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ZEROS AND SINGULAR POINTS FOR ONE-SIDED COQUATERNIONIC POLYNOMIALS WITH AN EXTENSION TO OTHER $\mathbb{R}^4$ ALGEBRAS*

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Abstract. For finding the zeros of a coquaternionic polynomial p of degree n, where p is given in standard form $p(z) = \sum c_j z^j$, the concept of a (real) companion polynomial q of degree $2n$, as introduced for quaternionic polynomials, is applied. If z_0 is a root of q, then, based on z_0, there is a simple formula for an element z with the property that $\overline{p(z)}p(z) = 0$, thus z is a singular point of p. Under certain conditions, the same z has the property that $p(z) = 0$, thus z is a zero of p. There is an algorithm for finding zeros and singular points of p. This algorithm will find all zeros z with the property that in the equivalence class to which z belongs, there are complex elements. For finding zeros which are not similar to complex numbers, Newton's method is applied, and a simple technique for computing the exact Jacobi matrix is presented. We also show, that there is no "Fundamental Theorem of Algebra" for coquaternions, but we state a conjecture that a "Weak Fundamental Theorem of Algebra" for coquaternions is valid. Several numerical examples are presented. It is also shown how to apply the given results to other algebras of $\mathbb{R}^4$ like tessarines, cotessarines, nectarines, conectarines, tangerines, cotangerines.

Key words. zeros of coquaternionic polynomials, zeros of polynomials in split quaternions, companion polynomial for coquaternionic polynomials, singular points for coquaternionic polynomials, Newton method for coquaternionic polynomials, exact Jacobi matrix for coquaternionic polynomials, "Weak Fundamental Theorem of Algebra" for coquaternions, zeros of polynomials in other $\mathbb{R}^4$ algebras (tessarines, cotessarines, nectarines, conectarines, tangerines, cotangerines)

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1. Introduction. In this paper we will use the notations $\mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{H}$ for the integers, the real number system, the complex numbers, and the quaternions, respectively. Coquaternions were introduced in 1849 by Sir James Cockle[1] (1819–1895) [3]. They may be regarded as elements of $\mathbb{R}^4$ of the form

$$a := a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}, \qquad a_1, a_2, a_3, a_4 \in \mathbb{R},$$

which we also abbreviate by $a = (a_1, a_2, a_3, a_4)$ and which obey the multiplication rules given in Table 1.1.

The algebra of coquaternions will be abbreviated by $\mathbb{H}_{\mathrm{coq}}$. The explicit multiplication rule for the product ab of two coquaternions $a = (a_1, a_2, a_3, a_4),\ b = (b_1, b_2, b_3, b_4)$ derived from Table 1.1 is

$$\begin{aligned} ab = {} & a_1b_1 - a_2b_2 + a_3b_3 + a_4b_4 \\ & + (a_1b_2 + a_2b_1 - a_3b_4 + a_4b_3)\mathbf{i} \\ & + (a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2)\mathbf{j} \\ & + (a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)\mathbf{k}, \end{aligned}$$

which implies that

$$a^2 = a_1^2 - a_2^2 + a_3^2 + a_4^2 + 2a_1(a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}). \tag{1.1}$$

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[1]For a biography see http://www.oocities.org/cocklebio/.

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TABLE 1.1
Multiplication table for coquaternions. Colored entries differ in sign from the quaternionic case.

	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	1	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	1

As elements of $\mathbb{R}^4$ we have $1 := (1,0,0,0)$, $\mathbf{i} := (0,1,0,0)$, $\mathbf{j} := (0,0,1,0)$, $\mathbf{k} := (0,0,0,1)$. If a coquaternion has the form $a = (a_1, 0, 0, 0)$, we will call it *real* and identify it with a_1. The real coquaternions and no others have the property that they commute with all coquaternions. In algebraic terms this means that the *center* of $\mathbb{H}_{\text{coq}}$ is $\mathbb{R}$. The first component a_1 of a coquaternion $a = (a_1, a_2, a_3, a_4)$ will be called *real part* of a and abbreviated by $\Re(a)$. The second component a_2 will be called the *imaginary part* of a and denoted by $\Im(a)$. Complex numbers $a_1 + a_2\mathbf{i}$ will be identified with $a := (a_1, a_2, 0, 0) \in \mathbb{H}_{\text{coq}}$ and vice versa, and the coquaternion a will then be called *complex*. We will use the notations

$$\text{abs}_2(a) := a_1^2 + a_2^2 - a_3^2 - a_4^2, \quad \overline{a} := \text{conj}(a) := (a_1, -a_2, -a_3, -a_4), \tag{1.2}$$

where both $\overline{a}$ and $\text{conj}(a)$ are called the *conjugate of* a, and it should be noted that abs_2 is not the square of a norm since abs_2 may take on negative values. The following additional properties hold as well:

$$a\overline{a} = \overline{a}a = \text{abs}_2(a), \quad \text{abs}_2(ab) = \text{abs}_2(ba) = \text{abs}_2(a)\text{abs}_2(b), \quad \Re(ab) = \Re(ba). \tag{1.3}$$

A coquaternion a will be called *invertible* if $\text{abs}_2(a) \neq 0$, and in this case

$$a^{-1} = \frac{\overline{a}}{\text{abs}_2(a)}, \quad \text{abs}_2(a^{-1}) = \frac{1}{\text{abs}_2(a)}.$$

A noninvertible coquaternion is also called *singular*. An invertible coquaternion is called *nonsingular*. A product $a_1 a_2 \cdots a_k,\ k \geq 1$, is singular if and only if one of the factors is singular. This follows from the second identity in (1.3). A more detailed survey of properties of coquaternions is given in [11]. Applications to physical problems are treated in [2, 6]. Information regarding general algebraic systems can be found in [1, 4, 7, 17].

Let

$$p(z) := \sum_{j=0}^{n} c_j z^j, \quad z,\ c_j \in, \mathbb{H}_{\text{coq}},\ \text{for } j = 0, 1, \ldots, n,\ \text{with } c_0 \neq 0,\ c_n \neq 0, \tag{1.4}$$

be a coquaternionic polynomial. The conditions on c_0, c_n in (1.4) ensure that $p(0) \neq 0$ and that the degree of p is not smaller than n. Because of the noncommutativity of the elements in $\mathbb{H}_{\text{coq}}$, we call this polynomial *one-sided* since there are other forms of polynomials with terms of the form $c_j z^j d_j$, called *two-sided*, or even with terms of the form $c_0 z c_1 z c_2 z \cdots c_{j-1} z c_j$, which are called a *monomials of degree* j. An arbitrary finite sum of monomials of any degree is called a *general coquaternionic polynomial*. Since we are dealing here only with one-sided coquaternionic polynomials, we will omit the word one-sided in the following. Solutions of $p(z) = 0$ will be called *zeros* of p. Before we start any general investigation, we will treat the most simple quadratic case since it sheds already some light on the general case.

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EXAMPLE 1.1. Let

$$p(z) := z^2 - c, \qquad z = (z_1, z_2, z_3, z_4), \;\; c = (c_1, c_2, c_3, c_4) \in \mathbb{H}_{\mathrm{coq}}, \tag{1.5}$$

and let us look for those z with $p(z) = 0$. We will call these z *square roots of* c and use the notation $\sqrt{c}$. The equation $p(z) = 0$ splits into four real equations (see (1.1))

$$z_1^2 - z_2^2 + z_3^2 + z_4^2 = c_1, \tag{1.6}$$

$$2z_1 z_j = c_j, \qquad j = 2, 3, 4. \tag{1.7}$$

(a) Let $c \neq 0$ be nonreal, i.e., $c_j \neq 0$ for at least one $j \in \{2, 3, 4\}$. Then, (1.7) implies that $z_1 \neq 0$ and

$$z_j = \frac{c_j}{2z_1}, \qquad j = 2, 3, 4,$$

$$z_1^2 - \left(\frac{c_2}{2z_1}\right)^2 + \left(\frac{c_3}{2z_1}\right)^2 + \left(\frac{c_4}{2z_1}\right)^2 = c_1.$$

The last equation can also be written as

$$z_1^4 - c_1 z_1^2 + \frac{-c_2^2 + c_3^2 + c_4^2}{4} = 0.$$

The standard solution formula yields

$$z_1^2 = \frac{1}{2}\left(c_1 \pm \sqrt{\operatorname{abs}_2(c)}\right). \tag{1.8}$$

Since $z_1 \neq 0$ must be real, the existence of a solution z_1 depends on the validity of the conditions

$$\text{(i)}\;\; \operatorname{abs}_2(c) \geq 0, \qquad \text{(ii)}\;\; c_1 - \sqrt{\operatorname{abs}_2(c)} > 0, \qquad \text{(iii)}\;\; c_1 + \sqrt{\operatorname{abs}_2(c)} > 0, \tag{1.9}$$

and this result is summarized in Lemma 1.4.
(b) Let $c = \Re(c)$, i.e., $c = (c_1, 0, 0, 0)$. In this case, (1.7) has several solutions:
  (b1) $z_1 = 0$ and z_j arbitrary, for $j = 2, 3, 4$, or
  (b2) $z_j = 0$, for $j = 2, 3, 4$, and z_1 arbitrary.
In case (b1), equations (1.6), (1.7) yield

$$z_1 = 0, \quad -z_2^2 + z_3^2 + z_4^2 = c_1. \tag{1.10}$$

In case (b2), equations (1.6), (1.7) yield

$$z_1^2 = c_1, \quad z_j = 0, \; j = 2, 3, 4. \tag{1.11}$$

Case (b2) will not have a solution if $c_1 < 0$, but case (b1) will have infinitely many solutions. For $c = 0$, for instance, we obtain from (1.10) that $\sqrt{c} = (0, z_2, z_3.z_4)$ with $-z_2^2 + z_3^2 + z_4^4 = 0$.

The just encountered phenomenon that polynomials may have infinitely many zeros is a typical feature for all coquaternionic polynomials with real coefficients.

THEOREM 1.2. *Let p be a coquaternionic polynomial as defined in (1.4) where all coefficients $c_j, j = 0, 1, \ldots, n$, are real. Then,*

$$p(z) = 0 \Rightarrow p(h^{-1}zh) = 0 \qquad \text{for all nonsingular } h \in \mathbb{H}_{\mathrm{coq}}.$$

In particular, if z is nonreal, p has infinitely many zeros.

Proof. From the fact that all real elements (and no others) commute with all elements in $\mathbb{H}_{\mathrm{coq}}$, we deduce

$$\begin{aligned} h^{-1}p(z)h = h^{-1}\left(\sum_{j=0}^{n} c_j z^j\right) h &= \sum_{j=0}^{n} c_j h^{-1} z^j h \\ &= \sum_{j=0}^{n} c_j (h^{-1}zh)^j = p(h^{-1}zh). \end{aligned}$$

The set $\{u \in \mathbb{H}_{\mathrm{coq}} : u = h^{-1}zh$ for all nonsingular $h \in \mathbb{H}_{\mathrm{coq}}\}$ contains either exactly one element if $z \in \mathbb{R}$, or, otherwise, it contains infinitely many elements. □

LEMMA 1.3. *Let p be given as in (1.5) with $c := (c_1, 0, 0, 0) \in \mathbb{H}_{\mathrm{coq}}$. Then there exist infinitely many zeros $z = (0, z_2, z_3, z_4)$ of p defined in (1.10). If $c_1 \geq 0$, there are up to two additional zeros $\sqrt{c} = (\pm\sqrt{c_1}, 0, 0, 0)$. For $c_1 < 0$, there are no additional zeros.*

Proof. Follows from the solution formulas (1.10), (1.11). □

LEMMA 1.4. *Let p be given as in (1.5) with $c \in \mathbb{H}_{\mathrm{coq}} \backslash \mathbb{R}$.*

(I) *If* (i) *of* (1.9) *is not valid, p has no zeros.*
(II) *If* (i) *is valid but* (ii) *and* (iii) *are not valid, p has no zeros.*
(III) *If* (i) *and* (ii) *are valid, p has up to four zeros.*
(IV) *If* (i) *and* (iii) *are valid, but* (ii) *is not valid, p has up to two zeros.*

Proof. In view of the solution formula (1.8) for z_1, which yields two solutions if the expression in parentheses is positive, the cases (I), (II), (IV) are clear. In (III), the validity of (ii) implies the validity of (iii). □

To summarize, infinitely many square roots will always occur if c is real and in no other cases. The cases (I), (II) of Lemma 1.4 are conditions such that there is no square root at all.

EXAMPLE 1.5. For each case mentioned in Lemma 1.4, there is an example:

$$\begin{aligned} &(\mathrm{I}): \ c := (1,2,3,4) \text{ has no square root since } \mathrm{abs}_2(c) = -20 < 0; \\ &(\mathrm{II}): \ c := (-2,1,2,0) \text{ has no square root since } \mathrm{abs}_2(c) = 1 > 0, \text{ but} \\ &\qquad c_1 - \sqrt{\mathrm{abs}_2(c)} = -3 < 0, \ c_1 + \sqrt{\mathrm{abs}_2(c)} = -1 < 0; \\ &(\mathrm{III}): \ c := (2,1,2,0) \text{ has four square roots since } \mathrm{abs}_2(c) = 1 > 0, \\ &\qquad c_1 - \sqrt{\mathrm{abs}_2(c)} = 1 > 0, \ c_1 + \sqrt{\mathrm{abs}_2(c)} = 3 > 0, \\ &\qquad \sqrt{c} = \pm\sqrt{2}\,(1/2, 1/2, 1, 0), \ \sqrt{c} = \pm\sqrt{6}\,(1/2, 1/6, 1/3, 0); \\ &(\mathrm{IV}): \ c := (4,6,5,1) \text{ has two square roots since } \mathrm{abs}_2(c) = 26 > 0, \\ &\qquad c_1 - \sqrt{\mathrm{abs}_2(c)} = 4 - \sqrt{26} < 0, \ a := c_1 + \sqrt{\mathrm{abs}_2(c)} = 4 + \sqrt{26} > 0, \\ &\qquad \sqrt{c} = \pm\sqrt{2a}\left(\frac{1}{2}, \frac{3}{a}, \frac{5}{2a}, \frac{1}{2a}\right). \end{aligned}$$

COROLLARY 1.6. *There is no "Fundamental Theorem of Algebra" for coquaternions.*

Proof. There is a well known "Fundamental Theorem of Algebra" for quaternions by Eilenberg and Niven from 1944 [5] ensuring that all general quaternionic polynomials of

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degree $n \geq 1$ have at least one zero provided that there is only one monomial term of the highest degree n. However, in the coquaternionic case, the first two parts of Lemma 1.4 show that there are coquaternionic, quadratic polynomials without any zeros. ∎

2. Similarity and quasi-similarity of coquaternions. In Lam [14, p. 52] and in Janovská and Opfer [11], we find that $\mathbb{H}_{\mathrm{coq}}$ is isomorphic to $\mathbb{R}^{2\times 2}$, where this isomorphism is defined by

$$\hat{\imath}(a) = \hat{\imath}(a_1, a_2, a_3, a_4) := \begin{bmatrix} a_1 + a_4 & a_2 + a_3 \\ -a_2 + a_3 & a_1 - a_4 \end{bmatrix}. \tag{2.1}$$

The mapping $\hat{\imath} : \mathbb{H}_{\mathrm{coq}} \to \mathbb{R}^{2\times 2}$ is invertible. Let $\mathbf{B} := \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be arbitrary. Then,

$$\hat{\imath}^{-1}(\mathbf{B}) = \frac{1}{2}\big(b_{11} + b_{22}, b_{12} - b_{21}, b_{12} + b_{21}, b_{11} - b_{22}\big).$$

Note that

$$\mathrm{abs}_2(a) = \det(\hat{\imath}(a)), \quad 2\Re(a) = \mathrm{tr}(\hat{\imath}(a)), \tag{2.2}$$

where $\det$ and tr stand for *determinant* and *trace*, respectively. We will transfer the concept of similarity of matrices to coquaternions.

DEFINITION 2.1. *Two coquaternions a, b will be called similar, denoted by $a \sim b$, if the corresponding matrices $\hat{\imath}(a), \hat{\imath}(b)$ are similar, or in other words, if there is a nonsingular (=invertible) coquaternion h such that $a = h^{-1}bh$. By*

$$[a] := \{b : b = h^{-1}ah, \textit{ for all invertible } h \in \mathbb{H}_{\mathrm{coq}}\}$$

we denote the equivalence class of all coquaternions which are similar to a.

The fact that this definition of similarity yields an *equivalence relation* was shown by Horn and Johnson [9, p. 45]. We have the following property:

$$a = \Re(a) \Leftrightarrow [a] = \{a\}, \tag{2.3}$$

which means that the equivalence class $[a]$ consists of one single element if and only if a is real.

LEMMA 2.2. *Let $a \sim b$. Then*

$$\Re(a) = \Re(b), \quad \mathrm{abs}_2(a) = \mathrm{abs}_2(b). \tag{2.4}$$

Proof. Both parts follow easily from conditions given in (1.3). ∎

In contrast to the quaternionic case, the conditions (2.4) are not sufficient for similarity. Take $a = (\alpha, 0, 0, 0),\ b = (\alpha, 5, 4, 3)$ for an arbitrary $\alpha \in \mathbb{R}$. Since $[a]$ consists of one single element only (see (2.3)), a, b are not similar. However, (2.4) is valid.

Similarity is a very useful tool when investigating properties of matrices such as the determination of the rank of a matrix. However, in this investigation we are mainly interested in a consequence of similarity, namely in the properties mentioned in (2.4) which in some cases are also valid for nonsimilar matrices or nonsimilar coquaternions.

DEFINITION 2.3. *Two coquaternions a, b are said to be* quasi-similar, *written as $a \overset{q}{\sim} b$, if (2.4) is valid.*

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LEMMA 2.4. *The relation* $\overset{q}{\sim}$ *is an equivalence relation.*

Proof. The three properties of equivalence relations, $a \overset{q}{\sim} a$ (reflexivity), $a \overset{q}{\sim} b \Leftrightarrow b \overset{q}{\sim} a$ (symmetry), $a \overset{q}{\sim} b,\ b \overset{q}{\sim} c \Rightarrow a \overset{q}{\sim} c$ (transitivity) are easily verified. ∎

The corresponding equivalence classes are denoted by

$$[a]_q := \{b : b \overset{q}{\sim} a\}.$$

We have the following simple properties:

$$a \sim b \Rightarrow a \overset{q}{\sim} b, \quad [a] \subset [a]_q.$$

Because of the first condition in (2.4), distinct real numbers are in different equivalence classes with respect to $\sim$ and to $\overset{q}{\sim}$. In a later section (Section 4) we will see that there is only a small difference between similarity and quasi-similarity.

3. Reformulation of coquaternionic polynomials via matrix equivalents. Let the matrix $\mathbf{A}\in\mathbb{R}^{2\times 2}$ be arbitrary. Then (see Horn and Johnson [9, p. 87]),

$$\mathbf{A}^j \in \langle \mathbf{I}, \mathbf{A}\rangle, \quad j \in \mathbb{N}\cup\{0\}, \tag{3.1}$$

where $\langle\cdots\rangle$ is the linear span of the elements between the brackets. The characteristic polynomial of $\mathbf{A}$ is (see (2.2))

$$\chi_{\mathbf{A}}(z) := z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2a_1 z + \mathrm{abs}_2(a).$$

The Cayley-Hamilton Theorem (see Horn and Johnson [9, p. 86]) implies the matrix identity $\mathbf{A}^2 = -\mathrm{abs}_2(a)\mathbf{I} + 2\Re(a)\mathbf{A}$. Because of the isomorphism between $\mathbb{R}^{2\times 2}$ and $\mathbb{H}_{\mathrm{coq}}$ and the relation (3.1), we have

$$z^j = \alpha_j + \beta_j z, \quad j = 0, 1, \ldots, \tag{3.2}$$

where the coefficients can be determined by the recursion

$$\alpha_0 = 1, \qquad \beta_0 = 0, \tag{3.3}$$

$$\alpha_{j+1} = -\mathrm{abs}_2(z)\beta_j, \qquad \beta_{j+1} = \alpha_j + 2\Re(z)\beta_j, \qquad j = 0, 1, \ldots \tag{3.4}$$

We observe that all coefficients α_j, β_j, $j \geq 0$, are real and, more important, they depend only on $\Re(z)$ and $\mathrm{abs}_2(z)$ and not fully on z.

LEMMA 3.1. *Let* $a, b \in \mathbb{H}_{\mathrm{coq}}$ *and* $a \overset{q}{\sim} b$*. Then,* a^j *and* b^j *have the same recursion coefficients* $\alpha_j, \beta_j,\ j \geq 0$ *as defined in* (3.2).

Proof. The quasi-similarity implies that the coefficients $\Re(z), \mathrm{abs}_2(z)$ occurring in (3.4) are the same for a and b. Thus, the recursion coefficients α_j, β_j are the same for a and b. ∎

There is another difference between coquaternions and quaternions. In the corresponding quaternionic equivalence classes $[a]$, there are always complex elements provided that $a \notin \mathbb{R}$; see [13].

LEMMA 3.2. *Let* $a = (a_1, a_2, a_3, a_4) \in \mathbb{H}_{\mathrm{coq}}\backslash\mathbb{C}$*. Then* $\Delta := [a]_q \cap \mathbb{C} \neq \emptyset$ *if and only if*

$$a_2^2 - a_3^2 - a_4^2 \geq 0.$$

In this case $\Delta = \{b^+, b^-\}$*, where* $b^{\pm} = a_1 \pm \sqrt{a_2^2 - a_3^2 - a_4^2}\,\mathbf{i}$.

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Proof. Let $b \in \Delta$. Then, $b := b_1 + b_2\mathbf{i}$, $a_1 = \Re(a) = \Re(b) = b_1$, and

$$a_1^2 + a_2^2 - a_3^2 - a_4^2 = \operatorname{abs}_2(a) = \operatorname{abs}_2(b) = b_1^2 + b_2^2 \Leftrightarrow b_2^2 = a_2^2 - a_3^2 - a_4^2 \geq 0.$$

By similar arguments, $a_2^2 - a_3^2 - a_4^2 < 0$ implies $\Delta = \emptyset$. □
Thus, in $[a := (1, 2, 3, 4)]_q$ there is no complex number.

We replace z^j in the definition of the polynomial p in (1.4) by the recursion formula for z^j and obtain

$$\begin{aligned} p(z) &:= \sum_{j=0}^{n} c_j z^j = \sum_{j=0}^{n} c_j(\alpha_j + \beta_j z) = \sum_{j=0}^{n} \alpha_j c_j + \left(\sum_{j=0}^{n} \beta_j c_j\right) z \\ &=: A(\Re(z), \operatorname{abs}_2(z)) + B(\Re(z), \operatorname{abs}_2(z))z. \end{aligned} \tag{3.5}$$

Here, the two newly defined quantities, A, B, depend only on $\Re(z)$ and $\operatorname{abs}_2(z)$. Therefore, we have included this dependence in parentheses.

LEMMA 3.3. *The quantities* $A(\Re(z), \operatorname{abs}_2(z))$, $B(\Re(z), \operatorname{abs}_2(z))$ *defined in* (3.5) *are constant on the quasi-similar equivalence class* $[z]_q$.

Proof. The properties (2.4) are valid for all elements in the same quasi-similar equivalence class $[z]_q$. □

Let $a = (a_1, a_2, a_3, a_4)$. In contrast to (2.3) we have

$$a \in [a_1]_q \Leftrightarrow a_2^2 - a_3^2 - a_4^2 = 0. \tag{3.6}$$

Thus, there is no quasi-similar equivalence class with only one element. Let $a = (a_1, 0, 0, 0)$. Then for all $b \in [a]_q$, we have according to (3.6) that $\operatorname{abs}_2(a) = \operatorname{abs}_2(b) = a_1^2$. Condition (3.6) also implies that there is no complex element with nonvanishing imaginary part in $[a_1]_q$.

LEMMA 3.4. *Let* $p(z) = 0$. *Then,*

$$p(z) = A(\Re(z_0), \operatorname{abs}_2(z_0)) + B(\Re(z_0), \operatorname{abs}_2(z_0))z = 0, \qquad \textit{for all } z_0 \in [z]_q, \tag{3.7}$$

$$Bz = 0 \Leftrightarrow A = 0, \qquad Bz = 0 \Rightarrow B \textit{ is singular}, \tag{3.8}$$

$$B = 0 \Rightarrow A = 0. \tag{3.9}$$

If $B = 0$ *or* $Bz = 0$, *then all* $z_0 \in [z]_q$ *are zeros of* p. *Here we have omitted the arguments of* A, B *in* (3.8), (3.9).

Proof. (3.7) follows from (3.5) and the second part of Lemma 2.2. In (3.8), $Bz = 0$ implies that B is singular because $z \neq 0$. In (3.9), $B = 0 \Rightarrow A = 0$ is obvious. The cases $B = 0$ and $Bz = 0$ imply $p(z_0) = 0$ for all z_0 in $[z]_q$. □

DEFINITION 3.5. *Let* $z \in \mathbb{H}_{\mathrm{coq}} \backslash \mathbb{R}$. *If* $p(z_0) = 0$ *for all* $z_0 \in [z]_q$, *then we say that* z generates a class of hyperbolic zeros *or* z is a hyperbolic zero. *If there is exactly one zero in* $[z]_q$, *we call this zero* isolated.

There is an instance of a hyperbolic zero in Example 1.1, equation (1.10). The conditions given there (second condition in (1.10)), expressed as an equation for a hyperbola $x^2 - y^2 = c$, are the reason for using the word *hyperbolic*.

EXAMPLE 3.6. Let p be a coquaternionic polynomial with only real coefficients. Then all nonreal zeros of p are hyperbolic. See Theorem 1.2.

LEMMA 3.7. *Let* u, v *be two distinct zeros of a coquaternionic polynomial* p *in the same quasi-similar equivalence class* $[u]_q = [v]_q$. *Let* $z \in [u]_q$. *Then,* B *is singular in the representation* $p(z) = A + Bz$ *(with the arguments of* A, B *omitted). Moreover,* $B = 0$ *if* $u - v$ *is nonsingular. If* $B = 0$, *all elements in* $[u]_q = [v]_q$ *are zeros of* p *and all zeros are hyperbolic.*

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Proof. It follows that $p(u) = A + Bu = 0$, $p(v) = A + Bv = 0$, with the consequence that $p(u) - p(v) = B(u - v) = 0$, because A, B are constant on $[u]_q$. Thus, B is singular. If $u - v$ is nonsingular, then it follows that $B = 0$. If $B = 0$, then the statement follows from Lemma 3.4. Two distinct zeros in the same equivalence class can never be real. Thus, $[u]_q$ consists of (infinitely many) hyperbolic zeros of p. ∎

We can refine the representation (3.5) of the coquaternionic polynomial p by applying the *column operator* $\mathrm{col} : \mathbb{H}_{\mathrm{coq}} \to \mathbb{R}^{4\times 1}$ defined by

$$\mathrm{col}(h) = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}, \tag{3.10}$$

where $h = (h_1, h_2, h_3, h_4) \in \mathbb{H}_{\mathrm{coq}}$. The following lemma is very useful in this context.

LEMMA 3.8. *Let B, z be two coquaternions.*

(α) *There is a real 4×4 matrix $\mathbf{C}$, such that*

$$\begin{aligned} &\mathrm{col}(Bz) = \mathbf{C}\,\mathrm{col}(z) \text{ and} \\ &\quad\quad \mathbf{C} = [\mathrm{col}(B), \mathrm{col}(B\mathbf{i}), \mathrm{col}(B\mathbf{j}), \mathrm{col}(B\mathbf{k})]. \end{aligned} \tag{3.11}$$

(β) *The matrix $\mathbf{C}$ is singular if and only if B is singular.*

(γ) *If $B \neq 0$ is singular, then $\mathrm{rank}(\mathbf{C}) = 2$.*

Proof. (α) and (β) follow from [11, equation (5.2) and Theorem 5.1]. (γ) Let $B \neq 0$ be singular. One can explicitly show that the last two columns of $\mathbf{C}$ can be expressed as a linear combination of the first two columns. ∎

The application of the col operator to (3.5) now yields

$$\mathrm{col}(p(z)) = \mathrm{col}(A) + \mathrm{col}(Bz) = \mathrm{col}(A) + \mathbf{C}\mathrm{col}(z),$$

where $\mathbf{C}$ is defined in Lemma 3.8.

THEOREM 3.9. *Let $[z_0]_q$ contain a zero of the coquaternionic polynomial p. Then all zeros of p in $[z_0]_q$ can be found by solving the linear inhomogeneous system*

$$\mathbf{C}\mathrm{col}(z) = -\mathrm{col}(A), \qquad \text{subject to } z \in [z_0]_q. \tag{3.12}$$

(i) *Let B be nonsingular. Then there is exactly one (isolated) zero $z \in [z_0]_p$, which can be computed by*

$$z = -B^{-1}A = -\frac{\mathrm{conj}(B)A}{\mathrm{abs}_2(B)} \Leftrightarrow \mathrm{col}(z) = -\mathbf{C}^{-1}\mathrm{col}(A), \tag{3.13}$$

where A, B is short for $A(\Re(z_0), \mathrm{abs}_2(z_0)), B(\Re(z_0), \mathrm{abs}_2(z_0))$, respectively, and where $\mathbf{C}$ is defined in (3.11).

(ii) *Let B be singular. Then the linear system given in* (3.12) *has a solution if the extended matrix $[\mathbf{C}, col(A)]$ has the same rank as $\mathbf{C}$.*

Proof. This follows from Lemma 3.8, from equation (3.7) in Lemma 3.4, and simple facts from linear algebra. ∎

4. Symmetric and normal coquaternions. A comparison of similarity and quasi-similarity. We will make a very short excursion to symmetric and normal coquaternions and show that, in most cases, similarity and quasi-similarity are identical notions.

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DEFINITION 4.1. *A coquaternion a is called* symmetric *or* normal *if the matrix $\hat{\imath}(a)$ defined in (2.1) has these properties, respectively. Thus, a is symmetric and normal if*

$$\hat{\imath}(a) = \hat{\imath}(a)^{\mathrm{T}}, \quad \hat{\imath}(a)\hat{\imath}(a)^{\mathrm{T}} - \hat{\imath}(a)^{\mathrm{T}}\hat{\imath}(a) = \mathbf{0},$$

respectively, where $\hat{\imath}(a)^{\mathrm{T}}$ denotes the transpose of $\hat{\imath}(a)$.

LEMMA 4.2. *A coquaternion a is symmetric if and only if $\Im(a) = 0$. A coquaternion a is normal if and only if $\Im(a) = 0$ or $a \in \mathbb{C}$.*

Proof. The condition for symmetry follows immediately from (2.1). Let $a \in \mathbb{H}_{\mathrm{coq}}$ be arbitrary and $a = (a_1, a_2, a_3.a_4)$. Then,

$$\hat{\imath}(a)\hat{\imath}(a)^{\mathrm{T}} - \hat{\imath}(a)^{\mathrm{T}}\hat{\imath}(a) = -4a_2 \begin{bmatrix} -a_3 & a_4 \\ a_4 & a_3 \end{bmatrix}. \qquad \square$$

Thus, a coquaternion is normal if and only if it is symmetric or complex. Let $a, b \in \mathbb{H}_{\mathrm{coq}}$ be normal. Then, the conditions in (2.4) are sufficient for similarity of a, b; see Horn and Johnson [9, p. 109, Problem 15]. Thus, $a \sim b \Leftrightarrow a \overset{q}{\sim} b$ if a and b are normal. Two normal coquaternions $a = (a_1, 0, a_3, a_4)$, $b = (b_1, b_2, 0, 0)$ are similar if and only if $a_1 = b_1$ and $b_2 = a_3 = a_4 = 0$, which means $a = b \in \mathbb{R}$. Two distinct complex numbers are similar (=quasi-similar) if and only if they are conjugate complex to each other. Two normal coquaternions $a = (a_1, 0, a_3, a_4)$, $b = (b_1, 0, b_3, b_4)$ are similar (=quasi-similar) if and only if

$$a_1 = b_1, \quad a_3^2 + a_4^2 = b_3^2 + b_4^2.$$

LEMMA 4.3. *Let $a, b \in \mathbb{H}_{\mathrm{coq}} \backslash \mathbb{R}$ and let (2.4) be valid. Then $a \sim b$.*

Proof. The assumption (2.4) implies that the two characteristic polynomials for $\hat{\imath}(a)$ and $\hat{\imath}(b)$ (see 2.1) are identical. The assumption $a \notin \mathbb{R}$ is in matrix terms equivalent to $\hat{\imath}(a) \neq \alpha\mathbf{I}$, $\alpha \in \mathbb{R}$, where $\mathbf{I}$ is the identity matrix in $\mathbb{R}^{2\times 2}$. In other words, $\hat{\imath}(a)$ does not belong to the center of $\mathbb{R}^{2\times 2}$. The two possible canonical Jordan normal forms of $\hat{\imath}(a)$ and $\hat{\imath}(b)$ are

$$\text{(i):} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, \qquad \text{(ii):} \begin{bmatrix} z & 1 \\ 0 & z \end{bmatrix}.$$

The assumptions imply that $x \neq y$, and thus all eigenvalues (in both forms) have geometric multiplicity one; see Horn and Johnson [9, p. 135]. This means that similarity $a \sim b$ is equivalent to the two conditions (2.4) and, thus, equivalent to quasi-similarity provided both $a, b \notin \mathbb{R}$. $\square$

5. Singular points of coquaternionic polynomials. The zero point in $\mathbb{R}, \mathbb{C}, \mathbb{H}$ is in all these three cases the only singular point in the sense that it has no inverse. This is one of the reasons why polynomials in these three spaces are investigated for its zeros. This applies also to eigenvalues in matrix spaces over $\mathbb{R}, \mathbb{C}$ because they are zeros of the corresponding characteristic polynomials. In the space $\mathbb{H}_{\mathrm{coq}}$, however, there are infinitely many singular points. Therefore, it is not surprising that we will encounter points z such that $p(z)$ is singular, i.e., $\mathrm{abs}_2(p(z)) = 0$, but $p(z) \neq 0$, where p is defined in (1.4). We will conjecture in Section 11, that all coquaternionic polynomials of degree $n \geq 1$ have at least one singular point.

DEFINITION 5.1. *Let p be given as in (1.4). We say that $z \in \mathbb{H}_{\mathrm{coq}}$ is a* singular point *for p if*

$$\mathrm{abs}_2(p(z)) := p(z)\overline{p(z)} = 0. \tag{5.1}$$

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It is clear, that a zero of p is also a singular point for p. In contrast to the polynomial equation $p(z)=0$, which may be regarded as four real equations in four real unknowns, equation (5.1) is only one equation in four real unknowns. Thus, it is not a polynomial equation.

EXAMPLE 5.2. Let $p(z) := z^2 - c$, where c is a given coquaternion. Then,

$$\operatorname{abs}_2(p(z)) = (z^2-c)\overline{(z^2-c)} = \operatorname{abs}_2(z^2) - 2\Re(c\overline{z}^2) + \operatorname{abs}_2(c).$$

Now, $\operatorname{abs}_2(z^2) = (\operatorname{abs}_2(z))^2$, which follows from (1.3). The middle term in the previous identity is

$$\Re(c\,\overline{z}^2) = \Re(z^2\,\overline{c}) = c_1(z_1^2 - z_2^2 + z_3^2 + z_4^2) + 2z_1(c_2z_2 - c_3z_3 - c_4z_4).$$

Altogether,

$$\begin{aligned}\operatorname{abs}_2(p(z)) = {}& (z_1^2+z_2^2-z_3^2-z_4^2)^2 - 2c_1(z_1^2 - z_2^2 + z_3^2 + z_4^2)\\ & - 4z_1(c_2z_2 - c_3z_3 - c_4z_4) + \operatorname{abs}_2(c).\end{aligned}$$

If we choose $c=(1,2,3,4)$ as in Example 1.5, part (I), and $z=(z_1,z_2,z_3,z_4)$ with

$$z_1 = z_2 = z_3 = z_4 = \pm\frac{1}{2}\sqrt{5},$$

then $z^2 = 2.5(1,1,1,1)$, $p(z) = z^2 - c = (1.5, 0.5, -0.5, -1.5)$ and $\operatorname{abs}_2(p(z)) = 0$. Thus, z is a singular point for $p(z) := z^2 - c$, though p is a polynomial without any zeros. The same applies to $c = (-2,1,2,0)$ in Example 1.5, part (II). In this case $z = 0.5(1,1,-1,1)$ yields $p(z) = z^2 - c = 0.5(5,-1,-5,1)$ and $\operatorname{abs}_2(p(z)) = 0$.

THEOREM 5.3. *Let $z_0 \in \mathbb{R}$ be a zero of a coquaternionic polynomial p and let $v \in [z_0]_q$. Then $\operatorname{abs}_2(p(v)) = 0$. Hence, v is a singular point for p.*

Proof. An element $v \in [z_0]_q$ has the form $v = z_0 + u$, where $u = (0, u_2, u_3, u_4)$ with $\operatorname{abs}_2(u) = u_2^2 - u_3^2 - u_4^2 = 0$. As a consequence $\Re(v) = \Re(z_0) = z_0$, $\operatorname{abs}_2(v) = \operatorname{abs}_2(z_0)$. If $p(z_0) = A(\Re(z_0), \operatorname{abs}_2(z_0)) + B(\Re(z_0), \operatorname{abs}_2(z_0))z_0 = 0$, then (using Lemma 3.3) $p(v) = A + B(z_0+u) = A + Bz_0 + Bu = Bu$ and $\operatorname{abs}_2(Bu) = \operatorname{abs}_2(B)\operatorname{abs}_2(u) = 0$. ∎

COROLLARY 5.4. *Let z_0 be a zero of a coquaternionic polynomial p and let $[z_0]_q$ contain a real element. Then all elements $v \in [z_0]_q$ are singular points for p.*

THEOREM 5.5. *Let there be a $z \in \mathbb{H}_{\mathrm{coq}}$ such that $A(\Re(z), \operatorname{abs}_2(z)) = 0$, and $B(\Re(z), \operatorname{abs}_2(z))$ is singular, where A, B are defined in (3.5). Then z is singular for p.*

Proof. In this situation we have

$$\begin{aligned}\operatorname{abs}_2(p(z)) &= p(z)\overline{p(z)} = (A+Bz)(\overline{A+Bz})\\ &= Bz\overline{Bz} = Bz\overline{z}\overline{B} = \operatorname{abs}_2(B)\operatorname{abs}_2(z) = 0. \qquad \square\end{aligned}$$

6. The companion polynomial. Theorem 3.9 describes a simple formula for computing a zero of a coquaternionic polynomial p provided that one knows an equivalence class $[z_0]_q \subset \mathbb{H}_{\mathrm{coq}}$ that contains a zero of p. This knowledge will be provided by the so-called companion polynomial of p, which will be introduced in this section and will be denoted by q. The concept of a companion polynomial is very successful in treating one-sided quaternionic polynomials; see [13]. It was originally already introduced by Niven in 1941 [16] and later used by Pogorui and Shapiro in 2004 [19] to find an alternative proof for the number of zeros of a one-sided quaternionic polynomials as given by Gordon and Motzkin

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in 1965 [8]. The companion polynomial q is a polynomial with real coefficients, the roots of which determine—under certain conditions—equivalence classes which contain zeros of p.

DEFINITION 6.1. *Let p be the polynomial of degree n given in (1.4). The polynomial q of degree $\leq 2n$ defined by*

$$q(z) := \sum_{j,k=0}^{n} \overline{c_j}\, c_k z^{j+k}, \qquad z \in \mathbb{C} \tag{6.1}$$

will be called the companion polynomial of p. *The zeros of q will be called* roots *of q.*

LEMMA 6.2. *The coefficients of q defined in (6.1) are all real, and q can be written as*

$$q(z) := \sum_{k=0}^{2n} b_k z^k, \quad b_k := \sum_{j=\max(0,k-n)}^{\min(k,n)} \overline{c_j}\, c_{k-j} \in \mathbb{R}, \qquad k = 0, 1, \ldots, 2n. \tag{6.2}$$

Proof. Equation (6.2) is obtained from (6.1) by putting $\kappa := j + k$, observing the restrictions $0 \leq \kappa \leq 2n$, $0 \leq j, k \leq n$, and in the end renaming κ as k. For a fixed $0 \leq k \leq 2n$, there are $\min\{k + 1, 2n + 1 - k\}$ terms in the representation of b_k given in (6.2). If k is odd, the number of terms is even, and in this case, if $\overline{c_j} c_{k-j}$ is one of the terms, then there is another, distinct term $\overline{c_{k-j}} c_j$, and the sum is real. If k is even, the number of terms is odd. We use the same argument as before and note that in this case there is an additional, single real term $\overline{c_{\frac{k}{2}}} c_{\frac{k}{2}}$. ☐

The companion polynomial q should only be regarded as a polynomial over $\mathbb{C}$, not over $\mathbb{H}_{\mathrm{coq}}$. The highest coefficient of the companion polynomial q is $b_{2n} = \overline{c_n} c_n = \mathrm{abs}_2(c_n)$. Thus, if c_n is singular, the degree of the companion polynomial is less than $2n$. If c_0 is singular, then the constant term of q is $b_0 = \mathrm{abs}_2(c_0) = 0$ and $q(0) = 0$. It is even possible that the companion polynomial q vanishes identically. The companion polynomial q can be computed by $q(z) := p(z)\overline{p(z)}$ assuming that $z \in \mathbb{R}$ and using that in this case, $z = \overline{z}$ commutes with all coefficients. This implies that a real zero of p appears as a double root of q.

EXAMPLE 6.3. Let $p(z) = bz^2 - c$. Then $q(z) = \mathrm{abs}_2(b)z^4 - 2\Re(b\overline{c})z^2 + \mathrm{abs}_2(c)$ and q vanishes if $\mathrm{abs}_2(b) = \mathrm{abs}_2(c) = \Re(b\overline{c}) = 0$. In this case

$$\mathrm{abs}_2(p(z)) = \mathrm{abs}_2(b)\mathrm{abs}_2(z^2) - 2\Re(bz^2\overline{c}) + \mathrm{abs}_2(c) = -2\Re(bz^2\overline{c}),$$

and $\mathrm{abs}_2(p(z))$ vanishes for all $z \in \mathbb{R}$. Now let $b = 1$. Then, for the roots z of q we have

$$z^2 = \Re(c) \pm \sqrt{-c_2^2 + c_3^2 + c_4^2}.$$

Put $\sigma := \pm\sqrt{-c_2^2 + c_3^2 + c_4^2}$ and assume that $\sigma \in \mathbb{R}$. Then, $z^2 = c_1 + \sigma \in \mathbb{R}$, and

$$\underbrace{c_1^2 - c_2^2 + c_3^2 + c_4^2 + 2c_1\sigma}_{\mathrm{abs}_2(z^2)} \underbrace{-2c_1^2 - 2c_1\sigma}_{-2\Re(z^2\overline{c})} + \underbrace{c_1^2 + c_2^2 - c_3^2 - c_4^2}_{\mathrm{abs}_2(c)} = 0.$$

Thus, the real roots z of q are singular points for p.

LEMMA 6.4. *Let $q(z) = 0$ for some $z \in \mathbb{R}$. Then z is a singular point for p. If q vanishes identically, then all $z \in \mathbb{R}$ are singular points for p.*

Proof. $q(z) = p(z)\overline{p(z)} = \mathrm{abs}_2(p(z)) = 0$ since $z \in \mathbb{R}$. ☐

LEMMA 6.5. *Let p have the form $p(z) = A\big(\Re(z), \mathrm{abs}_2(z)\big) + B\big(\Re(z), \mathrm{abs}_2(z)\big)z$; see (3.5). Then the companion polynomial q can be written as (omitting the arguments of A and B)*

$$q(z) = \mathrm{abs}_2(A) + 2\Re(\overline{B}A)z + \mathrm{abs}_2(B)z^2. \tag{6.3}$$

Proof: Let $z^j = \alpha_j + \beta_j z$; see (3.2)–(3.4). Then,

$$\begin{aligned} q(z) &= \sum_{j,k=0}^{n} \overline{c_j}\, c_k z^{j+k} = \sum_{j=0}^{n} \overline{c_j} \left(\sum_{k=0}^{n} c_k z^k \right) z^j = \sum_{j=0}^{n} \overline{c_j}\, (A + Bz) z^j \\ &= \sum_{j=0}^{n} \overline{c_j}\, (A + Bz)(\alpha_j + \beta_j z) \quad [\alpha_j, \beta_j \in \mathbb{R}] \\ &= \sum_{j=0}^{n} \alpha_j \overline{c_j}\, A + \sum_{j=0}^{n} \beta_j \overline{c_j}\, Az + \sum_{j=0}^{n} \alpha_j \overline{c_j}\, Bz + \sum_{j=0}^{n} \beta_j \overline{c_j}\, Bz^2 \\ &= \overline{A}A + \overline{B}Az + \overline{A}Bz + \overline{B}Bz^2. \qquad \square \end{aligned}$$

We will show that $z \overset{q}{\sim} z_0$ if $[z_0]_p$ contains a zero, which is computed by the formula given in Theorem 3.9.

LEMMA 6.6. *Let $z_0 = x + \mathbf{i}y$ be a root of q with $y \neq 0$ and $B(\Re(z_0), \mathrm{abs}_2(z_0))$ be nonsingular. Then for z defined in (3.13), we have $z \in [z_0]_q$, i.e.,*

$$\Re(z_0) = \Re(z), \quad \mathrm{abs}_2(z_0) = \mathrm{abs}_2(z).$$

Proof. Put $A := A(\Re(z_0), \mathrm{abs}_2(z_0)),\ B := B(\Re(z_0), \mathrm{abs}_2(z_0))$. Lemma 6.5 implies

$$q(z_0) = \mathrm{abs}_2(A) + 2\Re(\overline{B}A)z_0 + \mathrm{abs}_2(B)z_0^2 = 0.$$

Denote

$$v := \overline{B}A =: (v_1, v_2, v_3, v_4), \quad z_0 =: (x, y, 0, 0), \quad \text{where } y \neq 0.$$

Splitting $q(z_0)$ into real and imaginary part yields

$$\Re(q(z_0)) = \mathrm{abs}_2(A) + 2v_1 x + \mathrm{abs}_2(B)(x^2 - y^2) = 0, \tag{6.4}$$

$$\Im(q(z_0)) = 2\big(\Re(\overline{B}A) + \mathrm{abs}_2(B)x\big)y = 0 \Rightarrow v_1 = -\mathrm{abs}_2(B)x. \tag{6.5}$$

From the definition of z it follows that

$$\Re(z) = -\frac{\Re(\overline{B}A)}{\mathrm{abs}_2(B)} = -\frac{v_1}{\mathrm{abs}_2(B)} = x = \Re(z_0),$$

where the last part follows from (6.5). Thus, the first part is shown. Now from (3.13) we conclude that

$$\mathrm{abs}_2(z) = \mathrm{abs}_2(-B^{-1}A) = \frac{\mathrm{abs}_2(A)}{\mathrm{abs}_2(B)}. \tag{6.6}$$

If we insert $v_1 = -\mathrm{abs}_2(B)x$ from (6.5) into (6.4), then we obtain

$$\mathrm{abs}_2(A) - 2\mathrm{abs}_2(B)x^2 + \mathrm{abs}_2(B)(x^2 - y^2) = \mathrm{abs}_2(A) - \mathrm{abs}_2(B)\mathrm{abs}_2(z_0) = 0,$$

and together with (6.6) the second part, $\mathrm{abs}_2(z_0) = \mathrm{abs}_2(z)$, follows. $\square$

LEMMA 6.7. *Let the companion polynomial q have a pair of complex conjugate roots $z_0^{\pm}$. Then formula (3.13) yields the same value for z for both roots $z_0^{\pm}$.*

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Proof. The formula for z depends only on $\Re(z_0)$ and on $\mathrm{abs}_2(z_0)$, which are the same for both roots $z_0^{\pm}$. ∎

LEMMA 6.8. *Let p be a given coquaternionic polynomial (see (1.4)), and let q be the companion polynomial of p. Let $z_0 \in \mathbb{C}$ be a root of q, let $B(\Re(z_0), \mathrm{abs}_2(z_0))$ be nonsingular, and let z be determined by formula (3.13). Then,*

$$\mathrm{abs}_2(A) = \mathrm{abs}_2(Bz), \quad \Re(\overline{A}Bz) = -\mathrm{abs}_2(A). \tag{6.7}$$

Proof. The definition (3.13) of z implies that $Bz = -A$, which proves both parts. ∎

THEOREM 6.9. *Let p be a given coquaternionic polynomial (see (1.4)), and let q be the companion polynomial of p; see (6.1), (6.2). Let $z_0 \in \mathbb{C}$ be a root of q, let $B(\Re(z_0), \mathrm{abs}_2(z_0))$ be nonsingular, and let z be determined by formula (3.13). Then,*

$$\overline{p(z)}\, p(z) = \mathrm{abs}_2(p(z)) = 0. \tag{6.8}$$

If $z_0 \notin \mathbb{R}$, then

$$p(z) = 0. \tag{6.9}$$

Proof. We have

$$\begin{aligned}\overline{p(z)}\, p(z) &= \overline{(A + Bz)}(A + Bz) = \mathrm{abs}_2(A) + \overline{A}Bz + \overline{Bz}A + \mathrm{abs}_2(Bz)\\ &= \mathrm{abs}_2(A) + 2\Re(\overline{A}Bz) + \mathrm{abs}_2(Bz).\end{aligned}$$

Conditions (6.7) yield $\overline{p(z)}\, p(z) = 2\mathrm{abs}_2(A) - 2\mathrm{abs}_2(A) = 0$, which proves (6.8). In the second case, Lemma 6.6 is valid, and following (3.5), we find that

$$\begin{aligned}p(z) &= A(\Re(z), \mathrm{abs}_2(z)) + B(\Re(z), \mathrm{abs}_2(z))z\\ &= A(\Re(z), \mathrm{abs}_2(z))\\ &\quad + B(\Re(z), \mathrm{abs}_2(z))\left(-B(\Re(z_0), \mathrm{abs}_2(z_0))\right)^{-1} A(\Re(z_0), \mathrm{abs}_2(z_0))\\ &= A(\Re(z_0), \mathrm{abs}_2(z_0))\\ &\quad - B(\Re(z_0), \mathrm{abs}_2(z_0))(B(\Re(z_0), \mathrm{abs}_2(z_0)))^{-1} A(\Re(z_0), \mathrm{abs}_2(z_0))\\ &= 0,\end{aligned}$$

which proves (6.9). ∎

The assumption in Lemma 6.6 that z_0 is not real is essential. If $z_0 \in \mathbb{R}$, then we have equation (6.8). However, this does not exclude the case $p(z) = 0$.

THEOREM 6.10. *Let $p(z) = 0$, where p is defined in (1.4). Assume that*

$$\Delta := [z]_q \cap \mathbb{C} \neq \emptyset. \tag{6.10}$$

Then there is a $z_0 \in \Delta$ with $q(z_0) = 0$ and q as in Definition 6.1.

Proof. (a) Let $z \in \mathbb{R}$. Then $\Delta = \{z\}$ and $q(z) = \overline{p(z)}p(z) = 0$.

(b) Let $z \notin \mathbb{R}$ and $B(\Re(z), \mathrm{abs}_2(z)) = 0$ or $B(\Re(z), \mathrm{abs}_2(z))z = 0$. Then

$$A(\Re(z), \mathrm{abs}_2(z)) = 0 \quad \text{ and } \mathrm{abs}_2(B(\Re(z), \mathrm{abs}_2(z))) = 0$$

according to Lemma 3.4. Let $z_0 \in \Delta$. The companion polynomial in the representation (6.3) yields $q(z_0) = 0$.

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(c) Let $z \notin \mathbb{R}$ and B be nonsingular. In this case

$$p(z) = A(\Re(z), \mathrm{abs}_2(z)) + B(\Re(z), \mathrm{abs}_2(z))z = 0 \quad \text{and } z = -B^{-1}A,$$

omitting the arguments. The quadratic polynomial for q in the form (6.3) has real coefficients and can be solved by standard techniques with the zero

$$z_0 = -\frac{\Re(\overline{B}A)}{\mathrm{abs}_2(B)} + \frac{\mathbf{i}}{|\mathrm{abs}_2(B)|}\sqrt{\mathrm{abs}_2(AB) - (\Re(\overline{B}A))^2}. \tag{6.11}$$

The radicand is positive if $z \notin \mathbb{R}$. Equation (6.11) yields

$$\Re(z_0) = -\frac{\Re(\overline{B}A)}{\mathrm{abs}_2(B)} = \Re(z).$$

We use (6.11) again, and since $z_0 \in \mathbb{C}$, we have

$$\mathrm{abs}_2(z_0) = |z_0|^2 = \left(\frac{\Re(\overline{B}A)}{\mathrm{abs}_2(B)}\right)^2 + \frac{\mathrm{abs}_2(AB) - (\Re(\overline{B}A))^2}{(\mathrm{abs}_2(B))^2} = \frac{\mathrm{abs}_2(A)}{\mathrm{abs}_2(B)} = \mathrm{abs}_2(z).$$

Property (6.10) is governed by Lemma 3.2. □

The previous theorem tells us that we find all zeros of p by employing the companion polynomial provided that the zero has a complex number in its equivalence class. Or in other words, Theorem 6.10 says that all zeros z of p with the property $z_2^2 - z_3^2 - z_4^2 \geq 0$ can be found by applying the companion polynomial, but all others cannot be found. More precisely, the assumption $\Delta \neq \emptyset$ (see (6.10)) is equivalent to $c_2^2 = z_2^2 - z_3^2 - z_4^2 \geq 0$, which implies that $c_1 + c_2\mathbf{i} \in \mathbb{C}$, where $c_2 := \pm\sqrt{z_2^2 - z_3^2 - z_4^2}$. Thus, the complex zeros of q can be recovered from the zeros of p. If we have a look at Example 1.5, part (III), we see that all four square roots $\sqrt{c}$ given there do not satisfy condition (6.10), whereas the two roots $\sqrt{c}$ of part (IV) do satisfy (6.10).

EXAMPLE 6.11. Let us treat the most trivial case

$$p(z) := z - c, \quad c = (c_1, c_2, c_3, c_4).$$

In terms of (3.5), we have $p(z) = A + Bz$ with $A = -c$, $B = 1$. Both, A and B do not depend on $z \in \mathbb{H}_{\mathrm{coq}}$. The assumptions of Theorem 3.9 are met in this case since p always has a zero. Formula (3.13) yields the correct answer $z = c$. Let us now apply the companion polynomial $q(z) = z^2 - 2c_1z + \mathrm{abs}_2(c)$, the roots of which are $z_{1,2} = c_1 \pm \sqrt{-c_2^2 + c_3^2 + c_4^2}$. If we apply Theorem 6.9, we also obtain the correct answer, however, independent of the roots $z_{1,2}$.

EXAMPLE 6.12. Let the quadratic coquaternionic polynomial p be defined by the coefficients $c_0 = 1, c_1 = (-15, 6, -6, -25)/25, c_2 = 1$. Then, the companion polynomial

$$q(z) = z^4 - (6/5)z^3 + (43/25)z^2 - (6/5)z + 1$$

has two pairs of complex conjugate roots, $r_1 = a_1 \pm b_1\mathbf{i}, r_2 = a_2 \pm b_2\mathbf{i}$. For the first pair, we have $A_1 = (0,0,0,0), B_1 = (4/5, 6/25, -6/25, -4/5)$, and for the second pair, we have $A_2 = (0,0,0,0), B_2 = (-4/5, 6/25, -6/25, -4/5)$. Both, B_1, B_2 are singular and Theorem 5.5 applies.

Experiments with random integer coefficients for the coquaternionic polynomial show that cases where B is singular are very rare.

7. Application to the quadratic case. Let us treat the quadratic case

$$p(z) := c_2 z^2 + c_1 z + c_0, \qquad c_0, c_2 \neq 0. \tag{7.1}$$

Equation (3.5) applied to (7.1) yields

$$p(z) = c_0 - \mathrm{abs}_2(z) c_2 + (c_1 + 2\Re(z) c_2) z := A(\mathrm{abs}_2(z)) + B(\Re(z)) z.$$

The companion polynomial q of p is (omitting the arguments of A and B)

$$\begin{aligned} q(z) &= \mathrm{abs}_2(c_2) z^4 + 2\Re(c_1 \overline{c_2}) z^3 + \big(2\Re(c_0 \overline{c_2}) + \mathrm{abs}_2(c_1)\big) z^2 \\ &\quad + 2\Re(c_0 \overline{c_1}) z + \mathrm{abs}_2(c_0) \\ &= \mathrm{abs}_2(A) + 2\Re(\overline{B} A) z + \mathrm{abs}_2(B) z^2, \end{aligned}$$

where the second equation is derived from (6.3). The appearance of the factor c_2 at z^2 in (7.1) is only justified if $\mathrm{abs}_2(c_2) = 0$. In this case, q has degree ≤ 3, and the formalism just described has to be applied.

Let $\mathrm{abs}_2(c_2) \neq 0$. We may assume that $c_2 = 1$ and $\Re(c_1) = 0$. The latter condition can always be achieved by introducing the transformation $z = u - \frac{\Re(c_1)}{2}$. The transformed quadratic polynomial in u has the property that the real part of the coefficient at the linear term is zero; see also Niven [16]. Thus, if $\mathrm{abs}(c_2) \neq 0$, we can transform (7.1) into the simpler form

$$p(z) := z^2 + c_1 z + c_0, \quad c_0 \neq 0, \Re(c_1) = 0,$$

and if c_1 is real, we have $c_1 = 0$, and in this case the complete solution is described in the two Lemmas 1.3, 1.4. Let c_1 be nonreal. Then the above polynomial q specializes to

$$q(z) = z^4 + \big(2\Re(c_0) + \mathrm{abs}_2(c_1)\big) z^2 + 2\Re(c_0 \overline{c_1}) z + \mathrm{abs}_2(c_0). \tag{7.2}$$

If $c_0 \in \mathbb{R}$, then the linear term cancels and the roots z of q defined in (7.2) obey

$$z^2 = \frac{1}{2}\left(-2c_0 - \mathrm{abs}_2(c_1) \pm \sqrt{\mathrm{abs}_2(c_1)\big(\mathrm{abs}_2(c_1) + 4c_0\big)}\right).$$

For singular c_1 it follows that $z^2 = -c_0$. If c_0, c_1 are both nonreal, equation (7.2) has to be further investigated by the given means.

8. Newton's method for finding coquaternionic zeros and the determination of the exact Jacobi matrix. Let p be the coquaternionic polynomial (1.4). Since the method of using the companion polynomial q to find zeros of p is restricted to those zeros which share a complex number in their equivalence class, we turn to the computation of the zeros of p by Newton's method in order to find the remaining zeros, if any exist. Let $p'(z, h)$ be the (Fréchet-) derivative of p. Theorem 8.1 below describes how to find it. In short, Newton's method consists of solving the real, linear (4×4) system

$$\mathrm{col}(p(z_k)) + \mathrm{col}(p'(z_k, h)) = 0, \qquad z_{k+1} := z_k + h,\ k = 0, 1, \ldots,$$

for h, where z_0 is the *initial guess* and where col is defined in (3.10). If $\mathbf{J}(z)$ is the *Jacobi matrix* of the mapping $p : \mathbb{R}^4 \to \mathbb{R}^4$, then

$$\mathrm{col}(p'(z, h)) = \mathbf{J}(z) \mathrm{col}(h),$$

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where $\mathbf{J}(z)$ is a real (4×4) matrix. How do we find it? We refer to a paper by Lauterbach and Opfer [15].

THEOREM 8.1. *Let p be a given coquaternionic polynomial with coefficients $c_0, \ldots, c_n$, for $n \geq 1$. Define*

$$\lambda_j(z,h) := \sum_{\substack{k+\ell=j-1\\ k,\ell \geq 0}} z^k h z^\ell,\ j \geq 1, \quad L(z,h) := \sum_{j=1}^{n} c_j \lambda_j(z,h), \tag{8.1}$$

where $L : \mathbb{H}_{\mathrm{coq}} \to \mathbb{H}_{\mathrm{coq}}$ is a linear mapping over $\mathbb{R}$ with respect to h. Its matrix representation is the Jacobi matrix of p,

$$\mathbf{J}(z) := \sum_{j=1}^{n} \mathbf{M}_j(z) \in \mathbb{R}^{4\times 4}, \text{ where} \tag{8.2}$$

$$\mathbf{M}_j(z) := \sum_{\substack{k+\ell=j-1\\ k,\ell \geq 0}} \left[\mathrm{col}(z^k z^\ell), \mathrm{col}(z^k \mathbf{i} z^\ell), \mathrm{col}(z^k \mathbf{j} z^\ell), \mathrm{col}(z^k \mathbf{k} z^\ell)\right]. \tag{8.3}$$

Proof. The quantity $\lambda_j(z,h)$ is the derivative of z^j, for $j \geq 1$. It is a linear mapping over $\mathbb{R}$ with respect to h since real numbers (and no others) commute with coquaternions. Therefore the derivative of p is $p'(z,h) = L(z,h)$. The identity (8.3) for $\mathbf{M}_j(z)$ is equation (5.4) in [11]. ◻

The right hand side of $\lambda_j(z,h)$ is obtained by computing $(z+h)^j$ and deleting all terms which are not linear in h. The result is the derivative of z^j. To mention two examples, $\lambda_2(z,h) = zh + hz$, $\lambda_3(z,h) = z^2h + zhz + hz^2$. An application of the method just described to the quaternionic algebraic Riccati equation is treated in [10].

LEMMA 8.2. *Let $z, c_j \in \mathbb{Z}^4 \subset \mathbb{H}_{\mathrm{coq}}$, for $j = 1, 2, \ldots, n$. Then, $\mathbf{J}(z) \in \mathbb{Z}^{4\times 4}$.*

Proof. The formula for computing the Jacobi matrix $\mathbf{J}(z)$ involves only additions and multiplications of the data z, c_j, for $j = 1, 2, \ldots, n$; see (8.1)–(8.3). ◻

The property given in Lemma 8.2 is not shared by the numerical version of the Jacobi matrix which is columnwise computed by

$$\frac{p(z + \alpha\mathfrak{e}_j) - p(z)}{\alpha}, \qquad j = 1,2,3,4,\ \alpha \approx 10^{-6},$$

where $\mathfrak{e}_j \in \mathbb{H}_{\mathrm{coq}}$, $j = 1,2,3,4$, represent the four units of $\mathbb{H}_{\mathrm{coq}}$ and α is a real number of the order of the square root of the machine precision.

EXAMPLE 8.3. We take a cubic polynomial p with data from Example 10.3. Then, for the starting value $z_0 = (0,0,0,0)$, the Jacobi matrix is

$$\mathbf{J}(z_0) = \begin{bmatrix} 2 & -3 & 5 & 7 \\ 3 & 2 & 7 & -5 \\ 5 & 7 & 2 & -3 \\ 7 & -5 & 3 & 2 \end{bmatrix}.$$

After six Newton steps we arrive at the zero z which is listed in the third row of Table 10.3 and which satisfies $\|p(z)\| = 1.8609 \cdot 10^{-14}$, where $\|.\|$ is the Euclidean norm in $\mathbb{R}^4$.

9. The algorithms for finding zeros and singular points of coquaternionic polynomials. The techniques to find zeros and singular points for the coquaternionic polynomial p with the methods described in the foregoing sections is summarized by the following algorithms.

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ALGORITHM 9.1. Algorithm I for finding zeros and singular points for coquaternionic polynomials by means of the companion polynomial.

1. Let $c_0, c_1, \ldots, c_n$ be the coquaternionic coefficients of the polynomial p.
2. Compute the real coefficients $b_0, b_1, \ldots, b_{2n}$ of the companion polynomial q by equation (6.2).
3. If the companion polynomial vanishes, all points $z \in \mathbb{R}$ are singular points for p, stop here, otherwise continue.
4. Compute all (real and complex) roots of q. There are at most $2n$ roots.
5. Delete all complex roots with negative imaginary part. The remaining roots will be denoted by $r_1, r_2, \ldots, r_m$, $m \leq 2n$.
6. Define an integer vector ind (indicator) of length m and set all entries to zero.
   `for` $j = 1$ `to` m `do`
7. Compute A_j, B_j at the root r_j by using formulas (3.3), (3.4), and (3.5).
8. Compute $\rho_j = -A_j B_j^{-1}$ if B_j is nonsingular ($\mathrm{abs}_2(B_j) \neq 0$), otherwise put $\rho_j = r_j$ and identify ρ_j with a coquaternion.
9. Check whether $p(\rho_j) = 0$; in this case set $ind(j) = 2$ (ρ_j is a zero of p). Otherwise, check whether ρ_j is a singular point for p ($\mathrm{abs}_2(p(\rho_j)) = 0$). In this case set $ind(j) = 1$.

   `end for` j

REMARK 9.2. The result of Algorithm I is a vector of coquaternions ρ_j and an integer vector $ind_j,\ j = 1, 2, \ldots, m$ with

$$ind_j = \begin{cases} 0 & \text{if } \rho_j \text{ is neither a zero of } p \text{ nor a singular point for } p, \\ 1 & \text{if } \rho_j \text{ is a singular point for } p \text{ but not a zero of } p, \\ 2 & \text{if } \rho_j \text{ is a zero of } p. \end{cases}$$

In the above algorithm we have not paid special attention to the case when one of the zeros of p is real. In this case, a real double root appears in the set of roots of q. For the above algorithm to work smoothly, one should use a so-called *overloading technique*, which is available in several programming systems. Overloading means that the elementary arithmetic operations and functions can be extended to coquaternions keeping the standard notation such as $+, -, *, /$. In particular, one needs a subprogram with which one can evaluate coquaternionic polynomials at coquaternions. If the overloading technique has been implemented, then polynomial evaluation can be done simply by the application of Horner's scheme. For finding the roots of a real polynomial (such as the companion polynomial), standard programs are available. However, the standard program in MATLAB is called `roots` and it suffers from a severe loss in accuracy if there are multiple roots. A remedy is hinted in [13]. If one uses the overloading technique, typically, the definition for a vector of coquaternions c reads $c(j) =$coquaternion$([c_{j1}, c_{j2}, c_{j3}, c_{j4}]),\ j = 1, 2, \ldots$

ALGORITHM 9.3. Algorithm II for finding zeros of coquaternionic polynomials by means of Newton's method.

1. Let $c_0, c_1, \ldots, c_n$ be the coquaternionic coefficients of the polynomial p.
2. Define a list L of already known zeros of p, possibly empty at the beginning or filled with zeros obtained from Algorithm I.

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3. Define a number of trials no_trials, a maximal number of Newton iterations max_Newton, and a stopping criterion $crit\_stop$.
   `for` $j = 1$ `to` no_trials `do`
4. Define an initial guess $z = z_0$, compute $y = p(z)$, and $ynorm = \|y\|$ (Euclidean norm). Set $\ell = 0$.
   `while` $ynorm \geq crit\_stop$ and $\ell \leq max\_Newton$ `do`
5. $\ell = \ell + 1$; execute one Newton step: $z = Newton(z)$.
   Compute $y = p(z), ynorm = \|y\|$.
   `end while`
6. If $\ell < max\_Newton$ or $ynorm < crit\_stop$, check whether the last computed z is already contained in the list L. If not, add it to L.
   `end for` j

REMARK 9.4. The result of this algorithm will be a list L of zeros of p, which in general is not exhaustive. The starting value is—according to the experience of the authors—best selected by a random, integer choice, where the integers should be restricted to a small interval, say to $[-5, 5]$. It is possible (in rare cases) that the Jacobi matrix is singular. This can be easily overcome by choosing a new initial guess.

10. Numerical examples. We present some examples of cubic coquaternionic polynomials to show that the number of zeros and singular points computed by the companion polynomial and by Newton's methods is rather unpredictable. Nevertheless, there is some pattern we would like to show. The examples are ordered with respect to the number of zeros which one finds by using the companion polynomial. The corresponding zeros are printed in color.

EXAMPLE 10.1 (no zeros). The cubic coquaternionic polynomial p with coefficients $c_0 = (2, -2, 2, 3)$, $c_1 = (-4, -5, 1, 1)$, $c_2 = (-1, 0, -5, -1)$, $c_3 = (2, 2, -1, 0)$ has *no zeros* similar to complex numbers and six singular points. Applying Newton's method with the exact Jacobi matrix yields the eight zeros listed in Table 10.1. None of them is similar to a complex number.

EXAMPLE 10.2 (one zero). The cubic coquaternionic polynomial p with coefficients $c_0 = (1, -5, -2, 0)$, $c_1 = (3, 3, -2, 4)$, $c_2 = (-4, -3, -5, 2)$, $c_3 = (-3, -4, 1, -2)$ has *one zero* which is similar to a complex number, four singular points, and six zeros computed by Newton's method, not similar to a complex number. The seven zeros are listed in Table 10.2.

EXAMPLE 10.3 (two zeros). The cubic coquaternionic polynomial p with coefficients $c_0 = (7, 6, 5, 1)$, $c_1 = (2, 3, 5, 7)$, $c_2 = (4, -3, 2, 1)$, $c_3 = (1, 3, 2, 4)$ has *two zeros* which are similar to a complex number, two singular points, and one zero computed by Newton's method, listed in Table 10.3.

EXAMPLE 10.4 (three zeros). The cubic coquaternionic polynomial p with coefficients $c_0 = (0, 2, 0, 5)$, $c_1 = (0, 1, 0, 1)$, $c_2 = (-2, -4, 4, 1)$, $c_3 = (1, 0, 4, -2)$ has *three zeros* which are similar to a complex number, no singular points. The three zeros are listed in Table 10.4. No zeros were found by applying Newton's method (several thousand trials).

We made many more tests with coquaternionic polynomials of degree $n \geq 3$ which attained the maximal number n of zeros found by the companion polynomial. In all these cases we did not find additional zeros computed by Newton's method. This applies also to Example 1.5, part (IV).

CONJECTURE 10.5. *Let p be a coquaternionic polynomial of degree n which has n zeros which are similar to complex numbers. Then, there are no additional zeros not similar to complex numbers.*

Another observation is the following: if a coquaternionic polynomial p has many zeros which are not similar to complex numbers, then there are only few zeros which are similar to

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complex numbers. This applies, in particular, to Example 1.5, part (III). However, this is a very vague statement, which needs more attention.

TABLE 10.1
Eight zeros of a cubic coquaternionic polynomial p defined in Example 10.1, all found by Newton's method.

0.920792194877860,	-0.477350655832754,	2.428458796390070,	-1.654108298624764
2.450727144208431,	0.977395660928656,	0.317301767845470,	-1.652159929881599
0.038499359300800,	-0.459455816622210,	0.517633403455030,	0.178975644511005
0.040708445821269,	-0.839407205920705,	-0.328922433104592,	1.295950832229326
0.709019332932411,	-0.294621664264792,	-0.112477811982268,	0.461335678870540
-0.448518057687961,	1.536978387850394,	2.034978412068015,	0.749577465189058
0.410896918015976,	-0.063043960237222,	0.222327740640928,	0.635721599205228
-1.489226503509231,	-0.051244615268034,	0.422127971968205,	-0.252540209891112

TABLE 10.2
Seven zeros of a cubic coquaternionic polynomial p defined in Example 10.2, the last six found by Newton's method.

-0.084025738354299,	1.111175126311441,	-0.574783886624048,	0.584853095346396
-1.280365616247547,	0.020877114875100,	0.503907316675033,	2.157051290547817
-0.285608645398092,	1.407387895553819,	1.602481962888596,	-0.292825912129321
0.734696869093826,	-0.802514241229524,	-0.739507355478451,	0.370330803674946
-1.480332927529147,	-0.481980935905488,	0.945158732810532,	1.843761755812835
-2.300671130739401,	0.360373154160493,	-0.067402575042700,	1.246070549138632
-0.085641334116581,	3.501590113862619,	3.639869657498098,	0.231312656003601

TABLE 10.3
Three zeros of a cubic coquaternionic polynomial p of Example 10.3, the last one found by Newton's method.

-1.618852521797113,	6.463899263531390,	2.829324921055154,	5.651970856832540
0.418326476405790,	-1.691555573954496,	0.998887526357887,	0.395365114055260
-0.099473954608707,	-1.081012068817781,	-0.782231163978552,	-1.127180514797187

TABLE 10.4
Three zeros of a cubic coquaternionic polynomial p, defined in Example 10.4 all found by using the companion polynomial.

-1.466507448592167,	1.324915491617470,	1.123223813460332,	-0.564677198394439
0.781247091809576,	0.634161128551769,	-0.200695566535362,	0.065867128807512
-0.156844906375301,	-2.299180524759707,	1.304072974458774,	-1.766122605663109

11. On the number of zeros and singular points. We have already seen that coquaternionic polynomials may have no zeros; see Corollary 1.6. On the other hand there is a maximum number of zeros which contain complex numbers in the corresponding equivalence class.

THEOREM 11.1. *A coquaternionic polynomial p of degree $n \geq 1$ has at most n zeros in equivalence classes which contain complex numbers. The number n will be attained if all roots of the companion polynomial q are either pairs of complex conjugate numbers or pairs of real numbers and the corresponding quantities B are nonsingular.*

Proof. Let the companion polynomial q have degree $d \leq 2n$. It may have $2m_1 \leq d$ real double roots, $2m_2 \leq d$ complex roots, and $m_3 \leq d$ real single roots. The total number of roots is $2m_1 + 2m_2 + m_3 = d$. Only the real double roots and the complex roots may lead to

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a zero of p. Thus, $m_1 + m_2 = (d - m_3)/2 \le n$ is the maximum number of zeros of p. The maximum number is attained if $d = 2n$ and $m_3 = 0$. ∎

THEOREM 11.2. *Let p be a coquaternionic polynomial of degree n and let the companion polynomial q have degree $d \le 2n$. If n_z is the number of zeros of p and n_s is the number of singular points for p not identical with those zeros and where both zeros and singular points are computed from the roots of q, then $2n_z + n_s \le d$. In particular, $n_s \le 2n$, which means that there are at most $2n$ singular points for p that are different from zeros of p and which are computed by means of the companion polynomial q.*

Proof. Singular points for p are derived from single, real roots of q. See Theorem 6.9. ∎

Our experiments led us to the following conjecture.

CONJECTURE 11.3. *All coquaternionic polynomials p (defined in (1.4)) which do not reduce to a constant have singular points.*

Since $p(0) = c_0$ the conjecture is true if c_0 is singular. Let c_0 be nonsingular. In this case a proof could consist of showing that there is a $z \in \mathbb{H}_{\rm coq}$ such that

$$\operatorname{abs}_2(p(0))\operatorname{abs}_2(p(z)) = \operatorname{abs}_2(c_0)\operatorname{abs}_2(p(z)) \le 0. \tag{11.1}$$

In our tests we always found a very simple z such that (11.1) was valid. It was sufficient to choose either one of the four unit vectors (possibly multiplied by 2) for z or one of the square roots of $z = 0$ (see Example 1.1).

This conjecture (if true) could be called the "Weak Fundamental Theorem of Algebra" for coquaternions.

12. Extension to algebras in $\mathbb{R}^4$. If we go from the algebra of coquaternions to other algebras in $\mathbb{R}^4$, we observe many similarities with the coquaternionic case. The algebras to be considered are given in Table 12.1. The full multiplication table of the eight listed algebras can be obtained by multiplying the last three columns in Table 12.1 by $\mathbf{j}, \mathbf{k}, \mathbf{i}$, respectively. The table is obtained by allowing all eight combinations of signs ± 1 for the squares $\mathbf{i}^2, \mathbf{j}^2, \mathbf{k}^2$ and keeping the product $\mathbf{ij} = \mathbf{k}$ the same for all algebras. The names given to the algebras with numbers 2 to 4 are from Cockle, 1849 [3], the names for the algebras 5 to 8 are from Schmeikal, 2014 [21], who also establishes the connection to Clifford algebras in his paper. In a first draft, these algebras were called New Algebra 1 to New Algebra 4 by the present authors.

The first mentioned algebra, the algebra of quaternions, goes back to Hamilton, 1843. The problem of finding zeros of unilateral and bilateral polynomials in quaternionic variables has been treated by the authors already in [12, 13].

The 8 algebras separate into 4 noncommutative ones, namely those with numbers 1,2,5,6, and into four commutative ones, those with numbers 3,4,7,8. We note that the center of all 8 algebras is or contains $\mathbb{R}$, which means that the real numbers commute with all members of all algebras. In all eight cases we define the conjugate of an algebraic element a as in (1.2).

LEMMA 12.1. *Let $\mathcal{A}$ be one of the four noncommutative algebras of Table 12.1 (number 1, 2, 5, or 6) and $a \in \mathcal{A}$. Then the product $a\operatorname{conj}(a) = \operatorname{conj}(a)a$ is real and*

$$a^{-1} = \frac{\operatorname{conj}(a)}{\operatorname{conj}(a)a} \text{ if } \operatorname{conj}(a)a \ne 0. \tag{12.1}$$

Proof. As in (1.2) we put $\mathrm{abs}_2(a) := \mathrm{conj}(a)a$, and by applying the multiplication rules, we obtain

$$\mathrm{abs}_2(a) = \begin{cases} a_1^2 + a_2^2 + a_3^2 + a_4^2 & \text{for quaternions,} \\ a_1^2 + a_2^2 - a_3^2 - a_4^2 & \text{for coquaternions,} \\ a_1^2 - a_2^2 + a_3^2 - a_4^2 & \text{for nectarines,} \\ a_1^2 - a_2^2 - a_3^2 + a_4^2 & \text{for conectarines.} \end{cases} \tag{12.2}$$

Equation (12.1) follows by multiplying from the left or right by a and from the fact that $\mathrm{conj}(a)a$ is real. ∎

TABLE 12.1
Eight $\mathbb{R}^4$ algebras with multiplication rules.

No	Name of algebra	Short name	$\mathbf{i}^2$	$\mathbf{j}^2$	$\mathbf{k}^2$	$\mathbf{ij}$	$\mathbf{jk}$	$\mathbf{ki}$
1	Quaternions	$\mathbb{H}$	-1	-1	-1	$\mathbf{k}$	$\mathbf{i}$	$\mathbf{j}$
2	Coquaternions	$\mathbb{H}_{\mathrm{coq}}$	-1	1	1	$\mathbf{k}$	$-\mathbf{i}$	$\mathbf{j}$
3	Tessarines	$\mathbb{H}_{\mathrm{tes}}$	-1	1	-1	$\mathbf{k}$	$\mathbf{i}$	$-\mathbf{j}$
4	Cotessarines	$\mathbb{H}_{\mathrm{cotes}}$	1	1	1	$\mathbf{k}$	$\mathbf{i}$	$\mathbf{j}$
5	Nectarines	$\mathbb{H}_{\mathrm{nec}}$	1	-1	1	$\mathbf{k}$	$\mathbf{i}$	$-\mathbf{j}$
6	Conectarines	$\mathbb{H}_{\mathrm{con}}$	1	1	-1	$\mathbf{k}$	$-\mathbf{i}$	$-\mathbf{j}$
7	Tangerines	$\mathbb{H}_{\mathrm{tan}}$	1	-1	-1	$\mathbf{k}$	$-\mathbf{i}$	$\mathbf{j}$
8	Cotangerines	$\mathbb{H}_{\mathrm{cotan}}$	-1	-1	1	$\mathbf{k}$	$-\mathbf{i}$	$-\mathbf{j}$

It is clear that all 8 algebras contain $\mathbb{R}$ as a subalgebra by defining the set of elements of the form $a = (a_1, 0, 0, 0), a_1 \in \mathbb{R}$. However, this is not in general true for the field $\mathbb{C}$.

LEMMA 12.2. *The cotessarines $\mathbb{H}_{\mathrm{cotes}}$ (algebra number 4) do not contain the field of complex numbers $\mathbb{C}$ as a subalgebra. Let $z = (x, y) \in \mathbb{C}$. Then, $\mathbb{C}$ is a subalgebra of one of the remaining seven algebras $\mathcal{A}$ if $\mathcal{A}$ is reduced to the following form:*

$$z = (x,y) \to \begin{cases} (x, y, 0, 0) & \textit{for } \mathcal{A} = \textit{quaternions, or} \\ (x, 0, y, 0) & \textit{for } \mathcal{A} = \textit{quaternions, or} \\ (x, 0, 0, y) & \textit{for } \mathcal{A} = \textit{quaternions}, \\ (x, y, 0, 0) & \textit{for } \mathcal{A} = \textit{coquaternions}, \\ (x, y, 0, 0) & \textit{for } \mathcal{A} = \textit{tessarines, or}, \\ (x, 0, 0, y) & \textit{for } \mathcal{A} = \textit{tessarines}, \\ (x, 0, y, 0) & \textit{for } \mathcal{A} = \textit{nectarines}, \\ (x, 0, 0, y) & \textit{for } \mathcal{A} = \textit{conectarines}, \\ (x, 0, y, 0) & \textit{for } \mathcal{A} = \textit{tangerines or}, \\ (x, 0, 0, y) & \textit{for } \mathcal{A} = \textit{tangerines}, \\ (x, y, 0, 0) & \textit{for } \mathcal{A} = \textit{cotangerines or}, \\ (x, 0, y, 0) & \textit{for } \mathcal{A} = \textit{cotangerines}. \end{cases} \tag{12.3}$$

Proof. We set the imaginary part y of z at the positions where the squares $\mathbf{i}^2, \mathbf{j}^2, \mathbf{k}^2$ in Table 12.1 are equal to -1. However, there is not such a position in the case of cotessarines. ∎

Thus, in the noncommutative algebras coquaternions, nectarines, and conectarines, the complex numbers have the form $z = x + \mathbf{i}y, z = x + \mathbf{j}y, z = x + \mathbf{k}y$, respectively. In the

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quaternionic case, a complex number may have one of these three forms. The real numbers x, y are again called *real part* and *imaginary part* of the complex number z.

The task of finding zeros (or singular points) of polynomials p defined with coefficients from one of the noncommutative algebras (number $1, 2, 5, 6$) can now be achieved as described for coquaternions. The complex solutions of the companion polynomial must be inserted in the corresponding algebra according to the rules given in (12.3).

The fact that (quasi-) similarity classes may not contain complex numbers is prevalent in all noncommutative algebras apart from quaternions; compare Theorem 6.10 and Lemma 3.2.

LEMMA 12.3. *Let $\mathcal{A}$ be one of the noncommutative algebras and $a \in \mathbb{C}$ with real part a_1 and imaginary part a_2. In order that $b = (b_1, b_2, b_3, b_4) \in \mathcal{A}$ is quasi-similar to a, it is necessary and sufficient that $a_1 = b_1$ and*

$$a_2^2 = \left\{\begin{array}{ll} +b_2^2 + b_3^2 + b_4^2 & \textit{for quaternions} \\ +b_2^2 - b_3^2 - b_4^2 & \textit{for coquaternions} \\ -b_2^2 + b_3^2 - b_4^2 & \textit{for nectarines} \\ -b_2^2 - b_3^2 + b_4^2 & \textit{for conectarines} \end{array}\right\} \geq 0.$$

Proof. Follows from (12.2). ∎

For the commutative cases, the companion polynomial is not well defined since in these cases $\mathrm{conj}(a)a$ is not real. In addition, in commutative algebras all similarity classes shrink to one point. The commutative cases are treated in a subsequent section.

Let $\alpha \in \mathcal{A}$, and $\mathcal{A}$ one of the eight algebras, where $\alpha := (a, b, c, d)$. Since the mapping $l : \mathcal{A} \to \mathcal{A}$ defined by

$$l(x) := \alpha x, x \in \mathcal{A}$$

is linear in x (over $\mathbb{R}$), there must be a matrix $\mathbf{M} \in \mathbb{R}^{4\times 4}$, depending on $\mathcal{A}$, such that

$$\mathrm{col}(l(x)) = \mathbf{M}\mathrm{col}(x). \tag{12.4}$$

We will denote the four unit standard vectors again as

$$\mathfrak{e}_1 := 1 := (1, 0, 0, 0), \mathfrak{e}_2 := \mathbf{i} := (0, 1, 0, 0), \mathfrak{e}_3 := \mathbf{j} := (0, 0, 1, 0), \mathfrak{e}_4 := \mathbf{k} := (0, 0, 0, 1).$$

By putting $x = \mathfrak{e}_j, j = 1, 2, 3, 4$, in (12.4) we obtain the j-th column of $\mathbf{M}$, such that

$$\mathbf{M} := [\mathrm{col}(\alpha\mathfrak{e}_1), \mathrm{col}(\alpha\mathfrak{e}_2), \mathrm{col}(\alpha\mathfrak{e}_3), \mathrm{col}(\alpha\mathfrak{e}_4)],$$

and these eight matrices are given in Table 12.2 using the notation $\imath : \mathcal{A} \to \mathbb{R}^{4\times 4}$.

The inverses of elements of the noncommutative algebras can simply be computed by equation (12.1). The inverses of elements in one of the four commutative algebras can be computed by rules given in Table 12.3. The computations were facilitated by using `maple`.

The determinant of $\imath(\mathbb{H}_{\mathrm{cotes}})$ factors into the form

$$\det(\imath(\mathbb{H}_{\mathrm{cotes}})) = (a_1+a_2-a_3-a_4)(a_1-a_2-a_3+a_4)(a_1+a_2+a_3+a_4)(a_1-a_2+a_3-a_4),$$

such that $\det(\imath(\mathbb{H}_{\mathrm{cotes}})) = 0$ if and only if $|a_1 + a_2| = |a_3 + a_4|$ or $|a_1 - a_2| = |a_3 - a_4|$. Since all other determinants have the form $\det = x^2 - 4y^2$, they also factor into the form $\det = (x - 2y)(x + 2y)$, where the meaning of x, y has to be read from Table 12.3.

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TABLE 12.2
Representations of the above eight algebras in $\mathbb{R}^{4\times 4}$.

$$\imath(\mathbb{H}) = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}, \quad \imath(\mathbb{H}_{\mathrm{coq}}) = \begin{bmatrix} a & -b & c & d \\ b & a & d & -c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix},$$

$$\imath(\mathbb{H}_{\mathrm{tes}}) = \begin{bmatrix} a & -b & c & -d \\ b & a & d & c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix}, \quad \imath(\mathbb{H}_{\mathrm{cotes}}) = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix},$$

$$\imath(\mathbb{H}_{\mathrm{nec}}) = \begin{bmatrix} a & b & -c & d \\ b & a & -d & c \\ c & -d & a & b \\ d & -c & b & a \end{bmatrix}, \quad \imath(\mathbb{H}_{\mathrm{con}}) = \begin{bmatrix} a & b & c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & -c & b & a \end{bmatrix},$$

$$\imath(\mathbb{H}_{\mathrm{tan}}) = \begin{bmatrix} a & b & -c & -d \\ b & a & -d & -c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, \quad \imath(\mathbb{H}_{\mathrm{cotan}}) = \begin{bmatrix} a & -b & -c & d \\ b & a & -d & -c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix}.$$

TABLE 12.3
Rules for computing the inverses $a^{-1} = b/\det = (b_1, b_2, b_3, b_4)/\det$ of $a = (a_1, a_2, a_3, a_4) \in \mathcal{A}$ in the four commutative algebras $\mathcal{A}$.

No	Algebra	$a^{-1} = (b_1, b_2, b_3, b_4)/\det$
3	$\mathbb{H}_{\mathrm{tes}}$	$\det = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^2 - 4(a_1a_3 + a_2a_4)^2$ $b_1 = a_1(a_1^2 + a_2^2 - a_3^2 + a_4^2) - 2a_2a_3a_4$ $b_2 = -a_2(a_1^2 + a_2^2 + a_3^2 - a_4^2) + 2a_1a_3a_4$ $b_3 = a_3(-a_1^2 + a_2^2 + a_3^2 + a_4^2) - 2a_1a_2a_4$ $b_4 = -a_4(a_1^2 - a_2^2 + a_3^2 + a_4^2) + 2a_1a_2a_3$
4	$\mathbb{H}_{\mathrm{cotes}}$	$\det = (a_1^2 + a_2^2 - a_3^2 - a_4^2)^2 - 4(a_1a_2 - a_3a_4)^2$ $b_1 = a_1(a_1^2 - a_2^2 - a_3^2 - a_4^2) + 2a_2a_3a_4$ $b_2 = a_2(-a_1^2 + a_2^2 - a_3^2 - a_4^2) + 2a_1a_3a_4$ $b_3 = a_3(-a_1^2 - a_2^2 + a_3^2 - a_4^2) + 2a_1a_2a_4$ $b_4 = a_4(-a_1^2 - a_2^2 - a_3^2 + a_4^2) + 2a_1a_2a_3$
7	$\mathbb{H}_{\mathrm{tan}}$	$\det = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^2 - 4(a_1a_2 + a_3a_4)^2$ $b_1 = a_1(a_1^2 - a_2^2 + a_3^2 + a_4^2) - 2a_2a_3a_4$ $b_2 = a_2(-a_1^2 + a_2^2 + a_3^2 + a_4^2) - 2a_1a_3a_4$ $b_3 = -a_3(a_1^2 + a_2^2 + a_3^2 - a_4^2) + 2a_1a_2a_4$ $b_4 = -a_4(a_1^2 + a_2^2 - a_3^2 + a_4^2) + 2a_1a_2a_3$
8	$\mathbb{H}_{\mathrm{cotan}}$	$\det = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^2 - 4(a_1a_4 - a_2a_3)^2$ $b_1 = a_1(a_1^2 + a_2^2 + a_3^2 - a_4^2) + 2a_2a_3a_4$ $b_2 = -a_2(a_1^2 + a_2^2 - a_3^2 + a_4^2) - 2a_1a_3a_4$ $b_3 = -a_3(a_1^2 - a_2^2 + a_3^2 + a_4^2) - 2a_1a_2a_4$ $b_4 = a_4(-a_1^2 + a_2^2 + a_3^2 + a_4^2) + 2a_1a_2a_3$

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In (2.1) we have already seen, that the algebra of coquaternions is isomorphic to the algebra of all real 2×2 matrices. This is also true for the algebra of nectarines and conectarines, though the representation differs as follows (using $\hat{\imath} : \mathcal{A} \to \mathbb{R}^{2\times 2}$):

$$\hat{\imath}(\mathbb{H}_{\rm nec}) = \begin{bmatrix} a-d & b+c \\ b-c & a+d \end{bmatrix}, \quad \hat{\imath}(\mathbb{H}_{\rm con}) = \begin{bmatrix} a-c & b+d \\ b-d & a+c \end{bmatrix}. \tag{12.5}$$

This can be verified by putting $\alpha = (a, b, c, d) = \mathfrak{e}_j,\ j = 1, 2, 3, 4$, and checking the multiplication rules of Table 12.1.

The noncommutative algebras number $2, 5, 6$ (coquaternions, nectarines, conectarines) are all isomorphic to the set of real 2×2 matrices $\mathbb{R}^{2\times 2}$ with representations given in (2.1), (12.5). Thus, it is sufficient to study one of these algebras in order to study $\mathbb{R}^{2\times 2}$.

12.1. Polynomials with coefficients from commutative algebras. A polynomial p with coefficients from a commutative algebra $\mathcal{A}$ always has the form (1.4). We are interested in finding the zeros of a given p and their number for the commutative algebras presented in this section. This will be called the *polynomial problem*. We show that the three commutative algebras $\mathbb{H}_{\rm tes}, \mathbb{H}_{\rm tan}, \mathbb{H}_{\rm cotan}$ are isomorphic and recall a result of 1891 by Segre [22], which settles the polynomial problem for the three mentioned algebras.

LEMMA 12.4. *The three algebras $\mathbb{H}_{\rm tes}, \mathbb{H}_{\rm tan}, \mathbb{H}_{\rm cotan}$ are isomorphic.*

Proof. Let $a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ and define two mappings $\mathsf{e}_1 := \text{exchange}_1$, $\mathsf{e}_2 := \text{exchange}_2 : \mathbb{R}^4 \to \mathbb{R}^4$ by $\mathsf{e}_1(a) = (a_1, a_3, a_2, a_4)$, $\mathsf{e}_2(a) = (a_1, a_4, -a_3, a_2)$. Note, that the two mappings are bijective with $\mathsf{e}_j^{-1} = \mathsf{e}_j, j = 1, 2$. Besides $a \in \mathbb{R}^4$, let $b = (b_1, b_2, b_3, b_4) \in \mathbb{R}^4$.

(A) In $\mathbb{H}_{\rm tes}$ let $c = ab$. Then, in $\mathbb{H}_{\rm tan}$ we have $c = \mathsf{e}_1(\mathsf{e}_1(a)\mathsf{e}_1(b))$. Or, in an equivalent formulation, $\mathsf{e}_1(ab) = \mathsf{e}_1(a)\mathsf{e}_1(b)$. Thus, $\mathbb{H}_{\rm tes}$ and $\mathbb{H}_{\rm tan}$ are isomorphic.

(B) In $\mathbb{H}_{\rm tan}$ let $c = ab$. Then, in $\mathbb{H}_{\rm cotan}$ we have $c = \mathsf{e}_2(\mathsf{e}_2(a)\mathsf{e}_2(b))$. Thus, $\mathbb{H}_{\rm tan}$ and $\mathbb{H}_{\rm cotan}$ are isomorphic.

This implies that all three algebras are pairwise isomorphic

(C) In $\mathbb{H}_{\rm tes}$ let $c = ab$. Then in $\mathbb{H}_{\rm cotan}$ we have $c = \mathsf{e}_1(\mathsf{e}_2\big(\mathsf{e}_2(\mathsf{e}_1(a))\,\mathsf{e}_2(\mathsf{e}_1(b))\big))$. ◻

Segre [22] introduced in his paper *bicomplex numbers* (*numeri bicomplessi*) and used a definition (on p. 456) which is identical with the definition of the algebra $\mathbb{H}_{\rm cotan}$ (number 8 in Table 12.1) of this paper[2]. In all eight algebras we have $\mathbf{ij} = \mathbf{k}$. Therefore, $a = (a_1, a_2, a_3, a_4)$ can be expressed in the form

$$a = x + y\mathbf{j}, \quad x = a_1 + a_2\mathbf{i},\ y = a_3 + a_4\mathbf{i}, \tag{12.6}$$

and in $\mathbb{H}_{\rm tes}, \mathbb{H}_{\rm cotan}$ we have $\mathbf{i}^2 = -1$ such that x, y are complex numbers in the usual sense. The *direct sum* $\mathbb{C} \oplus \mathbb{C}$ is the set of pairs (x, y) of complex numbers with the standard vector space operations and a new multiplication

$$(x, y) \star (u, v) := (xu, yv), \quad x, y, u, v \in \mathbb{C}, \tag{12.7}$$

where xu, yv are the standard products of complex numbers.

THEOREM 12.5. *The direct sum $\mathbb{C} \oplus \mathbb{C}$ with the multiplication rule (12.7) is isomorphic with $\mathbb{H}_{\rm tes}, \mathbb{H}_{\rm tan}, \mathbb{H}_{\rm cotan}$.*

Proof. Let $a, b \in \mathbb{H}_{\rm tes}$ with the representations $a = x + y\mathbf{j}, b = u + v\mathbf{j}$; see (12.6). Then,

$$ab = xu + yv + (xv + yu)\mathbf{j}.$$

[2] See also http://en.wikipedia.org/wiki/Bicomplex_number.

Define the mapping

$$A : \mathbb{H}_{\mathrm{tes}} \to \mathbb{C} \oplus \mathbb{C} \text{ by } A(x + y\mathbf{j}) := (x + y, x - y). \tag{12.8}$$

The inverse A^{-1} exists for all $(r, s) \in \mathbb{C} \oplus \mathbb{C}$ and

$$x + y\mathbf{j} = A^{-1}(r, s) = \frac{1}{2}\left(\left[r_1 + s_1 + (r_2 + s_2)\mathbf{i}\right] + \left[r_1 - s_1 + (r_2 - s_2)\mathbf{i}\right]\mathbf{j}\right).$$

Thus, A is bijective. We have to show that $A(ab) = A(a) \star A(b)$. Now,

$$\begin{aligned} A(ab) &= (xu + yv + (xv + yu), xu + yv - (xv + yu)) \\ = A(a) \star A(b) &= ((x + y)(u + v), (x - y)(u - v)). \end{aligned}$$

The remaining part of the proof follows from Lemma 12.4. ◻

COROLLARY 12.6. *The number of zeros of a polynomial p of degree n in $\mathbb{H}_{\mathrm{tes}}, \mathbb{H}_{\mathrm{tan}}, \mathbb{H}_{\mathrm{cotan}}$ is at most n^2 and at least one.*

Proof. Let $p(z) = \sum_{\ell=0}^{n} a_\ell \star z^\ell$ be a polynomial in $\mathbb{C} \oplus \mathbb{C}$ and let $a_\ell = (b_\ell, c_\ell)$ and $z = (u, v)$. Then, $p(z) = \sum_{\ell=0}^{n} (b_\ell, c_\ell) \star (u^\ell, v^\ell) = (0, 0)$ splits into two complex polynomials $p_1(u) = \sum_{\ell=0}^{n} b_\ell u^\ell = 0, p_2(v) = \sum_{\ell=0}^{n} c_\ell v^\ell = 0$. If the zeros of p_1 are $u_1, u_2, \ldots, u_n$ and those of p_2 are $v_1, v_2, \ldots, v_n$, then the zeros of p are (u_r, v_s), $r, s = 1, 2, \ldots, n$. Because of the isomorphism between $\mathbb{C} \oplus \mathbb{C}$ and $\mathbb{H}_{\mathrm{tes}}, \mathbb{H}_{\mathrm{tan}}, \mathbb{H}_{\mathrm{cotan}}$, the statement of the theorem follows. ◻

For examples of zeros of polynomials in $\mathbb{H}_{\mathrm{cotan}}$, see [18].

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