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α-FRACTAL RATIONAL SPLINES FOR CONSTRAINED INTERPOLATION*

PUTHAN VEEDU VISWANATHAN† AND ARYA KUMAR BEDABRATA CHAND†

Abstract. This article is devoted to the development of a constructive approach to constrained interpolation problems from a fractal perspective. A general construction of an α-fractal function $s^\alpha \in \mathcal{C}^p$, the space of all p-times continuously differentiable functions, by a fractal perturbation of a traditional function $s \in \mathcal{C}^p$ using a finite sequence of base functions is introduced. The construction of smooth α-fractal functions described here allows us to embed shape parameters within the structure of differentiable fractal functions. As a consequence, it provides a unified approach to the fractal generalization of various traditional non-recursive rational splines studied in the field of shape preserving interpolation. In particular, we introduce a class of α-fractal rational cubic splines $s^\alpha \in \mathcal{C}^1$ and investigate its shape preserving aspects. It is shown that s^α converges to the original function $\Phi \in \mathcal{C}^2$ with respect to the $\mathcal{C}^1$-norm provided that a suitable mild condition is imposed on the scaling vector α. Besides adding a layer of flexibility, the constructed smooth α-fractal rational spline outperforms its classical non-recursive counterpart in approximating functions with derivatives of varying irregularity. Numerical examples are presented to demonstrate the practical importance of the shape preserving α-fractal rational cubic splines.

Key words. iterated function system, α-fractal function, rational cubic spline, convergence, convexity, monotonicity, positivity

AMS subject classifications. 28A80, 26A48, 26A51, 65D07, 41A20, 41A29, 41A05

1. Introduction. Fractal interpolation, a subject championed by Barnsley [1], is a new technique which has proven to be advantageous over traditional interpolation methods. The traditional interpolants such as polynomial, rational, trigonometric, and spline functions are always smooth or piecewise smooth. Fractal Interpolation Functions (FIFs) defined via a suitable Iterated Function System (IFS) possess the novelty of providing one of the very few methods that produce non-differentiable interpolants. Non-smooth FIFs are well suited for deterministic representations of complex real-world phenomena such as economic time series, weather data, bioelectric recordings, etc. Barnsley and Harrington [2] observed that FIFs are closed under the operation of integration and subsequently developed the calculus of fractal functions. Thus, these authors have initiated the construction of smooth FIFs and unfolded a striking relationship between the theory of fractal functions and splines. Overall, a FIF offers the flexibility of choosing either a smooth or a non-smooth approximant. Smooth FIFs can be utilized to generalize the classical interpolation and approximation techniques; see, for instance, [4, 5, 6, 8, 25, 26, 27, 28]. Furthermore, if experimental data are approximated by a $\mathcal{C}^p$-FIF f, then the fractal dimension of the graph of $f^{(p)}$ can be aptly used as an index for analyzing the underlying physical process.

Consequently, traditional interpolation theory and fractal theory together yield many possible approaches for interpolating given data by means of smooth functions. Unfortunately, there is no consensus on a “best” interpolant from the wealth of various possibilities. However, there are several desirable properties such as smoothness, approximation order, locality, fairness, and preservation of the inherent shape that are often expected from interpolants. By focusing on these properties and trade-offs between them, we may narrow down our search for a good interpolant. The problem of reproducing the qualitative properties inherent in the data not only eliminates some interpolants from consideration but also provides a realistic model for the intended physical situation. The subfield of interpolation/approximation

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†Department of Mathematics, Indian Institute of Technology Madras, Chennai-600036, India. (amritaviswa@gmail.com, chand@iitm.ac.in).

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wherein one deals with the problem of finding an interpolant s for which $s^{(k)}$ is nonnegative for some $k \in \mathbb{N} \cup \{0\}$, whenever $f^{(k)}$ is nonnegative for the data generating function f, is generally referred to as *shape preserving interpolation* or *isogeometric interpolation*. For $k = 0, 1,$ and 2, the problem reduces to preserving nonnegativity, monotonicity, and convexity, respectively.

Due to the everlasting demands from engineering, industrial, and scientific problems, the construction of shape preserving smooth interpolants is one of the major research areas of approximation theory and of computer aided design. There is a large body of literature devoted to shape preserving interpolation with traditional non-recursive interpolants; see, for instance, [3, 9, 10, 18, 30] and the references therein. Among various non-recursive shape preserving interpolants, the rational splines with shape or tension parameters are extensively used due to their simplicity and flexibility [11, 12, 19, 20]. However, many of these traditional shape preserving interpolation methods require the data to be generated from a continuous function which has derivatives of all orders except perhaps at a finite number of points in the interpolating interval. Consequently, these methods are less satisfactory for preserving the shape of given Hermite data wherein the variables representing the derivatives are modeled using functions of varying irregularity (from smooth to nowhere differentiable). Such data arise naturally and abundantly in nonlinear control systems (e.g., a pendulum-cart system) and in some fluid dynamics problems (e.g., the motion of a falling sphere in a non-Newtonian fluid) [21, 32]. Recursive subdivision schemes can produce shape preserving interpolants with fractality in the derivative of the interpolant. However, a quantification of the fractality of the derivative in terms of the parameters involved in the scheme is unavailable.

From an application's point of view, the development of shape preserving $\mathcal{C}^p$-FIFs is beneficial due to the following reasons: (i) they can recapture the traditional non-recursive shape preserving interpolants for suitable values of the IFS parameters, (ii) they provide shape properties of the interpolant and fractality of the derivatives, and (iii) the fractality can be controlled through the free parameters (scaling factors) of the IFS and can be quantified in terms of the fractal dimension allowing to compare and discriminate the experimental processes. On the other hand, the theoretical importance of developing shape preserving fractal functions lies in the fact that shape preserving interpolation and fractal interpolation are two methodologies that are evolving independently and in parallel, and hence there is a need to bridge this gap for one to benefit from the other. At the outset, we admit that due to the implicit and recursive nature of the fractal function, developing shape preserving polynomial FIFs will be more challenging than that of their classical counterparts. For an initial easy and elegant exposition of fractal interpolation techniques to shape preservation theory, rational FIFs with shape parameters act as a suitable vehicle.

For constructing smooth FIFs, we need to find an IFS satisfying the hypotheses of the Barnsley and Harrington theorem [2]. This may be difficult in some cases, especially when some specific boundary conditions are required. Based on the construction of $\mathcal{C}^0$-FIFs through a "base function" [1] and the Barnsley and Harrington theorem, Navascués and Sebastián [28] described a method for the construction of $\mathcal{C}^p$-FIFs, specifically polynomial FIFs. However, this single base function method is not suitable for the development of smooth rational FIFs with shape parameters. In Section 3.1, we generalize the construction of $\mathcal{C}^p$-FIFs using an α-fractal function technique with the help of a finite sequence of "base functions" in contrast to a single base function adopted in [1, 28]. Our present approach to the construction settles the issue of incorporating shape parameters into the structure of a fractal spline. Consequently, the construction of $\mathcal{C}^p$-continuous α-fractal splines enunciated in this article heralds a unified approach to the definition of fractal generalizations of various non-recursive shape preserving rational splines; see, for instance, [12, 19, 29, 31, 33]. Recently, the authors

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have investigated fractal versions of some of these rational splines using a constructive approach, thereby initiating the study of shape preserving fractal interpolation [7, 34, 35]. Note that the present approach is more general providing a common medium for these rational fractal splines and many more.

In Section 3.2, we particularize our construction to obtain an α-fractal function $s^\alpha \in \mathcal{C}^1$ corresponding to the traditional rational cubic spline s studied in detail in [31]. Our predilection to the choice of rational splines with linear denominator as an illustration for the process of generalizing the traditional shape preserving rational splines is attributed to the reasons of computational economy. Further, from the point of view of the magnitude of the optimal error coefficient, the spline with linear denominator can better approximate the function being interpolated than the rational interpolation with quadratic or cubic denominator [15]. A detailed study of the approximation property of the constructed α-fractal rational cubic spline when applied to the approximation of a function in class $\mathcal{C}^2$ is broached in Section 4. In Section 5.1, the constructed α-fractal rational cubic spline is further investigated and suitable conditions on the parameters are developed to preserve the convexity property of the given data. It is observed that, in general, it may not be possible to get a monotone fractal curve using the developed α-fractal rational cubic spline interpolation scheme unless the derivative parameters are chosen to satisfy some suitable conditions in addition to the necessary monotone conditions. Whence, our approach generalizes and corrects the monotonicity result quoted in [31]. Section 6 provides test examples where we compare the plots obtained by the proposed α-fractal rational cubic spline and its classical counterpart; the result is encouraging for the fractal spline class treated herein. We conclude the paper with some remarks and possible extensions in Section 7.

2. FIFs and α-fractal functions. In this section, we recall the concepts of a FIF and α-fractal functions, which are needed in the sequel. For a complete and rigorous treatment, we may refer the reader to [1, 2].

Let $\Delta := \{x_1, x_2, \dots, x_N\}$ be a partition of the real compact interval $I = [x_1, x_N]$ satisfying $x_1 < x_2 < \cdots < x_N$. Let a set of data points

$$\{(x_n, y_n) \in I \times \mathbb{R} : \ n = 1, 2, \dots, N\}$$

be given. For $n \in J = \{1, 2, \dots, N-1\}$, set $I_n = [x_n, x_{n+1}]$, and let $L_n : I \to I_n$ be affine maps defined by

$$L_n(x) = a_n x + c_n, \quad L_n(x_1) = x_n, \quad L_n(x_N) = x_{n+1}. \tag{2.1}$$

Let D be a large enough compact subset of $\mathbb{R}$. For $n \in J$, let $-1 < \alpha_n < 1$, and define $N-1$ continuous mappings $F_n : I \times D \to D$ such that

$$|F_n(x, y) - F_n(x, y^*)| \le |\alpha_n||y - y^*|, \quad F_n(x_1, y_1) = y_n, \quad F_n(x_N, y_N) = y_{n+1}. \tag{2.2}$$

Define functions $w_n : I \times D \to I \times D$ such that $w_n(x, y) = \big(L_n(x), F_n(x, y)\big)$, for all $n \in J$.

THEOREM 2.1 (Theorem 1, Barnsley [1]). *The Iterated Function System (IFS) $\mathcal{I} = \{I \times D, w_n : n \in J\}$ defined above admits a unique attractor G. Furthermore, G is the graph of a continuous function $f : I \to \mathbb{R}$ which obeys $f(x_n) = y_n$, $n = 1, 2, \dots, N$.*

The previous function is called a FIF corresponding to the IFS $\mathcal{I}$. Let the set $\mathcal{G} := \{f \in \mathcal{C}(I) \,|\, f(x_1) = y_1 \text{ and } f(x_N) = y_N\}$ be endowed with the uniform metric $d(f, g) = \max\{|f(x) - g(x)| : x \in I\}$. The IFS $\mathcal{I}$ induces an operator such that $T : \mathcal{G} \to \mathcal{G}$, $Tf(x) := F_n\big(L_n^{-1}(x), f \circ L_n^{-1}(x)\big)$, $x \in I_n, n \in J$. Note that T is a contraction on the complete metric space $(\mathcal{G}, d)$. Consequently, T possesses a unique fixed point on $\mathcal{G}$, i.e., there

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exists a unique $f \in \mathcal{G}$ such that $Tf(x) = f(x)$ for all $x \in I$. The function f turns out to be the FIF corresponding to $\mathcal{I}$ and it satisfies the functional equation

$$f(x) = F_n\big(L_n^{-1}(x), f \circ L_n^{-1}(x)\big), \quad \forall x \in I_n.$$

The FIFs that received extensive attention in the literature stem from the following IFS

$$w_n(x,y) = \big(L_n(x), F_n(x,y)\big), \quad L_n(x) = a_n x + c_n, \quad F_n(x,y) = \alpha_n y + q_n(x),$$

where q_n, $n \in J$, are suitably chosen continuous functions, commonly polynomials, that satisfy (2.2). The constant α_n is called a scaling factor of the transformation w_n, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1})$ is the scale vector of the IFS. Given $s \in \mathcal{C}(I)$, Barnsley [1] has constructed a function $q_n(x) = s \circ L_n(x) - \alpha_n b(x)$, where $s \not\equiv b \in \mathcal{C}(I)$ and where b satisfies $b(x_1) = s(x_1)$ and $b(x_N) = s(x_N)$. The corresponding FIF s^α obeys

$$s^\alpha(x) = s(x) + \alpha_n (s^\alpha - b) \circ L_n^{-1}(x), \quad \forall\, x \in I_n.$$

The graph $G(s^\alpha)$ of the function s^α is a union of transformed copies of itself, i.e., $G(s^\alpha) = \bigcup\limits_{n \in J} w_n(G(s^\alpha))$, and may have noninteger Hausdorff and Minkowski dimensions. Therefore, the function s^α can be treated as a "fractal perturbation" of s obtained via a *base function* b.

3. A general method for the construction of $\mathcal{C}^p$-continuous α-fractal functions. As mentioned in the introductory section, we observe that for the construction of smooth FIFs in the field of shape preserving interpolation, it is advantageous to define an α-fractal function s^α by perturbing a given continuous function s with the help of a finite sequence of base functions

$$B = \{b_n \in \mathcal{C}(I) \,|\, b_n(x_1) = s(x_1), b_n(x_N) = s(x_N), b_n \not\equiv s, n \in J\}$$

instead of a single base function b. That is, in the first place, we consider

$$q_n(x) = s \circ L_n(x) - \alpha_n b_n(x),$$

and the IFS

$$L_n(x) = a_n x + c_n, \;\; F_n(x,y) = \alpha_n y + s \circ L_n(x) - \alpha_n b_n(x), \quad x \in I, \; n \in J.$$

The corresponding α-fractal function $s^\alpha_{\Delta,B} = s^\alpha$ satisfies the functional equation

$$s^\alpha(x) = s(x) + \alpha_n (s^\alpha - b_n) \circ L_n^{-1}(x), \quad \forall\, x \in I_n. \tag{3.1}$$

Now we make the following definition which is reminiscent of the definition of α-fractal functions generated via a single base function; see [25, 26].

DEFINITION 3.1. *Let $\Delta := \{x_1, x_2, \dots, x_N\}$ be a partition of the interval $I = [x_1, x_N]$ such that $x_1 < x_2 < \cdots < x_N$ and $\alpha \in (-1,1)^{N-1}$ be a scale vector. The continuous function $s^\alpha_{\Delta,B} = s^\alpha$ defined in (3.1) is called an α-fractal function associated with s with respect to the partition Δ and the family B.*

3.1. Smooth α-fractal functions. Here we look for conditions to be satisfied by the functions in B and the scale vector α such that the α-fractal function s^α associated with s preserves the p-smoothness of s. To this end, at first we recall the following theorem that establishes the existence of differentiable FIFs (fractal splines).

THEOREM 3.2 (Theorem 2, Barnsley and Harrington [2]). *Let $I = [x_1, x_N]$ and $x_1 < x_2 < \cdots < x_N$ be a partition of I. Let $L_n(x) = a_n x + c_n$, $n \in J$, be affine maps satisfying (2.1), and let $F_n(x, y) = \alpha_n y + q_n(x)$, $n \in J$, satisfy (2.2). Suppose that for some integer $p \geq 0$, we have that $|\alpha_n| \leq \kappa a_n^p$, where $0 \leq \kappa < 1$ and $q_n \in \mathcal{C}^p(I)$, for all $n \in J$. Let*

$$F_{n,r}(x, y) = \frac{\alpha_n y + q_n^{(r)}(x)}{a_n^r}, \quad y_{1,r} = \frac{q_1^{(r)}(x_1)}{a_1^r - \alpha_1}, \quad y_{N,r} = \frac{q_{N-1}^{(r)}(x_N)}{a_{N-1}^r - \alpha_{N-1}}, \quad r = 1, 2, \ldots, p.$$

If $F_{n-1,r}(x_N, y_{N,r}) = F_{n,r}(x_1, y_{1,r})$ for $n = 2, 3, \ldots, N-1$ and $r = 1, 2, \ldots, p$, then the IFS $\{I \times \mathbb{R}, (L_n(x), F_n(x, y)) : n \in J\}$ determines a FIF $f \in \mathcal{C}^p(I)$, and $f^{(r)}$ is the FIF determined by $\{I \times \mathbb{R}, (L_n(x), F_{n,r}(x, y)) : n \in J\}$ for $r = 1, 2, \ldots, p$.

Let $s \in \mathcal{C}^p(I)$. In view of the previous theorem, we assume $|\alpha_n| \leq \kappa a_n^p$ for all $n \in J$ and for some $0 \leq \kappa < 1$. Our strategy is to impose conditions on the family of functions $B = \{b_n : n \in J\}$ such that the maps $F_n(x, y) = \alpha_n y + q_n(x) = \alpha_n y + s \circ L_n(x) - \alpha_n b_n(x)$, $n \in J$, satisfy the hypotheses of this theorem. The argument is patterned after the method of smooth FIFs developed in [28]. However, we work with a more general setting in the sense that the equality assumption on the scaling factors are not used, and a family of base functions B is employed instead of a single function b. As mentioned in the introductory section, the advantage gained by this slight modification is that, in addition to the polynomial splines, several standard rational splines that are extensively used in the field of shape preserving interpolation and approximation can also be generalized to fractal functions. This allows the intersection of two fields, the theory of fractal splines and shape preserving interpolation, which culminate with shape preserving fractal interpolation schemes.

Let us start with the decisive condition prescribed in the Barnsley-Harrington theorem, namely

$$F_{n-1,r}(x_N, y_{N,r}) = F_{n,r}(x_1, y_{1,r}), \quad n = 2, 3, \ldots, N-1, \ r = 1, 2, \ldots, p, \tag{3.2}$$

where $F_{n,r}(x, y) = \frac{\alpha_n y + q_n^{(r)}(x)}{a_n^r}$. For our choice of q_n, we have

$$q_n^{(r)}(x) = a_n^r s^{(r)}(L_n(x)) - \alpha_n b_n^{(r)}(x), \quad \text{for } r = 0, 1, 2, \ldots, p,$$

so that

$$\begin{aligned} a_{n-1}^r F_{n-1,r}(x_N, y_{N,r}) &= \frac{\alpha_{n-1}}{a_{N-1}^r - \alpha_{N-1}} \left[a_{N-1}^r s^{(r)}(x_N) - \alpha_{N-1} b_{N-1}^{(r)}(x_N) \right] \\ &\quad + a_{n-1}^r s^{(r)}(x_n) - \alpha_{n-1} b_{n-1}^{(r)}(x_N), \\ a_n^r F_{n,r}(x_1, y_{1,r}) &= \frac{\alpha_n}{a_1^r - \alpha_1} \left[a_1^r s^{(r)}(x_1) - \alpha_1 b_1^{(r)}(x_1) \right] \\ &\quad + a_n^r s^{(r)}(x_n) - \alpha_n b_n^{(r)}(x_1). \end{aligned} \tag{3.3}$$

In view of (3.3), the following conditions on the family $B = \{b_n : n \in J\}$ suffice to verify (3.2):

$$b_n^{(r)}(x_1) = s^{(r)}(x_1), \quad b_n^{(r)}(x_N) = s^{(r)}(x_N), \quad \text{for } r = 0, 1, \ldots, p, n \in J. \tag{3.4}$$

Thus, if we have a family of functions $B = \{b_n \in \mathcal{C}^p(I) : n \in J\}$ such that the derivatives up to p-th order of each of its members match with that of $s \in \mathcal{C}^p(I)$ at the end points

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of the interval, then the corresponding FIF s^α is in $\mathcal{C}^p(I)$ and satisfies $s^\alpha(x_n) = s(x_n)$. Furthermore, for $r = 1, 2, \ldots, p$, $(s^\alpha)^{(r)}$ is the FIF corresponding to the IFS

$$L_n(x) = a_n x + c_n, \quad F_{n,r}(x, y) = \frac{\alpha_n y + a_n^r s^{(r)}\big(L_n(x)\big) - \alpha_n b_n^{(r)}(x)}{a_n^r}.$$

Consequently, $(s^\alpha)^{(r)}$ satisfies the functional equation

$$(s^\alpha)^{(r)}(x) = s^{(r)}(x) + \frac{\alpha_n (s^\alpha - b_n)^{(r)} \circ L_n^{-1}(x)}{a_n^r}. \tag{3.5}$$

The above equation stipulates that the r-th derivative of the α-fractal function $s^\alpha_{\Delta,B}$ corresponding to s with respect to the scale vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1})$, the partition Δ, and the family of base functions $B = \{b_n : n \in J\}$ coincides with the fractal function of $s^{(r)}$ with respect to the scale vector $\tilde{\alpha} = (\frac{\alpha_1}{a_1^r}, \frac{\alpha_2}{a_2^r}, \ldots, \frac{\alpha_{N-1}}{a_{N-1}^r})$, the partition Δ, and the family $B_r = \{b_n^{(r)} : n \in J\}$, respectively, i.e., $(s^\alpha_{\Delta,B})^{(r)} = (s^{(r)})^{\tilde{\alpha}}_{\Delta,B_r}$. Using (3.5) and the conditions in (3.4) imposed on the family B, it can be verified that $(s^\alpha)^{(r)}(x_n) = s^{(r)}(x_n)$ for $n = 1, 2, \ldots, N$. That is, the r-th derivative of s^α agrees with the r-th derivative of s at the knot points.

THEOREM 3.3. *Suppose that for some integer $p \geq 0$, we have $|\alpha_n| \leq \kappa a_n^p$, for all $n \in J$ and $0 < \kappa < 1$. Let $|\alpha|_\infty = \max\{|\alpha_n| : n \in J\}$, $s \in \mathcal{C}^p$, and the family $B = \{b_n : n \in J\}$ obey the conditions prescribed in (3.4). The α-fractal function $s^\alpha \in \mathcal{C}^p(I)$ of s with respect to the partition Δ and the family B satisfies*

$$\|s^\alpha - s\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \max\big\{\|s - b_n\|_\infty : n \in J\big\},$$
$$\|(s^\alpha)^{(r)} - s^{(r)}\|_\infty \leq \frac{\kappa}{1-\kappa} \max\{\|s^{(r)} - b_n^{(r)}\|_\infty : n \in J\}, \quad r = 1, 2, \ldots, p.$$

Proof. We have the functional equation

$$s^\alpha(x) = s(x) + \alpha_n (s^\alpha - b_n) \circ L_n^{-1}(x), \quad \forall\, x \in I_n.$$

Consequently, for all $x \in I_n$,

$$|s^\alpha(x) - s(x)| \leq |\alpha_n| \|s^\alpha - b_n\|_\infty,$$

from which it follows that

$$\|s^\alpha - s\|_\infty \leq |\alpha|_\infty \max\big\{\|s^\alpha - b_n\|_\infty : n \in J\big\}.$$

According to the previous inequality,

$$\|s^\alpha - s\|_\infty \leq |\alpha|_\infty \big(\|s^\alpha - s\|_\infty + \max\{\|s - b_n\|_\infty : n \in J\}\big),$$

and thus

$$\|s^\alpha - s\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \max\big\{\|s - b_n\|_\infty : n \in J\big\}.$$

From (3.5), for $r = 1, 2, \ldots, p$, we have

$$(s^\alpha)^{(r)}(x) = s^{(r)}(x) + \frac{\alpha_n}{a_n^r}(s^\alpha - b_n)^{(r)} \circ L_n^{-1}(x), \quad \forall\, x \in I_n.$$

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Inasmuch as $0 < a_n = \frac{x_{n+1}-x_n}{x_N-x_1} < 1$, we have $a_n^p \le a_n^r$, for $r = 1, 2, \dots, p$. Hence,

$$|(s^\alpha)^{(r)}(x) - s^{(r)}(x)| \le \kappa \big|(s^\alpha - b_n)^{(r)}(L_n^{-1}(x))\big|, \quad \forall\, x \in I_n.$$

Calculations similar to that in the first part yield the second assertion. □

Let $s \in \mathcal{C}(I)$. Assume that the base functions b_n, $n \in J$, in the family B depend linearly on s. For instance, let b_n, $n \in J$, be given by $b_n = U_n s$, where the operators $U_n : \mathcal{C}(I) \to \mathcal{C}(I)$ are linear, bounded (with respect to the uniform norm on $\mathcal{C}(I)$) and satisfy $U_n s(x_1) = s(x_1)$, $U_n s(x_N) = s(x_N)$. Following [25, 27], we define the α-fractal operator $\mathcal{F}^\alpha \equiv \mathcal{F}^\alpha_{\Delta,B}$ as

$$\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I), \quad \mathcal{F}^\alpha(s) = s^\alpha.$$

DEFINITION 3.4. *Let $x_1 < x_2 < \cdots < x_N$ be fixed knots in the interval $I = [x_1, x_N]$. A linear operator $T : \mathcal{C}(I) \to \mathcal{C}(I)$ is said to be of interpolation type if for any $f \in \mathcal{C}(I)$, we have $Tf(x_n) = f(x_n)$, for all $n = 1, 2, \dots, N$.*

Next we study certain properties of the α-fractal operator $\mathcal{F}^\alpha$.

THEOREM 3.5.

(i) The fractal operator $\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I)$ is linear and bounded with respect to the uniform norm.

(ii) For a suitable value of the scale vector α, the operator $\mathcal{F}^\alpha$ is a simultaneous approximation and interpolation type operator.

(iii) If $\alpha = 0$, then $\mathcal{F}^\alpha$ is norm-preserving. In fact, it holds that $\mathcal{F}^0 \equiv I$.

(iv) For $|\alpha|_\infty < |U|^{-1}$, where $|U| = \max\{\|U_n\| : n \in J\}$ and $\|U_r\|$ is the operator norm $\|U_r\| := \sup\{\|U_r(f)\|_\infty : f \in \mathcal{C}(I), \|f\|_\infty \le 1\}$, $\mathcal{F}^\alpha$ is an injective, bounded, linear, and non-compact operator.

Proof. Let $\alpha \in (-1, 1)^{N-1}$. Let $s_1, s_2 \in \mathcal{C}(I)$ and $\lambda, \mu \in \mathbb{R}$. From (3.1), we have for all $x \in I_n$ that

$$\begin{aligned} s_1^\alpha(x) &= s_1(x) + \alpha_n (s_1^\alpha - U_n s_1) \circ L_n^{-1}(x), \\ s_2^\alpha(x) &= s_2(x) + \alpha_n (s_2^\alpha - U_n s_2) \circ L_n^{-1}(x). \end{aligned}$$

Therefore, from the linearity of U_n, we have

$$(\lambda s_1^\alpha + \mu s_2^\alpha)(x) = (\lambda s_1 + \mu s_2)(x) + \alpha_n \big(\lambda s_1^\alpha + \mu s_2^\alpha - U_n(\lambda s_1 + \mu s_2)\big) \circ L_n^{-1}(x).$$

From this equation we find that the function $\lambda s_1^\alpha + \mu s_2^\alpha$ is the fixed point of the Read-Bajraktarević operator $Tf(x) := (\lambda s_1 + \mu s_2)(x) + \alpha_n \big(f - U_n(\lambda s_1 + \mu s_2)\big) \circ L_n^{-1}(x)$. The uniqueness of the fixed point shows that $(\lambda s_1 + \mu s_2)^\alpha = \lambda s_1^\alpha + \mu s_2^\alpha$. That is, $\mathcal{F}^\alpha(\lambda s_1 + \mu s_2) = \lambda \mathcal{F}^\alpha(s_1) + \mu \mathcal{F}^\alpha(s_2)$ establishing the linearity of $\mathcal{F}^\alpha$.

From Theorem 3.3 we find that $\|\mathcal{F}^\alpha(s) - s\|_\infty \le \frac{|\alpha|_\infty}{1-|\alpha|_\infty} \max\{\|s - U_n s\|_\infty : n \in J\}$. Let $|U| := \max\{\|U_n\| : n \in J\}$. Using the boundedness of U_n, the former inequality implies

$$\|\mathcal{F}^\alpha(s)\|_\infty \le \Big(\frac{1 + |\alpha|_\infty |U|}{1 - |\alpha|_\infty}\Big) \|s\|_\infty,$$

which affirms that $\mathcal{F}^\alpha$ is bounded and the operator norm is bounded by $\|\mathcal{F}^\alpha\| \le \dfrac{1 + |\alpha|_\infty |U|}{1 - |\alpha|_\infty}$.

Let $s \in \mathcal{C}(I)$, $x_1 < x_2 < \cdots < x_N$ be distinct points in $I = [x_1, x_N]$, and $\epsilon > 0$. In view of the conditions imposed on the family $B = \{U_n s : n \in J\}$, it follows that

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$\mathcal{F}^\alpha s(x_n) = s^\alpha(x_n) = s(x_n)$, for $n = 1, 2, \dots, N$. That is, the operator $\mathcal{F}^\alpha$ is of interpolation type. Let $\alpha \in (-1,1)^{N-1}$ be such that $|\alpha|_\infty < \frac{\epsilon}{\epsilon + \|s\|_\infty (1+|U|)}$. Then it follows from Theorem 3.3 that $\|\mathcal{F}^\alpha(s) - s\|_\infty < \epsilon$. Consequently, $\mathcal{F}^\alpha$ is of approximation type. If $\alpha = 0 \in \mathbb{R}^{N-1}$ is chosen, then from equation (3.1), $s^\alpha = s$, for all $x \in I$. So $\mathcal{F}^0 \equiv I$.

Let $|\alpha|_\infty < |U|^{-1}$. Linearity and boundedness of the map $\mathcal{F}^\alpha$ follow from assertion (i). From Theorem 3.3 we have $\|s^\alpha - s\|_\infty \le |\alpha|_\infty \max\{\|s^\alpha - U_n s\|_\infty : n \in J\}$. After some routine calculations, this equation may be recast into the form $\frac{1-|U||\alpha|_\infty}{1+|\alpha|_\infty}\|s\|_\infty \le \|\mathcal{F}^\alpha(s)\|_\infty$. This shows that $\mathcal{F}^\alpha$ is bounded from below. Consequently, $\mathcal{F}^\alpha$ is injective and the inverse mapping $(\mathcal{F}^\alpha)^{-1} : \mathcal{F}^\alpha(\mathcal{C}(I)) \to \mathcal{C}(I)$ is bounded. From the injectivity of the linear map $\mathcal{F}^\alpha$, it follows that $\{1^\alpha, x^\alpha, (x^2)^\alpha, \dots\}$ is a linearly independent subset of $\mathcal{F}^\alpha(\mathcal{C}(I))$. The non-compactness of the operator $\mathcal{F}^\alpha$ can now be deduced using a result from basic operator theory that reads as follow: let X and Y be normed linear spaces and $A : X \to Y$ be an injective compact operator. Then $A^{-1} : A(X) \to X$ is bounded if and only if rank $A < \infty$. $\square$

REMARK 3.6. We can also consider the function space $\mathcal{C}^p(I)$ endowed with the $\mathcal{C}^p$-norm $\|f\|_{\mathcal{C}^p(I)} := \sum_{r=0}^{p} \|f^{(r)}\|_\infty$ and the operator $\mathcal{D}^\alpha : \mathcal{C}^p(I) \to \mathcal{C}^p(I)$ defined by $\mathcal{D}^\alpha(s) = s^\alpha$. Along the lines of Theorem 3.5, it can be proved that $\mathcal{D}^\alpha$ is a bounded linear map.

3.2. Construction of α-fractal rational cubic splines with shape parameters. Here, using the procedure developed in Section 3.1, we introduce a new class of α-fractal rational cubic splines $s^\alpha \in \mathcal{C}^1$ corresponding to the rational cubic splines $s \in \mathcal{C}^1$ studied in [14, 31]. The method of construction given in this section can be mimicked to obtain fractal generalizations of various rational cubic splines studied in the field of shape preserving interpolation.

Let a data set $\{(x_n, y_n, d_n) : n = 1, 2, \dots, N\}$, where $x_1 < x_2 < \dots < x_N$, be given. Here y_n and d_n, respectively, are the function value and the value of the first derivative at the knot x_n. If the derivatives at the knots are not given, we can estimate them by various approximation methods; see, for instance, [13]. A rational cubic spline $s \in \mathcal{C}^1(I)$ is defined in a piecewise manner as follows; see [14, 31] for details. For $\theta := \frac{x - x_1}{x_N - x_1}$, $x \in I$,

$$s\big(L_n(x)\big) = \frac{(1-\theta)^3 r_n y_n + \theta(1-\theta)^2 V_n + \theta^2(1-\theta) W_n + \theta^3 t_n y_{n+1}}{(1-\theta) r_n + \theta t_n}, \tag{3.6}$$

where

$$V_n = (2r_n + t_n) y_n + r_n h_n d_n, \quad W_n = (r_n + 2t_n) y_{n+1} - t_n h_n d_{n+1}, \quad h_n = x_{n+1} - x_n.$$

The free parameters r_n and t_n are selected to be strictly positive to ensure a strictly positive denominator, which in turn avoids a singularity of the rational expression occurring in (3.6). The parameters r_n and t_n can be varied to alter the shape of the interpolant and hence are called the shape parameters.

We note that the expression for s can be rewritten in the following form:

$$\begin{aligned} s\big(L_n(x)\big) &= \omega_1(\theta; r_n, t_n) y_n + \omega_2(\theta; r_n, t_n) y_{n+1} \\ &\quad + \omega_3(\theta; r_n, t_n) d_n + \omega_4(\theta; r_n, t_n) d_{n+1}, \end{aligned} \tag{3.7}$$

where

$$\omega_1(\theta; r_n, t_n) = \frac{(1-\theta)^3 r_n + \theta(1-\theta)^2(2r_n + t_n)}{(1-\theta) r_n + \theta t_n}, \quad \omega_3(\theta; r_n, t_n) = \frac{\theta(1-\theta)^2 r_n h_n}{(1-\theta) r_n + \theta t_n},$$
$$\omega_2(\theta; r_n, t_n) = \frac{\theta^2(1-\theta)(r_n + 2t_n) + \theta^3 t_n}{(1-\theta) r_n + \theta t_n}, \quad \omega_4(\theta; r_n, t_n) = -\frac{\theta^2(1-\theta) t_n h_n}{(1-\theta) r_n + \theta t_n},$$

are the rational cubic Hermite basis functions. The rational interpolant s satisfies the Hermite interpolation conditions $s(x_n) = y_n$ and $s^{(1)}(x_n) = d_n$, for $n = 1, 2, \dots, N$.

To develop the α-fractal rational cubic spline corresponding to s (cf. equation (3.6)), we set $|\alpha_n| \le \kappa a_n, 0 \le \kappa < 1$, and select a family $B = \{b_n \in \mathcal{C}^1(I) : n \in J\}$ satisfying the conditions $b_n(x_1) = y_1, b_n(x_N) = y_N, b_n^{(1)}(x_1) = d_1,$ and $b_n^{(1)}(x_N) = d_N$; cf. Section 3.1. There is a variety of choices for B. To define one such family, we take b_n to be a rational function of similar structure as that of the classical interpolant s. Our choice may be justified by the simplicity it offers for the final expression of the desired rational cubic spline FIF s^α. To be precise, for $x \in I = [x_1, x_N]$ and $\theta := \frac{x-x_1}{x_N-x_1}$, our specific choice for b_n is

$$b_n(x) = \frac{B_{1n}(1-\theta)^3 + B_{2n}\theta(1-\theta)^2 + B_{3n}\theta^2(1-\theta) + B_{4n}\theta^3}{(1-\theta)r_n + \theta t_n}, \tag{3.8}$$

where the coefficients B_{1n}, B_{2n}, B_{3n}, and B_{4n} are determined by the conditions $b_n(x_1)=y_1$, $b_n(x_N) = y_N$, $b_n^{(1)}(x_1) = d_1$, $b_n^{(1)}(x_N) = d_N$. Elementary computations yield

$$\begin{aligned} B_{1n} &= r_n y_1, & B_{2n} &= (2r_n + t_n)y_1 + r_n d_1(x_N - x_1), \\ B_{3n} &= (r_n + 2t_n)y_N - t_n d_N(x_N - x_1), & B_{4n} &= t_n y_N. \end{aligned}$$

We note that b_n can be reformulated as

$$b_n(x) = F_1(\theta; r_n, t_n)y_1 + F_2(\theta; r_n, t_n)y_N + D_1(\theta; r_n, t_n)d_1 + D_2(\theta; r_n, t_n)d_N, \tag{3.9}$$

where

$$\begin{aligned} F_1(\theta; r_n, t_n) &= \omega_1(\theta; r_n, t_n), & F_2(\theta; r_n, t_n) &= \omega_2(\theta; r_n, t_n), \\ D_1(\theta; r_n, t_n) &= \frac{\theta(1-\theta)^2 r_n(x_N - x_1)}{(1-\theta)r_n + \theta t_n}, & D_2(\theta; r_n, t_n) &= -\frac{\theta^2(1-\theta)t_n(x_N - x_1)}{(1-\theta)r_n + \theta t_n}. \end{aligned}$$

Consider the α-fractal rational cubic spline corresponding to s:

$$s^\alpha\big(L_n(x)\big) = \alpha_n s^\alpha(x) + s\big(L_n(x)\big) - \alpha_n b_n(x).$$

In view of (3.6) and (3.8), we have

$$s^\alpha\big(L_n(x)\big) = \alpha_n s^\alpha(x) + \frac{P_n(x)}{Q_n(x)}, \tag{3.10}$$

where

$$\begin{aligned} &P_n(x) \\ &= \{y_n - \alpha_n y_1\} r_n(1-\theta)^3 + \{y_{n+1} - \alpha_n y_N\} t_n\theta^3 \\ &\quad + \{(2r_n + t_n)y_n + r_n h_n d_n - \alpha_n [(2r_n + t_n)y_1 + r_n(x_N - x_1)d_1]\} \theta(1-\theta)^2 \\ &\quad + \{(r_n + 2t_n)y_{n+1} - t_n h_n d_{n+1} - \alpha_n [(r_n + 2t_n)y_N - t_n(x_N - x_1)d_N]\} \theta^2(1-\theta), \\ &Q_n(x) = (1-\theta)r_n + \theta t_n, \qquad n \in J,\ \theta = \frac{x - x_1}{x_N - x_1}. \end{aligned}$$

REMARK 3.7. Assuming d_1 and d_N to be exact first derivatives of s at the extreme knots x_1 and x_N, we define

$$\begin{aligned} b_n(x) = U_n s(x) = F_1(\theta; r_n, t_n)s(x_1) + F_2(\theta; r_n, t_n)s(x_N) \\ + D_1(\theta; r_n, t_n)s^{(1)}(x_1) + D_2(\theta; r_n, t_n)s^{(1)}(x_N). \end{aligned}$$

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For r_n and t_n independent of the data, U_n defines a linear operator on $\mathcal{C}^1(I)$. Furthermore, U_n is bounded with respect to the $\mathcal{C}^1$-norm on $\mathcal{C}^1(I)$ and

$$\|U_n\| \leq \sup_{x\in I,\ j=0,1} \left[\sum_{i=1}^{2} \left(|F_i^{(j)}(x)| + |D_i^{(j)}(x)|\right)\right].$$

REMARK 3.8. If $r_n = t_n$ for all $n \in J$, then the α-fractal rational cubic spline reduces to the $\mathcal{C}^1$-cubic Hermite FIF. For a detailed discussion of the Hermite spline FIFs of arbitrary degree and their convergence analysis, we refer to [6]. For $\alpha_n = 0$ and $r_n = t_n$, for all $n \in J$, the α-fractal rational cubic spline recovers the classical cubic Hermite interpolant.

REMARK 3.9. Consider a linear operator $T : \mathcal{C}(I) \to \mathcal{C}(I)$ which is of interpolation type. Now, if $f \in \mathcal{C}(I)$ is, for example, monotone (or convex) on I, it is easy to see that because of the interpolation conditions, in general, Tf cannot be monotone (or convex) on I. Hence $s^\alpha = \mathcal{F}^\alpha(s)$ is not, in general, monotone (or convex), even if s is so. However, it is a natural question whether parameters involved in the spline structure can be determined so that s^α is monotone or convex. We address this issue in the subsequent sections.

REMARK 3.10. Let us define $\Delta_n = \frac{y_{n+1}-y_n}{x_{n+1}-x_n}$, for $n \in J$. Assuming twice differentiability of the α-fractal spline s^α, the following are the functional equations corresponding to the first and second derivatives:

$$(s^\alpha)^{(1)}\big(L_n(x)\big) = \frac{\alpha_n}{a_n}(s^\alpha)^{(1)}(x) + \frac{M_{1n}(1-\theta)^3 + M_{2n}\theta(1-\theta)^2 + M_{3n}\theta^2(1-\theta) + M_{4n}\theta^3}{[r_n(1-\theta)+t_n\theta]^2}, \tag{3.11}$$

where

$$\begin{aligned}
M_{1n} &= r_n^2\left[d_n - \frac{\alpha_n}{h_n}d_1(x_N-x_1)\right],\\
M_{2n} &= (2r_n^2+4r_nt_n)\left[\Delta_n - \frac{\alpha_n}{h_n}(y_N-y_1)\right] - r_n^2\left[d_n - \frac{\alpha_n}{h_n}d_1(x_N-x_1)\right]\\
&\quad - 2r_nt_n\left[d_{n+1} - \frac{\alpha_n}{h_n}d_N(x_N-x_1)\right],\\
M_{3n} &= (2t_n^2+4r_nt_n)\left[\Delta_n - \frac{\alpha_n}{h_n}(y_N-y_1)\right] - 2r_nt_n\left[d_n - \frac{\alpha_n}{h_n}d_1(x_N-x_1)\right]\\
&\quad - t_n^2\left[d_{n+1} - \frac{\alpha_n}{h_n}d_N(x_N-x_1)\right],\\
M_{4n} &= t_n^2\left[d_{n+1} - \frac{\alpha_n}{h_n}d_N(x_N-x_1)\right],
\end{aligned}$$

and

$$(s^\alpha)^{(2)}\big(L_n(x)\big) = \frac{\alpha_n}{a_n^2}(s^\alpha)^{(2)}(x) + \frac{C_{1n}(1-\theta)^3 + C_{2n}\theta(1-\theta)^2 + C_{3n}\theta^2(1-\theta) + C_{4n}\theta^3}{[r_n(1-\theta)+t_n\theta]^3\,h_n}, \tag{3.12}$$

where

$$
\begin{aligned}
C_{1n} &= (2r_n^3 + 2r_n^2 t_n)\left\{\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1) - \left[d_n - \frac{\alpha_n}{h_n} d_1(x_N - x_1)\right]\right\}\\
&\quad - 2r_n^2 t_n \left\{d_{n+1} - \frac{\alpha_n}{h_n} d_N(x_N - x_1) - \left[\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1)\right]\right\},\\
C_{2n} &= 6r_n^2 t_n \left\{\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1) - \left[d_n - \frac{\alpha_n}{h_n} d_1(x_N - x_1)\right]\right\},\\
C_{3n} &= 6r_n t_n^2 \left\{d_{n+1} - \frac{\alpha_n}{h_n} d_N(x_N - x_1) - \left[\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1)\right]\right\},\\
C_{4n} &= (2t_n^3 + 2r_n t_n^2)\left\{d_{n+1} - \frac{\alpha_n}{h_n} d_N(x_N - x_1) - \left[\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1)\right]\right\}\\
&\quad - 2r_n t_n^2 \left\{\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1) - \left[d_n - \frac{\alpha_n}{h_n} d_1(x_N - x_1)\right]\right\}.
\end{aligned}
$$

These expressions will be used later in Sections 5.1–5.2 for studying the shape preserving aspects of the α-fractal rational cubic spline s^α.

4. Convergence of α-fractal rational cubic splines. In this section, we establish that the α-fractal rational spline s^α converges to the original function $f \in \mathcal{C}^2$ with respect to the $\mathcal{C}^1$-norm. We shall uncover this in a series of propositions and theorems.

PROPOSITION 4.1 (Theorem 1, Duan et al. [14]). *Given a function $f \in \mathcal{C}^2(I)$ and a data set $\{(x_n, y_n) : n = 1, 2, \dots, N\}$, $y_n = f(x_n)$. Let s be the corresponding rational cubic spline defined in (3.6). Then, for $x \in [x_n, x_{n+1}]$,*

$$|f(x) - s(x)| \le h_n^2 c_n \|f^{(2)}\|,$$

where $c_n = \max_{0\le\theta\le1} \Omega(\theta; r_n, t_n)$,

$$\Omega(\theta; r_n, t_n) = \frac{\theta^2(1-\theta)^2(r_n + t_n)^2}{[r_n + (r_n + t_n)\theta][r_n + 2t_n - (r_n + t_n)\theta]},$$

and $\|.\|$ is the uniform norm on $[x_n, x_{n+1}]$. Furthermore, for any given positive values of r_n and t_n, the error constant c_n satisfies $\frac{1}{16} \le c_n \le \frac{5\sqrt{5}-11}{2}$.

Using Propositions 3.3 and 4.1 we have the following theorem.

THEOREM 4.2. *Given a function $f \in \mathcal{C}^2(I)$ and a partition $\Delta = \{x_1, x_2, \dots, x_N\}$ of I satisfying $x_1 < x_2 < \dots < x_N$, let $s^{(\alpha)}$ be the α-fractal rational cubic spline that interpolates the values of the function f at the points of the partition Δ. Then*

$$
\begin{aligned}
\|f - s^\alpha\|_\infty &\le h^2 c\, \|f^{(2)}\|_\infty\\
&\quad + \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\Big\{|y|_\infty + \max\{|y_1|, |y_N|\} + \frac{1}{4}(h|d|_\infty + |I| \max\{|d_1|, |d_N|\})\Big\},
\end{aligned}
$$

where $|y|_\infty = \max\{|y_n| : 1 \le n \le N\}$, $|d|_\infty = \max\{|d_n| : 1 \le n \le N\}$, $|I| = x_N - x_1$, $h = \max\{h_n : n \in J\}$, and $c = \max\{c_n : n \in J\}$.

Proof. Let s be the classical rational cubic spline and s^α be the corresponding α-fractal function interpolating f at the points $x_1, x_2, \dots, x_N \in \Delta$. We have the triangle inequality,

$$\|f - s^\alpha\|_\infty \le \|f - s\|_\infty + \|s - s^\alpha\|_\infty. \tag{4.1}$$

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By Theorem 3.3,

$$\|s^\alpha - s\|_\infty \le \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\Big(\|s\|_\infty + \max\{\|b_n\|_\infty : n \in J\}\Big). \tag{4.2}$$

Next we establish upper bounds for $\|s\|_\infty$ and $\|b_n\|_\infty$ that depend only on the function values and the values of the derivatives at the knot points. For $x \in [x_n, x_{n+1}]$, $x = L_n(x')$, consider the classical rational cubic spline s (cf. equation (3.7))

$$s(x) = \omega_1(\theta; r_n, t_n) y_n + \omega_2(\theta; r_n, t_n) y_{n+1} + \omega_3(\theta; r_n, t_n) d_n + \omega_4(\theta; r_n, t_n) d_{n+1},$$

where $\theta = \frac{x'-x_1}{x_N - x_1}$. We note that $\omega_i(\theta; r_n, t_n) \ge 0$, for $i = 1, 2, 3$, and $\omega_4(\theta; r_n, t_n) \le 0$. Furthermore,

$$\omega_1(\theta; r_n, t_n) + \omega_2(\theta; r_n, t_n) = 1 \quad \text{and} \quad \omega_3(\theta; r_n, t_n) - \omega_4(\theta; r_n, t_n) = h_n\theta(1-\theta).$$

Thus,

$$|s(x)| \le \max\{|y_n|, |y_{n+1}|\} + \frac{h_n}{4}\max\{|d_n|, |d_{n+1}|\}.$$

Hence,

$$\|s\|_\infty \le |y|_\infty + \frac{h}{4}|d|_\infty. \tag{4.3}$$

Similarly, from the expression for b_n (cf. equation (3.9)), we obtain

$$\|b_n\|_\infty \le \max\{|y_1|, |y_N|\} + \frac{1}{4}|I|\max\{|d_1|, |d_N|\}. \tag{4.4}$$

Substituting bounds for $\|s\|_\infty$ and $\max\{\|b_n\|_\infty : n \in J\}$ obtained from (4.3) and (4.4) in (4.2), we find that

$$\|s^\alpha - s\|_\infty \le \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\Big\{|y|_\infty + \max\{|y_1|, |y_N|\} + \frac{1}{4}\big(h|d|_\infty + |I|\max\{|d_1|, |d_N|\}\big)\Big\}. \tag{4.5}$$

From Proposition 4.1 it follows that

$$|f(x) - s(x)| \le c_n h_n^2 \|f^{(2)}\|,$$

implying

$$\|f - s\|_\infty \le c\, h^2 \|f^{(2)}\|_\infty. \tag{4.6}$$

The inequality (4.1) coupled with (4.5) and (4.6) proves the theorem. ◻

PROPOSITION 4.3 (Theorem 1, Duan et al. [16]). *Let $f \in \mathcal{C}^2(I)$ be the function generating the data $\{(x_n, y_n) : n = 1, 2, \dots, N\}$ and s be the corresponding classical rational cubic spline. Then, on $[x_n, x_{n+1}]$, the error for the derivative function $s^{(1)}$ satisfies*

$$|f^{(1)}(x) - s^{(1)}(x)| \le h_n c_n^* \|f^{(2)}\|,$$

where $c_n^* = \max\{\chi(\theta; r_n, t_n) : 0 \le \theta \le 1\}$ with

$$\chi(\theta; r_n, t_n) = \begin{cases} \chi_1(\theta; r_n, t_n), & \text{if } 0 \le \theta \le \theta_* = \frac{3r_n - \sqrt{r_n^2 + 8r_n t_n}}{4(r_n - t_n)}, \\ \chi_2(\theta; r_n, t_n), & \text{if } \theta_* \le \theta \le \theta^* = \frac{4r_n - t_n - \sqrt{t_n^2 + 8r_n t_n}}{4(r_n - t_n)}, \\ \chi_3(\theta; r_n, t_n), & \text{if } \theta^* \le \theta \le 1, \end{cases}$$

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$$
\begin{aligned}
\chi_1(\theta; r_n, t_n) =&\theta \Big\{ \big[\theta(1-2\theta+2\theta^2)t_n^2 + 2(1-\theta)^2 r_n t_n + 2(1-\theta)^3 r_n^2\big]^2 \\
&\quad + \big[2(1-\theta)^2 r_n t_n + \theta(1-2\theta)\beta_n^2\big]^2 \Big\} \\
&\quad \times \Big\{ (1-\theta)\left[(1-\theta)r_n + \theta t_n\right]^2 \left[(1-\theta)r_n^2 + 2r_n t_n + \theta t_n^2\right] \Big\}^{-1} \\
&+ \frac{\theta(1-\theta)[(1-\theta)^3 r_n^2 + \theta^3 t_n^2]}{[(1-\theta)r_n + \theta t_n]^2}, \\
\chi_2(\theta; r_n, t_n) =&(1-\theta) \Big\{ \big[(1-\theta)(1-2\theta+2\theta^2) r_n^2 + 2\theta^2 r_n t_n + 2\theta^3 t_n^2\big]^2 \\
&\quad + \big[2\theta^2 r_n t_n + (1-\theta)(2\theta-1) r_n^2\big]^2 \Big\} \\
&\quad \times \Big\{ \theta \left[(1-\theta)r_n + \theta t_n\right]^2 \left[(1-\theta)r_n^2 + 2r_n t_n + \theta t_n^2\right] \Big\}^{-1} \\
&+ \theta \Big\{ \big[\theta(1-2\theta+2\theta^2)t_n^2 + 2(1-\theta)^2 r_n t_n + 2(1-\theta)^3 r_n^2\big]^2 \\
&\quad + \big[2(1-\theta)^2 r_n t_n + \theta(1-2\theta) t_n^2\big]^2 \Big\} \\
&\quad \times \Big\{ (1-\theta)\left[(1-\theta)r_n + \theta t_n\right]^2 \left[(1-\theta)r_n^2 + 2r_n t_n + \theta t_n^2\right] \Big\}^{-1}, \\
\chi_3(\theta; r_n, t_n) =&(1-\theta) \Big\{ \big[(1-\theta)(1-2\theta+2\theta^2) r_n^2 + 2\theta^2 r_n t_n + 2\theta^3 t_n^2\big]^2 \\
&\quad + \big[2\theta^2 r_n t_n + (1-\theta)(2\theta-1) r_n^2\big]^2 \Big\} \\
&\quad \times \Big\{ \theta \left[(1-\theta)r_n + \theta t_n\right]^2 \left[(1-\theta)r_n^2 + 2r_n t_n + \theta t_n^2\right] \Big\}^{-1} \\
&+ \frac{\theta(1-\theta)[(1-\theta)^3 r_n^2 + \theta^3 t_n^2]}{[(1-\theta)r_n + \theta t_n]^2}.
\end{aligned}
$$

THEOREM 4.4. *If $f \in \mathcal{C}^2(I)$ and s^α is the α-fractal rational cubic spline corresponding to the data $\{(x_n, y_n) : n = 1, 2, \dots, N\}$, $y_n = f(x_n)$, then*

$$
\|f^{(1)} - (s^\alpha)^{(1)}\|_\infty \le hc^* \|f^{(2)}\|_\infty + \frac{\kappa}{1-\kappa} \Big\{ \|s^{(1)}\|_\infty + \frac{\overline{\gamma}^2}{\underline{\delta}^2} \big[\max\{|d_1|, |d_N|\} + \frac{3|y_N - y_1|}{2|I|} \big] \Big\},
$$

where $\overline{\gamma} = \max_{n\in J}\{r_n, t_n\}$, $\underline{\delta} = \min_{n \in J}\{r_n, t_n\}$, and $c^* = \max\{c_n^* : n \in J\}$.

Proof. By the triangle inequality,

$$
\|f^{(1)} - (s^\alpha)^{(1)}\|_\infty \le \|f^{(1)} - s^{(1)}\|_\infty + \|s^{(1)} - (s^\alpha)^{(1)}\|_\infty. \tag{4.7}
$$

From Proposition 4.3 we have

$$
\|f^{(1)} - s^{(1)}\|_\infty \le hc^* \|f^{(2)}\|_\infty. \tag{4.8}
$$

By simple calculations it follows that

$$
\max_{n\in J} \|b_n^{(1)}\|_\infty \le \frac{\overline{\gamma}^2}{\underline{\delta}^2} \Big\{ \max\{|d_1|, |d_N|\} + \frac{3}{2} \frac{|y_N - y_1|}{x_N - x_1} \Big\}.
$$

Using the above estimate and Theorem 3.3, we obtain that

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$$\|s^{(1)} - (s^\alpha)^{(1)}\|_\infty \leq \frac{\kappa}{1-\kappa}\Big\{\|s^{(1)}\|_\infty + \frac{\overline{\gamma}^2}{\underline{\delta}^2}\big[\max\{|d_1|,|d_N|\} + \frac{3|y_N - y_1|}{2|I|}\big]\Big\}. \tag{4.9}$$

Substituting (4.8) and (4.9) in (4.7) completes the proof. □

The following theorem is a direct consequence of Theorems 4.2 and 4.4. It is worth to mention here that the derivative parameters are bounded because $f \in \mathcal{C}^2(I)$, and we assume that $\overline{\gamma}$, $\underline{\delta}$, and c^* are fixed.

THEOREM 4.5. *Let $f \in \mathcal{C}^2(I)$ be the original data generating function and let the scaling factors satisfy $|\alpha_n| \leq \kappa a_n$, for all $n \in J$, $0 < k < \frac{h}{|I|}$. Then s^α converges to f with respect to the $\mathcal{C}^1$-norm as the mesh size approaches zero.*

REMARK 4.6. In fact, we have the following more general result. If we consider scaling factors α_n such that $|\alpha_n| \leq \frac{h}{|I|}a_n^p$, for all $n \in J$, then as a consequence of Theorem 3.3, we obtain convergence of the α-fractal function s^α to the original function $f \in \mathcal{C}^p$ with respect to the $\mathcal{C}^p$-norm provided that s does so.

5. Constrained interpolation with α-fractal rational splines. In this section, we embrace the task of deriving conditions on the spline parameters so that the α-fractal rational cubic spline s^α is: (i) convex, (ii) convex and monotone. Some remarks on positive interpolation with s^α are given at the end of this section to cover all major aspects of shape preservation.

5.1. Convex interpolation. Consider a data set $\{(x_n, y_n) : n = 1, 2, \dots, N\}$. Suppose that the data set is convex, that is, $\Delta_1 < \Delta_2 < \cdots < \Delta_{n-1} < \Delta_n < \Delta_{n+1} < \cdots < \Delta_{N-1}$, where $\Delta_n := \frac{y_{n+1}-y_n}{x_{n+1}-x_n}$, for $n \in J$, as before. We identify suitable values for the parameters so that the corresponding α-fractal rational cubic spline s^α remains convex for a given set of convex data. A similar approach applies to a concave set of data.

THEOREM 5.1. *For a given set of data $\{(x_n, y_n) : n = 1, 2, \dots, N\}$ with derivatives (given or estimated) at the knot points satisfying the condition*

$$d_1 < \Delta_1 < \cdots < d_n < \Delta_n < d_{n+1} < \cdots < \Delta_{N-1} < d_N,$$

a convex α-fractal rational spline s^α involving shape parameters r_n and t_n exists provided the following conditions are satisfied.

$$0 \leq \alpha_n < \min\Big\{\frac{h_n(\Delta_n - d_n)}{y_N - y_1 - d_1(x_N - x_1)}, \frac{h_n(d_{n+1} - \Delta_n)}{d_N(x_N - x_1) - (y_N - y_1)}, a_n^2\Big\},$$

$$\frac{r_n}{t_n} = \frac{d_{n+1} - \frac{\alpha_n}{h_n}d_N(x_N - x_1) - [\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1)]}{\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1) - [d_n - \frac{\alpha_n}{h_n}d_1(x_N - x_1)]}, \quad n \in J.$$

Proof. Let us begin by recalling that, for the convexity of $f \in \mathcal{C}^1[x_1, x_N]$, it is sufficient to prove that $f^{(2)}(x^+)$ or $f^{(2)}(x^-)$ exist and are nonnegative (possibly $+\infty$) for all $x \in (x_1, x_N)$; see [24]. Informally, we have (see equation (3.12))

$$(s^\alpha)^{(2)}(L_n(x)) = \frac{\alpha_n}{a_n^2}(s^\alpha)^{(2)}(x) + \frac{C_{1n}(1-\theta)^3 + C_{2n}\theta(1-\theta)^2 + C_{3n}\theta^2(1-\theta) + C_{4n}\theta^3}{[r_n(1-\theta) + t_n\theta]^3 h_n}.$$

For the sake of convenience, let

$$R_n(x) \equiv R_n^*(\theta) = \frac{C_{1n}(1-\theta)^3 + C_{2n}\theta(1-\theta)^2 + C_{3n}\theta^2(1-\theta) + C_{4n}\theta^3}{[r_n(1-\theta) + t_n\theta]^3 h_n}.$$

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Using the fact that for $n \in J$, the maps $L_n : [x_1, x_N] \to [x_n, x_{n+1}]$ satisfy $L_n(x_1) = x_n$, $L_n(x_N) = x_{n+1}$, we obtain

$$(5.1)\quad \begin{aligned} &(s^\alpha)^{(2)}(x_1^+) = \frac{C_{11}}{r_1^3 h_1}\Big[1 - \frac{\alpha_1}{a_1^2}\Big]^{-1}, \qquad (s^\alpha)^{(2)}(x_N^-) = \frac{C_{4N-1}}{t_{N-1}^3 h_{N-1}}\Big[1 - \frac{\alpha_{N-1}}{a_{N-1}^2}\Big]^{-1},\\ &(s^\alpha)^{(2)}(x_n^+) = \frac{\alpha_n}{a_n^2}(s^\alpha)^{(2)}(x_1^+) + \frac{C_{1n}}{r_n^3 h_n}, \qquad n = 2, 3, \dots, N-1. \end{aligned}$$

Let $0 \le \alpha_n \le \kappa a_n^2$. Then from (5.1) it follows that if $C_{4N-1} \ge 0$ and $C_{1n} \ge 0$, for $n \in J$, then the second derivatives (from the right) at the knots x_n, for $n \in J$, and the second derivatives (from the left) at x_N are nonnegative. For the knot x_m, $m \in J$, we have

$$(5.2)\qquad s^{(2)}\big(L_n(x_m)^+\big) = \frac{\alpha_m}{a_m^2} s^{(2)}(x_m^+) + R_n(x_m).$$

Assuming $C_{1n} \ge 0$, for all $n \in J$, (5.2) suggests that $s^{(2)}\big(L_n(x_m)^+\big) \ge 0$ is satisfied, provided that $R_n(x_m) \ge 0$. Also $R_n(x_m) \ge 0$ is satisfied if $C_{jn} \ge 0$, for $j = 1, 2, 3, 4$. By the three chord lemma for convex functions, a strictly convex data set necessarily satisfies $d_1 < \frac{y_N - y_1}{x_N - x_1} < d_N$. The selection of the free parameters α_n so that they satisfy

$$(5.3)\qquad \alpha_n < \frac{h_n(\Delta_n - d_n)}{(y_N - y_1) - d_1(x_N - x_1)} \quad \text{and} \quad \alpha_n < \frac{h_n(d_{n+1} - \Delta_n)}{d_N(x_N - x_1) - (y_N - y_1)},$$

for all $n \in J$, ensures the nonnegativity of C_{2n} and C_{3n}, respectively. In view of (3.12) and (5.3), by some algebraic manipulations, it is easy to see that $C_{1n} \ge 0$ and $C_{4n} \ge 0$ is satisfied if

$$\frac{r_n}{t_n} = \frac{d_{n+1} - \frac{\alpha_n}{h_n} d_N(x_N - x_1) - [\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1)]}{\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1) - [d_n - \frac{\alpha_n}{h_n} d_1(x_N - x_1)]}.$$

Thus, the conditions on the scaling factors and the shape parameters prescribed in the theorem guarantee that $C_{jn} \ge 0$, for $j = 1, 2, 3, 4, n \in J$. Since $(s^\alpha)^{(2)}(x^+)$, $(s^\alpha)^{(2)}(x^-)$ are determined iteratively, $(s^\alpha)^{(2)}\big(L_n(x_m)^+\big) \ge 0$ holds for the maps L_n, $n \in J$, and at the knots $x_m, m \in J$, and $s^\alpha(x_N^-) \ge 0$ assures that $(s^\alpha)^{(2)}(x^+) \ge 0$ or $(s^\alpha)^{(2)}(x^-) \ge 0$, for all $x \in (x_1, x_N)$. □

REMARK 5.2. By taking $\alpha_n = 0$, for all $n \in J$, the convexity theorem for the classical rational cubic spline interpolant (3.6) [31, Theorem 3.2] follows as a straightforward consequence of the convexity theorem for the α-fractal cubic spline stated above. Had we imposed the condition $t_n > r_n > 0$ on the shape parameters as in [31], then the obtained convexity condition for the classical rational cubic spline, namely $\frac{r_n}{t_n} = \frac{d_{n+1} - \Delta_n}{\Delta_n - d_n}$, might not have been consistent with the additional condition $t_n > r_n$. It seems that this issue is not addressed in [31].

5.2. Convex and monotone interpolation. Reviving the spirit of the previous subsection, now we illustrate that the convexity preserving scheme developed therein is suitable for solving the convex and monotone interpolation problem.

THEOREM 5.3. *For a given set* $\{(x_n, y_n) : n = 1, 2, \dots, N\}$ *of monotonically increasing convex data with the values of the derivatives at the knots satisfying the conditions*

$$(5.4)\qquad 0 \le d_1 < \Delta_1 < \cdots < d_n < \Delta_n < d_{n+1} < \cdots < \Delta_{N-1} < d_N,$$

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the following conditions on the scaling and the shape parameters ensure that the corresponding α-fractal rational cubic spline is monotonically increasing and convex:

$$0 \le \alpha_n < \min\left\{\frac{h_n(\Delta_n - d_n)}{y_N - y_1 - d_1(x_N - x_1)}, \frac{h_n(d_{n+1} - \Delta_n)}{d_N(x_N - x_1) - (y_N - y_1)}, a_n^2\right\},$$
$$\frac{r_n}{t_n} = \frac{d_{n+1} - \frac{\alpha_n}{h_n} d_N(x_N - x_1) - [\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1)]}{\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1) - [d_n - \frac{\alpha_n}{h_n} d_1(x_N - x_1)]}.$$

Proof. In Theorem 5.1 we have already proved that for scaling factors and the shape parameters as in this theorem, s^α is convex. Since the fractal function $(s^\alpha)^{(1)}$ is constructed iteratively, in order to prove $(s^\alpha)^{(1)}(x) \ge 0$, for all $x \in I$, it is enough to prove that $(s^\alpha)^{(1)}\big(L_n(.)\big) \ge 0$, for all $n \in J$, whenever $(s^\alpha)^{(1)}(.) \ge 0$. The stated assumptions on α_n imply

$$d_n - \frac{\alpha_n}{h_n} d_1(x_N - x_1) < \Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1) < d_{n+1} - \frac{\alpha_n}{h_n} d_N(x_N - x_1) \tag{5.5}$$

and

$$\alpha_n < a_n,\ d_n \ge 0, \forall\, n \implies \alpha_n d_1 < \frac{h_n}{x_N - x_1} d_1 \implies d_n - \frac{\alpha_n}{h_n} d_1(x_N - x_1) > d_n - d_1.$$

Since $d_1 < d_n$, we have $d_n - \frac{\alpha_n}{h_n} d_1(x_N - x_1) > 0$. Whence, (5.5) yields the inequalities $\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1) > 0$ and $d_{n+1} - \frac{\alpha_n}{h_n} d_N(x_N - x_1) > 0$. From (3.11) we obtain

$$(s^\alpha)^{(1)}\big(L_n(x)\big) = \frac{\alpha_n}{a_n}(s^\alpha)^{(1)}(x) + \frac{M_{1n}(1-\theta)^3 + M_{2n}\theta(1-\theta)^2 + M_{3n}\theta^2(1-\theta) + M_{4n}\theta^3}{[r_n(1-\theta) + t_n\theta]^2}.$$

Now $d_n - \frac{\alpha_n}{h_n} d_1(x_N - x_1) \ge 0$ and $d_{n+1} - \frac{\alpha_n}{h_n} d_N(x_N - x_1) \ge 0$ imply M_{1n} and $M_{4n} \ge 0$. From some simple calculations, we infer that M_{2n} and M_{3n} are nonnegative if

$$r_n\left\{\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1) - \left[d_n - \frac{\alpha_n}{h_n} d_1(x_N - x_1)\right]\right\} \ge t_n\left\{d_{n+1} - \frac{\alpha_n}{h_n} d_N(x_N - x_1) - \left[\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1)\right]\right\}.$$

Therefore, the conditions on the shape parameters stated in the theorem ensure that $M_{2n} \ge 0$ and $M_{3n} \ge 0$. This demonstrates the monotonicity of the rational cubic spline FIF s^α. ◻

REMARK 5.4. A moment of reflection on the proof of the foregoing theorem shows that for given data satisfying only the monotonicity condition $d_n \ge 0$, $n = 1, 2, \dots, N$, the following conditions on the scaling factors and the shape parameters are sufficient to ensure the monotonicity of the α-fractal rational cubic spline:

$$\begin{aligned} &0 \le \alpha_n < \min\left\{\frac{h_n d_n}{d_1(x_N - x_1)}, \frac{h_n d_{n+1}}{d_N(x_N - x_1)}, \frac{h_n \Delta_n}{y_N - y_1}, a_n\right\}, \\ &r_n\left\{\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1) - \left[d_n - \frac{\alpha_n}{h_n} d_1(x_N - x_1)\right]\right\} \\ &\qquad \ge t_n\left\{d_{n+1} - \frac{\alpha_n}{h_n} d_N(x_N - x_1) - \left[\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1)\right]\right\}. \end{aligned} \tag{5.6}$$

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However, in general, there may not exist nonnegative shape parameters satisfying the above inequality, for instance, consider the case where

$$\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1) - [d_n - \frac{\alpha_n}{h_n} d_1 (x_N - x_1)] < 0 \qquad \text{and}$$
$$d_{n+1} - \frac{\alpha_n}{h_n} d_N (x_N - x_1) - [\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1)] > 0,$$

and hence we may not get a monotone α-fractal rational cubic spline. However, if the derivative parameters are selected so as to satisfy (5.4), $d_1 < \frac{y_N - y_1}{x_N - x_1} < d_N$, and the additional condition $\alpha_n < \min\{\frac{h_n(\Delta_n - d_n)}{y_N - y_1 - d_1(x_N - x_1)}, \frac{h_n(d_{n+1} - \Delta_n)}{d_N(x_N - x_1) - (y_N - y_1)}\}$ is imposed, then we can select r_n and t_n satisfying (5.6) by taking $\frac{r_n}{t_n} \geq \dfrac{d_{n+1} - \frac{\alpha_n}{h_n} d_N (x_N - x_1) - [\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1)]}{\Delta_n - \frac{\alpha_n}{h_n}(y_N - y_1) - [d_n - \frac{\alpha_n}{h_n} d_1 (x_N - x_1)]}$ to solve the monotonicity interpolation.

Analogously, for the monotonicity of s, the condition $r_n(\Delta_n - d_n) \geq t_n(d_{n+1} - \Delta_n)$ is sufficient. As in the fractal case, there may not exist nonnegative r_n and t_n satisfying the above mentioned inequality. However, if the given/estimated derivative values satisfy the condition prescribed in (5.4), which is stronger than the necessary monotonicity condition $d_n \geq 0$, then we can select $r_n > 0$ and $t_n > 0$ satisfying $r_n(\Delta_n - d_n) \geq t_n(d_{n+1} - \Delta_n)$ by demanding that $\frac{r_n}{t_n} \geq \frac{d_{n+1} - \Delta_n}{\Delta_n - d_n}$. This observation corrects the sufficient condition for the monotonicity of s (cf. equation (3.6)) studied in [31].

5.3. Positive interpolation. Given a data set $\{(x_n, y_n) : n = 1, 2, \ldots, N\}$ with $y_n > 0$, it is of interest to know whether the parameters involved in the α-fractal rational cubic spline s^α can be chosen such that $s^\alpha(x) > 0$ for all $x \in I = [x_1, x_N]$. Assume $\alpha_n \geq 0$, for all $n \in J$. Then from (3.10) and the iterative nature of the fractal interpolant, it follows that for $s^\alpha(x) > 0$, for all $x \in I$, it is sufficient to have $P_n(x) > 0$ for all $x \in I$, $n \in J$. Now the condition $P_n(x) > 0$ holds if

$$\begin{aligned} y_n - \alpha_n y_1 &> 0, \\ y_{n+1} - \alpha_{n+1} y_N &> 0, \\ (2r_n + t_n) y_n + r_n h_n d_n - \alpha_n \left[(2r_n + t_n) y_1 + r_n (x_N - x_1) d_1\right] &> 0, \\ (r_n + 2t_n) y_{n+1} - t_n h_n d_{n+1} - \alpha_n \left[(r_n + 2t_n) y_N - t_n (x_N - x_1) d_N\right] &> 0. \end{aligned} \tag{5.7}$$

For the first two inequalities to hold, it suffices to take $\alpha_n < \min\left\{\frac{y_n}{y_1}, \frac{y_{n+1}}{y_N}\right\}$. Thus, our search reduces to that of finding $r_n > 0$ and $t_n > 0$ such that third and fourth inequalities in (5.7) hold. It is not hard to verify that this is true if

$$\begin{aligned} t_n(y_n - \alpha_n y_1) + r_n h_n \left(d_n - \frac{\alpha_n}{h_n} d_1 (x_N - x_1)\right) &> 0, \\ r_n(y_{n+1} - \alpha_n y_N) - t_n h_n \left(d_{n+1} - \frac{\alpha_n}{h_n} d_N (x_N - x_1)\right) &> 0. \end{aligned} \tag{5.8}$$

Now to find a solution to these inequalities satisfying $r_n > 0, t_n > 0$, we impose an additional condition on the derivatives, namely $d_n \geq 0$ for all $n = 1, 2, \ldots, N$. Assuming d_1 and d_N to be non-zero, (5.8) can be satisfied by taking

$$\alpha_n < \min\left\{\frac{h_n d_n}{d_1(x_N - x_1)}, \frac{h_n d_{n+1}}{d_N(x_N - x_1)}\right\}, \qquad \frac{r_n}{t_n} > \frac{h_n[d_{n+1} - \frac{\alpha_n}{h_n}(x_N - x_1) d_N]}{y_{n+1} - \alpha_n y_N}.$$

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One may be interested in solving the inequalities in (5.8) in the absence of the additional condition on the derivative values, and we can heuristically argue that there may not exist values $r_n > 0$, $t_n > 0$ satisfying (5.8). Thus, maintaining positivity without the additional assumption on the derivatives values is doubtful. In conclusion, given a positive data set, where the derivative values satisfy $d_n \geq 0$, $n = 1, 2, \dots, N$, a positive/nonnegative rational cubic spline FIF can be constructed by choosing the parameters such that

$$\begin{aligned}
&\frac{r_n}{t_n} > \frac{h_n[d_{n+1} - \frac{\alpha_n}{h_n}(x_N - x_1)d_N]}{y_{n+1} - \alpha_n y_N},\\
&0 \leq \alpha_n < \min\Big\{a_n, \frac{y_n}{y_1}, \frac{y_{n+1}}{y_N}, \frac{h_n d_n}{d_1(x_N - x_1)}, \frac{h_n d_{n+1}}{d_N(x_N - x_1)}\Big\}, \qquad n \in J.
\end{aligned}$$

The above discussion reveals the following facts about the shape preserving properties of the α-fractal rational cubic spline introduced in this paper: (i) it is well suited for preserving convexity of the given data, (ii) given monotonically increasing data $\{(x_n, y_n) : n=1, 2, \dots, N\}$, which satisfy the necessary condition $d_n \geq 0$, suitable choices of the scaling factors and the shape parameters produce monotone α-fractal rational cubic splines provided the derivative parameters satisfy the additional condition

$$d_1 < \Delta_1 < \cdots < d_n < \Delta_n < d_{n+1} < \cdots < \Delta_{N-1} < d_N,$$

(iii) given positive data, $\{(x_n, y_n) : n = 1, 2, \dots, N\}$, the suitable choices of the parameters generate positive α-fractal rational cubic splines s^α provided the derivative parameters satisfy $d_n \geq 0$. Without these additional conditions on the derivative parameters, it is not certain whether or not the proposed α-fractal rational cubic spline s^α with linear denominator satisfactorily solves the monotonicity and positivity preservation problems. However, as mentioned elsewhere, our approach can be used to obtain a fractal generalization of the traditional rational cubic splines that solves all the three fundamental shape preserving problems; see, for instance, [29].

6. Numerical examples. The aim of this section is to illustrate the rational cubic spline fractal interpolation scheme and its shape features by some examples. We want to notice that in all the examples, the free shape parameters t_n, $n \in J$, are taken to be unity. Since the rational IFS scheme requires the derivative parameters as input, we shall describe an approximation method for their estimation.

Let a data set $\{(x_n, y_n) : n = 1, 2, \dots, N\}$ be given. To estimate the values of the derivatives at the knot points, the three point difference approximation for the arithmetic mean method (amm) can be used, which is expressed by the following equations:

$$d_n = \frac{h_n \Delta_{n-1} + h_{n-1}\Delta_n}{h_{n-1} + h_n}, \qquad n = 2, 3, \dots, N-1,$$

with end conditions

$$\begin{aligned}
d_1 &= \Big(1 + \frac{h_1}{h_2}\Big)\Delta_1 - \frac{h_1}{h_2}\Delta_{3,1}, & \Delta_{3,1} &= \frac{y_3 - y_1}{x_3 - x_1},\\
d_N &= \Big(1 + \frac{h_{N-1}}{h_{N-2}}\Big)\Delta_{N-1} - \frac{h_{N-1}}{h_{N-2}}\Delta_{N,N-2}, & \Delta_{N,N-2} &= \frac{y_N - y_{N-2}}{x_N - x_{N-2}}.
\end{aligned}$$

The nonlinear approximation by the geometric mean method (gmm) is given by

$$d_n = {\Delta_{n-1}}^{\frac{h_n}{h_n + h_{n-1}}} {\Delta_n}^{\frac{h_{n-1}}{h_n + h_{n-1}}}, \qquad n = 2, 3, \dots, N-1,$$

with end conditions

$$d_1 = {\Delta_1}^{(1+\frac{h_1}{h_2})} {\Delta_{3,1}}^{\frac{-h_1}{h_2}}, \qquad d_N = {\Delta_{N-1}}^{(1+\frac{h_{N-1}}{h_{N-2}})} {\Delta_{N,N-2}}^{\frac{-h_{N-1}}{h_{N-2}}}.$$

The gmm works well for monotonically increasing data but not necessarily for general data. On the contrary, the amm can be applied to general data and turns out to be well suited for convex data.

To illustrate convex interpolation with $s^\alpha \in \mathcal{C}^1$ and to study the effects of perturbation of the scaling factors in the resulting convex α-fractal rational cubic spline, we consider simple convex data $\mathcal{D}_1 = \{(1,1),(1.5,0.7),(3,1.7)\}$. The derivative parameters are estimated using the amm. To obtain a convex α-fractal rational cubic spline s^α, we take the scaling factors as $\alpha_1 = 0.06$ and $\alpha_2 = 0.39$; see Theorem 5.1. The constrained shape parameters r_n, $n = 1, 2$, are calculated with the help of Theorem 5.1, and the corresponding convex α-fractal rational spline $s^\alpha \in \mathcal{C}^1$ and its first derivative $(s^\alpha)^{(1)}$ are displayed in Figure 6.1a. Due to the implicit and recursive nature of the α-fractal function, each curve segment between two knot points will have global properties inherited from the entire set of interpolating points. Thus, theoretically, a perturbation in a scaling factor or shape parameter pertaining to a particular subinterval may influence the shape of the entire curve. To study this in practice, we modify some specific scaling factors in Figure 6.1a whilst maintaining the convexity condition.

Firstly, we change α_1 to 0.01 keeping the other scaling factors as in Figure 6.1a. The constrained shape parameters r_n are calculated using Theorem 5.1, and the corresponding convex α-fractal rational cubic spline s^α and its first derivative are displayed in Figure 6.1b. It can be observed that in comparison with Figure 6.1a, s^α in Figure 6.1b changes only in the first subinterval. Next, we modify $\alpha_2 = 0.03$ keeping all other scaling factors and shape parameters as in Figure 6.1a. The constrained shape parameters are calculated, and the resulting spline is shown in Figure 6.1c. In this case, as far as the α-fractal function s^α is concerned, apparent changes occur only in the second subinterval. Thus, from a theoretical standpoint, a perturbation of a particular component of a scale vector α may ripple through the entire interpolating interval, but practically, it has prominent influence only in the corresponding subinterval. To be precise, since the completely local convexity preserving classical rational cubic scheme emerges as a special case when the scaling factors are taken to be zero, the proposed fractal scheme is locally or globally depending on the values of the scaling factors. With a null scaling vector, the shape parameters r_n, $n = 1, 2$, are calculated according to Theorem 5.1. This retrieves a convexity preserving classical rational cubic spline interpolant s displayed in Figure 6.1d. Thus, the scaling factors not only provide a layer of flexibility in adjusting the shape of the interpolant but also control the fractality of the derivative of the interpolant. Furthermore, one would like to quantify the irregularity, for which the fractal dimension can be employed. As $|\alpha_n|$ is increased from zero, the dimension of the graph of the fractal function increases [1]. The fractal dimension of the derivative of the smooth α-fractal functions constitutes a numerical characterization of a signal. Recall that if a real function is smooth, the fractal dimension of its graph is one. In this case, this parameter cannot be used as a quantifying parameter for the complexity of a signal. Due to the presence of varying irregularities in the first derivatives, the proposed α-fractal rational cubic splines have potential applications in areas wherein the data set has convexity in the measured variable and fractality in the variable representing the derivative, for instance, for data arising in connection with nonlinear motions occurring in electro-mechanical systems [17].

To illustrate Theorem 5.3, we consider the convex and monotonically increasing data $\mathcal{D}_2 = \{(-2,0.25),(-1,1),(-0.3,11.11),(-0.2,25)\}$. The derivative values are estimated using gmm to satisfy the necessary condition. Taking $\alpha_1 = 0.01$, $\alpha_2 = 0.15$, and $\alpha_3 = 0.003$, the constrained shape parameters r_n are calculated by the formulas in Theorem 5.3. With

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these values of the parameters, the corresponding IFS code is iterated to generate Figure 6.2a, which represents a convex and monotone fractal curve.

For simplicity of presentation, we have considered a data set which is convex on the entire interpolation interval. However, with a simple modification, the present interpolation scheme can be adapted for generating a fractal curve that is co-convex with the given data. Let us illustrate this with an example. Consider the data set

$$\begin{aligned}\mathcal{D}_3 &= \Big\{(x_n, y_n, d_n) : n = 1, 2, \ldots, 8\Big\}\\ &= \Big\{(0,0,5), (1,3,1.342), (2,3.6,0.346), (3,3.8,0.1058),\\ &\qquad (4,4.1,0.6408), (5,5.5,1.54), (6,7.2,1.74), (7,9,1.854)\Big\}.\end{aligned}$$

We divide the interpolation interval $I = [0,7]$ into two subintervals, namely $I_1 = [0,3]$ and $I_2 = [3,7]$ such that the data have the same type of convexity (convex or concave) throughout that subinterval. The concavity preserving algorithm with the scale vector $\alpha^1 = (\alpha_1^1, \alpha_2^1, \alpha_3^1)$, where $\alpha_1^1 = 0.1$, $\alpha_2^1 = 0.06$, $\alpha_3^1 = 0.01$, produce a concave α-fractal rational cubic spline s^{α^1} on I_1. In a similar manner, the convexity preserving algorithm with the scale vector $\alpha^2 = (\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2)$ with $\alpha_1^2 = 0.04$, $\alpha_2^2 = 0.06$, $\alpha_3^2 = \alpha_4^2 = 0.01$ produces a convex α-fractal function s^{α^2} on I_2. Define a FIF s^α by $s^\alpha|_{I_i} = s^{\alpha^i}$ for $i = 1, 2$. Note that the Hermite interpolation conditions on s^{α^i}, $i = 1, 2$, provide $\mathcal{C}^1$-smoothness for s^α. The fractal function s^α given in Figure 6.2b is co-convex with the given data set and has a point of inflection at $x = 3$. We would like to remark that in case of too few data points being available in a subinterval for the iteration of the IFS scheme, we insert node points such that the inserted node is consistent with the desired shape.

7. Concluding remarks and possible extensions. Smooth FIFs provide an advance in the technique of approximation since various classical real-data interpolation problems can be generalized by means of these maps. However, methods for constructing smooth fractal interpolants available in the fractal-related literature are not satisfactory for generalizing the traditional non-recursive rational spline interpolants with shape parameters that are used for shape preserving interpolation. By a modification of the procedure in [1, 28], a general method is proposed in the present work for the construction of $\mathcal{C}^p$-continuous α-fractal functions. The construction of smooth α-fractal functions given in this paper provides a unified approach for the generalization of various traditional non-recursive shape preserving rational spline interpolants.

Utilizing our procedure, we have constructed α-fractal rational cubic splines with two families of shape parameters. With a mild condition on the scaling factors, the present α-fractal rational cubic spline possesses convergence properties analogous to its classical counterpart. The created fractal function is investigated for use in convexity preserving interpolation. Our convexity preserving α-fractal function scheme generalizes the convexity preserving classical interpolation developed in [31]. For a restricted class of data, the developed α-fractal rational cubic spline can produce monotone and positive interpolants as well. It is noted that the shape preserving classical interpolation schemes remain suitable only for interpolating data generated from functions with smooth (except possibly at a finite number of points) first or second derivatives, whereas the proposed fractal generalization works well for functions with smooth or non-smooth derivatives. The fractal dimension of the derivative may be used as a quantifying parameter to study the complexity of a signal and to make comparisons. Thus, the present method supersedes its classical counterpart and finds practical utility in areas such as geometric modeling, curve design, and nonlinear phenomena.

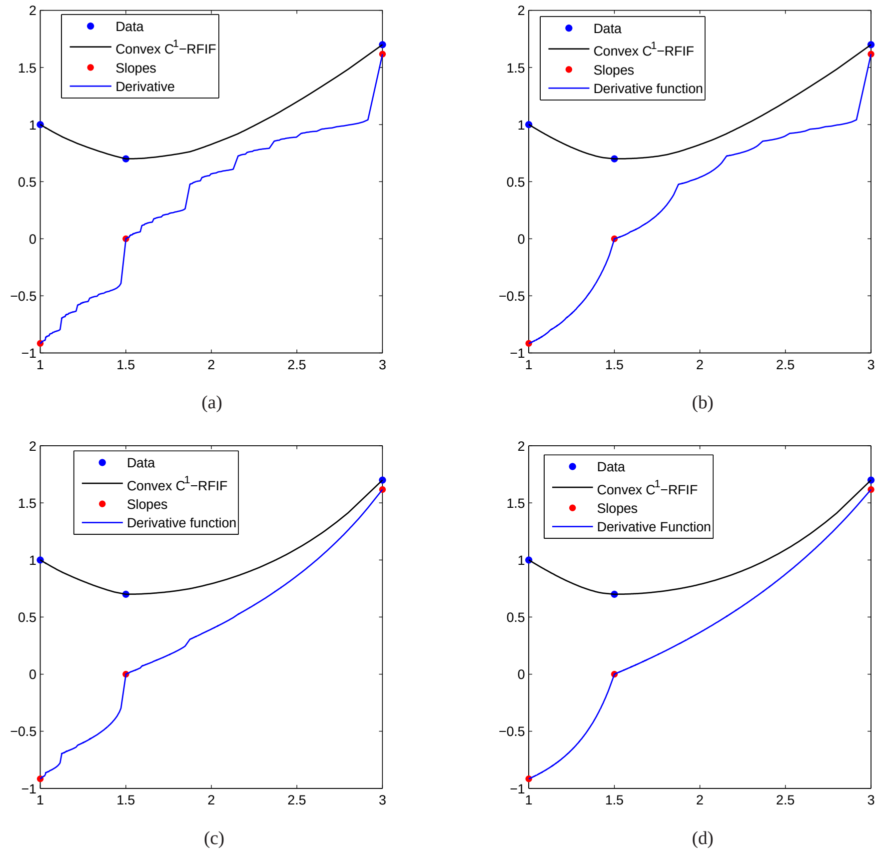

FIG. 6.1. *Convexity preserving α-fractal rational cubic splines and their first derivatives.*

Challenges that remain to be addressed are the following. By definition, our α-fractal rational cubic spline is in the class $\mathcal{C}^1$. Following the procedure described in [8], the continuity of an α-fractal spline can be enhanced to $\mathcal{C}^2$ by finding the derivative parameters via the solution of a suitable linear system of equations. However, the convexity problem was solved with the α-fractal $\mathcal{C}^1$-rational cubic spline. It is natural to query on the possibility of constructing a $\mathcal{C}^2$-continuous convex α-fractal rational cubic spline. A close observation of our discussion will imply that the aforementioned problem basically leads to the problem of solving a constrained nonlinear system of equations.

Though the presence of the parameters yield much flexibility, at times there may be a curse of choice, and the user may encounter problems of selecting the "optimal" ones. A couple of strategies that are likely to be useful to settle the issue of optimality are as follows. It is common to select a preferable shape preserving interpolant by minimizing a choice functional subject to the constraints arising from the shape requirement. A widely used one is the Holladay functional or an approximation thereof. From the point of view of fractal approximation theory, this is an "inverse problem" which reads as: given a function (or set of sampled values), recover the IFS parameters generating this function. Levkovich's work [22], in which contraction affine mappings generating a given function is obtained based on the connection between the maxima skeleton of the wavelet transform of the function and positions of the fixed points of the mappings in question, may provide basic tools for settling this issue. However, for adapting this, in the first place, the method has to be modified and

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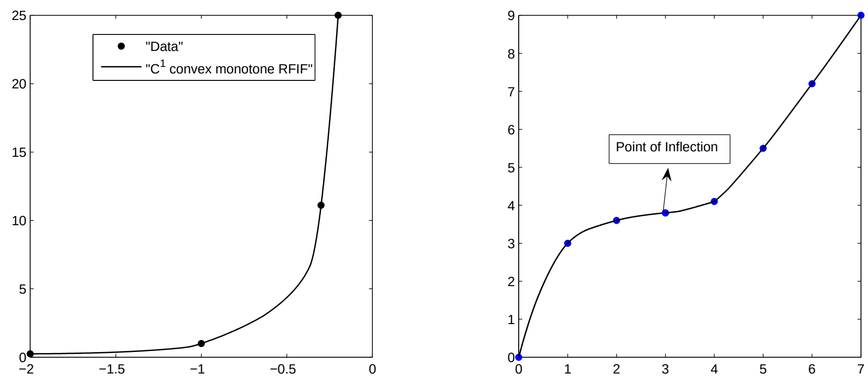

(a) *Convex monotonically increasing α-fractal rational cubic spline.*

(b) *A fractal ogee with point of inflection at $x = 3$.*

FIG. 6.2. *Convexity preserving α-fractal cubic splines.*

extended to cover the non-affine setting. Alternatively, following Lutton et al. [23], it should be possible to use genetic algorithms for solving these types of inverse problems.

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