Electronic Transactions on Numerical Analysis.
Volume 39, pp. 333-339, 2012.

ISSN 1068-9613.

ETNA
Kent State University
http://etna.math.kent.edu

CASCADIC MULTIGRID PRECONDITIONER FOR ELLIPTIC EQUATIONS WITH JUMP COEFFICIENTS*

ZHIYONG LIU† AND YINNIAN HE†

Abstract. This paper provides a proof of robustness of the cascadic multigrid preconditioner for the linear finite element approximation of second order elliptic problems with strongly discontinuous coefficients. As a result, we prove that the convergence rate of the conjugate gradient method with cascadic multigrid preconditioner is uniform with respect to large jumps and mesh sizes.

Key words. Jump coefficients, conjugate gradient, condition number, cascadic multigrid.

AMS subject classifications. 65N30, 65N55, 65F10.

1. Introduction.

In this paper, we will discuss the cascadic multigrid preconditioned conjugate gradient method for the linear finite element approximation of the second order elliptic boundary value problem

$$\begin{cases} -\nabla \cdot (\omega \nabla u) = f, & \text{in } \Omega, \\ u = g_D, & \text{on } \Gamma_D, \\ -\omega \dfrac{\partial u}{\partial n} = g_N, & \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

where $\Omega \in R^d$ $(d = 1, 2, 3)$ is a polygonal or polyhedral domain with Dirichlet boundary Γ_D and Neumann boundary Γ_N. The coefficient $\omega = \omega(x)$ is a positive and piecewise constant function. More precisely, we assume that there are M open disjointed polygonal or polyhedral regions Ω_m $(m = 1, \ldots, M)$ satisfying $\bigcup_{m=1}^{M} \overline{\Omega}_m = \overline{\Omega}$ with

$$\omega|_{\Omega_m} = \omega_m, \quad m = 1, \ldots, M,$$

where each $\omega_m > 0$ is a constant. The analysis can be carried through to a more general case when $\omega(x)$ varies moderately in each subdomain.

We assume that the subdomains Ω_m, $m = 1, \ldots, M$, are given and fixed, but may possibly have complicated geometry. We are concerned with the robustness of the preconditioned conjugate gradient method with regard to both the mesh size and jump coefficients. This model problem is relevant to many applications, such as groundwater flow [1, 9], fluid pressure prediction [13], electromagnetics [7], semiconductor modeling [4], electrical power network modeling [8], and fuel cell modeling [14, 15], where the coefficients have large discontinuities across interfaces between subdomains with different material properties.

The goal of this paper is to provide a proof of the robustness of the cascadic multigrid preconditioner (CMG-PCG). In this paper, we improve the condition number bound for CMG-PCG to $C_1/(1 - C_2 h^2)$.

The rest of the paper is organized as follows. In Section 2, we introduce some basic notation, the PCG algorithm, and some theoretical foundations. In Section 3, we introduce the cascadic multigrid method and analyze the condition number of the CMG preconditioner. Section 4 contains our conclusions. Following [16], $x \lesssim y$ means $x \leq Cy$.

*Received October 20, 2011. Accepted July 18, 2012. Published online on October 13, 2012. Recommended by U. Langer.

†Center for Computational Geosciences, School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, P. R. China (zhiyongliu1983@163.com, heyn@mail.xjtu.edu.cn). This work was supported by the Natural Science Foundations of China (No.10971166).

ETNA
Kent State University
http://etna.math.kent.edu

2. Preliminaries.

2.1. Notation. We introduce the bilinear form

$$a(u,v) = \sum_{m=1}^{M} \omega_m (\nabla u, \nabla v)_{L^2(\Omega_m)}, \quad u, v \in H_D^1(\Omega),$$

where $H_D^1(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$, and define the H^1-norm and seminorm with respect to any subregion Ω_m by

$$|u|_{1,\Omega_m} = \|\nabla u\|_{0,\Omega_m}, \quad \|u\|_{1,\Omega_m} = (\|u\|_{0,\Omega_m}^2 + |u|_{1,\Omega_m}^2)^{\frac{1}{2}}.$$

Thus,

$$a(u,u) = \sum_{m=1}^{M} \omega_m |u|_{1,\Omega_m}^2 := |u|_{1,\omega}^2 .$$

We also need the weighted L^2-inner product

$$(u,v)_{0,\omega} = \sum_{m=1}^{M} \omega_m (u,v)_{L^2(\Omega_m)}$$

and the weighted L^2- and H^1-norms

$$\|u\|_{0,\omega} = (u,u)_{0,\omega}^{\frac{1}{2}}, \quad \|u\|_{1,\omega} = (\|u\|_{0,\omega}^2 + |u|_{1,\omega}^2)^{\frac{1}{2}}.$$

2.2. The discrete systems. Given a quasi-uniform triangulation $\mathcal{T}_h$ with mesh size h, let

$$\mathcal{V}_h = \{v \in H_D^1(\Omega) : v|_\tau \in \mathcal{P}_1(\tau), \forall \tau \in \mathcal{T}_h\}$$

be the piecewise linear finite element space, where $\mathcal{P}_1$ denotes the set of linear polynomials. The finite element approximation of (1.1) (with homogeneous Dirichlet boundary) is the function $u \in \mathcal{V}_h$, such that

$$a(u,v) = (f,v) + \int_{\Gamma_N} g_N v, \quad \forall v \in \mathcal{V}_h.$$

We define a linear symmetric positive definite operator $A : \mathcal{V}_h \to \mathcal{V}_h$ by

$$(Au, v)_{0,\omega} = a(u,v).$$

The related inner product and the induced energy norm are denoted by

$$(\cdot,\cdot)_A := a(\cdot,\cdot), \quad \|\cdot\|_A := \sqrt{a(\cdot,\cdot)}.$$

Then, we have the following operator equation

$$Au = F. \tag{2.1}$$

Indeed, (2.1) can be reduced to a linear system of equations with coefficient matrix

$$\mathcal{A} = (a_{ij})_{n\times n}, \quad a_{ij} = a(\phi_i,\phi_j) = \int_\Omega \omega \, \nabla\phi_i \cdot \nabla\phi_j \ .$$

Here, $\{\phi_i\}_{i=1}^n$ are natural nodal basis in the space V_h. For the sake of simplicity, in the following sections we still view A as the coefficient matrix.

ETNA
Kent State University
http://etna.math.kent.edu

2.3. Preconditioned conjugate gradient (PCG) method. The well known conjugate gradient method is the basis of the preconditioning technique to be studied in this paper. The PCG method can be viewed as a conjugate gradient method applied to the preconditioned systems

$$BAu = BF.$$

Here, B is a symmetric positive definite operator, known as a preconditioner of A. Note that BA is symmetric with respect to the inner product $(\cdot,\cdot)_{B^{-1}}$ (or $(\cdot,\cdot)_A$). Regarding the implementation of the PCG algorithm, we refer to monographs [2, 10, 11].

Let $u_k, k = 0, 1, \ldots,$ be the solution sequence of the PCG algorithm. It is well known that

$$\|u - u_k\|_A \leq 2\left(\frac{\sqrt{k(BA)} - 1}{\sqrt{k(BA)} + 1}\right)^k \|u - u_0\|_A, \tag{2.2}$$

which implies that the PCG method generally converges faster the smaller the condition number $k(BA)$ is.

Consider now the right-preconditioned method for the systems (2.1),

$$ABx = F, \quad u \equiv Bx.$$

Simple calculations give

$$\begin{aligned}(u,u)_A &= (x,x)_{B^TAB},\\ (u,BAu)_A &= (x,ABx)_{B^TAB},\\ (BAu,BAu)_A &= (ABx,ABx)_{B^TAB}\ .\end{aligned}$$

So, the convergence rate estimate (2.2) is accurate also for the right-preconditioned method.

3. Cascadic multigrid preconditioner. In this section, we introduce the cascadic multigrid preconditioner. The cascadic conjugate-gradient method (CCG-algorithm, in short) was proposed by P. Deuflhard [5] and developed by V. V. Shaidurov [12]. In 1996, F. A. Bornemann [3] extended it to the case where the CG iteration on each refinement level is replaced by some general smoother, like the traditional candidates symmetric Gauss-Seidel, SSOR or damped Jacobi iteration. They call such “one-way multigrid” methods cascadic multigrid methods. In this paper, we will apply the ideas of both P. Deuflhard and F. A. Bornemann. We only use CG iteration at the finest level while using traditional iteration methods on other levels. That is to say, we will obtain a better preconditioner at the finest level through computation on some coarse levels. And then, by (2.2), we can prove the convergence rate of the cascadic multigrid method.

3.1. Some notation. For problem (1.1) (with homogeneous Dirichlet boundary), given a nested family of triangulations, we have the linear finite element spaces

$$V_0 \subset V_1 \subset \cdots \subset V_L := V \subset H_0^1(\Omega).$$

The finite element approximations at level ℓ are given by $u \in V_\ell$ such that

$$a(u,v) = (f,v) + \int_{\Gamma_N} g_N v, \quad \forall v \in V_\ell\ . \tag{3.1}$$

The cascadic multigrid method for (1.1) can be defined as follows:

ETNA
Kent State University
http://etna.math.kent.edu

- smoothing m_0 times of (3.1) on the coarsest level with a given initial guess $u_0^* = u_0$, to obtain an initial guess for the next finer level through interpolation;

$$\dots$$

- smoothing m_L times of (3.1) on the finest level, to obtain a final approximation to the solution.

Following [3], denoting the basic iterative procedure on each level by the operator $\mathscr{T}$, the cascadic multigrid method can be rewritten as:

1. $u_0^* = u_0$;
2. $u_\ell^* = \mathscr{T}_{\ell,m_\ell} u_{\ell-1}^*$, $\ell = 1, 2, \dots, L$.

Here $\mathscr{T}_{\ell,m_\ell}$ denotes m_ℓ steps of the basic iteration applied at level ℓ.

We consider the following type of basic iteration for the given problem at level ℓ, with initial vector $v \in V_\ell$:

$$u - \mathscr{T}_{\ell,m_\ell} v = R_{\ell,m_\ell}(u - v), \tag{3.2}$$

with a linear mapping $R_{\ell,m_\ell} : V_\ell \to V_\ell$ for the error propagation. We call the basic iteration an energy reducing smoother, if it obeys the smoothing properties

$$\|R_{\ell,m_\ell} v\|_A \le c \frac{h_\ell^{-1}}{m_\ell^r} \|v\|_{L^2} \tag{3.3}$$

and

$$\|R_{\ell,m_\ell} v\|_A \le \|v\|_A, \tag{3.4}$$

for any $v \in V_\ell$.

Here $0 < r \le 1$, and m_ℓ is the number of steps of the basic iteration applied at level ℓ. As shown in [3] and [6], the symmetric Gauss-Seidel, SSOR and damped Jacobi iteration are smoothers in the sense of (3.3) and (3.4), with parameter $r = 1/2$. A detailed proof of (3.3) and (3.4) is provided in [6].

For the sake of simplicity, we write $\mathscr{T}_{\ell,m_\ell}$ and R_{ℓ,m_ℓ} as $\mathscr{T}_\ell$ and R_ℓ respectively. Then the right cascadic multigrid preconditioner B can be defined as:

$$B = \prod_{\ell=1}^{L} \mathscr{T}_\ell \, .$$

3.2. Eigenvalue analysis of BA. The analysis of the cascadic multigrid preconditioner relies on the following three lemmas.

LEMMA 3.1. *The linear operator*

$$P = \prod_{\ell=1}^{L-1} \mathscr{T}_\ell A$$

is bounded on V.

Proof. We only need to prove that P is continuous at 0. For any $v_n \in V_\ell \subset V$ such that $v_n \to 0$, from (3.2) we have

$$u - \mathscr{T}_\ell v_n = R_\ell(u - v_n),$$
$$u - \mathscr{T}_\ell 0 = R_\ell(u - 0),$$

ETNA
Kent State University
http://etna.math.kent.edu

which implies

$$\mathscr{T}_\ell v_n = R_\ell v_n \ .$$

Using (3.4), we have

$$0 \le \|\mathscr{T}_\ell v_n\|_A = \|R_\ell v_n\|_A \le \|v_n\|_A \to 0,$$

which implies $\mathscr{T}_\ell v_n \to 0$. Since $\mathscr{T}_\ell$ is bounded, P is bounded too. □

LEMMA 3.2. *Let D, L, and L^T be the diagonal, lower triangular, and upper triangluar part of A, respectively. Then $(D^{-1}u, u)_A \lesssim h^2(u, u)_A$, for any $u \in V$. Here $h = h_L$ is the mesh size at the finest level.*

Proof. Since ϕ_i is a piecewise linear function,

$$a_{ii} = \int_\Omega \omega\,|\nabla\phi_i|^2 = \overline{\omega}_i \int_{\Omega_i} |\nabla\phi_i|^2 \gtrsim \overline{\omega}_i h^{d-2}.$$

Following [17], we have

$$(Du, u)_A \gtrsim h^{d-2} \sum_{i=1}^{n} \overline{\omega}_i |u_i|^2_{1,\omega} \gtrsim h^{-2}(u, u)_A \ . \qquad \square$$

LEMMA 3.3. *If R_ℓ satisfies (3.4), then for any $u \in V$ there exist a constant $m > 0$ such that*

$$(R_\ell u, u)_A \le m(D^{-1}u, u)_A \ .$$

Proof. If not, then for any natural number n there exists $v_n \in V$ such that

$$(R_\ell v_n, v_n)_A > n(D^{-1}v_n, v_n)_A \ .$$

Take $u_n = v_n/\|v_n\|_A \in V$, so that $\|u_n\|_A = 1$. By dividing both sides of the above inequality by $\|v_n\|_A^2$, we obtain

$$(D^{-1}u_n, u_n)_A < \frac{1}{n}(R_\ell u_n, u_n)_A \le \frac{1}{n}\|R_\ell u_n\|_A\|u_n\|_A \le \frac{1}{n} \ .$$

Since V is complete, $u_n \to u_0$ in V. So $u_0 \in V$ and $\|u_0\|_A = 1$. Taking the limit of the above inequality, we have

$$\frac{(D^{-1}u_0, u_0)_A}{(u_0, u_0)_A} < 0.$$

This is a contradiction with the definition of a_{ii}. □

THEOREM 3.4. *There exist constants $C_1 > 0$ and $C_2 > 0$, which depend only on the connectivity of the mesh, such that*

$$\lambda_{\max}(AB) \le C_1, \quad \lambda_{\min}(AB) \gtrsim 1 - C_2 h^2 \ .$$

Proof. To prove the upper bound, we use (3.2), (3.4), the Schwarz inequality, and Lemma 3.1. Letting $v = Pu$, for any $u \in V$, we obtain

$$\begin{aligned}\frac{(u, BAu)_A}{(u,u)_A} &= \frac{(u, \mathscr{T}_L v)_A}{(u,u)_A} = \frac{(u,u)_A - (u, R_L u)_A + (u, R_L v)_A}{(u,u)_A} \\ &\leq 1 + \frac{(u, R_L v)_A}{(u,u)_A} \leq 1 + \frac{(u,u)_A^{1/2}(R_L v, R_L v)_A^{1/2}}{(u,u)_A} \\ &= 1 + \left\{\frac{(R_L v, R_L v)_A}{(u,u)_A}\right\}^{1/2} \leq 1 + \left\{\frac{(v,v)_A}{(u,u)_A}\right\}^{1/2} \\ &= 1 + \frac{\|Pu\|_A}{\|u\|_A} \leq C_1.\end{aligned}$$

To prove the lower bound, we use (3.2), Lemma 3.2, and Lemma 3.3

$$\begin{aligned}\frac{(u, BAu)_A}{(u,u)_A} &= \frac{(u, \mathscr{T}_L v)_A}{(u,u)_A} = \frac{(u,u)_A - (u, R_L u)_A + (u, R_L v)_A}{(u,u)_A} \\ &\geq 1 - \frac{m(D^{-1}u, u)_A}{(u,u)_A} \gtrsim 1 - C_2 h^2. \qquad \square\end{aligned}$$

REMARK 3.5. From Theorem 3.4, we know that

$$k(AB) \leq \frac{C_1}{1 - C_2 h^2}\ .$$

When $h \to 0$, $k(AB) \leq C$.

The following theorem states that the CMG-PCG algorithm introduced above behaves much better than other methods.

THEOREM 3.6. *For the CMG-PCG algorithm, the convergence rate estimate* (2.2) *becomes*

$$\frac{\|u - u_k\|_A}{\|u - u_0\|_A} \lesssim \left(\frac{C_1 - 1 + C_2 h^2}{C_1 + 1 - C_2 h^2}\right)^k,$$

where C_1 and C_2 are constants independent of coefficients and mesh size. The number of iterations needed to satisfy $\frac{\|u-u_k\|_A}{\|u-u_0\|_A} < \epsilon$, for a given tolerance $\varepsilon \in (0,1)$, satisfies

$$k \geq \frac{\log(\varepsilon)}{\log \frac{C_1 - 1 + C_2 h^2}{C_1 + 1 - C_2 h^2}}\ .$$

4. Conclusions. In this paper, we provided a proof of robustness of the cascadic multigrid preconditioner for the linear finite element approximation of second order elliptic problems with strongly discontinuous coefficients. We analyzed the eigenvalues of the CMG-preconditioner and found that the condition number of the preconditioned systems can be bounded by $C_1/(1 - C_2 h^2)$. The convergence rate of the CMG-PCG method is uniform with respect to the jump coefficients and mesh size.

Acknowledgement. The authors are grateful to the referees who made many valuable comments on an earlier version of this paper.

REFERENCES

[1] R. E. ALCOUFFE, A. BRANDT, J. E. DENDY, JR., AND J. W. PAINTER, *The multi-grid method for the diffusion equation with strongly discontinuous coefficients*, SIAM J. Sci. Stat. Comput., 2 (1981), pp. 430–454.

[2] O. AXELSSON, *Iterative Solution Methods*, Cambridge University Press, Cambridge, 1994.

[3] F. A. BORNEMANN AND P. DEUFLHARD, *The cascadic multigrid method for elliptic problems*, Numer. Math., 75 (1996), pp. 135–152.

[4] R. K. COOMER AND I. G. GRAHAM, *Massively parallel methods for semiconductor device modelling*, Computing, 56 (1996), pp. 1–27.

[5] P. DEUFLHARD, *Cascadic conjugate gradient methods for elliptic partial differential equations I. Algorithm and numerical results*, Tech. Report SC 93-23, Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB), 1993.

[6] W. HACKBUSCH, *Multi-Grid Methods and Applications*, Springer, Berlin, 1985.

[7] B. HEISE AND M. KUHN, *Parallel solvers for linear and nonlinear exterior magnetic field problems based upon coupled FE/BE formulations*, Computing, 56 (1996), pp. 237–258.

[8] V. E. HOWLE, S. A. VAVASIS, *An iterative method for solving complex-symmetric systems arising in electrical power modeling*, SIAM J. Matrix Anal. Appl., 26 (2005), pp. 1150–1178.

[9] C. E. KEES, C. T. MILLER, E. W. JENKINS, AND C. T. KELLEY, *Versatile two-level Schwarz preconditioners for multiphase flow*, Comput. Geosci., 7 (2003), pp. 91–114.

[10] P. OSWALD, *On the robustness of the BPX-preconditioner with respect to jumps in the coefficients*, Math. Comp., 68 (1999), pp. 633–650.

[11] Y. SAAD, *Iterative methods for sparse linear systems*, SIAM, Philadelphia, 2003.

[12] V. V. SHAIDUROV, *Some estimates of the rate of convergence for the cascadic conjugate-gradient method*, Comput. Math. Appl., 31 (1996), pp. 161–171.

[13] C. VUIK, A. SEGAL, AND J. A. MEIJERINK, *An efficient preconditioned CG method for the solution of a class of layered problems with extreme contrasts in the coefficients*, J. Comput. Phys., 152 (1999), pp. 385–403.

[14] C. WANG, *Fundamental models for fuel cell engineering*, Chem. Rev., 104 (2004), pp. 4727–4766.

[15] Z. WANG, C. WANG, AND K. CHEN, *Two phase flow and transport in the air cathode of proton exchange membrane fuel cells*, J. Power Sources, 94 (2001), pp. 40–50.

[16] J. XU, *Iterative methods by space decomposition and subspace correction*, SIAM Rev., 34 (1992), pp. 581–613.

[17] J. XU AND Y. ZHU, *Uniform convergent multigrid methods for elliptic problems with strongly discontinuous coefficients*, Math. Models Methods Appl. Sci., 18 (2008), pp. 77–106.