matrix polynomials, this notion is important in control theory and other engineering applications; see, e.g., [7, 8, 10, 11, 12, 13, 20, 25, 26].

Formula (1.3) has, in fact, a wider meaning: for an arbitrary quadratic matrix polynomial (1.2), it represents the distance to the set of quadratic matrix polynomials which have singularities on the unit circle, i.e.,

$$d = \min_{\omega\in\mathbb{R}} \sigma_{\min}(Q(e^{i\omega})). \tag{1.4}$$

The present work extends the investigation on the estimation of the distance d started in [17]. We assume some familiarity with the results of that paper. While the main result of [17] is a proof of the fact that the structure-preserving methods can provide reliable lower bounds for the distance to a contour, the present paper justifies the use of the so-called Laub trick as a structure-preserving method and recommends a deflation in addition to the Laub trick. It also introduces an indicator function $\chi(\sigma)$ which suitably characterizes the distance of the eigenvalues to the unit circle.

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The outline of this paper is as follows: in Section 2 the distance problem is recast as a palindromic eigenvalue problem having or having not an eigenvalue on the unit circle. Section 3 introduces and justifies the use of the indicator function $\chi(\sigma)$. Section 4 studies the Laub trick [18, 24], which is used to compute the anti-triangular Schur form of a matrix using the standard QZ algorithm [19]. It is shown that this transformation can be done reliably despite rounding errors. Section 5 summarizes the algorithms including a deflation procedure which refines the anti-triangular Schur form to decide whether the computed eigenvalues are on the unit circle and consequently estimates the sought distance using a bisection method. Comparisons with the MATLAB optimization function fminbnd and other numerical illustrations are presented in Section 6. Concluding remarks are given in Section 7.

2. Reduction to a palindromic linear matrix pencil. Recent advances in eigenvalue problems with palindromic structure motivated us to transform the distance eigenvalue problem (1.4) as follows.

First note that

$$\sigma_{\text{up}} = \min\left\{\sigma_{\min}(A_0 + A_1 + A_2), \sigma_{\min}(A_0 - A_1 + A_2)\right\} \tag{2.1}$$

is a rough upper bound for d. For each $\sigma \in [d, \sigma_{\text{up}}]$, there exist a suitable $\lambda \in \mathbb{C}$ on the unit circle $|\lambda| = 1$ and singular vectors u and v such that

$$\left(A_0 + \lambda A_1 + \lambda^2 A_2\right) u = \sigma v \qquad \text{and} \qquad \left(A_0^* + \bar{\lambda} A_1^* + \bar{\lambda}^2 A_2^*\right) v = \sigma u.$$

The equivalent equalities $\left(A_0 + \lambda A_1 + \lambda^2 A_2\right) \lambda u = \lambda\sigma v$ and $\left(\lambda^2 A_0^* + \lambda A_1^* + A_2^*\right) v = \lambda^2 \sigma u$ can be gathered as

$$\begin{bmatrix} 0 & A_2^* + \lambda A_1^* + \lambda^2 A_0^* \\ A_0 + \lambda A_1 + \lambda^2 A_2 & 0 \end{bmatrix} \begin{bmatrix} \lambda u \\ v \end{bmatrix} = \lambda \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} \lambda u \\ v \end{bmatrix}.$$

Hence,

$$\left[\begin{bmatrix} 0 & A_2^* \\ A_0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} -\sigma I & A_1^* \\ A_1 & -\sigma I \end{bmatrix} + \lambda^2 \begin{bmatrix} 0 & A_0^* \\ A_2 & 0 \end{bmatrix}\right] \begin{bmatrix} \lambda u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Denoting

$$\mathcal{A}_k = \begin{bmatrix} 0 & A_{2-k}^* \\ A_k & 0 \end{bmatrix}, \quad k = 0, 1, 2, \quad \text{and} \quad w = \begin{bmatrix} \lambda u \\ v \end{bmatrix},$$

we arrive at the eigenvalue problem $\mathcal{P}(\lambda) w = 0$, where $\mathcal{P}(\lambda)$ is the quadratic matrix polynomial

$$\mathcal{P}(\lambda) = \mathcal{A}_0 + \lambda\left(\mathcal{A}_1 - \sigma I\right) + \lambda^2 \mathcal{A}_2,$$

which depends on the parameter σ. Note that $\mathcal{P}(\lambda)$ is palindromic because $\mathcal{A}_1 = \mathcal{A}_1^*$ and $\mathcal{A}_2 = \mathcal{A}_0^*$. As a consequence, its spectrum is symmetric with respect to the unit circle.

The distance d defines the partition $[0, \sigma_{\text{up}}] = [0, d) \cup [d, \sigma_{\text{up}}]$ such that $\mathcal{P}(\lambda)$ has a singularity on the unit circle if $\sigma \in [d, \sigma_{\text{up}}]$ ($\sigma < d$). We continue with a transformation of $\mathcal{P}(\lambda)$ into a linear pencil of double size which preserves the palindromic structure:

$$X + \lambda X^* \quad \text{with} \quad X = \begin{bmatrix} \mathcal{A}_0 & \mathcal{A}_1 - \sigma I \\ 0 & \mathcal{A}_0 \end{bmatrix}. \tag{2.2}$$

To avoid cumbersome notation, we will not exhibit the dependence of X on σ. The transformation (2.2) will be referred to as "Toeplitz reduction". Note that the equality

$$X + \lambda^2 X^* = \begin{bmatrix} I & 0 \\ \lambda I & I \end{bmatrix} \begin{bmatrix} \mathcal{P}(\lambda) & \mathcal{A}_1 - \sigma I \\ 0 & \mathcal{P}(-\lambda) \end{bmatrix} \begin{bmatrix} I & 0 \\ \lambda I & I \end{bmatrix}^{-1}$$

proves that the eigenvalues of $X + \lambda X^*$ are squares of those of $\mathcal{P}(\lambda)$. Moreover, it was shown in [17] that

$$\min_{\omega\in\mathbb{R}} \sigma_{\min}(X + X^* e^{i\omega}) = \begin{cases} d - \sigma, & \text{when } 0 \le \sigma < d, \\ 0, & \text{when } d \le \sigma. \end{cases} \tag{2.3}$$

3. An indicator function via the generalized Schur form. Let us consider the generalized Schur form of the palindromic pencil $X + \lambda X^*$ computed by the QZ algorithm in floating point arithmetic

$$Q^* \left(X + \Delta_x\right) Z = T_x, \qquad Q^* \left(X^* + \Delta_{x^*}\right) Z = T_{x^*}, \tag{3.1}$$

where T_x and T_{x^*} are upper triangular, Q and Z are unitary, and the backward errors Δ_x and Δ_{x^*} are of sufficiently small norm

$$\delta = \max\left\{\|\Delta_x\|_2, \|\Delta_{x^*}\|_2\right\} = \mathcal{O}\left(\epsilon_{\text{machine}}\right) \|X\|_2 . \tag{3.2}$$

We introduce the indicator function

$$\chi(\sigma) = \min_k \min_{\omega\in\mathbb{R}} \left|(T_x)_{kk} + e^{i\omega}(T_{x^*})_{kk}\right| , \tag{3.3}$$

where $(T)_{kk}$ designates the kth diagonal element of the matrix T. It is obvious that

$$\chi(\sigma) = \min_k \left|\, |(T_x)_{kk}| - |(T_{x^*})_{kk}| \,\right| .$$

PROPOSITION 3.1. *We have*

$$\chi(\sigma) \ge \max(0, d - \sigma - 2\delta).$$

Proof. Since for any triangular matrix T it holds that

$$\sigma_{\min}(T) \le \min_k |(T)_{kk}| ,$$

we have the inequality

$$\sigma_{\min}(T_x + e^{i\omega} T_{x^*}) \le \min_k \left|(T_x)_{kk} + e^{i\omega}(T_{x^*})_{kk}\right| , \qquad \forall \omega \in \mathbb{R}.$$

It follows that

$$\min_{\omega\in\mathbb{R}} \sigma_{\min}(T_x + e^{i\omega} T_{x^*}) \le \min_{\omega\in\mathbb{R}} \min_k \left|(T_x)_{kk} + e^{i\omega}(T_{x^*})_{kk}\right| = \chi(\sigma).$$

Moreover, (3.1) implies that

$$\sigma_{\min}(T_x + e^{i\omega} T_{x^*}) \ge \sigma_{\min}(X + e^{i\omega} X^*) - 2\delta,$$

and therefore

$$\chi(\sigma) \ge \sigma_{\min}(X + e^{i\omega} X^*) - 2\delta.$$

Applying (2.3) we arrive at the desired estimate $\chi(\sigma) \ge \max(0, d - \sigma - 2\delta)$. ∎

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The following practical upper bound

$$d \le \sigma + 2\delta + \chi(\sigma) \tag{3.4}$$

is a corollary of Proposition 3.1. Note that the upper bound (3.4) does not require structure-preserving methods for the computation of the Schur form. The next sections are devoted to reliable lower bounds for d.

4. On the Laub trick. Structure preserving eigenvalue methods especially devised for the palindromic pencil (2.2) are mostly based on the anti-triangular Schur form of a matrix [15, 24]. They include the URV-type methods [22, 24], QR-type methods with the Laub trick [23, 24], and Jacobi-type methods [24]. Another idea based on structured doubling algorithms is pursued in [6]. All these algorithms suffer from the presence of eigenvalues on the unit circle. Nevertheless, we show below that when the pencil $X + \lambda X^*$ has no eigenvalues on the unit circle, the Laub trick followed, if necessary, by a deflation procedure is satisfactory for our purposes.

The Laub trick for Hamiltonian and symplectic matrices is described, e.g., in [18]. The first step in the palindromic version of the Laub trick is based on the QZ algorithm, and the following proposition shows that despite rounding errors, some columns of Q and Z remain almost orthogonal.

THEOREM 4.1. *Assume that the pencil $X + \lambda X^*$ of order $2n$ has no eigenvalues on the unit circle, and consider its computed generalized Schur form (3.1) with a reordering of the eigenvalues in non-decreasing order of magnitude and the backward errors Δ_x and Δ_{x^*} that satisfy (3.2) and $2\delta < d - \sigma$.*

Denote by Z_1 and Q_1 the first n columns of Z and Q and recall that

$$d - \sigma = \min_{|\lambda|=1} \sigma_{\min}\left(X + \lambda X^*\right), \qquad \textit{when } \sigma \le d.$$

Then

$$\|Z_1^* Q_1\|_2 \le \min\left\{\frac{4\delta(\|X\|_2 + \delta)}{(d-\sigma-2\delta)^2}, 1\right\}.$$

Proof. First note that since the pencil $X + \lambda X^*$ has no eigenvalues on the unit circle, it follows that $\sigma < d$; see Section 2. Therefore $0 < d - \sigma = \min_{|\lambda|=1} \sigma_{\min}\left(X + \lambda X^*\right)$.

Let us denote by S and R the $n \times n$ upper triangular matrices formed by the first n rows and columns of T_x and T_{x^*}, respectively. Since the pencil $X + \lambda X^*$ is palindromic and the eigenvalues are arranged in non-decreasing order of magnitude, the eigenvalues of $S + \lambda R$ lie in the open unit disk, and, in particular, R is nonsingular. Moreover, from (3.1) we obtain

$$\left\|(X + \lambda X^* + \Delta_x + \lambda\Delta_{x^*})^{-1}\right\|_2 = \left\|(T_x + \lambda T_{x^*})^{-1}\right\|_2 \ge \left\|(S + \lambda R)^{-1}\right\|_2$$

and hence,

$$\begin{aligned}\max_{|\lambda|=1}\left\|(S+\lambda R)^{-1}\right\|_2 &\le \frac{\|\left(X+\lambda X^*\right)^{-1}\|_2}{1 - \|\left(X+\lambda X^*\right)^{-1}\|_2 \, \|\Delta_x + \lambda\Delta_{x^*}\|_2} \\ &\le \frac{1}{d - \sigma - 2\delta}.\end{aligned}$$

Also, from (3.1) we have

$$XZ_1 = Q_1 S - \Delta_x Z_1, \qquad X^* Z_1 = Q_1 R - \Delta_{x^*} Z_1,$$

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and a premultiplication on the left by Z_1^* gives

$$(Z_1^*Q_1)S - R^*(Q_1^*Z_1) = \Delta, \tag{4.1}$$
$$S^*(Q_1^*Z_1) - (Z_1^*Q_1)R = \Delta^*, \tag{4.2}$$

with $\Delta = Z_1^*\Delta_x Z_1 - Z_1^*(\Delta_{x^*})^* Z_1$. Note that $\|\Delta\|_2 = \|\Delta^*\|_2 \leq 2\delta$ and that equation (4.2) is simply the conjugate transpose of (4.1).

To eliminate the matrix $Q_1^*Z_1$ from (4.1) and (4.2), we multiply (4.1) from the left by $S^*(R^{-1})^*$ and from the right by R^{-1}, multiply (4.2) from the right by R^{-1}, and add the resulting equations. This leads to the following matrix equation for $Z_1^*Q_1$:

$$Z_1^*Q_1 - (R^{-1}S)^*\,(Z_1^*Q_1)\,SR^{-1} = -\left[(R^{-1}S)^*\Delta + \Delta^*\right]R^{-1}. \tag{4.3}$$

Since the eigenvalues of $(R^{-1}S)^*$ and SR^{-1} lie in the open unit disk, the unique solution of (4.3) is given by (see, e.g., [9])

$$Z_1^*Q_1 = \frac{-1}{2\pi}\int_0^{2\pi}\left((R^{-1}S)^* - e^{-i\theta}I\right)^{-1}\left((R^{-1}S)^*\Delta + \Delta^*\right)R^{-1}\left(SR^{-1} - e^{i\theta}I\right)^{-1}d\theta,$$

which simplifies to

$$Z_1^*Q_1 = -S^*Y - R^*Y^*, \qquad \text{where}$$
$$Y = \frac{1}{2\pi}\int_0^{2\pi}\left(S^* - e^{-i\theta}R^*\right)^{-1}\Delta\left(S - e^{i\theta}R\right)^{-1}d\theta.$$

The proof follows by taking the norm and noting that $\|R^*\| \leq \|X\|_2 + \delta$, $\|S^*\| \leq \|X\|_2 + \delta$, $\|Y\|_2 \leq 2\delta \cdot \max_\theta \|\left(S - e^{i\theta}R\right)^{-1}\|_2^2$, and $\max_\theta \|\left(S - e^{i\theta}R\right)^{-1}\|_2 \leq \frac{1}{d-\sigma-2\delta}$. $\square$

Using the notation of Theorem 4.1, let $U = [Z_1, Q_1 J]$, where J is the anti-diagonal unit matrix of order n. Then

$$U^*XU = T + \Delta_1, \tag{4.4}$$
$$U^*U = I + \Delta_2, \tag{4.5}$$

with

$$T = \begin{bmatrix} 0 & R^*J \\ JS & J(Q_1^*XQ_1)J \end{bmatrix}, \tag{4.6}$$
$$\Delta_1 = \begin{bmatrix} (Z_1^*Q_1)S - Z_1^*\Delta_x Z_1 & -Z_1^*(\Delta_{x^*})^*Q_1 J \\ -JQ_1^*\Delta_x Z_1 & 0 \end{bmatrix},$$
$$\Delta_2 = \Delta_2^* = \begin{bmatrix} 0 & Z_1^*Q_1 J \\ JQ_1^*Z_1 & 0 \end{bmatrix}.$$

Note that R^*J and JS are lower anti-triangular and that

$$\begin{aligned}\|\Delta_1\| &\leq \|Z_1^*Q_1\|_2\|S\|_2 + 2\max(\|\Delta_x\|_2, \|\Delta_{x^*}\|_2) \\ &\leq \frac{4\delta(\|X\|_2+\delta)^2}{(d-\sigma-2\delta)^2} + 2\delta,\end{aligned} \tag{4.7}$$

$$\|\Delta_2\|_2 = \|Z_1^*Q_1\|_2 \leq \frac{4\delta(\|X\|_2+\delta)}{(d-\sigma-2\delta)^2}. \tag{4.8}$$

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It follows that the matrix U is close to being unitary and U^*XU is close to being lower block anti-triangular provided that $\|X\|_2/(d-\sigma)$ is not large.

In accordance with (3.4) we have, approximately, the bound $\sigma \geq d$ if $\chi(\sigma)$ is small. We expect that $\sigma < d$ holds approximately when $\chi(\sigma)$ is not small. The precise lower bound (4.11) is justified as follows.

From (4.5), it is easy to see that the matrix $\hat{U} = U\,(I+\Delta_2)^{-\frac{1}{2}}$ is unitary and that

$$(I+\Delta_2)^{-\frac{1}{2}} = I + \Delta_3, \qquad \text{with} \quad \|\Delta_3\|_2 \leq \frac{\|\Delta_2\|_2}{2(1-\|\Delta_2\|_2)}\,. \tag{4.9}$$

Then (4.4) becomes

$$\hat{U}^* X \hat{U} = (I+\Delta_3)(T+\Delta_1)(I+\Delta_3) = T + \Delta_4, \tag{4.10}$$

with

$$\Delta_4 = \Delta_1 + T\Delta_3 + \Delta_3 T + \Delta_1\Delta_3 + \Delta_3\Delta_1 + \Delta_3 T\Delta_3 + \Delta_3\Delta_1\Delta_3.$$

In view of (4.9), we have

$$\|\Delta_4\|_2 \leq \frac{\|\Delta_1\|_2 + \|\Delta_2\|_2\|T\|_2}{(1-\|\Delta_2\|_2)^2},$$

and from (4.4), (4.5), (4.7), and (4.8), we have at first order in $\|Z_1^* Q_1\|_2$ and δ:

$$\frac{\|\Delta_1\|_2 + \|\Delta_2\|_2\|T\|_2}{(1-\|\Delta_2\|_2)^2} \approx 2\,(\|Z_1^*Q_1\|_2\|X\|_2 + \delta)\,.$$

Now from (4.10), we have

$$\hat{U}^*\left(e^{-i\omega}X + e^{i\omega}X^*\right)\hat{U} = \left(e^{-i\omega}T + e^{i\omega}T^*\right) + \left(e^{-i\omega}\Delta_4 + e^{i\omega}\Delta_4^*\right),$$

and a result on palindromic perturbations of palindromic pencils (see [17, Section 4]) tells us that

$$\sigma \leq d + 2\|\Delta_4\|_2. \tag{4.11}$$

5. Algorithms. The arguments of Section 4 justify the Laub trick. Namely, when the palindromic pencil (2.2) has no eigenvalues on the unit circle, the anti-triangular form can be computed via the QZ algorithm despite rounding errors. However, the presence of eigenvalues near the unit circle makes this computation difficult, and a refinement procedure should be used in this case. The numerical procedures are summarized in Algorithm 1 and Algorithm 2 below.

Algorithm 1 The Laub trick.

Input: $2n \times 2n$ matrix X.
Output: Unitary matrix U, block anti-triangular $T = U^*XU$, vector r of n residual norms.
  Compute a QZ factorization of the pencil $X + \lambda X^*$ with the eigenvalues ordered in non-decreasing magnitude such that $Q^*XZ = T_x$ and $Q^*X^*Z = T_{x^*}$.
  Compose $U = \begin{bmatrix} Z(:,1\!:\!n) & Q(:,n\!:\!-1\!:\!1)\end{bmatrix}$ and orthonormalize the columns of U.
  Set $T = U^*XU$.
  for $k = 1,\dots,n$ **do**
    Set $r_k = \sqrt{\sum_{ij}|T_{ij}|^2}$, where the summation is taken over the indices satisfying $1 \leq j \leq k, 1 \leq i \leq 2n-j$ or $1 \leq i \leq k, 1 \leq j \leq 2n-i$.
  end for

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Algorithm 2 The Laub trick with deflation.

Input: $2n \times 2n$ matrix X and tolerance tol.
Output: Unitary matrix U, block-anti-triangular $T = U^*XU$.
  Call [U,T,r] = laub(X).
  Compute the number k of residuals r, which are less than tol, and set $i = k$.
  while $k > 0$ **do**
    Call [V,T,r] = laub $(T(k+1 : end-k, k+1 : end-k))$.
    $U(:, i+1 : 2n-i) = U(:, i+1 : 2n-i)V$.
    Compute the number k of residuals r, which are less than tol, and set $i = i + k$.
  end while
  $T = U^*XU$.

Algorithm 1 is adopted from [23]. The MATLAB notation $Z(:, 1 : n)$ and $Q(:, n : -1 : 1)$ denotes the first n columns of Z and the first n columns of Q in reverse order. The residual r measures the gap between U^*XU and its lower anti-triangular part. More precisely, the kth component of r contains the Frobenius norm of the first k rows and columns of the strictly upper anti-triangular part of U^*XU. If the pencil $X + \lambda X^*$ has no eigenvalues near the unit circle, then the vector r has small components and U^*XU has the desired lower anti-triangular form. The presence of eigenvalues near the unit circle means that the dominant eigenvalues of SR^{-1} are near the unit circle; see the proof of Theorem 4.1. This translates into a large value of $\|Z_1^*Q_1\|_2$, as formula (4.8) shows. The matrix $Z_1^*Q_1$ has tiny entries in its leading principal part, which correspond to the eigenvalues well separated from the unit circle, and larger entries elsewhere thus causing an increase in the last components of r. In this case, we propose to re-apply Algorithm 1 only to the columns of U which contribute to the increase of the components of r. Such an operation is repeated recursively until the last components of r are small. The resulting matrix U^*XU has a block lower anti-triangular form. The presence of upper diagonal elements is due to the presence of eigenvalues near or on the unit circle. A formal description is given in Algorithm 2.

Algorithm 2 correctly computes the anti-triangular form for the pencil $X + \lambda X^*$ which has no eigenvalues near the unit circle. In the following algorithm, Algorithm 3, it is implicitly combined with a bisection to estimate the distance d.

Algorithm 3 Bisection.

Input: $m \times m$ matrices A_0, A_1, A_2, and a tolerance parameter tol.
Output: α and β such that either $\beta/1.001 \le \alpha \le d \le \beta$ or $0 = \alpha \le d \le \beta \le 1.001\,\mathsf{tol}$.
  $\alpha = 0$, $\beta = \min\{\sigma_{\min}(A_0 + A_1 + A_2), \sigma_{\min}(A_0 - A_1 + A_2)\}$.
  while $\beta > 1.001 \max(\mathsf{tol}, \alpha)$ **do**
    $d = \sqrt{\beta \cdot \max(\mathsf{tol}, \alpha)}$.
    if (2.2) has an eigenvalue on the unit circle **then**
      $\beta = d$,
    else
      $\alpha = d$.
    end if
  end while

Algorithm 3 is written in the style of [5]. It estimates the distance d within a factor of 1.001. The upper bound (2.1) provides a first estimate for d and this bound is then refined by the decision taken on the eigenvalues of (2.2). The problem of computing the eigenvalues of (2.2) is reduced to the one for $T + \lambda T^*$, where T is as defined in (4.6) and computed by Algorithm 2.

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The computed bounds for d must be tuned to include the effect of roundoff errors. Thus, the computed upper bound d_1 should be increased by the value $2\delta + \chi(\sigma)$ with $\sigma = d_1$ as shown in (3.4). Concerning the correction of the computed lower bound d_2, the best way is to compute the matrix $\hat{U}^* X \hat{U}$ from (4.13) for $\sigma = d_2$, then compute the 2-norm of its part above the anti-diagonal. Let us denote this by δ_4. This value δ_4 yields $\|\Delta_4\|_2$ satisfying (4.11). The computed lower bound d_2 should be decreased by the value $2\delta_4$.

6. Numerical tests. We present in this section results of numerical experiments with the method summarized in Algorithm 3, where the anti-triangular form of X is computed by Algorithm 2. In all numerical tests, the parameter tol equals $10^{-14}\left\|\begin{bmatrix}A_0 & A_1 & A_2\end{bmatrix}\right\|_2$. We also show comparisons with the MATLAB function fminbnd, which finds a minimum of the functional

$$\theta \ni [0, 2\pi] \mapsto \sigma_{\min}\left(A_0 e^{-i\theta} + A_1 + A_2 e^{i\theta}\right).$$

EXMAPLE 6.1. Consider the quadratic matrix polynomial $Q(\lambda)$ with coefficients

$$A_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 3.5 & 1 & 1 & 1 & 1 \\ 1 & 3.5 & 1 & 1 & 1 \\ 1 & 1 & 3.5 & 1 & 1 \\ 1 & 1 & 1 & 3.5 & 1 \\ 1 & 1 & 1 & 1 & 3.5 \end{bmatrix}, \quad A_2 = A_0^*.$$

Algorithm 3 yields $\alpha = \beta = 4.246 \times 10^{-2}$. The function fminbnd yields $d = 4.246 \times 10^{-2}$.

Figure 6.1 illustrates the fact that the function χ defined in (3.3) is large in the interval $(0, d)$ and small in the interval $[d, \sigma_{\mathrm{up}}]$; see the discussion at the end of Section 3.

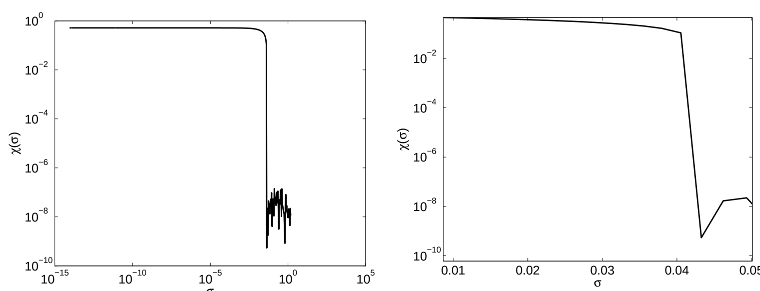

FIG. 6.1. *Behavior of the function $\chi(\sigma)$ for Example 6.1.*

EXMAPLE 6.2. In this test case, the quadratic matrix polynomial is of size 3 and is constructed as follows:

$$A_k = Q_l T_k Q_r, \qquad k = 0, 1, 2,$$

where the elements of Q_l and Q_r are chosen randomly with zero means and standard deviations one, T_k is strictly upper triangular with 1 on its strictly upper triangular part. The diagonal elements of T_2 are all equal to 1, and those of T_0 and T_1 are chosen so that $T_0(k,k) = \rho^2/(1+\epsilon_k)$ and $T_1(k,k) = (\rho^2 + T_0(k,k))/(1+\epsilon_k)$, for $k = 1, 2, 3$, with $\epsilon_1 = 10^{-5}$, $\epsilon_2 = 10^{-4}$, $\epsilon_3 = 10^{-3}$, and ρ is a parameter to be varied. The quadratic matrix polynomial thus constructed has all its eigenvalues inside the circle of center 0 and radius ρ.

Table 6.1 displays results for two different values of ρ. Figure 6.2 shows the behavior of the function χ.

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TABLE 6.1
Results for Example 6.2.

Method	ρ	Estimates of distance
algorithm3	0.9	$[3.38 \times 10^{-7}\ ,\ 3.38 \times 10^{-7}]$
fminbnd	0.9	3.38×10^{-7}
algorithm3	1.001	$[4.62 \times 10^{-13}\ ,\ 1.38 \times 10^{-12}]$
fminbnd	1.001	1.38×10^{-12}

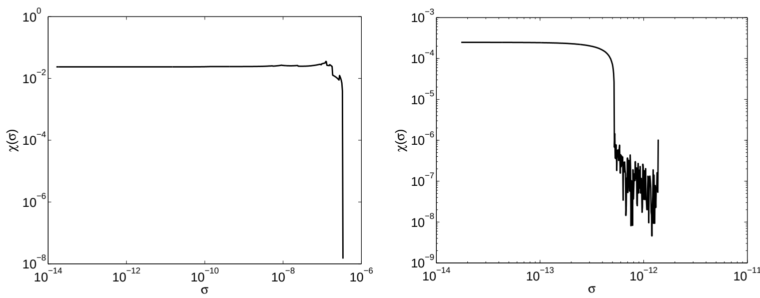

FIG. 6.2. *Behavior of the function* $\chi(\sigma)$ *for Example 6.2. Left:* $\rho = 0.9$*. Right:* $\rho = 1.001$.

TABLE 6.2
Results from Algorithm 2 for Example 6.2.

σ	$\rho = 0.9$		$\rho = 1.001$	
	$iter$	r	$iter$	r
10^{-14}	4	9.82×10^{-14}	5	6.65×10^{-14}
10^{-12}	3	1.48×10^{-13}	0	9.09×10^{-3}
10^{-10}	3	1.28×10^{-13}	4	6.86×10^{-3}
10^{-8}	3	4.66×10^{-14}	0	6.63×10^{-2}
10^{-6}	2	9.33×10^{-2}	1	1.42×10^{-1}
10^{-4}	1	2.24×10^{-1}	1	1.09×10^{-1}
10^{-2}	1	5.03×10^{-1}	1	5.11×10^{-1}

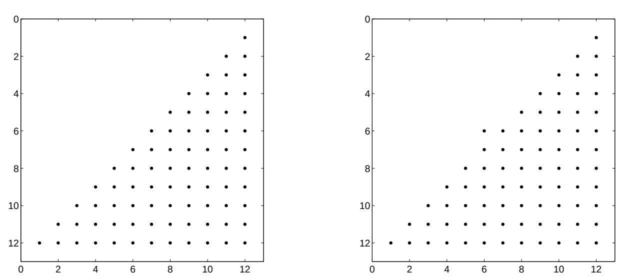

FIG. 6.3. *Antitriangular form of* X *for Example 6.2 with* $\rho = 0.9$*. Left:* $\sigma = 10^{-10}$*. Right:* $\sigma = 10^{-4}$.

Table 6.2 displays information provided by Algorithm 2. In this table, the Frobenius norm of the upper anti-triangular part of $T = U^*XU$ is denoted by r, and the number of refinement steps needed to reduce X to a block anti-triangular form is denoted by $iter$. Small (large) values of r indicate that $T = U^*XU$ is reduced to an anti-triangular (block-anti-triangular) form. An illustration is given in Figure 6.3. In the latter case, the quadratic pencil $\mathcal{P}(\lambda)$ has an eigenvalue near or on the unit circle.

7. Concluding remarks. The tests examples presented in the previous section and several numerical tests not reported here have shown that the bisection method described in Algorithm 3 often gives very good estimates of the distance. At the heart of this algorithm are the QZ algorithm, the Laub trick, and a refinement that enhances the reduction to (block) anti-triangular form. The resulting algorithm takes into account to some extent the palindromic structure and benefits from the error analysis for the palindromic reduction (2.2) developed in [17]. Variants of Algorithm 3 have been tested where, instead of Algorithm 2, the QZ method and methods developed in [6, 15, 16] were used to compute the eigenvalues of (2.2). With a few exceptions, these methods delivered results comparable to those given by the proposed method. However, they are either unstructured and/or computationally expensive or lack a stability analysis. The MATLAB method `fminbnd` has the advantage of being fast but may stagnate in a local minimum.

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