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# AN ITERATIVE SUBSTRUCTURING ALGORITHM FOR A $C^0$ INTERIOR PENALTY METHOD*

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**Abstract.** We study an iterative substructuring algorithm for a $C^0$ interior penalty method for the biharmonic problem. This algorithm is based on a Bramble-Pasciak-Schatz preconditioner. The condition number of the preconditioned Schur complement operator is shown to be bounded by $C\left(1+\ln(\frac{H}{h})\right)^2$, where $h$ is the mesh size of the triangulation, $H$ represents the typical diameter of the nonoverlapping subdomains, and the positive constant $C$ is independent of $h$, $H$, and the number of subdomains. Corroborating numerical results are also presented.

**Key words.** biharmonic problem, iterative substructuring, domain decomposition, $C^0$ interior penalty methods, discontinuous Galerkin methods

**AMS subject classification.** 65N55, 65N30

**1. Introduction.** Consider the following weak formulation of a fourth order model problem on a bounded polygonal domain $\Omega$ in $\mathbb{R}^2$.

Find $u \in H_0^2(\Omega)$ such that

$$\int_\Omega \nabla^2 u : \nabla^2 v\, dx = \int_\Omega f v\, dx \tag{1.1}$$

for all $v \in H_0^2(\Omega)$, where $f \in L_2(\Omega)$ and $\nabla^2 w : \nabla^2 v = \sum_{i,j=1}^2 \frac{\partial^2 w}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j}$ is the inner product of the Hessian matrices of the functions $w$ and $v$.

The model problem (1.1) can be solved by $C^0$ interior penalty methods [10, 17, 25, 29]. For simplicity we assume that $\Omega$ has a quasi-uniform triangulation $\mathcal{T}_h$ consisting of rectangles, and we take $V_h \subset H_0^1(\Omega)$ to be the $\mathbb{Q}_2$ Lagrange finite element space associated with $\mathcal{T}_h$. The discrete problem for (1.1) is to find $u_h \in V_h$ such that

$$\mathcal{A}_h(u_h, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_h, \tag{1.2}$$

where

$$\begin{aligned} \mathcal{A}_h(u_h, v) &= \sum_{D\in\mathcal{T}_h} \int_D \nabla^2 u_h : \nabla^2 v\, dx \\ &\quad + \sum_{e\in\mathcal{E}_h} \int_e \left( \left\{\!\!\left\{ \frac{\partial^2 u_h}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[ \frac{\partial v}{\partial n} \right]\!\!\right] + \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[ \frac{\partial u_h}{\partial n} \right]\!\!\right] \right) ds \\ &\quad + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \int_e \left[\!\!\left[ \frac{\partial u_h}{\partial n} \right]\!\!\right] \left[\!\!\left[ \frac{\partial v}{\partial n} \right]\!\!\right] ds, \end{aligned} \tag{1.3}$$

$\mathcal{E}_h$ is the set of all edges of $\mathcal{T}_h$, $|e|$ is the length of the edge $e$, and $\sigma > 0$ is a penalty parameter. The jump $[\![\cdot]\!]$ and the average $\{\!\{\cdot\}\!\}$ are defined as follows.

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If $e$ is an interior edge of $\mathcal{T}_h$ shared by two elements $D_-$ and $D_+$ of $\mathcal{T}_h$, and $n_e$ is the unit normal vector pointing from $D_-$ to $D_+$, then we define on $e$

$$\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] = \frac{\partial v_+}{\partial n_e} - \frac{\partial v_-}{\partial n_e} \qquad \text{and} \qquad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v_+}{\partial n_e^2} + \frac{\partial^2 v_-}{\partial n_e^2}\right),$$

where $v_\pm = v\big|_{D_\pm}$. Note that the values of the jumps and averages are independent of the choices of $D_\pm$. For an edge $e$ on the boundary of $\Omega$, we take $n_e$ to be the outward pointing unit normal vector and define

$$\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] = -\frac{\partial v}{\partial n_e} \qquad \text{and} \qquad \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} = \frac{\partial^2 v}{\partial n_e^2}.$$

The $C^0$ interior penalty method is consistent in the sense that

$$\mathcal{A}_h(u,v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_h.$$

Moreover, for $\sigma > 0$ sufficiently large (which is assumed to be the case), there exist positive constants $C_1$ and $C_2$ independent of $h$ such that

$$C_1 \mathcal{A}_h(v,v) \le |v|^2_{H^2(\Omega,\mathcal{T}_h)} \le C_2 \mathcal{A}_h(v,v) \qquad \forall\, v \in V_h, \tag{1.4}$$

where

$$|v|^2_{H^2(\Omega,\mathcal{T}_h)} = \sum_{D\in\mathcal{T}_h} |v|^2_{H^2(D)} + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \left\| [\![ \tfrac{\partial v}{\partial n} ]\!] \right\|^2_{L_2(e)}.$$

Consequently, the error $\|u - u_h\|_{H^2(\Omega,\mathcal{T}_h)}$ is quasi-optimal [17].

$C^0$ interior penalty methods, which belong to the class of discontinuous Galerkin methods, have certain advantages over the usual finite element methods for fourth order problems. They are simpler than $C^1$ finite element methods. They come in a natural hierarchy (which is not the case for classical nonconforming finite element methods), and they preserve the symmetric positive definite property of the continuous problem (which is not the case for mixed finite element methods). They have also been applied to many other fourth order problems [11, 12, 18, 25, 33, 38, 39].

As an approximation of a fourth order differential operator, the condition number of the discrete problem grows at the rate of $h^{-4}$; cf. [31]. Thus a good preconditioner is essential for solving the discrete problem efficiently and accurately. Previously we have shown in [19] that the two-level additive Schwarz preconditioner for classical finite element methods [24] can be extended to $C^0$ interior penalty methods with similar performance. In this paper we will extend the Bramble-Pasciak-Schatz preconditioner [8] to $C^0$ interior penalty methods and show that the preconditioned system satisfies similar condition number estimates as in the case of classical finite element methods. This extension requires a new treatment of the degrees of freedom on the interface of the subdomains, which is discussed in Section 2. The techniques developed in this paper can be applied to $C^0$ interior penalty methods on general domains with simplicial triangulations, and they are also useful for other discontinuous Galerkin methods for fourth order problems [4, 34]. We note that domain decomposition algorithms for other discontinuous Galerkin methods can be found in [1, 2, 3, 5, 13, 22, 23, 26, 27, 30].

The rest of this paper is organized as follows. We introduce the iterative substructuring algorithm in Section 2. In Section 3 we construct a trace norm that plays a key role in the analysis of the preconditioned system. The condition number estimates are then derived in Section 4, and numerical results are presented in Section 5. Appendix A contains the proof of a lemma that is crucial for the analysis in Section 4.

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**2. An iterative substructuring algorithm.** We begin with a nonoverlapping domain decomposition of $\Omega$ consisting of rectangular (open) subdomains $\Omega_1, \Omega_2, \dots, \Omega_J$ aligned with $\mathcal{T}_h$ such that

$$\begin{aligned} \Omega_i \cap \Omega_j &= \emptyset && \text{if} \quad i \neq j, \\ \bar{\Omega} &= \bigcup_{j=1}^{J} \bar{\Omega}_j, \\ \partial\Omega_j \cap \partial\Omega_l &= \emptyset, \text{ a vertex, or an edge} && \text{if} \quad j \neq l. \end{aligned}$$

We assume the subdomains are shape regular and denote the typical diameter of the subdomains by $H$. The interface of the subdomains is the set $\Gamma = \bigcup_{j=1}^{J} \Gamma_j$, where $\Gamma_j = \partial\Omega_j$.

REMARK 2.1. Note that $\partial\Omega$ is part of the interface because the boundary condition for the normal derivative is only enforced weakly through the penalty term in (1.3).

The off-interface space $V_h(\Omega \setminus \Gamma) \subset V_h$ is defined by

$$V_h(\Omega \setminus \Gamma) = \{v \in V_h : v \text{ vanishes to first order on } \Gamma\},$$

i.e., $v \in V_h$ belongs to $V_h(\Omega \setminus \Gamma)$ if and only if $v$ and its normal derivative vanish on $\Gamma$. Since the condition that the normal derivative of $v$ vanishes on $\Gamma$ is implicit in terms of the standard degrees of freedom (dofs) of the $\mathbb{Q}_2$ finite element, it is more convenient for both implementation and analysis to modify the dofs for $V_h$ as follows.

(i) For an element $D$ away from the interface $\Gamma$, we keep the standard dofs, namely the values of $v \in V_h$ at the four vertices of $D$, at the four midpoints along $\partial D$, and at the center of $D$ (cf. the left-hand side of Figure 2.1).

(ii) For an element $D$ that is away from the corners of the subdomains but has an edge $e$ on $\Gamma$, we take the dofs to be the values of $v$ and its normal derivative at the vertices and the midpoint of $e$ and the values of $v$ at the vertices and midpoint of the edge parallel to $e$ (cf. the middle of Figure 2.1).

(iii) Finally, suppose a corner of the subdomain is also a vertex $p$ of an element $D$ and $e_1$ and $e_2$ are the two edges of $D$ that share $p$ as a common vertex (i.e., $e_1, e_2 \subset \Gamma$). In this case we take the dofs to be the value of $v$ at $p$, the values of its first order derivatives and second order mixed derivative at $p$, the values of $v$ at the other three vertices of $D$, and the values of the normal derivative of $v$ at the endpoints of $e_1$ and $e_2$ that are different from $p$ (cf. the right-hand side of Figure 2.1).

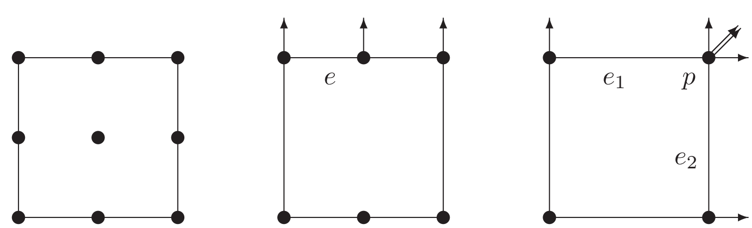


FIG. 2.1. *Dofs for the $\mathbb{Q}_2$ element.*

The dofs for the three cases are depicted in Figure 2.1, where the solid dot $\bullet$ denotes the pointwise evaluation of the shape functions, the arrow $\uparrow$ denotes the pointwise evaluation of the directional derivatives of the shape functions, and the double arrow $\nearrow\!\!\!\nearrow$ denotes the

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pointwise evaluation of the mixed second order derivative of the shape functions. It is easy to check that in each case a biquadratic polynomial is uniquely determined by the dofs.

REMARK 2.2. If one of the edges of $D$ is on the boundary of the subdomain, then the values of $v$ and $\frac{\partial v}{\partial n}$ are uniquely determined by the dofs associated with the nodes on that edge (cf. the middle and the right-hand side in Figure 2.1).

The modified (global) dofs for $V_h$ are depicted on the left of Figure 2.2 for a square divided into four subdomains.

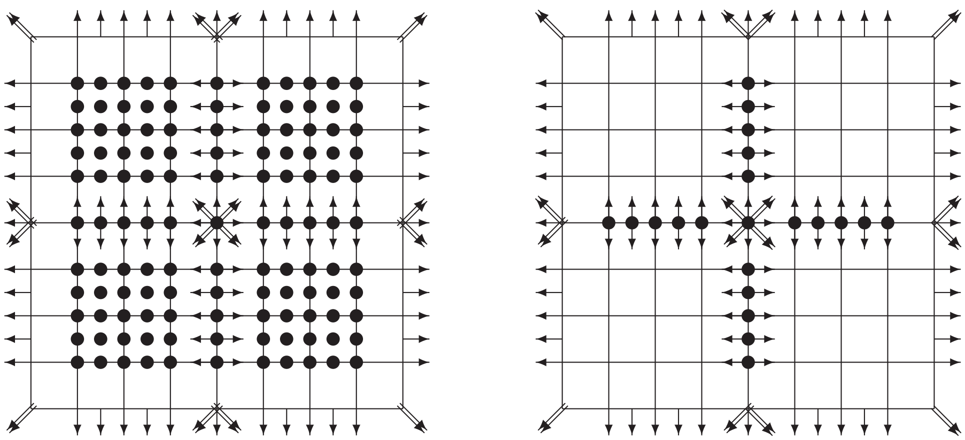

FIG. 2.2. *Modified dofs for* $V_h$ *and* $V_h(\Gamma)$.

Let $v \in V_h$. The dofs of $v$ associated with the nodes that are not on $\Gamma$ are standard. The dofs of $v$ associated with the nodes on $\Gamma$ can be divided into the following cases.

(i) There are three dofs associated with a node on $\Gamma$ that is interior to $\Omega$ and not the corner of any subdomain, namely the value of $v$ and the values of the normal derivatives of $v$ from the two sides.

(ii) At a node on $\partial\Omega$ that is not the corner of any subdomain, there is only one dof, namely the value of the normal derivative of $v$.

(iii) There is also only one dof at a node that is one of the corners of $\Omega$, namely the value of the mixed second order derivative $\frac{\partial^2 v}{\partial x_1 \partial x_2}$.

(iv) At a node on $\Gamma \cap \partial\Omega$ that is the common corner of two subdomains, there are three dofs, namely the value of the normal derivative of $v$ and the values of the two mixed second order derivatives of $v$ from the two subdomains.

(v) There are nine dofs associated with a node on $\Gamma$ that is the common vertex of four subdomains: the value of $v$, the values of $\frac{\partial v}{\partial x_1}$ from left and right, the values of $\frac{\partial v}{\partial x_2}$ from below and above, and the values of the mixed second order derivatives of $v$ from the four subdomains.

In terms of the new dofs, $v \in V_h(\Omega \setminus \Gamma)$ if and only if the dofs of $v$ along $\Gamma$ are identically 0. We will use these new dofs for $V_h$ in the rest of the paper.

REMARK 2.3. Since $V_h$ is a subspace of $H_0^1(\Omega)$, the dofs represented by solid dots on $\partial\Omega$ are not included in the global dofs. On the other hand, the normal derivative and mixed second order derivative of a finite element function in $V_h$ are not constrained along $\partial\Omega$ and therefore the dofs represented by arrows and double arrows along $\partial\Omega$ are included in the global dofs.

Next we define the interface space $V_h(\Gamma)$ to be the orthogonal complement of $V_h(\Omega \setminus \Gamma)$ with respect to $\mathcal{A}_h(\cdot, \cdot)$, i.e.,

$$V_h(\Gamma) = \{v \in V_h : \mathcal{A}_h(v, w) = 0,\ \forall\, w \in V_h(\Omega \setminus \Gamma)\}.$$

The functions in $V_h(\Gamma)$ will be referred to as discrete biharmonic functions. They are uniquely determined by the dofs associated with $\Gamma$ (cf. the right-hand side of Figure 2.2 for the case where a square is divided into four subdomains). The discrete biharmonic functions enjoy the following minimum energy property.

LEMMA 2.4. *We have*

$$\mathcal{A}_h(v,v) \le \mathcal{A}_h(w,w)$$

*for any* $v \in V_h(\Gamma)$ *and* $w \in V_h$ *that have identical dofs along* $\Gamma$.

*Proof*. Since $w - v \in V_h(\Omega \setminus \Gamma)$, we have by orthogonality

$$\begin{aligned}\mathcal{A}_h(w,w) &= \mathcal{A}_h((w-v)+v,(w-v)+v)\\ &= \mathcal{A}_h(w-v,w-v) + \mathcal{A}_h(v,v) \ge \mathcal{A}_h(v,v). \qquad \square\end{aligned}$$

The solution of the discrete problem (1.2) can be decomposed as

$$u_h = \dot{u}_h + \bar{u}_h,$$

where $\dot{u}_h \in V_h(\Omega\setminus\Gamma)$ and $\bar{u}_h \in V_h(\Gamma)$, and then (1.2) is equivalent to the following problem.

Find $\dot{u}_h \in V_h(\Omega \setminus \Gamma)$ and $\bar{u}_h \in V_h(\Gamma)$ such that

$$\begin{aligned}\mathcal{A}_h(\dot{u}_h, v) &= \int_\Omega f v\,dx \qquad \forall\, v \in V_h(\Omega\setminus\Gamma),\\ \mathcal{A}_h(\bar{u}_h, v) &= \int_\Omega f v\,dx \qquad \forall\, v \in V_h(\Gamma).\end{aligned} \tag{2.1}$$

Let $V_h(\Omega_j)$ be the space of $\mathbb{Q}_2$ finite element functions on $\Omega_j$ that vanish to first order on $\partial\Omega_j$, i.e., it is the restriction of $V_h(\Omega \setminus \Gamma)$ to $\Omega_j$. Then $\dot{u}_{h,j} = \dot{u}_h\big|_{\Omega_j} \in V_h(\Omega_j)$ and we have

$$\mathcal{A}_h(\dot{u}_{h,j}, v) = \int_\Omega f\tilde{v}\,dx \qquad \forall\, v \in V_h(\Omega_j), \tag{2.2}$$

where $\tilde{v} \in V_h$ is the trivial extension of $v$. Therefore, for $1 \le j \le J$, $\dot{u}_{h,j}$ can be computed by solving the subdomain problems (2.2) in parallel, and it only remains to construct an efficient solver for (2.1).

Let $S_h : V_h(\Gamma) \longrightarrow V_h(\Gamma)'$ be the Schur complement operator defined by

$$\langle S_h v_1, v_2\rangle = \mathcal{A}_h(v_1, v_2) \qquad \forall\, v_1, v_2 \in V_h(\Gamma), \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form between a vector space and its dual. We can rewrite (2.1) as

$$S_h \bar{u}_h = f_h, \tag{2.4}$$

where $f_h \in V_h(\Gamma)'$ is defined by $\langle f_h, v\rangle = \int_\Omega f v\,dx$ for all $v \in V_h(\Gamma)$. The last ingredient of the iterative substructuring algorithm is provided by a preconditioner for $S_h$ introduced by Bramble-Pasciak-Schatz [8] for classical finite element methods. Equation (2.4) can then be solved efficiently by the preconditioned conjugate gradient method.

The Bramble-Pasciak-Schatz (BPS) preconditioner involves local edge spaces and a global coarse space. Let $E_1$, $E_2$, …, $E_L$ be the (closed) edges of the subdomains. The edge space $V_\ell$ ($\subset V_h(\Gamma)$) associated with the edge $E_\ell$ is defined as follows. A discrete biharmonic function $v$ belongs to $V_\ell$ if and only if

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(i) $v$ vanishes identically outside the subdomains that contain $E_\ell$ as a boundary edge,
(ii) the dofs of $v$ at the nodes on $\Gamma \setminus E_\ell$ are identically 0.

Thus the discrete biharmonic functions in an edge space are determined by the dofs depicted in Figure 2.3, where on the left we have an edge shared by two subdomains and on the right we have an edge on $\partial\Omega$ that belongs to the boundary of only one subdomain.

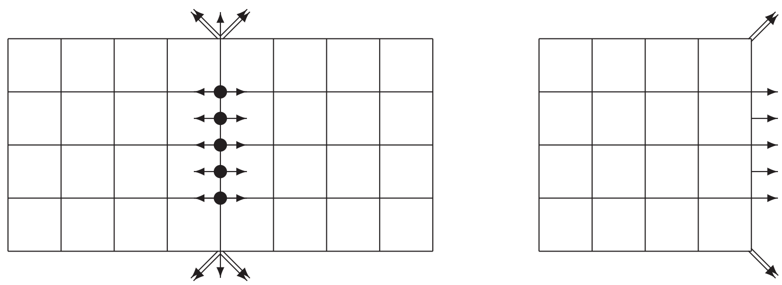

FIG. 2.3. *Dofs for edge spaces.*

The edge space $V_\ell$ is connected to $V_h(\Gamma)$ by the natural injection $I_\ell$, and there is an SPD operator $S_\ell : V_\ell \longrightarrow V_\ell'$ defined by

$$\langle S_\ell v, w\rangle = \mathcal{A}_h(v,w) \qquad \forall v,w \in V_\ell. \tag{2.5}$$

For the BPS preconditioner, the global communication among subdomains is provided by the coarse space $V_0 = V_H \subset H_0^1(\Omega)$, which is the $\mathbb{Q}_1$ Lagrange finite element space associated with the subdomains $\Omega_1, \dots, \Omega_J$. (The dofs for the $\mathbb{Q}_1$ Lagrange finite element are depicted on the left-hand side of Figure 2.4.) We define $S_0 : V_H \longrightarrow V_H'$ by

$$\langle S_0 v, w\rangle = \mathcal{A}_H(v,w) \qquad \forall\, v,w \in V_H, \tag{2.6}$$

where $\mathcal{A}_H$ is the analog of $\mathcal{A}_h$.

The connection between $V_H$ and $V_h(\Gamma)$ is given by an operator $I_0$ constructed by the following procedure. Let $\hat{V}_H \subset H_0^2(\Omega)$ be the $\mathbb{Q}_3$ Bogner-Fox-Schmit finite element space associated with $\mathcal{T}_H$. (The dofs for this $C^1$ element are depicted in the middle of Figure 2.4.) First we define an enriching operator $\mathbb{E}_H : V_H \longrightarrow \hat{V}_H$ by averaging, i.e., we define the dof of $\mathbb{E}_H v$ at a node to be the average of the dofs of $v$ at the same node from all the subdomains sharing that node. More precisely, we take

$$\begin{aligned}
(\mathbb{E}_H v)(p) &= v(p),\\
\nabla(\mathbb{E}_H v)(p) &= \frac{1}{4}\sum_{\Omega_j \in \mathcal{T}_{H,p}} \nabla v_j(p),\\
\frac{\partial^2(\mathbb{E}_H v)}{\partial x_1 \partial x_2}(p) &= \frac{1}{4}\sum_{\Omega_j \in \mathcal{T}_{H,p}} \frac{\partial^2 v_j}{\partial x_1 \partial x_2}(p),
\end{aligned}$$

where $p$ is any subdomain vertex in the interior of $\Omega$, $\mathcal{T}_{H,p}$ is the set of the four subdomains sharing $p$ as a vertex, and $v_j = v\big|_{\Omega_j}$. The following result can be easily obtained by a direct calculation; cf. [9, 17, 20] for similar estimates.

LEMMA 2.5. *There exists a positive constant $C_3$ depending only on the shape regularity of $\mathcal{T}_H$ such that*

$$|\mathbb{E}_H v|_{H^2(\Omega)} \le C_3 \sqrt{\mathcal{A}_H(v,v)} \qquad \forall\, v \in V_H.$$

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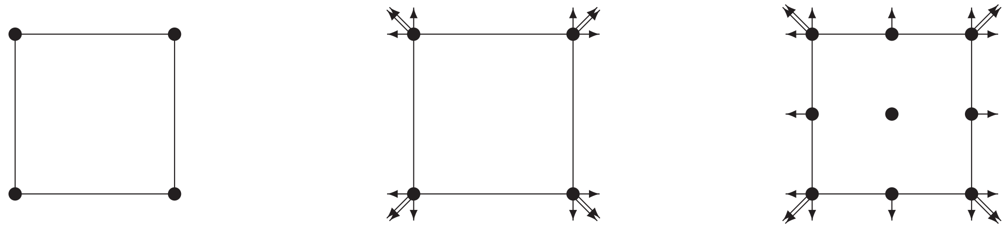

FIG. 2.4. *$H^1$ conforming $\mathbb{Q}_1$ Lagrange finite element and $H^2$-conforming Bogner-Fox-Schmit elements ($\mathbb{Q}_3$ and $\mathbb{Q}_4$).*

We take $I_0 v \in V_h(\Gamma)$ to be the discrete biharmonic function whose dofs on $\Gamma$ (cf. the right-hand side of Figure 2.2) are identical with the corresponding dofs of $\mathbb{E}_H v$.

REMARK 2.6. If we define the dofs of $I_0 v$ directly from $v$, then the performance of the preconditioner will be adversely affected by the different scalings that appear in the penalty terms for $\mathcal{A}_H(\cdot,\cdot)$ and $\mathcal{A}_h(\cdot,\cdot)$. This problem is avoided by $I_0$ defined above because $\mathbb{E}_H v \in H_0^2(\Omega)$ and the penalty term associated with $\mathcal{A}_h(\cdot,\cdot)$ has no effect on $I_0 v$.

We can now define the BPS preconditioner $B_{BPS} : V_h(\Gamma)' \longrightarrow V_h(\Gamma)$ by

$$B_{BPS} = I_0 S_0^{-1} I_0^t + \sum_{\ell=1}^{L} I_\ell S_\ell^{-1} I_\ell^t,$$

where $I_\ell^t : V_h(\Gamma)' \longrightarrow V_\ell'$ is the transpose of $I_\ell : V_\ell \longrightarrow V_h(\Gamma)$, i.e.,

$$\langle I_\ell^t \phi, v\rangle = \langle \phi, I_\ell v\rangle \qquad \forall\, v \in V_\ell,\ \phi \in V_h(\Gamma)'.$$

It is easy to see that $V_h(\Gamma) = \sum_{\ell=0}^{L} I_\ell V_\ell$. It then follows from the theory of additive Schwarz preconditioners [6, 14, 24, 28, 32, 35, 36, 37, 40, 41] that the eigenvalues of $B_{BPS} S_h$ are positive and that the maximum and minimum eigenvalues of $B_{BPS} S_h$ are characterized by the following formulas:

$$\lambda_{\max}(B_{BPS} S_h) = \max_{\substack{v \in V_h(\Gamma)\\ v \neq 0}} \frac{\langle S_h v, v\rangle}{\displaystyle\min_{\substack{v=\sum_{\ell=0}^{L} I_\ell v_\ell\\ v_\ell \in V_\ell}} \sum_{\ell=0}^{L} \langle S_\ell v_\ell, v_\ell\rangle}, \tag{2.7}$$

$$\lambda_{\min}(B_{BPS} S_h) = \min_{\substack{v \in V_h(\Gamma)\\ v \neq 0}} \frac{\langle S_h v, v\rangle}{\displaystyle\min_{\substack{v=\sum_{\ell=0}^{L} I_\ell v_\ell\\ v_\ell \in V_\ell}} \sum_{\ell=0}^{L} \langle S_\ell v_\ell, v_\ell\rangle}. \tag{2.8}$$

**3. A trace norm.** In this section we construct a trace norm on $V_h(\Gamma)$ that only involves integrals defined on $\Gamma$, and which is equivalent to the energy norm $\sqrt{\mathcal{A}_h(\cdot,\cdot)}$. It will play an important role in the derivation of a lower bound for $\lambda_{\min}(B_{BPS} S_h)$.

To avoid the proliferation of constants, from now on we use the notation $A \lesssim B$ to represent the statement $A \leq$ (constant) $\times B$, where the positive constant does not depend on $h$, $H$, and $J$. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

Let $V_{h,j}$, $1 \leq j \leq J$, be the restrictions of $V_h$ to the subdomain $\Omega_j$, i.e., it is the $\mathbb{Q}_2$ finite element space associated with $\mathcal{T}_{h,j}$ (the restriction of $\mathcal{T}_h$ to $\Omega_j$) whose members vanish

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on $\partial\Omega \cap \partial\Omega_j$. We introduce a seminorm $\|\cdot\|_{H^2(\Omega_j,\mathcal{T}_{h,j})}$ on $V_{h,j}$ defined by

$$\|v\|^2_{H^2(\Omega_j,\mathcal{T}_{h,j})} = \sum_{\substack{D\in\mathcal{T}_h\\ D\subset\Omega_j}} |v|^2_{H^2(D)} + \sum_{\substack{e\in\mathcal{E}_h\\ e\subset\Omega_j}} \frac{1}{|e|} \left\| \left[\!\left[ \frac{\partial v}{\partial n} \right]\!\right] \right\|^2_{L_2(e)} \qquad \forall\, v \in V_{h,j}.$$

We can then write

$$|v|^2_{H^2(\Omega,\mathcal{T}_h)} = \sum_{\substack{e\in\mathcal{E}_h\\ e\subset\Gamma}} \frac{1}{|e|} \left\| \left[\!\left[ \frac{\partial v}{\partial n} \right]\!\right] \right\|^2_{L_2(e)} + \sum_{j=1}^{J} \|v_j\|^2_{H^2(\Omega_j,\mathcal{T}_{h,j})} \qquad \forall\, v \in V_h, \tag{3.1}$$

where $v_j = v\big|_{\Omega_j}$.

Let $\tilde{V}_{h,j}$ be the $\mathbb{Q}_4$ Bogner-Fox-Schmit finite element space on $\Omega_j$ associated with $\mathcal{T}_{h,j}$ such that its members vanish on $\partial\Omega \cap \partial\Omega_j$. (The dofs for this $C^1$ element are depicted on the right-hand side of Figure 2.4.) Our construction of the trace norm on $V_h(\Gamma)$ uses the enriching map $\mathbb{E}_j : V_{h,j} \longrightarrow \tilde{V}_{h,j}$ defined by averaging: at any node of $\tilde{V}_{h,j}$, we assign a dof of $\mathbb{E}_j v$ to be the average of the corresponding dofs of $v$ from the elements that share that node. More precisely, for a given $v \in V_{h,j}$, the dofs of $\mathbb{E}_j v \in \tilde{V}_{h,j}$ are defined as follows.

(i) $\mathbb{E}_j v$ equals $v$ at all nodes (vertices, midpoints, centers) of $\mathcal{T}_{h,j}$.

(ii) At an interior vertex of $\mathcal{T}_{h,j}$, $\nabla(\mathbb{E}_j v)$ (respectively $\frac{\partial^2(\mathbb{E}_j v)}{\partial x_1 \partial x_2}$) is the average of $\nabla v$ (respectively $\frac{\partial^2 v}{\partial x_1 \partial x_2}$) at that vertex from the four elements sharing $p$ as a common vertex.

(iii) At a vertex of $\mathcal{T}_{h,j}$ on $\partial\Omega_j$ that is not a corner of $\Omega_j$, $\frac{\partial(\mathbb{E}_j v)}{\partial n} = \frac{\partial v}{\partial n}$ while the tangential (respectively mixed second order) derivative of $\mathbb{E}_j v$ is the average of the tangential (respectively mixed second order) derivatives of $v$ from the two elements sharing $p$ as a common vertex.

(iv) At the midpoint of an interior edge, the normal derivative of $\mathbb{E}_j v$ is the average of the two normal derivatives of $v$ (from the two sides) at that midpoint.

(v) At the midpoint of an edge on $\partial\Omega_j$, the normal derivative $\mathbb{E}_j v$ equals the normal derivative of $v$.

(vi) The dofs of $\mathbb{E}_j v$ at the four corners of $\Omega_j$ are identical with the dofs of $v$ at the corners.

REMARK 3.1. In view of Remark 2.2, the dofs of $\mathbb{E}_j v$ on $\partial\Omega_j$ are determined by the dofs of $v$ on $\partial\Omega_j$.

The following result again can be obtained by a direct calculation.

LEMMA 3.2. *We have, for* $1 \le j \le J$,

$$|\mathbb{E}_j v|_{H^2(\Omega_j)} \lesssim \|v\|_{H^2(\Omega_j,\mathcal{T}_{h,j})} \qquad \forall\, v \in V_{h,j}. \tag{3.2}$$

We can also define a map $\mathbb{F}_j : \tilde{V}_{h,j} \longrightarrow V_{h,j}$ by assigning the dofs of $\mathbb{F}_j v \in V_{h,j}$ to be identical with the corresponding dofs of $v \in \tilde{V}_{h,j}$. The following result can be derived by a simple element-wise calculation.

LEMMA 3.3. *We have, for* $1 \le j \le J$,

$$\|\mathbb{F}_j w\|_{H^2(\Omega_j,\mathcal{T}_{h,j})} \lesssim |w|_{H^2(\Omega_j)} \qquad \forall\, w \in \tilde{V}_{h,j}.$$

From the definitions of $\mathbb{E}_j$ and $\mathbb{F}_j$, it is easy to see that $\mathbb{F}_j(\mathbb{E}_j v) = v$ for all $v \in V_{h,j}$. The lemma below follows directly from Lemma 3.2 and Lemma 3.3.

LEMMA 3.4. *We have, for* $1 \le j \le J$,

$$\|v\|_{H^2(\Omega_j,\mathcal{T}_{h,j})} \approx |\mathbb{E}_j v|_{H^2(\Omega_j)} \qquad \forall\, v \in V_{h,j}.$$

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Given any $v_j \in V_{h,j}$, we define the functions $\mathcal{D}_1 v_j$ and $\mathcal{D}_2 v_j$ on $\partial\Omega_j$ by

$$\mathcal{D}_1 v_j = \frac{\partial(\mathbb{E}_j v_j)}{\partial x_1}\Big|_{\partial\Omega_j} \quad \text{and} \quad \mathcal{D}_2 v_j = \frac{\partial(\mathbb{E}_j v_j)}{\partial x_2}\Big|_{\partial\Omega_j}. \tag{3.3}$$

In view of Remark 3.1, the functions $\mathcal{D}_1 v$ and $\mathcal{D}_2 v$ can be computed from the dofs of $v$ associated with $\Gamma_j$. Recall that the Sobolev seminorm $H^{1/2}(\partial\Omega_j)$ is given by

$$|w|^2_{H^{1/2}(\partial\Omega_j)} = \int_{\partial\Omega_j}\int_{\partial\Omega_j} \frac{|w(x)-w(y)|^2}{|x-y|^2}\, ds(x)ds(y).$$

The following result shows that on the space $V_h(\Gamma)$, the energy norm $\sqrt{\mathcal{A}_h(\cdot,\cdot)}$ is equivalent to a trace norm that only involves integrals defined on $\Gamma$. Its proof is given in Appendix A.

LEMMA 3.5. *We have*

$$\mathcal{A}_h(v,v) \approx \sum_{\substack{e\in\mathcal{E}_h\\ e\subset\Gamma}} \frac{1}{|e|}\,\big\|\,[\![\tfrac{\partial v}{\partial n}]\!]\,\big\|^2_{L_2(e)} + \sum_{j=1}^{J}\Big(|\mathcal{D}_1 v_j|^2_{H^{1/2}(\partial\Omega_j)} + |\mathcal{D}_2 v_j|^2_{H^{1/2}(\partial\Omega_j)}\Big) \tag{3.4}$$

*for all $v \in V_h(\Gamma)$, where $v_j$ is the restriction of $v$ to $\Omega_j$ for $1 \le j \le J$.*

**4. Condition number estimates.** First we consider an upper bound for the eigenvalues of the operator $B_{\scriptscriptstyle BPS} S_h$.

LEMMA 4.1. *The maximum eigenvalue of $B_{\scriptscriptstyle BPS} S_h$ satisfies the following estimate:*

$$\lambda_{\max}(B_{\scriptscriptstyle BPS} S_h) \lesssim 1. \tag{4.1}$$

*Proof*. Let $v \in V_h(\Gamma)$ be arbitrary, and let $v_\ell \in V_\ell$ for $0 \le \ell \le L$ satisfy

$$v = \sum_{\ell=0}^{L} I_\ell v_\ell. \tag{4.2}$$

It follows from (2.3) and the Cauchy-Schwarz inequality that

$$\langle S_h v, v\rangle = \mathcal{A}_h\Big(\sum_{\ell=0}^{L} I_\ell v_\ell, \sum_{k=0}^{L} I_k v_k\Big) \lesssim \mathcal{A}_h(I_0 v, I_0 v) + \mathcal{A}_h\Big(\sum_{\ell=1}^{L} I_\ell v_\ell, \sum_{k=1}^{L} I_k v_k\Big). \tag{4.3}$$

Let $z \in V_h$ be defined by $z\big|_{\Omega_j} = \mathbb{F}_j\big(\mathbb{E}_H v_0\big|_{\Omega_j}\big)$. Then $z$ and $I_0 v$ have identical dofs along $\Gamma$ and hence

$$\begin{aligned}\mathcal{A}_h(I_0 v_0, I_0 v_0) &\le \mathcal{A}_h(z,z) \approx |z|^2_{H^2(\Omega,\mathcal{T}_h)} \\ &= \sum_{j=1}^{J} \|z_j\|^2_{H^2(\Omega_j,\mathcal{T}_{h,j})} \lesssim |\mathbb{E}_H v_0|^2_{H^2(\Omega)} \lesssim \langle S_0 v_0, v_0\rangle\end{aligned} \tag{4.4}$$

by Lemma 2.4, (1.4), (3.1), Lemma 3.3, Lemma 2.5, and (2.6). Here we have also used the fact that $[\![\frac{\partial z}{\partial n}]\!] = 0$ on $\Gamma$. Finally since $\mathcal{A}_h(I_\ell v_\ell, I_k v_k) = 0$ unless the subdomains $\Omega_\ell$ and $\Omega_k$ are sufficiently close, we have by (2.5)

$$\mathcal{A}_h\Big(\sum_{\ell=1}^{L} I_\ell v_\ell, \sum_{k=1}^{L} I_k v_k\Big) \lesssim \sum_{\ell=1}^{L} \mathcal{A}_h(I_\ell v_\ell, I_\ell v_\ell) = \sum_{\ell=1}^{L}\langle S_\ell v_\ell, v_\ell\rangle. \tag{4.5}$$

Putting the estimates (4.3)–(4.5) together, we find $\langle S_h v, v\rangle \lesssim \sum_{\ell=0}^{L}\langle S_\ell v_\ell, v_\ell\rangle$ and therefore

$$\langle S_h v, v\rangle \lesssim \min_{\substack{v=\sum_{\ell=0}^{L} I_\ell v_\ell\\ v_\ell\in V_\ell}} \sum_{\ell=0}^{L}\langle S_\ell v_\ell, v_\ell\rangle \qquad \forall\, v \in V_h(\Gamma). \tag{4.6}$$

The bound (4.1) then follows from (2.7) and (4.6). $\square$

In order to obtain a lower bound for the eigenvalues of $B_{BPS}S_h$, we need to construct a particular decomposition (4.2) for any given $v \in V_h(\Gamma)$ so that the energy of the functions $v_\ell \in V_\ell$ can be estimated in terms of the energy of $v$.

First of all, $v_0 \in V_H$ is defined by the condition that $v_0(p) = v(p)$ at the vertices of $\mathcal{T}_H$, i.e., at the corners of the subdomains $\Omega_1, \ldots, \Omega_J$. We can treat $V_0$ as the $\mathbb{Q}_1$ interpolant of the function $\mathbb{E}_h v \in H_0^2(\Omega)$, where $\mathbb{E}_h : V_h \longrightarrow \tilde{V}_h \subset H_0^2(\Omega)$ is defined using averaging and the $\mathbb{Q}_4$ Bogner-Fox-Schmit finite element space $\tilde{V}_h$. The operator $\mathbb{E}_h$, which is an analog of $\mathbb{E}_H : v_H \longrightarrow \hat{V}_H$, satisfies (by a direct calculation) the following analog of the estimate in Lemma 2.5

$$|\mathbb{E}_h v|_{H^2(\Omega)} \lesssim |v|_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall\, v \in V_h. \tag{4.7}$$

REMARK 4.2. The operators $\mathbb{E}_h : V_h \longrightarrow \tilde{V}_h$ and $\mathbb{E}_j : V_{h,j} \longrightarrow \tilde{V}_{h,j}$ are not related.

LEMMA 4.3. *The following estimate holds*

$$\langle S_0 v_0, v_0\rangle \lesssim \langle S_h v, v\rangle \qquad \forall\, v \in V_h(\Gamma). \tag{4.8}$$

*Proof*. By the standard interpolation error estimate for the $\mathbb{Q}_1$ element, we have

$$\|v_0 - \mathbb{E}_h v\|_{L_2(\Omega_j)} + H|v_0 - \mathbb{E}_h v|_{H^1(\Omega_j)} + H^2|v_0 - \mathbb{E}_h v|_{H^2(\Omega_j)} \lesssim H^2|\mathbb{E}_h v|_{H^2(\Omega_j)} \tag{4.9}$$

for $1 \le j \le J$. Let $E$ belong to $\mathcal{E}_H$, the set of the edges of the subdomains. It follows from (4.9) and the trace theorem with scaling that

$$\begin{aligned}
\frac{1}{|E|}\left\|\left[\!\left[\tfrac{\partial v_0}{\partial n}\right]\!\right]\right\|_{L_2(E)}^2 &= \frac{1}{|E|}\left\|\left[\!\left[\tfrac{\partial (v_0-\mathbb{E}_h v)}{\partial n}\right]\!\right]\right\|_{L_2(E)}^2\\
&\lesssim \sum_{\Omega_j\in\mathcal{T}_{H,E}}\left[H^{-2}|v_0-\mathbb{E}_h v|_{H^1(\Omega_j)}^2 + |v_0-\mathbb{E}_h v|_{H^2(\Omega_j)}^2\right]\\
&\lesssim \sum_{\Omega_j\in\mathcal{T}_{H,E}} |\mathbb{E}_h v|_{H^2(\Omega_j)}^2,
\end{aligned} \tag{4.10}$$

where $\mathcal{T}_{H,E}$ is the set of the subdomains sharing $E$ as a common edge.

Summing up (4.9) over $\Omega_j \in \mathcal{T}_H$ and (4.10) over $E \in \mathcal{E}_H$, we find by (1.4), (2.6), and (4.7),

$$\begin{aligned}
\langle S_0 v_0, v_0\rangle \approx |v_0|_{H^2(\Omega,\mathcal{T}_H)}^2 &= \sum_{j=1}^{J}|v_0|_{H^2(\Omega_j)}^2 + \sum_{E\in\mathcal{E}_H}\frac{1}{|E|}\left\|\left[\!\left[\tfrac{\partial v_0}{\partial n}\right]\!\right]\right\|_{L_2(E)}^2\\
&\lesssim |\mathbb{E}_h v|_{H^2(\Omega)}^2 \lesssim |v|_{H^2(\Omega,\mathcal{T}_h)}^2 \approx \mathcal{A}_h(v,v) = \langle S_h v, v\rangle. \qquad \square
\end{aligned}$$

Let $w = v - I_0 v_0$. It follows from (4.4) and (4.8) that

$$|w|_{H^2(\Omega,\mathcal{T}_h)}^2 \approx \mathcal{A}_h(w,w) \lesssim \mathcal{A}_h(v,v) + \mathcal{A}_h(I_0v_0, I_0v_0) \lesssim \mathcal{A}_h(v,v) = \langle S_h v, v\rangle. \tag{4.11}$$

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We also have a discrete Sobolev inequality.

LEMMA 4.4. *We have, for* $1 \le j \le J$ *and* $w_j = w\big|_{\Omega_j} = (v - I_0 v_0)\big|_{\Omega_j}$,

$$\|\nabla \mathbb{E}_j w_j\|_{L_\infty(\partial\Omega_j)} \lesssim \big(1 + \ln(\tfrac{H}{h})\big)^{\frac{1}{2}} |\mathbb{E}_j w_j|_{H^2(\Omega_j)}.$$

*Proof*. Since $\nabla \mathbb{E}_j w_j \in H^1(\Omega_j)$, by a standard discrete Sobolev inequality [8, 16], we have

$$\|\nabla \mathbb{E}_j w_j\|_{L_\infty(\partial\Omega_j)} \lesssim \big(1 + \ln(\tfrac{H}{h})\big)^{\frac{1}{2}} \big(H^{-1}\|\nabla \mathbb{E}_j w_j\|_{L_2(\partial\Omega_j)} + |\nabla \mathbb{E}_j w_j|_{H^{1/2}(\Omega_j)}\big).$$

Furthermore, since $\mathbb{E}_j w_j = w_j = 0$ at the corners of $\Omega_j$, we also have [15, Lemma 4.8]

$$H^{-1}\|\nabla \mathbb{E}_j w_j\|_{L_2(\partial\Omega_j)} + |\nabla \mathbb{E}_j w_j|_{H^{1/2}(\Omega_j)} \lesssim |\mathbb{E}_j w_j|_{H^2(\Omega_j)}. \qquad \square$$

Now we choose $v_\ell \in V_\ell$, for $1 \le \ell \le L$, so that (4.2) holds, i.e., $w = \sum_{\ell=1}^{L} v_\ell$. By comparing the dofs for $V_h(\Gamma)$ (cf. the right-hand side of Figure 2.2) and the dofs for the edge spaces (cf. Figure 2.3), we see that the dofs of $v_\ell$ are uniquely determined by the corresponding dofs of $w$ except the mixed second order derivatives at a common corner of four subdomains. At such a node we choose the mixed second order derivative of $v_\ell$ to be $\frac{1}{2}$ of the corresponding mixed second order derivative of $w$.

It follows from Lemma 3.5 that

$$\begin{aligned}\sum_{\ell=1}^{L}\langle S_\ell v_\ell, v_\ell\rangle \approx \sum_{\ell=1}^{L}\Bigg[&\sum_{\substack{e\in\mathcal{E}_h\\ e\subset\Gamma}} \big\|\big[\!\big[\tfrac{\partial v_\ell}{\partial n}\big]\!\big]\big\|^2_{L_2(e)} \\ &+ \sum_{\Omega_k\in\mathcal{T}_{H,E_\ell}} \Big(|\mathcal{D}_1 v_\ell|^2_{H^{1/2}(\partial\Omega_k)} + |\mathcal{D}_2 v_\ell|^2_{H^{1/2}(\partial\Omega_k)}\Big)\Bigg],\end{aligned} \tag{4.12}$$

where $\mathcal{T}_{H,E_\ell}$ is the set of the subdomains that share $E_\ell$ as a common edge.

We begin by estimating the first sum on the right-hand side of (4.12).

LEMMA 4.5. *We have*

$$\sum_{\ell=1}^{L}\sum_{\substack{e\in\mathcal{E}_h\\ e\subset\Gamma}} \big\|\big[\!\big[\tfrac{\partial v_\ell}{\partial n}\big]\!\big]\big\|^2_{L_2(e)} \lesssim \big(1 + \ln(\tfrac{H}{h})\big)\, |w|^2_{H^2(\Omega,\mathcal{T}_h)}.$$

*Proof*. We will focus on the estimate for $v_\ell$ associated with an interior vertical (closed) edge $E_\ell$ (cf. the left-hand side of Figure 2.3). The cases of horizontal edges and boundary edges can be handled in a similar fashion.

Let $\Omega_{j_1}$ and $\Omega_{j_2}$ be the two subdomains sharing $E_\ell$ as a common edge and $e$ be a (closed) edge in $\mathcal{E}_h$ and $e \subset \partial\Omega_{j_1} \cup \partial\Omega_{j_2}$. There are several possibilities.

(i) If $e$ does not intersect $E_\ell$, then $\big[\!\big[\frac{\partial v_\ell}{\partial n}\big]\!\big] = 0$ because the normal derivative of $v_\ell$ is identically 0 from both sides of $e$.

(ii) If $e \subset E_\ell$ but does not touch either endpoints of $E_\ell$, then $\big[\!\big[\frac{\partial v_\ell}{\partial n}\big]\!\big] = \big[\!\big[\frac{\partial w}{\partial n}\big]\!\big]$ on $e$.

(iii) If $e \subset E_\ell$ does touch the endpoint $p$ of $E_\ell$, then $\big[\!\big[\frac{\partial v_\ell}{\partial n}\big]\!\big] \ne \big[\!\big[\frac{\partial w}{\partial n}\big]\!\big]$ on $e$ because the derivatives $\frac{\partial v_{\ell,1}}{\partial x_1}$ and $\frac{\partial v_{\ell,2}}{\partial x_1}$ have been set to 0 at $p$ and the mixed second order derivatives $\frac{\partial^2 v_{\ell,1}}{\partial x_1 \partial x_2}(p)$ and $\frac{\partial^2 v_{\ell,2}}{\partial x_1 \partial x_2}(p)$ equal one half of the corresponding mixed second order derivatives of $w$ at $p$. In this case we have by scaling

$$\frac{1}{|e|}\,\big\|\big[\!\big[\tfrac{\partial v_\ell}{\partial n}\big]\!\big]\big\|^2_{L_2(e)} \lesssim \frac{1}{|e|}\,\big\|\big[\!\big[\tfrac{\partial w}{\partial n}\big]\!\big]\big\|^2_{L_2(e)}.$$

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(iv) If $e$ is one of the four horizontal edges that touch $p$, say $e \subset \partial\Omega_{j_1}$, then we have $\left|[\![\frac{\partial v_\ell}{\partial n}]\!]\right| = \left|\frac{\partial v_{\ell,1}}{\partial x_2}\right|$ on $e$ because $v_\ell$ is identically 0 on the other side. Since $\frac{\partial v_{\ell,1}}{\partial x_2}$ on $e$ is determined by the values of $\frac{\partial w_{\ell,1}}{\partial x_2}$ and $\frac{\partial^2 w_{\ell,1}}{\partial x_1 \partial x_2}$ at $p$, we have by scaling and a standard inverse estimate

$$\begin{aligned}\left|\left[\!\!\left[\frac{\partial v_\ell}{\partial n}\right]\!\!\right]\right| &\lesssim \left|\frac{\partial w_{\ell,1}}{\partial x_2}(p)\right| + |e|\left|\frac{\partial^2 w_{\ell,1}}{\partial x_1 \partial x_2}(p)\right| \\ &= \left|\frac{\partial \mathbb{E}_{j_1} w_{\ell,1}}{\partial x_2}(p)\right| + |e|\left|\frac{\partial^2 \mathbb{E}_{j_1} w_{\ell,1}}{\partial x_1 \partial x_2}(p)\right| \\ &\lesssim \left\|\frac{\partial \mathbb{E}_{j_1} w_{\ell,1}}{\partial x_2}\right\|_{L_\infty(e)}\end{aligned}$$

on the edge $e$. Note that we have used the defining properties (iii) and (vi) of $\mathbb{E}_j$ that appear just before Remark 3.1.

Summing up the contributions over all the cases, we find

$$\begin{aligned}\sum_{\ell=1}^{L}\sum_{\substack{e\in\mathcal{E}_h\\ e\subset\Gamma}}\left\|\left[\!\!\left[\frac{\partial v_\ell}{\partial n}\right]\!\!\right]\right\|^2_{L_2(e)} &\lesssim \frac{1}{|e|}\sum_{\substack{e\in\mathcal{E}_h\\ e\subset\Gamma}}\left\|\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\right\|^2_{L_2(e)} + \sum_{j=1}^{J}|\nabla \mathbb{E}_j w_j|^2_{L_\infty(\Omega_j)} \\ &\lesssim \frac{1}{|e|}\sum_{\substack{e\in\mathcal{E}_h\\ e\subset\Gamma}}\left\|\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\right\|^2_{L_2(e)} + \left(1+\ln(\tfrac{H}{h})\right)\sum_{j=1}^{J}|\mathbb{E}_j w_j|^2_{H^2(\Omega_j)} \\ &\lesssim \frac{1}{|e|}\sum_{\substack{e\in\mathcal{E}_h\\ e\subset\Gamma}}\left\|\left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right]\right\|^2_{L_2(e)} + \left(1+\ln(\tfrac{H}{h})\right)\sum_{j=1}^{J}|\!|\!| w_j |\!|\!|^2_{H^2(\Omega_j,\mathcal{T}_{h,j})} \\ &\lesssim \left(1+\ln(\tfrac{H}{h})\right)|w|^2_{H^2(\Omega,\mathcal{T}_h)}\end{aligned}$$

by using Lemma 4.4, (3.2), and (3.1). ∎

We now turn to the second sum on the right-hand side of (4.12).

LEMMA 4.6. *We have*

$$\begin{aligned}&\sum_{\ell=1}^{L}\sum_{\Omega_k\in\mathcal{T}_{H,E_\ell}}\left(|\mathcal{D}_1 v_\ell|^2_{H^{1/2}(\partial\Omega_k)} + |\mathcal{D}_2 v_\ell|^2_{H^{1/2}(\partial\Omega_k)}\right) \\ &\qquad \lesssim \left(1+\ln(\tfrac{H}{h})\right)^2\sum_{j=1}^{J}|\!|\!| w_j |\!|\!|^2_{H^2(\Omega_j,\mathcal{T}_{h,j})}.\end{aligned}\tag{4.13}$$

*Proof*. This time we will focus on a horizontal edge $E_\ell$ (cf. Figure 4.1). Let $\Omega_k$ be a subdomain that shares $E_\ell$ as a common edge, $v_{\ell,k} = v_\ell\big|_{\Omega_k}$ and $w_k = w\big|_{\Omega_k}$. First we consider $\mathcal{D}_1 v_{\ell,k}$ on $\partial\Omega_k$. Since $v_{\ell,k}$ and $w_k$ have identical dofs that define them as piecewise quartic polynomials on $E_\ell$ in the $x_1$ variable (cf. Figure 4.1), we have $v_{\ell,k} = w_k$ on $E_\ell$ and hence

$$\mathcal{D}_1 v_{\ell,k} = \frac{\partial \mathbb{E}_k v_{\ell,k}}{\partial x_1} = \frac{\partial \mathbb{E}_k w_k}{\partial x_1} \qquad \text{on} \quad E_\ell.\tag{4.14}$$

The dofs of $\mathcal{D}_1 v_{\ell,k}\big|_{\partial\Omega_k}$ are identically zero outside $E_\ell$ except those at the endpoints and midpoints of $e_5$ and $e_7$ (cf. Figure 4.1). It follows that

$$\mathcal{D}_1 v_{\ell,k} = 0 \qquad \text{on} \quad \partial\Omega_k \setminus (e_6 \cup e_5 \cup E_\ell \cup e_7 \cup e_8).\tag{4.15}$$

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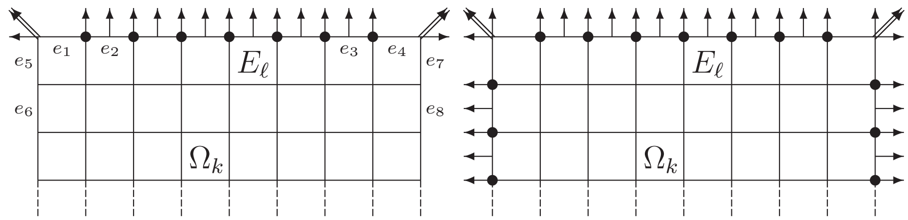


FIG. 4.1. *Dofs for $v_{\ell,k}$ (left) and $w_k$ (right) on $\Omega_k \in \mathcal{T}_{H,E_\ell}$.*

Moreover, the dofs of the piecewise quartic polynomial $\mathcal{D}_1 v_{\ell,k}$ (in the $x_2$ variable) at these nodes are determined by the values of $\frac{\partial w_k}{\partial x_1} = \frac{\partial \mathbb{E}_k w_k}{\partial x_1}$ and $\frac{\partial^2 w_k}{\partial x_2 \partial x_1} = \frac{\partial^2 \mathbb{E}_k w_k}{\partial x_2 \partial x_1}$ at the endpoints of $E_\ell$. Therefore, by scaling, we have

$$\left\| \frac{\partial \mathbb{E}_k v_{\ell,k}}{\partial x_1} \right\|_{L_\infty(e_5 \cup e_6 \cup e_7 \cup e_8)} \lesssim \left\| \frac{\partial \mathbb{E}_k w_k}{\partial x_1} \right\|_{L_\infty(\partial\Omega_k \setminus E_\ell)}, \tag{4.16}$$

and hence, in view of (4.14),

$$\|\mathcal{D}_1 v_{\ell,k}\|_{L_\infty(\partial\Omega_k)} \lesssim \left\| \frac{\partial \mathbb{E}_k w_k}{\partial x_1} \right\|_{L_\infty(\partial\Omega_k)}. \tag{4.17}$$

Let $E_{\ell,k} = e_6 \cup e_5 \cup E_\ell \cup e_7 \cup e_8$. By (4.15) and a standard estimate for truncated piecewise polynomials (cf. [8, Section 3], [37, Section 4.6], [14, Section 7.5]), we have

$$|\mathcal{D}_1 v_{\ell,k}|^2_{H^{1/2}(\partial\Omega_k)} \lesssim |\mathcal{D}_1 v_{\ell,k}|^2_{H^{1/2}(E_{\ell,k})} + \big(1 + \ln(\tfrac{H}{h})\big) \|\mathcal{D}_1 v_{\ell,k}\|^2_{L_\infty(E_{\ell,k})}. \tag{4.18}$$

Furthermore, we have by the relations (4.14)–(4.16) and scaling

$$\begin{aligned} &|\mathcal{D}_1 v_{\ell,k}|_{H^{1/2}(E_{\ell,k})} \\ &\quad \le \left| \mathcal{D}_1 v_{\ell,k} - \frac{\partial \mathbb{E}_k w_k}{\partial x_1} \right|_{H^{1/2}(E_{\ell,k})} + \left| \frac{\partial \mathbb{E}_k w_k}{\partial x_1} \right|_{H^{1/2}(E_{\ell,k})} \\ &\quad \lesssim \left\| \frac{\partial \mathbb{E}_k w_k}{\partial x_1} \right\|_{L_\infty(\partial\Omega_k)} + \left| \frac{\partial \mathbb{E}_k w_k}{\partial x_1} \right|_{H^{1/2}(\partial\Omega_k)}. \end{aligned} \tag{4.19}$$

Combining (4.17)–(4.19), Lemma 4.4, the trace theorem, and Lemma 3.2, we conclude that

$$\begin{aligned} &|\mathcal{D}_1 v_{\ell,k}|^2_{H^{1/2}(\partial\Omega_k)} \\ &\quad \lesssim \left| \frac{\partial \mathbb{E}_k w_k}{\partial x_1} \right|^2_{H^{1/2}(\Omega_k)} + \big(1 + \ln(\tfrac{H}{h})\big) \left\| \frac{\partial \mathbb{E}_k w_k}{\partial x_1} \right\|^2_{L\infty(\partial\Omega_k)} \\ &\quad \lesssim \big(1 + \ln(\tfrac{H}{h})\big)^2 |\mathbb{E}_k w_k|^2_{H^2(\Omega_k)} \lesssim \big(1 + \ln(\tfrac{H}{h})\big)^2 |\!|\!| w_k |\!|\!|^2_{H^2(\Omega_k, \mathcal{T}_{h,k})}. \end{aligned} \tag{4.20}$$

Next we consider $\mathcal{D}_2 v_{\ell,k} = \frac{\partial \mathbb{E}_k v_{\ell,k}}{\partial x_2}$ on $\partial\Omega_k$. The dofs of the piecewise quartic polynomial $\frac{\partial \mathbb{E}_k v_{\ell,k}}{\partial x_2}$ on $E_\ell$ (in the $x_1$ variable) are identical with those for the piecewise quartic polynomial $\frac{\partial \mathbb{E}_k w_k}{\partial x_2}$ except at the vertices and midpoints of $e_1$ and $e_4$ (cf. Figure 4.1). It follows that

$$\mathcal{D}_2 v_{\ell,k} = \frac{\partial \mathbb{E}_k v_{\ell,k}}{\partial x_2} = \frac{\partial \mathbb{E}_k w_k}{\partial x_2} \quad \text{on} \quad E_\ell \setminus (e_1 \cup e_2 \cup e_3 \cup e_4). \tag{4.21}$$

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Moreover, the difference between $\frac{\partial \mathbb{E}_k v_{\ell,k}}{\partial x_2}$ and $\frac{\partial \mathbb{E}_k w_k}{\partial x_2}$ on $e_1 \cup e_2 \cup e_3 \cup e_4$ is determined by the values of $\frac{\partial w_k}{\partial x_2} = \frac{\partial \mathbb{E}_k w_k}{\partial x_2}$ and $\frac{\partial^2 w_k}{\partial x_1 \partial x_2} = \frac{\partial^2 \mathbb{E}_k w_k}{\partial x_1 \partial x_2}$ at the two endpoints of $E_\ell$. Therefore we have

$$\left\| \mathcal{D}_2 v_{\ell,k} - \frac{\partial \mathbb{E}_k w_k}{\partial x_2} \right\|_{L_\infty(e_1 \cup_2 \cup e_3 \cup e_4)} \lesssim \left\| \frac{\partial \mathbb{E}_k w_k}{\partial x_2} \right\|_{L_\infty(E_\ell)}. \tag{4.22}$$

Finally we observe that

(4.23) the function $\mathcal{D}_2 v_{\ell,k} = \frac{\partial \mathbb{E}_k v_{\ell,k}}{\partial x_2} = 0$ on $\partial\Omega_k \setminus E_\ell$.

Using (4.21)–(4.23) and arguments similar to the ones for the derivation of (4.20), we have

$$\begin{aligned} |\mathcal{D}_2 v_{\ell,k}|^2 &\lesssim |\mathcal{D}_2 v_{\ell,k}|^2_{H^{1/2}(E_\ell)} + \big(1+\ln(\tfrac{H}{h})\big) \|\mathcal{D}_2 v_{\ell,k}\|^2_{L_\infty(E_\ell)} \\ &\lesssim \left| \frac{\partial \mathbb{E}_k w_k}{\partial x_2} \right|^2_{H^{1/2}(\partial\Omega_k)} + \big(1+\ln(\tfrac{H}{h})\big) \left\| \frac{\partial \mathbb{E}_k w_k}{\partial x_2} \right\|^2_{L_\infty(\partial\Omega_k)} \\ &\lesssim \big(1+\ln(\tfrac{H}{h})\big)^2 |\mathbb{E}_k w_k|^2_{H^2(\Omega_k)} \lesssim \big(1+\ln(\tfrac{H}{h})\big)^2 \|w_k\|^2_{H^2(\Omega_k, \mathcal{T}_{h,k})}. \end{aligned} \tag{4.24}$$

The estimate (4.13) follows by summing up (4.20) and (4.24) over the edges $E_1, \ldots, E_L$. ∎

We can now establish a lower bound for the eigenvalues of $B_{\scriptscriptstyle BPS} S_h$.

LEMMA 4.7. *The minimum eigenvalue of $B_{\scriptscriptstyle BPS} S_h$ satisfies the following estimate:*

$$\lambda_{\min}(B_{\scriptscriptstyle BPS} S_h) \gtrsim \big(1+\ln(\tfrac{H}{h})\big)^{-2}.$$

*Proof*. Let $v \in V_h(\Gamma)$ be arbitrary, and let $v_\ell \in V_\ell$ for $0 \le \ell \le L$ be the particular decomposition of $v$ that we have constructed. It follows from (4.12), Lemma 4.5, Lemma 4.6, (3.1), and (4.11), that

$$\begin{aligned} \sum_{\ell=0}^{L} \langle S_\ell v_\ell, v_\ell \rangle &\lesssim \big(1+\ln(\tfrac{H}{h})\big) |w|^2_{H^2(\Omega,\mathcal{T}_h)} + \big(1+\ln(\tfrac{H}{h})\big)^2 \sum_{j=1}^{J} \|w_j\|^2_{H^2(\Omega_j,\mathcal{T}_{h,j})} \\ &\lesssim \big(1+\ln(\tfrac{H}{h})\big)^2 |w|^2_{H^2(\Omega,\mathcal{T}_h)} \approx \big(1+\ln(\tfrac{H}{h})\big)^2 \langle S_h v, v \rangle \end{aligned}$$

and hence

$$\min_{\substack{v=\sum_{\ell=0}^{L} I_\ell v_\ell \\ v_\ell \in V_\ell}} \sum_{\ell=0}^{L} \langle S_\ell v_\ell, v_\ell \rangle \lesssim \big(1+\ln(\tfrac{H}{h})\big)^2 \langle S_h v, v \rangle,$$

which together with (2.8) implies the lower bound. ∎

Lemma 4.1 and Lemma 4.7 immediately imply the following bound on the condition number of the preconditioned system $B_{\scriptscriptstyle BPS} S_h$.

THEOREM 4.8. *We have*

$$\kappa(B_{\scriptscriptstyle BPS} S_h) = \frac{\lambda_{\max}(B_{\scriptscriptstyle BPS} S_h)}{\lambda_{\min}(B_{\scriptscriptstyle BPS} S_h)} \le C\big(1+\ln(\tfrac{H}{h})\big)^2,$$

*where the positive constant $C$ is independent of $h$, $H$, and $J$.*

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**5. Numerical results.** In this section, we report some numerical results for our model problem on the unit square. We take the penalty parameter $\sigma$ in $\mathcal{A}_h, \mathcal{A}_H$, and $\mathcal{A}_\ell$ to be $5$ in the numerical experiments, and we compute the maximum eigenvalue, the minimum eigenvalue, and the condition number of $B_{BPS}S_h$ for different values of $h$, $H$ and $J$.

For each choice of $h$, $H$, and $J$, we generate a vector $v_h \in V_h(\Gamma)$ randomly as our exact solution and compute the right-hand side $g$. Then we apply the preconditioned conjugate gradient algorithm to the linear system $S_h z = g$ with the Bramble-Pasciak-Schatz preconditioner and $0$ as the initial value. The iteration is stopped when the energy norm error is reduced by a factor of $10^{-6}$ and the minimum and maximum eigenvalues are estimated by the Lanczos algorithm. The average results over 5 random choices of $v_h$ are reported in the tables below.

REMARK 5.1. Since we are solving a fourth order problem, the condition number of $S_h$ is very large for small $h$. This is the reason why we use a more stringent stopping criterion than the usual criterion based on the residual error.

The results for the eigenvalues and condition numbers for $4$ subdomains, $16$ subdomains, and $64$ subdomains are reported in Table 5.1, Table 5.2, and Table 5.3, respectively. They agree with the estimates in Lemma 4.1, Lemma 4.7, and Theorem 4.8. The average number of iterations in these computations are presented in Table 5.4, where the scalability of the preconditioner can be observed.

TABLE 5.1
*Eigenvalues and condition numbers for $H = 1/2$ (4 subdomains).*

| | $\lambda_{\max}(B_{BPS}S_h)$ | $\lambda_{\min}(B_{BPS}S_h)$ | $\kappa(B_{BPS}S_h)$ | $\sqrt{\kappa(B_{BPS}S_h)}$ |
|---|---|---|---|---|
| h=1/4 | 6.6170 | 0.4945 | 13.3825 | 3.6582 |
| h=1/8 | 6.5345 | 0.2617 | 24.9672 | 4.9967 |
| h=1/16 | 6.5354 | 0.1675 | 39.0163 | 6.2463 |
| h=1/32 | 6.5359 | 0.1157 | 56.5020 | 7.5168 |
| h=1/64 | 6.5360 | 0.0845 | 77.3800 | 8.7966 |

TABLE 5.2
*Eigenvalues and condition numbers for $H = 1/4$ ($16$ subdomains).*

| | $\lambda_{\max}(B_{BPS}S_h)$ | $\lambda_{\min}(B_{BPS}S_h)$ | $\kappa(B_{BPS}S_h)$ | $\sqrt{\kappa(B_{BPS}S_h)}$ |
|---|---|---|---|---|
| h=1/8 | 6.8434 | 0.2235 | 30.6210 | 5.5336 |
| h=1/16 | 6.6952 | 0.1387 | 48.2550 | 6.9466 |
| h=1/32 | 6.6847 | 0.0978 | 68.3611 | 8.2681 |
| h=1/64 | 6.6808 | 0.0725 | 92.1217 | 9.5980 |

TABLE 5.3
*Eigenvalues and condition numbers for $H = 1/8$ ($64$ subdomains).*

| | $\lambda_{\max}(B_{BPS}S_h)$ | $\lambda_{\min}(B_{BPS}S_h)$ | $\kappa(B_{BPS}S_h)$ | $\sqrt{\kappa(B_{BPS}S_h)}$ |
|---|---|---|---|---|
| h=1/16 | 6.8785 | 0.1742 | 39.4859 | 6.2838 |
| h=1/32 | 6.7239 | 0.1173 | 57.3270 | 7.5715 |
| h=1/64 | 6.7102 | 0.0825 | 81.3200 | 9.0178 |

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TABLE 5.4
*Average number of iterations for reducing the energy norm error by a factor of* $10^{-6}$.

| | $H = 1/2$ | $H = 1/4$ | $H = 1/8$ | $H = 1/16$ | $H = 1/32$ |
|---|---|---|---|---|---|
| $H/h = 2$ | 22 | 37 | 43 | 43 | 43 |
| $H/h = 4$ | 21 | 36 | 41 | 41 | — |
| $H/h = 8$ | 20 | 38 | 42 | — | — |
| $H/h = 16$ | 21 | 39 | — | — | — |
| $H/h = 32$ | 22 | — | — | — | — |

To illustrate the practical performance of the preconditioner, we present in Table 5.5 the number of iterations required to reduce the energy error by a factor of $10^{-2}$ for various $h$ and $H$.

TABLE 5.5
*Average number of iterations for reducing the energy norm error by a factor of* $10^{-2}$.

| | $H = 1/2$ | $H = 1/4$ | $H = 1/8$ | $H = 1/16$ | $H = 1/32$ |
|---|---|---|---|---|---|
| $H/h = 2$ | 9 | 13 | 14 | 14 | 14 |
| $H/h = 4$ | 9 | 10 | 11 | 11 | — |
| $H/h = 8$ | 9 | 10 | 10 | — | — |
| $H/h = 16$ | 8 | 8 | — | — | — |
| $H/h = 32$ | 7 | — | — | — | — |

**Appendix A. Proof of Lemma 3.5.** We need two technical results for the proof of Lemma 3.5. The first one is a trace theorem proven in [15, Lemmas 4.1–4.3].

LEMMA A.1. *We have, for* $1 \le j \le J$,

$$|\nabla w|_{H^{1/2}(\partial\Omega_j)} \lesssim |w|_{H^2(\Omega_j)} \qquad \forall\, w \in H^2(\Omega_j).$$

*Furthermore, given any* $w \in H^2(\Omega_j)$, *there exists* $\tilde{w} \in H^2(\Omega_j)$ *such that*

$$\tilde{w}\big|_{\partial\Omega_j} = w\big|_{\partial\Omega_j}, \quad \nabla\tilde{w}\big|_{\partial\Omega_j} = \nabla w\big|_{\partial\Omega_j} \quad \textit{and} \quad |\tilde{w}|_{H^2(\Omega_j)} \lesssim |\nabla w|_{H^{1/2}(\partial\Omega_j)}.$$

The second result concerns a $\mathbb{Q}_4$ Bogner-Fox-Schmit quasi-interpolant for a function $\zeta \in H^2(\Omega_j)$. Suppose that for each edge $e \in \mathcal{E}_h$ such that $e \subset \bar{\Omega}_j$, a unit normal vector $n_e$ has been chosen. Let $\zeta_e \in \mathbb{P}_4(e)$ and $\zeta_e^* \in \mathbb{P}_4(e)$ be the $L_2(e)$ projections of $\zeta|_e$ and $\frac{\partial\zeta}{\partial n_e}\big|_e$, respectively. We then assign the dofs of an quasi-interpolant $v_\zeta$ in the $\mathbb{Q}_4$ Bogner-Fox-Schmit space associated with $\mathcal{T}_{h,j}$ as follows.

If $m$ is the midpoint of an edge $e \in \mathcal{E}_{h,j}$ (the set of the edges of $\mathcal{T}_{h,j}$), we define

$$v_\zeta(m) = \zeta(m) \quad \text{and} \quad \big(\nabla v_\zeta(m)\big) \cdot n_e = \zeta_e^*(m). \tag{A.1}$$

If $p$ is a vertex in $\mathcal{T}_{h,j}$, then we choose an edge $e \in \mathcal{E}_{h,j}$ with $p$ as an endpoint and define

$$v_\zeta(p) = \zeta(p), \qquad t_e \cdot \big(\nabla^2 v_\zeta(p)\big) n_e = (\zeta_e^*)'(p), \tag{A.2}$$

and $(\nabla v_\zeta)(p)$ to be the vector satisfying

$$(\nabla v_\zeta)(p) \cdot n_e = \zeta_e^*(p), \tag{A.3}$$

$$(\nabla v_\zeta)(p) \cdot t_e = \zeta_e'(p), \tag{A.4}$$

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where $t_e$ is a unit tangent vector of $e$ and $\zeta_e'$ (respectively $(\zeta_e^*)'$) is the derivative of $\zeta_e$ (respectively $\zeta_e^*$) in the direction of $t_e$. Finally, for the center $c$ of an element $D \in \mathcal{T}_h$, we define

$$v_\zeta(c) = \zeta(c). \tag{A.5}$$

Note that the choice of the dofs of $v_\zeta$ at a vertex $p$ is not unique since there are many edges sharing $p$ as a common endpoint. In order to control the behavior of $v_\zeta$ on $\partial\Omega_j$ for any $p$ on $\partial\Omega_j$, we choose $e$ to be an edge on $\partial\Omega_j$. Furthermore, we choose $e$ to be an edge on $\partial\Omega \cap \partial\Omega_j$ if $p$ belongs to $\partial\Omega \cap \partial\Omega_j$. (Admissible edges represented by thick lines for various vertices represented by bullets are depicted in Figure A.1.)

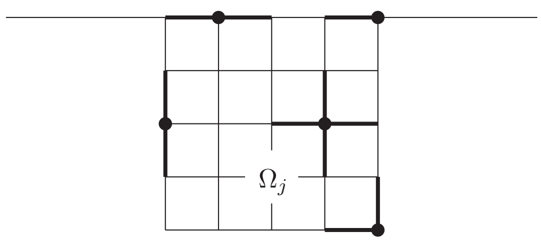


FIG. A.1. *Admissible edges in the definition of the dofs of $v_\zeta$ at vertices.*

REMARK A.2. If both $\zeta$ and $\frac{\partial \zeta}{\partial n}$ belong to $\mathbb{P}_4(e)$ on all the boundary edges $e$ on $\partial\Omega_j$, then $\zeta_e = \zeta$ and $\zeta_e^* = \frac{\partial \zeta}{\partial n_e}$ on all the boundary edges, which implies $v_\zeta = \zeta$ to first order on $\partial\Omega$.

REMARK A.3. If $\zeta = 0$ on $\partial\Omega \cap \partial\Omega_j$, then $v_\zeta = 0$ on $\partial\Omega \cap \partial\Omega_j$ and hence $v_\zeta \in V_{h,j}$.

LEMMA A.4. *We have, for* $1 \le j \le J$,

$$|v_\zeta|_{H^2(\Omega_j)} \lesssim |\zeta|_{H^2(\Omega_j)} \qquad \forall\, \zeta \in H^2(\Omega_j). \tag{A.6}$$

*Proof*. Let $D_* \in \mathcal{T}_{h,j}$ be arbitrary, and let $e_1$, $e_2$ be the two edges of $D_*$ sharing $p$ as a common endpoint. Suppose $(\nabla v_{\zeta,e_1})(p)$ and $\frac{\partial^2 v_{\zeta,e_1}}{\partial x_1 \partial x_2}(p)$ are defined by (A.2)–(A.4) using $\zeta_{e_1}$ and $\zeta_{e_1}^*$ and $(\nabla v_{\zeta,e_2})(p)$ and $\frac{\partial^2 v_{\zeta,e_2}}{\partial x_1 \partial x_2}(p)$ are defined by (A.2)–(A.4) using $\zeta_{e_2}$ and $\zeta_{e_2}^*$.

If $\zeta \in \mathbb{P}_1(D_*)$, then it is clear that

$$(\nabla v_{\zeta,e_1})(p) = (\nabla v_{\zeta,e_2})(p) \quad \text{and} \quad \frac{\partial^2 v_{\zeta,e_1}}{\partial x_1 \partial x_2}(p) = \frac{\partial^2 v_{\zeta,e_2}}{\partial x_1 \partial x_2}(p).$$

Hence, by the Bramble-Hilbert Lemma [7] and scaling, we have

$$|(\nabla v_{\zeta,e_1})(p) - (\nabla v_{\zeta,e_2})(p)|^2 \lesssim |\zeta|^2_{H^2(D_*)} \tag{A.7}$$

and

$$\left| \frac{\partial^2 v_{\zeta,e_1}}{\partial x_1 \partial x_2}(p) - \frac{\partial^2 v_{\zeta,e_2}}{\partial x_1 \partial x_2}(p) \right|^2 \lesssim (\operatorname{diam} D_*)^{-2} |\zeta|^2_{H^2(D_*)}. \tag{A.8}$$

The estimates (A.7) and (A.8) measure the effect of choosing different edges (from the same element) in the definition of the quasi-interpolant $v_\zeta$.

Suppose now the interpolant $v_{\zeta,D_*}$ of $\zeta$ is defined on $D_*$ by (A.1)–(A.5) in a particular way, namely the dofs of $v_{\zeta,D_*}$ at a vertex $p$ of $D_*$ are defined by using the edge that precedes $p$ in the counterclockwise direction. The triangle inequality and a standard inverse estimate [21, 14] imply that

$$\begin{aligned} |v_\zeta|^2_{H^2(D_*)} &\lesssim |v_{\zeta,D_*}|^2_{H^2(D_*)} + |v_\zeta - v_{\zeta,D_*}|^2_{H^2(D_*)} \\ &\lesssim |v_{\zeta,D_*}|^2_{H^2(D_*)} + (\operatorname{diam} D_*)^{-4}\|v_\zeta - v_{\zeta,D_*}\|^2_{L_2(D_*)}. \end{aligned} \tag{A.9}$$

First we claim that

$$|v_{\zeta,D_*}|_{H^2(D_*)} \lesssim |\zeta|_{H^2(D_*)}. \tag{A.10}$$

Indeed, since all the dofs of $v_{\zeta,D_*}$ defined by (A.1)–(A.5) are bounded by $\|\zeta\|_{H^2(D_*)}$, the seminorm $|v_{\zeta,D_*}|_{H^2(D_*)}$ is bounded by a multiple of $\|\zeta\|_{H^2(D_*)}$. Moreover, $v_{\zeta,D_*} = \zeta$ if $\zeta \in \mathbb{P}_1(D_*)$, and the seminorm $|v_{\zeta,D_*}|_{H^2(D_*)}$ is invariant under addition of linear polynomials. Therefore the estimate (A.10) follows from the Bramble-Hilbert Lemma and scaling.

Secondly it follows from the definitions of $v_\zeta$ and $v_{\zeta,D_*}$ and (A.7)–(A.8) that

$$\begin{aligned} \|v_\zeta - v_{\zeta,D_*}\|^2_{L_2(D_*)} \lesssim &(\operatorname{diam} D_*)^4 \sum_{p\in D_*} |\nabla v_\zeta(p) - \nabla v_{\zeta,D_*}(p)|^2 \\ &+ (\operatorname{diam} D_*)^6 \sum_{p\in D_*} \left|\frac{\partial^2 v_\zeta}{\partial x_1 \partial x_2}(p) - \frac{\partial^2 v_{\zeta,D_*}}{\partial x_1 \partial x_2}(p)\right|^2 \\ \lesssim &(\operatorname{diam} D_*)^4 |\zeta|^2_{H^2(S(D_*))}, \end{aligned} \tag{A.11}$$

where $S(D_*)$ is the union of all $D \in \mathcal{T}_{h,j}$ that share at least one common vertex with $D_*$. Combining (A.9)–(A.11), we have

$$|v_\zeta|^2_{H^2(D_*)} \lesssim |\zeta|^2_{H^2(S(D_*))}. \tag{A.12}$$

The estimate (A.6) is obtained by summing up (A.12) over all $D_* \in \mathcal{T}_{h,j}$. $\square$

*Proof of Lemma 3.5*. Let $v \in V_h(\Gamma)$ be arbitrary, $v_j = v\big|_{\Omega_j} \in V_{h,j}$ and $w_j = \mathbb{E}_j v_j \in \tilde{V}_{h,j}$. It follows from (3.3), Lemma A.1, and Lemma 3.4 that

$$\begin{aligned} |\mathcal{D}_1 v_j|_{H^{1/2}(\partial\Omega_j)} + |\mathcal{D}_2 v_j|_{H^{1/2}(\partial\Omega_j)} &= |\nabla w_j|_{H^{1/2}(\partial\Omega_j)} \\ &\lesssim |w_j|_{H^2(\Omega_j)} \approx \|\!|v_j|\!\|_{H^2(\Omega_j,\mathcal{T}_{h,j})}. \end{aligned} \tag{A.13}$$

Combining (A.13), (3.1), and (1.4), we find

$$\begin{aligned} &\sum_{\substack{e\in\mathcal{E}_h \\ e\subset\Gamma}} \frac{1}{|e|}\left\|[\![\tfrac{\partial v}{\partial n}]\!]\right\|^2_{L_2(e)} + \sum_{j=1}^J \Big(|\mathcal{D}_1 v_j|^2_{H^{1/2}(\partial\Omega_j)} + |\mathcal{D}_2 v_j|^2_{H^{1/2}(\partial\Omega_j)}\Big) \\ &\qquad \lesssim \sum_{\substack{e\in\mathcal{E}_h \\ e\subset\Gamma}} \frac{1}{|e|}\left\|[\![\tfrac{\partial v}{\partial n}]\!]\right\|^2_{L_2(e)} + \sum_{j=1}^J \|\!|v|\!\|^2_{H^2(\Omega_j,\mathcal{T}_{h,j})} \\ &\qquad = |v|^2_{H^2(\Omega,\mathcal{T}_h)} \approx \mathcal{A}_h(v,v). \end{aligned} \tag{A.14}$$

On the other hand, it follows from Lemma A.1 that there exist functions $\zeta_j \in H^2(\Omega_j)$ for $1 \le j \le J$ such that

$$\zeta_j\big|_{\partial\Omega_j} = w_j\big|_{\partial\Omega_j} \quad \text{and} \quad \nabla\zeta_j\big|_{\partial\Omega_j} = \nabla w_j\big|_{\partial\Omega_j}, \tag{A.15}$$

$$|\zeta_j|_{H^2(\Omega_j)} \lesssim |\nabla w_j|_{H^{1/2}(\partial\Omega_j)}. \tag{A.16}$$

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Let $v_{\zeta_j} \in V_{h,j}$ be a $\mathbb{Q}_4$ Bogner-Fox-Schmit quasi-interpolant of $\zeta_j$. In view of (A.15) and Remark A.2, we have

$$v_{\zeta_j}\big|_{\partial\Omega_j} = \zeta_j\big|_{\partial\Omega_j} = \mathbb{E}_j v_j\big|_{\partial\Omega_j} \quad \text{and} \quad \nabla v_{\zeta_j}\big|_{\partial\Omega_j} = \nabla\zeta_j\big|_{\partial\Omega_j} = \nabla(\mathbb{E}_j v_j)\big|_{\partial\Omega_j}. \tag{A.17}$$

Let $z_j = \mathbb{F}_j v_{\zeta_j} \in V_{h,j}$. It follows from the definition of $\mathbb{F}_j$, Lemma 3.3, Lemma A.4, (A.16), and (A.17) that

$$z_j\big|_{\partial\Omega_j} = v_j\big|_{\partial\Omega_j} \quad \text{and} \quad \nabla z_j\big|_{\partial\Omega_j} = \nabla v_j\big|_{\partial\Omega_j}, \tag{A.18}$$

$$\|z_j\|_{H^2(\Omega_j,\mathcal{T}_{h,j})} \lesssim |v_{\zeta_j}|_{H^2(\Omega_j)} \lesssim |\zeta_j|_{H^2(\Omega_j)} \lesssim |\nabla w_j|_{H^{1/2}(\partial\Omega_j)}. \tag{A.19}$$

Now we take $z \in V_h$ such that $z\big|_{\Omega_j} = z_j$. It follows from (A.18) that $z = v$ up to first order on $\Gamma$. Therefore we can apply Lemma 2.4, (1.4), (3.1), (A.18), (A.19), and (3.3) to obtain

$$\begin{aligned} \mathcal{A}_h(v,v) &\le \mathcal{A}_h(z,z) \approx |z|^2_{H^2(\Omega,\mathcal{T}_h)} \\ &= \sum_{\substack{e\in\mathcal{E}_h\\ e\subset\Gamma}} \frac{1}{|e|}\left\|\left[\!\left[\tfrac{\partial z}{\partial n}\right]\!\right]\right\|^2_{L_2(e)} + \sum_{j=1}^{J} \|z_j\|^2_{H^2(\Omega_j,\mathcal{T}_{h,j})} \\ &\lesssim \sum_{\substack{e\in\mathcal{E}_h\\ e\subset\Gamma}} \frac{1}{|e|}\left\|\left[\!\left[\tfrac{\partial v}{\partial n}\right]\!\right]\right\|^2_{L_2(e)} + \sum_{j=1}^{J}\Big(|\mathcal{D}_1 v_j|^2_{H^{1/2}(\partial\Omega_j)} + |\mathcal{D}_2 v_j|^2_{H^{1/2}(\partial\Omega_j)}\Big). \end{aligned} \tag{A.20}$$

The equivalence (3.4) follows from (A.14) and (A.20). $\square$

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