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POLYNOMIALS AND VANDERMONDE MATRICES OVER THE FIELD OF QUATERNIONS *

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Abstract. It is known that the space of real valued, continuous functions $\mathrm{C}(B)$ over a multidimensional compact domain $B \subset \mathbb{R}^k$, $k \geq 2$ does not admit Haar spaces, which means that interpolation problems in finite dimensional subspaces V of $\mathrm{C}(B)$ may not have a solutions in $\mathrm{C}(B)$. The corresponding standard short and elegant proof does not apply to complex valued functions over $B \subset \mathbb{C}$. Nevertheless, in this situation Haar spaces $V \subset \mathrm{C}(B)$ exist. We are concerned here with the case of quaternionic valued, continuous functions $\mathrm{C}(B)$ where $B \subset \mathbb{H}$ and $\mathbb{H}$ denotes the skew field of quaternions. Again, the proof is not applicable. However, we show that the interpolation problem is not unisolvent, by constructing quaternionic entries for a Vandermonde matrix $\mathbf{V}$ such that $\mathbf{V}$ will be singular for all orders $n > 2$. In addition, there is a section on the exclusion and inclusion of all zeros in certain balls in $\mathbb{H}$ for general quaternionic polynomials.

Key words. Quaternionic interpolation polynomials, Vandermonde matrix in quaternions, location of zeros of quaternionic polynomials

AMS subject classifications. 11R52, 12E15, 12Y05, 65D05

1. Introduction. Let B be a compact topological Hausdorff space and $X := \mathrm{C}(B)$ the normed vector space of all real valued, continuous functions defined on B with norm $||f|| := \max_{x\in B} |f(x)|$. Consider the set of all n-dimensional subspaces of X with $n \geq 2$ (let B contain sufficiently many points). We investigate whether there is a *Haar space* in this set. This is a space V with the following property: Given arbitrary, but pairwise distinct points $t_j \in B$ and arbitrary real numbers u_j, there is a unique $v \in V$, such that $v(t_j) = u_j,\ j = 1, 2, \ldots, n$. Thus, in Haar spaces of dimension n all interpolation problems in the above sense can be solved uniquely, regardless of the choice of t_j and u_j, $j = 1, 2, \ldots, n$. The only restriction on the t_j is that they be pairwise distinct. This type of space is also called *unisolvent*. The prototype of a Haar space is Π_{n-1}, the space of all real polynomials of degree at most $n - 1$ on a compact interval of positive length. A counterexample is the span $\langle x, x^2, \cdots, x^n\rangle$ on a compact interval containing the origin. The fact that Haar spaces do not exist if B is a subset of $\mathbb{R}^k$ with $k \geq 2$ is known for a long time, Haar [6, p. 311]. For a proof we refer to the original paper. The essential ingredients of the proof are properties of the determinant of the matrix which describes the interpolation problem and the intermediate value theorem for real valued, continuous functions.

Since there is no intermediate value theorem for complex valued functions, the proof does not carry over to the case $B \subset \mathbb{C}$, though B may be regarded as two dimensional in this case. However, as is also well known, $\mathrm{C}(B)$ contains Haar spaces if $B \subset \mathbb{C}$ and if $\mathrm{C}(B)$ is now the space of complex valued functions on B. The set of complex polynomials, also denoted by Π_{n-1} is again a prototype. A more precise information on what subsets of $\mathbb{C}$ allow the definition of Haar spaces is given by Mairhuber [12]. In the quaternionic case all essential ingredients of the proof are missing. There is no determinant, Fan [2], and no intermediate value theorem. The quaternionic case is the topic of the next sections.

A comprehensive bibliography on quaternions ordered with respect to subjects has been published by Gsponer [5].

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2. Quaternionic polynomials. By $\mathbb{H}$ we denote the (skew) field of quaternions. A polynomial on $\mathbb{H}$ is already a very complicated item. A *monomial of degree* $j \geq 0$ is a mapping $m_j : \mathbb{H} \to \mathbb{H}$ defined by

$$m_j(x) := a_{0j}xa_{1j}xa_{2j}x \cdots a_{j-1,j}xa_{jj}, \quad x, a_{0j}, a_{1j}, \ldots, a_{jj} \in \mathbb{H}. \tag{2.1}$$

A *polynomial of degree* n is any finite sum of monomials of degree $\leq n$. Therefore, the space of all polynomials of degree $\leq n$ has no finite dimension. According to Eilenberg and Niven [1] a polynomial p of degree ≥ 1 with the property that the monomial with the highest degree in p occurs exactly once, has at least one zero. This is called *The Fundamental Theorem of Algebra for Quaternions* by the two mentioned authors. And there is no hope that the restriction on the monomial with the highest degree can be weakened, since $p(x) := ax^n - x^n a - 1$ has no zero. This follows by application of the real part $\Re$ which is linear and commutative, hence, $\Re p(x) = -1$, implying that p cannot have a zero. This example is taken from Pumplün and Walcher [14], which also contains a review and expansion of some results on the number of zeros of polynomials on quaternions $\mathbb{H}$. The Fundamental Theorem can be applied to the polynomial $p(x) := (x-a)^2 = x^2 - xa - ax + a^2$, $a \in \mathbb{H}\backslash\{\mathbb{R}\}$ and shows that, in general, we cannot expect more than one zero. This applies even to polynomials without repetition of monomials of the same degree. Examples are $p(x) := x^2 - x(\mathbf{i}+\mathbf{j}) + \mathbf{k}$ and $q(x) := x^2 - (\mathbf{i}+\mathbf{j})x + \mathbf{k}$ where $\mathbf{i} := (0,1,0,0)$, $\mathbf{j} := (0,0,1,0)$, $\mathbf{k} := (0,0,0,1)$. We have $p(\mathbf{i}) = q(\mathbf{j}) = 0$ and there are no other roots.[1]

3. Quaternionic simple polynomials. We will turn our attention to polynomials of one of the following types:

$$p_l(x) := a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \tag{3.1}$$

$$p_r(x) := a_0 + xa_1 + x^2a_2 + \cdots + x^na_n, \quad a_0, a_1, \ldots, a_n;\ x \in \mathbb{H}. \tag{3.2}$$

We will call these polynomials *simple*. If the coefficients are real, the two types coincide. They also coincide with the polynomials of general type if all coefficients are real. Thus, a real polynomial (i. e. having real coefficients) is always simple. There is a tight connection between p_l and p_r, which is explained in Janovská and Opfer [10].

THEOREM 3.1. *Let p be a real polynomial. If p has (as a polynomial over $\mathbb{C}$) only real zeros, then p as a polynomial over $\mathbb{H}$ has no other zeros. If p has (as a polynomial over $\mathbb{C}$) also complex zeros, then, p as a polynomial over $\mathbb{H}$ has infinitely many zeros. More precisely: Let z be one of the complex zeros of p, then, $h^{-1}zh$ is also a zero for all $h \in \mathbb{H}\backslash\{0\}$. There are no other zeros.*

Proof. It is easy to see that $p(h^{-1}zh) = h^{-1}p(z)h$ for all real polynomials and all $h \in \mathbb{H}\backslash\{0\}$; see also Janovská and Opfer [8] and Pumplün and Walcher [14]. □

The mapping

$$x \to h^{-1}xh, \quad h \in \mathbb{H}\backslash\{0\}$$

defines an equivalence relation in $\mathbb{H}$ with equivalence classes

$$[x] := \{u := h^{-1}xh : h \in \mathbb{H}\backslash\{0\}\}. \tag{3.3}$$

The following lemma makes it easy to recognize equivalent elements.

LEMMA 3.2. *Two quaternions a, b are equivalent (in the sense $a = h^{-1}bh$ for some $h \neq 0$) if and only if*

$$\Re a = \Re b \quad \textit{and} \quad |a| = |b|. \tag{3.4}$$

[1] The first example was communicated by Fabio Vlacci, Firenze, Italy.

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Proof. Janovská and Opfer [7]. □

By using (3.3), (3.4) and $x := (x_1, x_2, x_3, x_4)$ we can also write

$$[x] = \{(x_1, u_2, u_3, u_4) \in \mathbb{H} : u_2^2 + u_3^2 + u_4^2 = r^2 := x_2^2 + x_3^2 + x_4^2\}.$$

This is apparently a sphere in $\mathbb{H}$ where the first component, x_1, is fixed. If $x \in \mathbb{R}$ then we have $[x] = \{x\}$ which means that $[x]$ contains exactly the element x. Let $z \in \mathbb{C}$. Then, the complex conjugate $\overline{z}$ is also belonging to $[z]$, and if z is nonreal, then $[z]$ contains infinitely many elements. We can put Theorem 3.1 into a simpler form.

COROLLARY 3.3. *Let p be a real polynomial over $\mathbb{H}$ of degree n. Then the set of all zeros can be partitioned into at most n equivalence classes.*

For simple polynomials (with quaternionic coefficients) the zeros fall in two classes. Let z be a nonreal zero. Then, either all elements of the equivalence class $[z]$ consist of zeros, or apart from z there is no zero in $[z]$. In the first case the zero z (and all zeros in $[z]$ as well) is called a *spherical zero* and in the second case the zero is called an *isolated zero*. If a zero is real, then it will also be called isolated. See Pogorui and Shapiro [13] and Janovská and Opfer [10] for details. The next theorem contains a statement on the number of zeros of a simple, quaternionic polynomial. For a proof see also [10, 13].

THEOREM 3.4. *Let p be a simple polynomial over $\mathbb{H}$ of degree n. Then, p has $n_1 \geq 0$ isolated zeros and $n_2 \geq 0$ equivalence classes of zeros with $1 \leq n_1 + n_2 \leq n$.*

That means, that Corollary 3.3 is also valid for simple polynomials. We called the polynomials defined in (3.1) and in (3.2) *simple*. In the literature one finds also other words for simple. There is an Italian group, Gentile et al. [3, 4], who refers to these polynomials as *regular*, and there are two other groups, a Portuguese one, Serôdio et al. [15]), and a Brazilian one, de Leo et al. [11], who refer to these polynomials as *unilateral*. It should be remarked, that both the Italian and the Brazilian groups did not take notice of the mentioned paper by Pogorui and Shapiro [13].

4. Location of zeros of quaternionic polynomials. For complex polynomials there are some theorems saying that all roots are outside a certain disk centered at the origin, and that all roots are inside some other disk also centered at the origin. We will show that analogous results hold for simple and even for almost all types of quaternionic polynomials. Without loss of generality we may assume that $a_n = 1$ and $a_0 \neq 0$ in the simple polynomials (3.1), (3.2) if we are interested in their zeros.

THEOREM 4.1. *Let p be a simple polynomial over $\mathbb{H}$ of degree n with $a_n \neq 0$, and $a_0 \neq 0$. Then, the open ball $\{z \in \mathbb{H} : |z| < r\}$ does not contain any zero of p, where r is the only positive root of the real polynomial*

$$\tilde{p}(x) := \sum_{j=1}^{n} |a_j| x^j - |a_0|.$$

Proof. We have $\tilde{p}(0) = -|a_0| < 0$ and $\tilde{p}(x) > 0$ for sufficiently large $x > 0$. In addition, $\tilde{p}'(x) > 0$ for all $x > 0$, implying that $\tilde{p}$ is strictly increasing for all $x > 0$. Thus, there is exactly one positive zero, denoted by r. Let $p := p_l$ and $p(x) = 0$, hence

$$p(x) = a_0 + \sum_{j=1}^{n} a_j x^j = 0 \Rightarrow |a_0| = |\sum_{j=1}^{n} a_j x^j| \leq \sum_{j=1}^{n} |a_j||x|^j \Rightarrow \tilde{p}(|x|) \geq 0 \Rightarrow |x| \geq r.$$

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The proof for $p := p_r$ is the same. ☐

THEOREM 4.2. *Let p be a simple polynomial over $\mathbb{H}$ of degree n with $a_n = 1$ and $a_0 \neq 0$. Then, all zeros of p are contained in the ball*

$$\{z \in \mathbb{H} : |z| \leq R\} \text{ where } R := \max\{1, \sum_{j=0}^{n-1} |a_j|\}.$$

Proof. Let $p := p_l$ and x be a zero of p and assume the contrary, hence $|x| > R$, in particular, $|x| > 1$. Then,

$$0 = |p(x)| = |x^n + \sum_{j=0}^{n-1} a_j x^j| \geq |x^n| - |\sum_{j=0}^{n-1} a_j x^j| \geq$$

$$|x^n| - \sum_{j=0}^{n-1} |a_j||x^j| = |x^n| - |x^{n-1}| \sum_{j=0}^{n-1} \frac{|a_j|}{|x^{n-j-1}|} \geq |x^{n-1}|(|x| - \sum_{j=0}^{n-1} |a_j|) > 0,$$

a contradiction. Thus, $|x| \leq R$. The proof for $p = p_r$ is the same. ☐

EXAMPLE 4.3. The two simple, quadratic polynomials

$$p_l(x) := x^2 + \mathbf{j}x + \mathbf{i}, \quad p_r(x) := x^2 + x\mathbf{j} + \mathbf{i}$$

have the following roots:

$$\text{Roots of } p_l: \; x_1 := 0.5(-1, 1, -1, -1), x_2 := 0.5(1, -1, -1, -1).$$
$$\text{Roots of } p_r: \; x_1 := 0.5(-1, 1, -1, \;\; 1), x_2 := 0.5(1, -1, -1, \;\; 1).$$

All roots have absolute value one. Applying Theorem 4.1 yields $r := \frac{\sqrt{5}-1}{2} \approx 0.618$, and Theorem 4.2 yields $R := 2$.

Both bounds, r, R, are sharp for $p(x) := x^n + 1$. Interestingly, Theorem 4.1 can be carried over to general polynomials.

COROLLARY 4.4. *Let $p(x) := \sum_{j=0}^{n} \mu_j(x)$ be a polynomial of degree n over $\mathbb{H}$, where each μ_j is a finite sum of monomials of degree j defined in 2.1:*

$$\mu_j := \sum_{k=1}^{n_j} m_j^{(k)}, \quad m_j^{(k)}(x) := a_{0j}^{(k)} x a_{1j}^{(k)} x \cdots x a_{jj}^{(k)}, \quad n_j \in \mathbb{N}, \; n_0 = 1,$$

(4.1)

$$|A_j| := \sum_{k=1}^{n_j} |a_{0j}^{(k)} a_{1j}^{(k)} \cdots a_{jj}^{(k)}|, \quad j = 0, 1, \ldots, n,$$

with $|A_n| \neq 0, |A_0| \neq 0$. Then, there is no zero of p located in the open ball

$$\{z \in \mathbb{H} : |z| < \hat{r}\},$$

where $\hat{r}$ is the only positive zero of the real polynomial

$$\hat{p}(x) := \sum_{j=1}^{n} |A_j| x^j - |A_0|.$$

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Proof. Repeat the proof of Theorem 4.1 replacing $a_j x^j$ with $\mu_j(x)$. ∎

For the proof it is apparently sufficient to assume that $|A_j| \neq 0$ for one $1 \leq j \leq n$.

In order to generalize Theorem 4.2 we need to make the assumption that the highest degree monomial occurs only once.

COROLLARY 4.5. *Let $p(x) := \sum_{j=0}^{n} \mu_j(x)$ be a polynomial of degree n over $\mathbb{H}$, where each μ_j is a finite sum of monomials of degree j defined in (2.1), with the exception that μ_n consists only of a single monomial of degree n. Define $|A_j|$ as in (4.1) and assume that $|A_n| = 1, |A_0| \neq 0$. Then, all zeros of p are contained in the ball*

$$\{z \in \mathbb{H} : |z| \leq \hat{R}\}, \text{ where } \hat{R} := \max\{1, \sum_{j=0}^{n-1} |A_j|\}.$$

Proof. Repeat the proof of Theorem 4.2 and replace $a_j x^j$ with μ_j. ∎

EXAMPLE 4.6. Let $p(x) := x^2 - ax - xa + a^2$ with $a \in \mathbb{H}\backslash\{\mathbb{R}\}$. The polynomial p has the single zero $x = a$. Corollary 4.4 yields $\hat{p}(x) := x^2 + 2|a| - |a|^2$. The only positive root is $\hat{r} := (\sqrt{2} - 1)|a|$. Corollary 4.5 yields $\hat{R} := \max\{1, (|a| + 1)^2 - 1\}$. The smallest $\hat{R} = 1$ is obtained for $|a| := \sqrt{2} - 1 \approx 0.41$ which yields $\hat{r} := 3 - 2\sqrt{2} \approx 0.17$.

5. The interpolation problem and the Vandermonde matrix. Let $B \subset \mathbb{H}$ be a compact set and $X := \mathrm{C}(B)$ the space of all quaternion valued, continuous functions defined on B. The general question is whether there are Haar spaces $V \subset \mathrm{C}(B)$. We shall show, that the polynomial space composed of simple polynomials is not a Haar space. For this purpose let us study two interpolation problems: Given arbitrary, but pairwise distinct points $t_j \in \mathbb{H}$ and arbitrary numbers $u_j \in \mathbb{H}, j = 0, 1, \ldots, n$, we are interested in whether the interpolation problems

$$p_l(t_j) = u_j, \tag{5.1}$$

$$p_r(t_j) := u_j, \quad j = 0, 1, \ldots, n, \tag{5.2}$$

have a solution, where p_l, p_r are simple polynomials of degree n, defined by (3.1) and (3.2), respectively. The following matrix $\mathbf{V}$ will be called *Vandermonde matrix*:

$$\mathbf{V} := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_n \\ \vdots & \vdots & \vdots & \vdots \\ t_0^n & t_1^n & \cdots & t_n^n \end{bmatrix} \in \mathbb{H}^{(n+1)\times(n+1)}.$$

The two problems (5.1), (5.2) are equivalent to the following two matrix problems, respectively:

$$\mathbf{a}^{\mathrm{T}}\mathbf{V} = \mathbf{u}^{\mathrm{T}}, \quad \mathbf{V}^{\mathrm{T}}\mathbf{a} = \mathbf{u},$$

where

$$\mathbf{a}^{\mathrm{T}} := (a_0, a_1, \ldots, a_n), \quad \mathbf{u}^{\mathrm{T}} := (u_0, u_1, \ldots, u_n).$$

The main point here is, that these problems are different in general. Since quaternionic matrices have no determinants, Fan [2], the singularity of a matrix must be defined in a more elementary way.

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6. Short excursion to linear systems over $\mathbb{H}$. We will give some elementary properties of matrices with quaternions as entries and mappings defined by matrices; see Zhang [16] for an overview of such mappings and for possible decompositions for such matrices. For more general types of linear mappings; cf. Janovská and Opfer [9].

DEFINITION 6.1. Let $\mathbf{A} \in \mathbb{H}^{p\times q}$. The maximal number of right independent columns of $\mathbf{A}$ will be called *right column rank* of $\mathbf{A}$. Let $p = q$. The square matrix $\mathbf{A}$ will be called *nonsingular*, if the right column rank is maximal, i. e. $\mathrm{rank}(\mathbf{A}) = p$, where *rank* is to be understood as right column rank. The mapping $f : \mathbb{H}^p \to \mathbb{H}^p$ defined by

$$f(\mathbf{x}) := \mathbf{A}\mathbf{x} \tag{6.1}$$

will be called *nonsingular* if $\mathbf{A}$ is nonsingular.

In the same fashion *left column rank* and *left, right row rank* of $\mathbf{A}$ are defined. A standard theorem in non commutative linear algebra is: The right column rank coincides with the left row rank and the left column rank coincides with the right row rank. Thus, a quaternionic matrix has two ranks, the right and the left column rank. The above definition already suggests that the right column rank will be more important than the left column rank, since in $\mathbf{A}\mathbf{x}$ the components of $\mathbf{x}$ are always on the right of the matrix elements of $\mathbf{A}$.

THEOREM 6.2. *Let $\mathbf{A} \in \mathbb{H}^{p\times p}$ be a square matrix. The mapping f as defined in (6.1) is singular (i. e. not nonsingular) if and only if the homogeneous system $f(\mathbf{x}) = \mathbf{0}$ has non trivial solutions. The system $f(\mathbf{x}) = \mathbf{c}$ has a unique solution for all $\mathbf{c} \in \mathbb{H}^p$ if and only if f is nonsingular. The mapping g defined by $g(\mathbf{x}) := \mathbf{x}^{\mathrm{T}}\mathbf{A}$ is singular if and only if f is singular, where $\mathbf{x}^{\mathrm{T}}$ denotes the transpose of $\mathbf{x}$.*

Proof. The mapping f defined in (6.1) may be regarded as a right linear combination of the columns of $\mathbf{A}$. The mapping g may be regarded as a left linear combination of the rows of $\mathbf{A}$, and the right column rank and the left row rank coincide. The remaining part is easy. $\square$

DEFINITION 6.3. Let $\mathbf{A} \in \mathbb{H}^{p\times p}$. The right column rank of $\mathbf{A}$ will be called *rank* of $\mathbf{A}$.

That $f(x) := \mathbf{A}\mathbf{x}$ and $f(x)^{\mathrm{T}} := \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ do not define the same mapping (apart from transposition) will be shown by the following example; see Zhang [16].

EXAMPLE 6.4. Let

$$\mathbf{A} := \begin{bmatrix} 1 & \mathbf{i} \\ \mathbf{j} & \mathbf{k} \end{bmatrix}.$$

This matrix has right column rank 2 and left column rank 1. The latter statement can be easily verified by multiplying the first column of $\mathbf{A}$ from the left by $\mathbf{i}$. The result is the second column. The transpose $\mathbf{A}^{\mathrm{T}}$ has right column rank 1 and left column rank 2. Thus, $\mathbf{A}$ is nonsingular, whereas $\mathbf{A}^{\mathrm{T}}$ is singular.

7. Vandermonde matrices, continued. We return now to the Vandermonde matrix and the corresponding interpolation problems (5.1), (5.2). It is clear that the Vandermonde matrix and its transpose are nonsingular for $n \le 1$. In order to show that it is possible to find singular Vandermonde matrices for general $n \ge 3$ the idea is the following: Try to find pairwise distinct points $t_0, t_1, \ldots, t_n$ such that the sum of the first and penultimate row equals the sum of the second and last row. If this is possible, the left and right row rank are not maximal, which implies that the rank is not maximal. Hence, $\mathbf{V}$ and $\mathbf{V}^{\mathrm{T}}$ are singular.

THEOREM 7.1. *Let $n = 2$. Define $t_0 := \mathbf{i} := (0, 1, 0, 0)$, $t_1 := \mathbf{j} := (0, 0, 1, 0)$, $t_2 := \mathbf{k} := (0, 0, 0, 1)$, and*

$$\mathbf{V} := \begin{bmatrix} 1 & 1 & 1 \\ t_0 & t_1 & t_2 \\ t_0^2 & t_1^2 & t_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -1 \end{bmatrix}. \tag{7.1}$$

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Let $n \geq 3$ and $h \in \mathbb{H}\backslash\{\mathbb{C}\}$ be arbitrary. Define the following Vandermonde matrix $\mathbf{V}$: Put $t_0 = 1$ and set

$$\mathbf{V} := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & e_1 & e_2 & \cdots & e_{n-1} & h^{-1}e_{n-1}h \\ 1 & e_1^2 & e_2^2 & \cdots & e_{n-1}^2 & h^{-1}e_{n-1}^2h \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & -e_1 & -e_2 & \cdots & -e_{n-1} & -h^{-1}e_{n-1}h \end{bmatrix} \in \mathbb{H}^{(n+1)\times(n+1)},$$

where $e_1, e_2, \ldots, e_{n-1}$ are the real or complex roots of

$$t^{n-1} + 1 = 0. \tag{7.2}$$

If -1 is one of the roots, then let $e_1 = -1$, otherwise, choose any enumeration of the roots. Then $\mathbf{V}$ and its transpose $\mathbf{V}^{\mathrm{T}}$ are singular.

Proof. For $n = 2$ it is obvious that t_0, t_1, t_2 are pairwise distinct and that $\mathbf{V}$ defined in (7.1) and $\mathbf{V}^{\mathrm{T}}$ are singular. Let $n \geq 3$. It is clear that the second row of $\mathbf{V}$ contains only pairwise distinct entries, in particular, $t_n := h^{-1}e_{n-1}h \notin \mathbb{C}$. The formula (7.2) implies that the first and penultimate row sum to $(2, 0, \ldots, 0)$. Formula (7.2) implies $t^n + t = 0$ for all roots, which implies that row 2 and the last row also sum to $(2, 0, \ldots, 0)$. Thus, the (right and left) row rank are not maximal. It follows that the rank is not maximal and in all cases $n \geq 2$ the Vandermonde matrix $\mathbf{V}$ and its transpose $\mathbf{V}^{\mathrm{T}}$ are singular. □

We observe that for all selected points (second row of Vandermonde's matrix) the absolute value is one. That means, we can restrict our considerations to the unit ball $B := \{z \in \mathbb{H} : |z| \leq 1\}$ or to the unit sphere $\partial B := \{z \in \mathbb{H} : |z| = 1\}$.

8. Unisolvency and the number of zeros. In the theory of real or complex valued continuous functions the existence of Haar spaces of dimension n is equivalent to the fact that the elements in the Haar space do not have more than $n-1$ zeros with the only exception of the zero function. We will see that even in quaternionic spaces the situation is analogue.

THEOREM 8.1. *Let $V \subset \mathrm{C}(B)$ be a vector space with (left or right) dimension n, where the set $\mathrm{C}(B)$ is the set of quaternion valued, continuous functions on B, and $B \subset \mathbb{H}$ a compact set. The space V is a Haar space if and only if all functions in $V\backslash\{0\}$ have at most $n-1$ zeros.*

Proof. (a) Assume V is a Haar space. Then, $v(t_j) = 0, j = 1, 2, \ldots, n$ for pairwise distinct $t_j \in B$ implies $v = 0$. Thus, any $v \neq 0$ can have at most $n-1$ zeros in B. (b) Assume V is not a Haar space. Then there are pairwise distinct points t_j and values $u_j, j = 1, 2 \ldots, n$, such that the interpolation problem $v(t_j) = u_j, j = 1, 2 \ldots, n$, has no or two different solutions v_1, v_2. In the latter case, $v := v_1 - v_2$ is not the zero function but has n zeros at the given n points t_j. If the interpolation problem has no solution, then the homogeneous problem $v(t_j) = 0, j = 1, 2 \ldots, n$ must have a non trivial solution. In all cases there is a non zero function v with at least n zeros. □

The fact that polynomials have too many zeros is, thus, responsible for the fact that polynomials do not form a Haar space. The question, whether there are Haar spaces in the quaternionic $\mathrm{C}(B)$ remains open.

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