Electronic Transactions on Numerical Analysis.
Volume 37, pp. 105-112, 2010.

ISSN 1068-9613.

ETNA
Kent State University
http://etna.math.kent.edu

ON THE APPROXIMATION OF ANALYTIC FUNCTIONS BY THE q-BERNSTEIN POLYNOMIALS IN THE CASE $q > 1$*

SOFIYA OSTROVSKA†

Abstract. Since for $q > 1$, the q-Bernstein polynomials $B_{n,q}$ are not positive linear operators on $C[0, 1]$, the investigation of their convergence properties turns out to be much more difficult than that in the case $0 < q < 1$. In this paper, new results on the approximation of continuous functions by the q-Bernstein polynomials in the case $q > 1$ are presented. It is shown that if $f \in C[0, 1]$ and admits an analytic continuation $f(z)$ into $\{z : |z| < a\}$, then $B_{n,q}(f; z) \to f(z)$ as $n \to \infty$, uniformly on any compact set in $\{z : |z| < a\}$.

Key words. q-integers, q-binomial coefficients, q-Bernstein polynomials, uniform convergence

AMS subject classifications. 41A10, 30E10

1. Introduction.

Let $q > 0$. For any $n \in \mathbb{Z}_+$, the *q-integer* $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1} \ (n \in \mathbb{N}), \ [0]_q := 0,$$

and the *q-factorial* $[n]_q!$ by

$$[n]_q! := [1]_q[2]_q \dots [n]_q \ (n = 1, 2, \dots), \ [0]_q! := 1.$$

For integers $0 \le k \le n$, the *q-binomial* coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Clearly, for $q = 1$,

$$[n]_1 = n, \ \ [n]_1! = n!, \ \ \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

DEFINITION 1.1. *Let $f : [0, 1] \to \mathbb{C}$. The q-Bernstein polynomials of f are defined by*

$$B_{n,q}(f; z) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q; z), \quad n \in \mathbb{N},$$

where

$$p_{nk}(q; z) := \begin{bmatrix} n \\ k \end{bmatrix}_q z^k \prod_{j=0}^{n-k-1} \left(1 - q^j z\right), \quad k = 0, 1, \dots n. \tag{1.1}$$

Note that for $q = 1$, we recover the classical Bernstein polynomials.

During the last ten years, the q-Bernstein polynomials have attracted a lot of interest and have been studied from different angles along with some generalizations and modifications by a number of researchers. A comprehensive review of results on q-Bernstein polynomials together with some open problems and an extensive bibliography on the subject is given

*Received February 22, 2008. Accepted for publication November 30, 2009. Published online April 7, 2010. Recommended by V. Andriyevskyy.

†Department of Mathematics, Atilim University, Incek 06836 Ankara, Turkey (ostrovsk@atilim.edu.tr).

ETNA
Kent State University
http://etna.math.kent.edu

in [11]. A two-parametric generalization of q-Bernstein polynomials, called ω, q-Bernstein polynomials, was studied in [8, 21], while an analogue of the Bernstein-Durrmeyer operator with respect to q-Bernstein polynomials was investigated in [3]. The probabilistic aspects of the theory of q-Bernstein polynomials were studied in [1, 5].

It is known (cf. [11] and references therein) that some properties of the classical Bernstein polynomials remain valid for the q-Bernstein polynomials. Among those are the end-point interpolation property, the shape-preserving properties in the case $0 < q < 1$, and the representation via divided differences. Just as the classical Bernstein polynomials, the q-Bernstein polynomials reproduce linear functions, and they are degree-reducing on the set of polynomials. In contrast, the convergence properties of the q-Bernstein polynomials basically *vary* from those of the classical ones. Moreover, the cases $0 < q < 1$ and $q > 1$ in terms of convergence are not similar to each other. This lack of similarity is attributed to the fact that for $0 < q < 1$, the q-Bernstein polynomials are *positive* linear operators, whereas for $q > 1$, the positivity does not hold. Consequently, the convergence of q-Bernstein polynomials in the case $0 < q < 1$ has been studied in detail, including the rate of convergence, Korovkin-type theorem, saturation results, and the properties of the limit q-Bernstein operator (see [6, 12, 16, 17, 18, 19]), while there are still many open problems related to the case $q > 1$. Currently, there are only two papers, namely [10, 20], dealing with the case $q > 1$ in a systematic way. In addition, some results on the behavior of iterates (cf. [22]) and specific results on the exemplary classes of functions (cf. [13, 14]) are available. It should be emphasized that the investigation of the convergence in the case $q > 1$ has revealed some astonishing phenomena not observed previously for $0 < q \leq 1$. For instance (see [14]), while the q-Bernstein polynomials of the Cauchy kernel $f_a(z) := 1/(z - a),\ a \in \mathbb{C} \setminus [0, 1]$, uniformly approximate f_a on any compact set in $\{z : |z| < |a|\}$, the sequence $\{B_{n,q}(f_a; z)\}$ is not even uniformly bounded on any set J having an accumulation point in $\{z : |z| > |a|\}$. The available results show that, even though for $q > 1$ in some cases the approximation with the q-Bernstein polynomials in $C[0, 1]$ may be *faster* than with the classical ones (see [10, Theorem 6]), there exist analytic functions on [0,1] whose sequences of q-Bernstein polynomials are divergent. This situation is in no way possible for $0 < q \leq 1$. The problem to describe the class of functions in $C[0, 1]$ which are uniformly approximated by their q-Bernstein polynomials in the case $q > 1$ is yet to be solved. It is exactly the unexpected behavior of the q-Bernstein polynomials with respect to convergence that makes the study of their convergence an interesting and challenging one.

In this paper, we present new results on the approximation by q-Bernstein polynomials in the case $q > 1$, which are concerned with the approximation of functions which are analytic at 0.

2. Statement of results. The results of the present paper are related to the approximation of functions which are continuous on $[0, 1]$ and possess an analytic continuation into a disk $\{z : |z| < a\},\ a > 0$, by their q-Bernstein polynomials in the case $q > 1$. From here on we assume that $q > 1$ is fixed.

The key role in our considerations is played by the following estimate.

THEOREM 2.1. *Let $f(x)$ be bounded on $[0, 1]$ and admit an analytic continuation $f(z)$ into a closed disk $\{z : |z| \leq \rho\},\ \rho > 0$. If*

$$B_{n,q}(f; z) = \sum_{k=0}^{n} c_{kn} z^k,$$

then the following estimate holds,

$$|c_{kn}| \leq \frac{C}{\rho^k}, \tag{2.1}$$

ETNA
Kent State University
http://etna.math.kent.edu

where $C = C_{f,q,\rho}$ is independent of both k and n.

REMARK 2.2. In [13] and [14] the estimate (2.1) was proven for $f(x) = \ln(x+a)$ and $f_a(x) = 1/(x+a)$ with the help of explicit formulae for the coefficients c_{kn}. Here, we prove the estimate regardless of a specific function.

The next assertion constitutes the main result of the paper.

THEOREM 2.3. *If $f(x)$ is bounded on $[0,1]$ and admits an analytic continuation $f(z)$ into a disk $\{z : |z| < a\}$, $a > 0$, then*

$$B_{n,q}(f; z) \to f(z) \;\; as \;\; n \to \infty$$

uniformly on any compact set $K \subset \{z : |z| < a\}$.

COROLLARY 2.4. [*10*] *If f admits an analytic continuation as an entire function $f(z)$, then*

$$B_{n,q}(f; z) \to f(z) \;\; as \;\; n \to \infty$$

uniformly on any compact set in $\mathbb{C}$.

REMARK 2.5. It is worth pointing out that for $0 < a < 1$, the statement of Theorem 2.3 does not depend on the values of f outside of $[0,a]$ as long as f is bounded on $[0,1]$, while the polynomials $B_{n,q}(f; z)$ certainly do.

EXAMPLE 2.6. In general, a function satisfying the conditions of Theorem 2.3 may not be uniformly approximated by its q-Bernstein polynomials on any interval within $[a,1]$ as the following simple example reveals. Let

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in [0, 1/q], \\ x - 1/q & \text{for} \quad x \in (1/q, 1]. \end{cases}$$

Obviously, $B_{n,q}(f; z) = z^n$ and it is clear that $B_{n,q}(f; x)$ does not approximate f on any interval outside $[0, 1/q]$.

Theorem 2.3 generalizes some previously known results on the approximation of analytic functions by their q-Bernstein polynomials. It has to be mentioned that, while the case $a > 1$ can be treated by the methods used in [10], the case $0 < a \leq 1$ requires a different approach similar to the one given in Theorem 2.1.

3. Proofs of the theorems. We use the representation of the q-Bernstein polynomials given in [10, formulae (6) and (7)],

$$B_{n,q}(f; z) = \sum_{k=0}^{n} \lambda_{kn} f\left[0; \frac{1}{[n]_q}; \ldots; \frac{[k]_q}{[n]_q}\right] z^k,$$

where $f\left[x_0; x_1; \ldots; x_k\right]$ denotes the divided difference of f,

$$f\left[x_0\right] = f(x_0), \quad f\left[x_0; x_1\right] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \ldots,$$

$$f\left[x_0; x_1; \ldots; x_k\right] = \frac{f\left[x_1; \ldots; x_k\right] - f\left[x_0; \ldots; x_{k-1}\right]}{x_k - x_0},$$

and λ_{kn} are given by

$$\lambda_{0n} = \lambda_{1n} = 1, \quad \lambda_{kn} = \prod_{j=1}^{k-1}\left(1 - \frac{[j]_q}{[n]_q}\right), \quad k = 2, \ldots, n.$$

ETNA
Kent State University
http://etna.math.kent.edu

REMARK 3.1. It was shown in [10] that $\lambda_{kn}, k = 0, 1, \ldots, n$, are eigenvalues of the q-Bernstein operator $B_{n,q}$. For $q = 1$, we obtain the eigenvalues of the Bernstein operator, whose eigenstructure together with applications was studied in [2] and [7]. Some results of [2] were extended to the q-Bernstein polynomials in [10].

If f is an analytic function, then (cf., e.g., [9, § 2.7, p. 44]) the divided differences of f can be expressed as

$$f[x_0; x_1; \ldots; x_k] = \frac{1}{2\pi i} \oint_{\mathcal{L}} \frac{f(\zeta)\, d\zeta}{(\zeta - x_0)\ldots(\zeta - x_k)}, \tag{3.1}$$

where $\mathcal{L}$ is a contour encircling $x_0, \ldots, x_k$ and f is assumed to be analytic on and within $\mathcal{L}$.

For a function $f(z)$ analytic in $\{z : |z| \le r\}$, we use the standard notation,

$$M(r; f) := \max_{|z| \le r} |f(z)|.$$

In the sequel, we need the following lemma proven in [14].

LEMMA 3.2. *Let* $q > 1, 0 \ne \rho \notin \{q^{-m}\}_{m=0}^{\infty}$. *Then*

$$\lim_{n\to\infty} \prod_{j=1}^{n-k} \left(1 - \frac{[j]_q}{\rho [n]_q}\right) = \prod_{j=k}^{\infty} \left(1 - \frac{1}{\rho q^j}\right).$$

Proof of Theorem 2.1. Without loss of generality, we assume that $\rho \notin \{q^{-k}\}_{k=0}^{\infty}$ (we may increase ρ slightly if necessary).

(i) First, let $0 < \rho < 1$. We set $j := \min\{k : q^{-k} < \rho\}$. Clearly, for $k \le n - j$, we have

$$\frac{[k]_q}{[n]_q} \le \frac{[n-j]_q}{[n]_q} < q^{-j} < \rho,$$

whence by virtue of (3.1), we obtain

$$f\left[0; \frac{1}{[n]_q}; \ldots; \frac{[k]_q}{[n]_q}\right] = \frac{1}{2\pi i} \oint_{|\zeta|=\rho} \frac{f(\zeta)\, d\zeta}{\zeta^{k+1}\left(1 - \frac{1}{\zeta [n]_q}\right)\ldots\left(1 - \frac{[k_q]}{\zeta [n]_q}\right)}. \tag{3.2}$$

Notice that for $|\zeta| = \rho$ and $l = 0, 1, \ldots, n - j$, we have

$$\left|1 - \frac{[l]_q}{\zeta [n]_q}\right| \ge \left|1 - \frac{[l]_q}{\rho [n]_q}\right| = 1 - \frac{[l]_q}{\rho [n]_q},$$

since $\frac{[l]_q}{[n]_q} < \rho = |\zeta|$. Therefore, for $k \le n - j$, we obtain

$$\begin{aligned}\left|\left(1 - \frac{1}{\zeta [n]_q}\right)\ldots\left(1 - \frac{[k]_q}{\zeta [n]_q}\right)\right| &\ge \left(1 - \frac{1}{\rho [n]_q}\right)\ldots\left(1 - \frac{[k]_q}{\rho [n]_q}\right)\\ &\ge \left(1 - \frac{1}{\rho [n]_q}\right)\ldots\left(1 - \frac{[n-j]_q}{\rho [n]_q}\right).\end{aligned}$$

By Lemma 3.2, as $n \to \infty$, we have

$$\left(1 - \frac{1}{\rho [n]_q}\right)\ldots\left(1 - \frac{[n-j]_q}{\rho [n]_q}\right) \longrightarrow \prod_{s=j}^{\infty}\left(1 - \frac{1}{\rho q^s}\right) \ne 0,$$

ETNA
Kent State University
http://etna.math.kent.edu

whence for $|\zeta| = \rho$,

$$\left|\left(1 - \frac{1}{\zeta[n]_q}\right) \dots \left(1 - \frac{[k]_q}{\zeta[n]_q}\right)\right| \geq C_1 = C_{\rho,q} > 0$$

for all n. Applying (3.2), we obtain

$$|c_{kn}| \leq \left| f\left[0; \frac{1}{[n]_q}; \dots ; \frac{[k]_q}{[n]_q}\right]\right| \leq \frac{1}{2\pi} \cdot \frac{2\pi\rho M(\rho; f)}{C_1 \cdot \rho^{k+1}} =: \frac{C_2}{\rho^k}$$

with $C_2 = C_{\rho,f,q}$.

Now we have to estimate the coefficients c_{kn} for $k > n - j$, that is, to consider the case $\frac{[k]_q}{[n]_q} > \rho$. We use the following formula (see [4, Chap. 4, § 7, p. 121)]),

$$f[x_0; x_1; \dots ; x_k] = \sum_{s=0}^{k} \frac{f(x_k)}{(x_s - x_0) \dots (x_s - x_{s-1})(x_s - x_{s+1}) \dots (x_s - x_k)}.$$

Therefore,

$$\begin{aligned} f\left[0; \frac{1}{[n]_q}; \dots ; \frac{[k]_q}{[n]_q}\right] &= \sum_{s=0}^{k} \frac{f\left(\frac{[s]_q}{[n]_q}\right)}{\frac{[s]_q}{[n]_q}\left(\frac{[s]_q}{[n]_q} - \frac{1}{[n]_q}\right) \dots \left(\frac{[s]_q}{[n]_q} - \frac{[s-1]_q}{[n]_q}\right)\left(\frac{[s]_q}{[n]_q} - \frac{[s+1]_q}{[n]_q}\right)\left(\frac{[s]_q}{[n]_q} - \frac{[k]_q}{[n]_q}\right)} \\ &= \sum_{s=0}^{n-j} + \sum_{s=n-j+1}^{k} . \end{aligned}$$

By the Residue Theorem,

$$\sum_{s=0}^{n-j} = \frac{1}{2\pi i}\oint_{|\zeta|=\rho} \frac{f(\zeta)d\zeta}{\zeta\left(\zeta - \frac{1}{[n]_q}\right) \dots \left(\zeta - \frac{[k]_q}{[n]_q}\right)} = \frac{1}{2\pi i}\oint_{|\zeta|=\rho} \frac{f(\zeta)d\zeta}{\zeta^{k+1}\left(1 - \frac{1}{\zeta[n]_q}\right) \dots \left(1 - \frac{[k]_q}{\zeta[n]_q}\right)}.$$

To estimate the last integral, we set $k := n - t,\ 0 \leq t \leq j - 1$, and consider (with $|\zeta| = \rho$)

$$\left|\left(1 - \frac{1}{\zeta[n]_q}\right) \dots \left(1 - \frac{[n-t]_q}{\zeta[n]_q}\right)\right| \geq \left|\left(1 - \frac{1}{\rho[n]_q}\right) \dots \left(1 - \frac{[n-t]_q}{\rho[n]_q}\right)\right| \geq C_t > 0,$$

by virtue of Lemma 3.2.

Let

$$\min_{0 \leq t \leq j-1} C_t =: C_3 > 0.$$

Clearly, $C_3 = C_{q,\rho}$. We derive

$$\left|\sum_{s=0}^{n-j}\right| \leq \frac{1}{2\pi}\left|\oint_{|\zeta|=\rho} \frac{f(\zeta)d\zeta}{\zeta\left(\zeta - \frac{1}{[n]_q}\right) \dots \left(\zeta - \frac{[k]_q}{[n]_q}\right)}\right| \leq \frac{1}{2\pi} \cdot \frac{M(\rho; f)}{\rho^k C_3} =: \frac{C_4}{\rho^k}. \tag{3.3}$$

ETNA
Kent State University
http://etna.math.kent.edu

To estimate $\sum_{s=n-j+1}^{k} = \sum_{n-j+1}^{n-t}$, we first consider

$$\begin{aligned}
&\left|\frac{[s]_q}{[n]_q}\cdot\frac{[s]_q-1}{[n]_q}\cdots\frac{[s]_q-[s-1]_q}{[n]_q}\,\frac{[s]_q-[s+1]_q}{[n]_q}\cdots\frac{[s]_q-[n-t]_q}{[n]_q}\right|^{-1}\\
&\quad=\frac{[n]_q^{n-t}\cdot(q-1)^{n-t}}{(q^s-1)(q^s-q)\dots(q^s-q^{s-1})(q^{s+1}-q^s)\dots(q^{n-t}-q^s)}\\
&\quad=\frac{(q^n-1)^{n-t}}{q^{s^2}(1-\frac{1}{q})\dots(1-\frac{1}{q^s})q^{(s+1)+\dots+(n-t)}(1-\frac{1}{q})\dots(1-\frac{1}{q^{n-t-s}})}\\
&\quad\le\prod_{j=1}^{\infty}\left(1-\frac{1}{q^j}\right)^{-2}\frac{q^{n(n-t)}}{q^{s^2}\cdot q^{(s+1)\dots+(n-t)}}\\
&\quad=\prod_{j=1}^{\infty}\left(1-\frac{1}{q^j}\right)^{-2}\cdot q^{-(t^2-t)/2}\cdot q^{(n^2-s^2)/2-(n-s)/2}\le C_5q^{n(n-s)},
\end{aligned}$$

where $C_5 = C_{\rho,q}$. Since $s \geq n-j+1$, it follows that

$$q^{n(n-s)} \leq q^{n(j-1)} \leq \frac{1}{\rho^n}.$$

Setting

$$M := \max_{x\in[0,1]} |f(x)|,$$

we obtain

$$\left|\sum_{s=n-j+1}^{k}\right| \le M\sum_{s=n-j+1}^{n-t}\frac{C_5}{\rho^n}\le C_6\frac{j-t}{\rho^n}\le\frac{C_6 j\rho^{-t}}{\rho^{n-t}}\le\frac{C_6 j\rho^{-j}}{\rho^{n-t}}=:\frac{C_7}{\rho^k}, \tag{3.4}$$

with $C_7 = C_{q,f}$.

Finally, juxtaposing (3.3) and (3.4), we obtain the required estimate.

(ii) The case $\rho > 1$ is much easier. Indeed, using (3.1) and Lemma 3.2, we write

$$\begin{aligned}
c_{kn}&\le\left|f\left[0;\frac{1}{[n]_q};\dots;\frac{[k]_q}{[n]_q}\right]\right|\le\frac{1}{2\pi}\left|\oint_{|\zeta|=\rho}\frac{f(\zeta)\,d\zeta}{\zeta^{k+1}\left(1-\frac{1}{\zeta[n]}\right)\dots\left(1-\frac{[k]}{\zeta[n]}\right)}\right|\\
&\le\frac{1}{2\pi}\cdot\frac{2\pi\rho M(\rho;f)}{C_8\rho^{k+1}}=:\frac{C_9}{\rho^k},
\end{aligned}$$

and the proof is complete. □

Proof of Theorem 2.3. To prove the theorem, we need the following modification of [10, Lemma 1].

Let $q > 1$ and $f : [0,1] \to \mathbb{C}$ be a bounded function, such that $f \in C[0,a],\ 0 < a \leq 1$. Then

$$\lim_{n\to\infty} B_{n,q}\left(f;q^{-m}\right) = f\left(q^{-m}\right) \quad\text{for all}\quad q^{-m}\in[0,a]. \tag{3.5}$$

Indeed, we write

$$B_{n,q}(f;z) = \sum_{k=0}^{n} f\left(\frac{[n-k]_q}{[n]_q}\right) p_{n,n-k}(q;z)$$

and observe that

$$p_{n,n-k}\left(q;q^{-m}\right)=0 \ \text{ for } \ m<k,$$

and

$$\lim_{n\to\infty} p_{n,n-k}\left(q;q^{-m}\right)=0 \ \text{ for } \ m>k,$$

while

$$\lim_{n\to\infty} p_{n,n-m}\left(q;q^{-m}\right)=1.$$

Since f is continuous at q^{-m}, the statement follows.

Let $K\subset\{z:|z|<a\}$ be a compact set. Choose $0<\eta<\rho<a$ in such a way that $K\subset\{z:|z|<\eta\}$. Theorem 2.1 implies that for $|z|\leq\eta$, we have

$$|B_{n,q}(f;z)|\leq\sum_{k=0}^{n}\frac{C}{\rho^k}\eta^k\leq\frac{C}{1-\eta/\rho}.$$

That is, the sequence $\{B_{n,q}(f;z)\}$ is uniformly bounded in $\{z:|z|<\eta\}$. In addition, by (3.5) the sequence converges to the function f, analytic in $\{z:|z|<\eta\}$ on the set $\{q^{-m}\}$ having an accumulation point in $\{z:|z|<\eta\}$. By the Vitali Theorem (see, e.g., [15, Chap. V, Sec. 5.2]), the sequence converges to f on any compact set in $\{z:|z|<\eta\}$, and in particular on K. □

Acknowledgments. I would like to express my sincere gratitude to P. Danesh from the Atilim University Academic Writing and Advisory Centre for his assistance in the preparation of the manuscript.

REFERENCES

[1] CH. A. CHARALAMBIDES, *The q-Bernstein basis as a q-binomial distribution*, J. Statist. Plann. Inference, to appear, 2010.
[2] S. COOPER AND S. WALDRON, *The eigenstructure of the Bernstein operator*, J. Approx. Theory, 105 (2000), pp. 133–165.
[3] M.-M. DERRIENNIC, *Modified Bernstein polynomials and Jacobi polynomials in q-calculus*, Rend. Circ. Mat. Palermo (2) Suppl., 76 (2005), pp. 269–290.
[4] R. DEVORE AND G. G. LORENTZ, *Constructive Approximation*, Springer, Berlin, 1993.
[5] A. IL'INSKII, *A probabilistic approach to q-polynomial coefficients, Euler and Stirling numbers I*, Mat. Fiz. Anal. Geom., 11 (2004), pp. 434–448.
[6] A. IL'INSKII AND S. OSTROVSKA, *Convergence of generalized Bernstein polynomials*, J. Approx. Theory, 116 (2002), pp. 100–112.
[7] U. ITAI, *On the eigenstructure of the Bernstein kernel*, Electron. Trans. Numer. Anal., 25 (2006), pp. 431–438.
http://etna.math.kent.edu/vol.25.2006/pp431-438.dir.
[8] S. LEWANOWICZ AND P. WOŹNY, *Generalized Bernstein polynomials*, BIT, 44 (2004), pp. 63–78.
[9] G. G. LORENTZ, *Bernstein Polynomials*, Chelsea, New York, 1986.
[10] S. OSTROVSKA, *q-Bernstein polynomials and their iterates*, J. Approx. Theory, 123 (2003), pp. 232–255.
[11] ———, *The first decade of the q-Bernstein polynomials: results and perspectives*, J. Math. Anal. Approx. Theory, 2 (2007), pp. 35–51.
[12] ———, *On the improvement of analytic properties under the limit q-Bernstein operator*, J. Approx. Theory, 138 (2006), pp. 37–53.
[13] ———, *The approximation of logarithmic function by q-Bernstein polynomials in the case $q>1$*, Numer. Algorithms, 44 (2007), pp. 69–82.
[14] ———, *q-Bernstein polynomials of the Cauchy kernel*, Appl. Math. Comput., 198 (2008), pp. 261–270.
[15] E. C. TITCHMARSH, *Theory of Functions*, Oxford University Press, Oxford, UK, 1986.

[16] V. S. VIDENSKII, *On some classes of q-parametric positive operators*, Oper. Theory Adv. Appl., 158 (2005), pp. 213–222.
[17] ———, *On the polynomials with respect to the generalized Bernstein basis*, in Problems of Modern Mathematics and Mathematical Education, Hertzen readings, St.-Petersburg, 2005, pp. 130–134. (In Russian).
[18] H. WANG, *Korovkin-type theorem and application*, J. Approx. Theory, 132 (2005), pp. 258–264.
[19] ———, *Voronovskaya type formulas and saturation of convergence for q-Bernstein polynomials for* $0 < q < 1$, J. Approx. Theory, 145 (2007), pp. 182–195.
[20] H. WANG AND X. Z. WU, *Saturation of convergence for q-Bernstein polynomials in the case $q \geq 1$*, J. Math. Anal. Appl., 337 (2008), pp. 744–750.
[21] H. WANG, *Properties of convergence for ω, q-Bernstein polynomials*, J. Math. Anal. Appl., 340 (2008), pp. 1096–1108.
[22] XUEYAN XIANG, QIAN HE, AND WENSHAN YANG, *Convergence rate for iterates of q-Bernstein polynomials*, Anal. Theory Appl., 23 (2007), pp. 243–254.