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GAUSSIAN DIRECT QUADRATURE METHODS FOR DOUBLE DELAY VOLTERRA INTEGRAL EQUATIONS*

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Abstract. In this paper we consider Volterra integral equations with two constant delays. We construct Direct Quadrature methods based on Gaussian formulas, combined with a suitable interpolation technique. We study the convergence and the stability properties of the methods and we carry out some numerical experiments that confirm our theoretical results.

Key words. Volterra integral equations, Direct Quadrature method, Gaussian quadrature formulas, convergence, stability

AMS subject classifications. 65R20

1. Introduction. In this paper we consider double delays Volterra integral equations (VIEs) of the type

$$y(t) = f(t) + \int_{t-\tau_2}^{t-\tau_1} k(t-\tau)g(y(\tau))d\tau, \quad t \in [\tau_2, T], \tag{1.1}$$

with $y(t) = \phi(t)$, $t \in [0, \tau_2]$, $\tau_1, \tau_2 \in \mathbb{R}_+$, where $\phi(t)$ is a known function such that

$$\phi(\tau_2) = f(\tau_2) + \int_0^{\tau_2-\tau_1} k(\tau_2-\tau)g(\phi(\tau))d\tau. \tag{1.2}$$

We assume that the functions $f(t)$, $k(t)$, and $\phi(t)$ are at least continuous on $[0, T]$, on $[\tau_1, \tau_2]$ and on $[0, \tau_2]$, respectively, and that $g(y)$ satisfies the Lipschitz condition. These assumptions ensure existence, uniqueness, and continuity of the solution of (1.1) [7]. By successively differentiating (1.1) it is easy to verify that $y^{(l)}$, $l = 1, 2, \ldots$ presents some points, $\theta_1, ..., \theta_Z$, of primary discontinuities ($\theta_1 := \tau_2$ for y'; $\theta_1 := \tau_2,\ \theta_2 := \tau_2 + \tau_1,\ \theta_3 := 2\tau_2$ for y'';...), and it is continuous in $]l\tau_2, T]$.

Double delay VIEs arise in the mathematical modeling of population dynamics, whose present history depends only on a finite and variable part of the past history. For example, equations of the form (1.1) model the growth of a population structured by age with a finite life span [1, 4].

The numerical treatment of (1.1) has been carried out only recently and in the specialized literature a few papers can be found on this topic. To our knowledge, the only numerical methods for the equation (1.1) have been constructed in [6, 7], where Direct Quadrature (DQ) methods based on Newton-Cotes formulas have been proposed.

The aim of our research is to extend the class of numerical methods for solving equation (1.1) to DQ methods based on Gaussian quadrature formulas. It is known that Gaussian formulas ensure a higher order of accuracy and have better stability properties than Newton-Cotes formulas [2]. On the other hand, the use of DQ methods based on Gaussian formulas

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produces several problems. As a matter of fact, such methods require the knowledge of the solution at some points not belonging to the mesh. In order to overcome this difficulty we have chosen to use an interpolation technique.

We have studied the convergence properties of the constructed methods and we have proven that their order of convergence is the minimum between the order of convergence of the Gaussian formula and the degree of accuracy of the interpolating polynomial.

The study of numerical stability of our methods has been carried out on the following test equation, introduced in [7]:

$$y(t) = 1 + \int_{t-\tau_2}^{t-\tau_1} (\lambda + \mu(t-s))y(s)ds, \qquad t \in [\tau_2, T],$$

$\lambda,\ \mu \in \mathbb{R}$. We have found sufficient conditions under which the numerical solution produced by our method shows the same behavior as the analytical one. In particular we have determined the bound and the limiting value of the numerical solution.

Section 2 contains the construction of the method and in Section 3 the convergence analysis is carried out. The numerical stability of our method is treated in Section 4. In Section 5, we report some numerical experiments that confirm the theoretical results stated in Sections 3 and 4. In Section 6, some concluding remarks and future developments are reported.

2. The method. Let $\Pi_N = \{t_j : 0 < t_0 < t_1 < \cdots < t_N = T\}$ be a partition of the time interval $[0, T]$ with constant stepsize $h = t_{j+1} - t_j$, $j = 0, \ldots, N-1$, and assume that there exist n_1 and n_2, positive integers, such that

$$h = \frac{\tau_1}{n_1} = \frac{\tau_2}{n_2}. \tag{2.1}$$

In the following we denote by y_j an approximation to the exact solution $y(t_j)$ of (1.1). Let $\{\xi_k\}_{k=1}^m$ and $\{\omega_k\}_{k=1}^m$ be the nodes and the weights of an m-point Gaussian quadrature formula on $[0, 1]$. Then, the m-point Gaussian quadrature formula on $[0, h]$ has nodes $\{\xi_k h\}_{k=1}^m$ and weights $\{h\omega_k\}_{k=1}^m$,

$$\int_0^h \Phi(\xi)d\xi \approx h\sum_{k=1}^m \omega_k \Phi(\xi_k h), \tag{2.2}$$

where $\Phi(\xi)$ is any continuous integrand function.

The integral equation (1.1) at the mesh points is

$$y(t_j) = f(t_j) + \int_{t_j-\tau_2}^{t_j-\tau_1} k(t_j-\tau)g(y(\tau))d\tau, \qquad j = n_2+1, \ldots, N. \tag{2.3}$$

The integral in (2.3) can be written as

$$\int_{t_j-\tau_2}^{t_j-\tau_1} k(t_j-\tau)g(y(\tau))d\tau = \sum_{r=1}^{n_{21}} \int_0^h k(q_r h - \tau)g(y(t_{j-q_r}+\tau))d\tau, \tag{2.4}$$

where $n_{21} := n_2 - n_1$, $q_r := n_2 - r + 1$. We discretize each of the integrals on $[0, h]$ by the quadrature rule (2.2), thus obtaining

$$y(t_j) \approx f(t_j) + h\sum_{r=1}^{n_{21}}\sum_{k=1}^m \omega_k k(q_r h - \xi_k h)g(y(t_{j-q_r} + \xi_k h)), \quad j = n_2+1, \ldots, N. \tag{2.5}$$

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The points, $t_{j-q_r} + \xi_k h$, $k = 1, \ldots, m$, $r = 1, \ldots, n_{21}$, do not belong to the mesh Π_N. In order to overcome this problem, we adopt an interpolation technique, similar to that used in [5] to discretize a linear delay integro-differential equation. We construct the Lagrange interpolating polynomial $\mathcal{P}(x)$ of degree s for the data points,

$$(t_{j-q_r-s_-}, y_{j-q_r-s_-}), \ldots, (t_{j-q_r}, y_{j-q_r}), \ldots, (t_{j-q_r+s_+}, y_{j-q_r+s_+}),$$

with $s := s_- + s_+$, and $s_-, s_+ \in \mathbb{N}$, that is,

$$\mathcal{P}(x) = \sum_{l=-s_-}^{s_+} \mathcal{L}_l(x) y_{j-q_r+l}, \tag{2.6}$$

where $\mathcal{L}_l$ is the lth fundamental Lagrange polynomial with nodes $t_{j-q_r-s_-}, \ldots, t_{j-q_r+s_+}$. We replace $y(t_{j-q_r} + \xi_k h)$ by $\mathcal{P}(t_{j-q_r} + \xi_k h)$ in (2.5), thus obtaining the numerical method,

$$\begin{aligned} &y_j = f(t_j) + h \sum_{r=1}^{n_{21}} \sum_{k=1}^{m} \omega_k k(q_r h - \xi_k h) g(\mathcal{P}(t_{j-q_r} + \xi_k h)), \\ &j = n_2 + 1, \ldots, N. \end{aligned} \tag{2.7}$$

In the following it will be useful to observe that

$$\mathcal{L}_l(t_{j-q_r} + xh) = P_l(x),$$

where P_l is the lth fundamental Lagrange polynomial determined by the nodes, $-s_-, \ldots, s_+$, namely,

$$P_l(x) = \prod_{\substack{i=-s_- \\ i \neq l}}^{s_+} \frac{x - i}{l - i}. \tag{2.8}$$

Thus the method (2.7) can be written equivalently as

$$\begin{aligned} &y_j = f(t_j) + h \sum_{r=1}^{n_{21}} \sum_{k=1}^{m} \omega_k k(q_r h - \xi_k h) g\left(\sum_{l=-s_-}^{s_+} P_l(\xi_k) y_{j-q_r+l} \right), \\ &j = n_2 + 1, \ldots, N. \end{aligned} \tag{2.9}$$

The method (2.7) depends on the parameters s_-, s_+ and m, which have to satisfy some suitable conditions for its applicability. First, in order not to require values of the solution outside $[0, T]$, we have to require that

$$s_- \leq 1.$$

In addiction, a necessary and sufficient condition to avoid the use of future mesh points (where the numerical solution is not yet known) is

$$s_+ \leq n_1 + 1. \tag{2.10}$$

Thus, we have $0 \leq s_- \leq 1$ and $0 \leq s_+ \leq n_1 + 1$. Finally, we observe that, if (2.10) holds and $s_+ \neq n_1 + 1$, then the method (2.7) is explicit.

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3. Convergence analysis. In the previous section we have constructed our method by using two kinds of approximations, one arising from the discretization of the integral in (1.1) by the Gaussian quadrature formula (2.2), and another one arising from the computation of the unknown values of the solution by the interpolating polynomial (2.6). Therefore, in the convergence analysis, we have to take into account the contributions to the errors of both approximations.

In the following we assume that h satisfies the condition (2.1), which implies that the discontinuity points $\theta_1, ..., \theta_Z$ of order $\leq s+1$ are all included in the mesh Π_N. Furthermore, we assume that

$$\begin{aligned} &\forall j, r \;\; \exists z: \text{ either } t_{j-q_r-s_-}, ..., t_{j-q_r+s_+} \in [\theta_z, \theta_{z+1}] \quad \text{or} \\ &t_{j-q_r-s_-} \geq \theta_Z \quad \text{or} \quad t_{j-q_r+s_+} \leq \theta_1 \end{aligned} \tag{3.1}$$

hold. Condition (3.1) may be satisfied by a suitable choice of s_- and s_+.

Now we are able to prove the following theorem.

THEOREM 3.1. *Let y_j be the numerical solution of (1.1) obtained by the method (2.7) with $0 \leq s_- \leq 1$ and $0 \leq s_+ \leq n_1 + 1$. Let $p = \min(2m, s+1)$ and $q = \max(2m, s+1)$, where $2m$ is the order of the quadrature formula (2.2) and s is the degree of the interpolating polynomial (2.6). Assume that $f \in C^{s+1}([0,T])$, $k \in C^q([\tau_1, \tau_2])$, $\phi \in C^{s+1}([0, \tau_2])$, $g \in C^q(\mathbb{R})$, and g satisfies the Lipschitz condition. Then, for sufficiently small step size h, the error $e_j = y(t_j) - y_j$ satisfies*

$$\max_{1 \leq j \leq N} |e_j| \leq Ch^p,$$

for some finite C not depending on h.

REMARK 3.2. From the smoothness hypotheses on ϕ, f, and k, the exact solution $y(t)$ of (1.1) is at least $s+1$ times continuously differentiable on $[\theta_z, \theta_z + 1]$, $z = 1, ..., Z-1$, and on $[0, \theta_1]$ and $[\theta_Z, T]$. From the expression for $y^{(\nu)}(t)$, $\nu = 0, ..., s+1$, obtained by successively differentiating (1.1) with respect to t, it is readily seen that both the left and right limits of $y^{(\nu)}(t)$ as $t \longrightarrow \theta_z$ exist and are finite.

Proof of Theorem 3.1. Take $t_j \in [\theta_z, \theta_{z+1}]$, with $z \in 1, \dots, Z-1$. We have

$$e_j = \sum_{r=1}^{n_{21}} \left[\int_0^h k(q_r h - \tau) g(y(t_{j-q_r} + \tau)) d\tau - h \sum_{k=1}^{m} w_k k(q_r h - \xi_k h) g(\mathcal{P}(t_{j-q_r} + \xi_k h)) \right].$$

We can rewrite the error as $e_j = \sum_{r=1}^{n_{21}} (B_{jr} + D_{jr})$, where

$$B_{jr} := \int_0^h k(q_r h - \tau) g(y(t_{j-q_r} + \tau)) d\tau - \int_0^h k(q_r h - \tau) g(\mathcal{P}(t_{j-q_r} + \tau)) d\tau$$

and

$$D_{jr} := \int_0^h k(q_r h - \tau) g(\mathcal{P}(t_{j-q_r} + \tau)) d\tau - h \sum_{k=1}^{m} w_k k(q_r h - \xi_k h) g(\mathcal{P}(t_{j-q_r} + \xi_k h)).$$

Thus, letting L be the Lipschitz constant for g, we have

$$\begin{aligned} |B_{jr}| \leq L &\int_0^h |k(q_r h - \tau)| \, |I_{jr}(\tau)| \, d\tau \\ &+ L \int_0^h |k(q_r h - \tau)| \left| \sum_{l=-s_-}^{s_+} P_l(\tau/h)(y(t_{j-q_r+l}) - y_{j-q_r+l}) \right| d\tau, \end{aligned}$$

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where

$$I_{jr}(\tau) = y(t_{j-q_r} + \tau) - \sum_{l=-s_-}^{s_+} P_l(\tau/h) y(t_{j-q_r+l})$$

is the interpolation error at $t_{j-q_r} + \tau$. By condition (3.1) and Remark 3.2, there exists $C_s > 0$, such that

$$|I_{jr}(\tau)| \leq C_s h^{s+1}, \qquad \forall \tau \in [0, h].$$

We observe that the constant C_s when s is large has an exponential behaviour, since it depends on the Lebesgue constant. Therefore,

$$|B_{jr}| \leq \bar{C}_s h^{s+2} + C_1 h \sum_{l=-s_-}^{s_+} |e_{j-q_r+l}|, \tag{3.2}$$

where $\bar{C}_s = L\, C_s \max_{t\in[\tau_1,\tau_2]} |k(t)|$ and $C_1 = L \max_{l\in\{-s_-,\ldots,s_+\}} \max_{x\in[0,1]} |P_l(x)| \max_{t\in[\tau_1,\tau_2]} |k(t)|$. D_{jr} is the Gauss-Legendre quadrature error in $[0, h]$ for the function $k(q_r h - \tau) g(\mathcal{P}(t_{j-q_r} + \tau)$. Then, there exists $\tilde{C}_m > 0$ such that [3, (2.7.12), p. 98]

$$|D_{jr}| \leq \tilde{C}_m h^{2m+1}. \tag{3.3}$$

By (3.2) and (3.3) it follows that

$$\begin{aligned}
|e_j| &\leq \sum_{r=1}^{n_{21}} \left(\bar{C}_s h^{s+2} + C_1 h \sum_{l=-s_-}^{s_+} |e_{j-q_r+l}| + \tilde{C}_m h^{2m+1} \right) \\
&= (\tau_2 - \tau_1) \left(\bar{C}_s h^{s+1} + \tilde{C}_m h^{2m} \right) + C_1 h \sum_{l=-s_-}^{s_+} |e_{j-q_r+l}| \\
&\leq C_{s,m} h^p + C_1 h \sum_{i=0}^{j} |e_i|.
\end{aligned}$$

Therefore,

$$|e_j| \leq \frac{C_{s,m}}{1 - C_1 h}\, h^p + \frac{C_1}{1 - C_1 h} h \sum_{i=0}^{j-1} |e_i|.$$

Now we apply the Gronwall-type inequality [2, p. 41], and, since there are no starting errors, we get

$$|e_j| \leq \frac{C_{s,m}}{1 - C_1 h}\, h^p \, e^{\frac{C_1}{1-C_1 h} T}.$$

Therefore, $e_j = O(h^p)$ as $h \to 0$ and the theorem follows. ☐

Theorem 3.1 allows us to choose the parameters of the method by balancing efficiency and accuracy, as shown in the following example.

EXAMPLE 3.3. The best way to achieve order 2 is the 1-node Gaussian quadrature formula ($m = 1$) combined with the linear interpolation ($s = 1$). The choices $s_- = 0$, $s_+ = 1$ yield

$$\omega_1 = 1, \qquad \xi_1 = 1/2, \tag{3.4}$$

$$\mathcal{P}(x) = \frac{t_{j-q_r+1} - x}{h} y_{j-q_r} + \frac{x - t_{j-q_r}}{h} y_{j-q_r+1}. \tag{3.5}$$

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4. Numerical stability. Our next task is to study the stability properties of our numerical methods with respect to test equation introduced in [7],

$$y(t) = 1 + \int_{t-\tau_2}^{t-\tau_1} (\lambda + \mu(t-\tau))y(\tau)d\tau, \qquad t \in [\tau_2, T], \tag{4.1}$$

$\lambda,\ \mu \in \mathbb{R}$, according to the following definition.

DEFINITION 4.1. *A numerical method is stable with respect to (4.1) when its application to (4.1) gives a numerical solution behaving like the continous one.*

It is known that the stability analysis for test equations is the starting point for the investigation of the stability properties of the method for more general equations.

We make the following definitions:

$$\rho := \lambda(\tau_2 - \tau_1) + \frac{\mu}{2}(\tau_2^2 - \tau_1^2), \tag{4.2}$$

$$\begin{aligned} \alpha &:= -\frac{1}{2\mu}(\lambda + \mu\tau_1)^2, \qquad \beta := \frac{1}{2\mu}(\lambda + \mu\tau_2)^2, \\ \Phi &:= \max_{[0,\tau_2]} |\phi(t)|. \end{aligned} \tag{4.3}$$

In the theorem below we summarize some theoretical results about the bound and the limiting value of the analytical solution of (4.1) [7].

THEOREM 4.2. *Assume that one of the following sets of conditions holds:*

a) $(\lambda + \mu\tau_1)(\lambda + \mu\tau_2) \geq 0$, $|\rho| < 1$,

b) $(\lambda + \mu\tau_1)(\lambda + \mu\tau_2) \leq 0$, $|\alpha| + |\beta| < 1$.

Then $y(t)$ is bounded for all $t \geq 0$ and

$$|y(t)| \leq \frac{1}{1 - (|\alpha| + |\beta|)} + \Phi.$$

Moreover,

$$\lim_{t\to+\infty} y(t) = \frac{1}{1-\rho}.$$

The numerical solution of (4.1) obtained by the method (2.9) is

$$y_j = 1 + h \sum_{r=1}^{n_{21}} \sum_{k=1}^{m} \omega_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l=-s_-}^{s_+} P_l(\xi_k) y_{j-q_r+l}, \tag{4.4}$$

$j = n_2 + 1, \ldots, N$. Let us define the sets of indices

$$R_1 = \{(r,k) : \lambda + \mu h(q_r - \xi_k) \leq 0\}, \quad R_2 = \{(r,k) : \lambda + \mu h(q_r - \xi_k) > 0\}, \tag{4.5}$$

and

$$\begin{aligned} L_1 &:= \{l : P_l(\xi_k) \leq 0,\ k = 1, ..., m\} = \{(-1), 2, 4, \ldots, 2l, \ldots\} \\ L_2 &:= \{l : P_l(\xi_k) > 0,\ k = 1, ..., m\} = \{0, 1, 3, \ldots, 2l+1, \ldots\}. \end{aligned} \tag{4.6}$$

According to formula (2.8) all the zeros of $P_l(x)$ are integers. Therefore, in (0,1) $P_l(x)$ cannot vanish, that is $P_l(x)$ has a constant sign in $(0,1)$. Hence, $P_l(x)$ assumes the same sign

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at all the nodes ξ_k, $k = 1, ..., m$ (which belong to $(0, 1)$; see formula (2.2)) and equalities (4.6) follow. Moreover, we will use

$$\begin{aligned} \alpha(h) &= h \sum_{r,k \in R_1} w_k(\lambda + \mu h(q_r - \xi_k)), \\ \beta(h) &= h \sum_{r,k \in R_2} w_k(\lambda + \mu h(q_r - \xi_k)). \end{aligned} \tag{4.7}$$

We set

$$\sigma := \max_{k=1,\ldots,m} \sum_{l \in L_1} |P_l(\xi_k)|. \tag{4.8}$$

Since $\sum_{l=-s_-}^{s_+} P_l(\xi_k) = 1$, $\forall k = 1, \ldots, m$, we obtain

$$\sum_{l=-s_-}^{s_+} |P_l(\xi_k)| \le 2\sigma + 1, \qquad \forall k = 1, \ldots, m. \tag{4.9}$$

Our first result establishes some sufficient conditions for the boundedness of the numerical solution (4.4).

THEOREM 4.3. *Assume that one of the following set of conditions holds:*

a) $(\lambda + \mu\tau_1)(\lambda + \mu\tau_2) \ge 0$, $(2\sigma + 1)|\rho| < 1$,

b) $(\lambda + \mu\tau_1)(\lambda + \mu\tau_2) \le 0$, $(2\sigma + 1)(|\alpha(h)| + |\beta(h)|) < 1$.

Then

$$|y_j| \le \frac{1}{1 - (2\sigma + 1)(|\alpha(h)| + |\beta(h)|)} + \Phi, \quad j = 1, ..., N. \tag{4.10}$$

Proof. We rewrite (4.4) as

$$\begin{aligned} y_j &= 1 + \delta h \sum_{r=1}^{n_{21}} \sum_{k=1}^{m} w_k(\lambda + \mu h(q_r - \xi_k)) P_{q_r}(\xi_k) y_j \\ &\quad + h \sum_{r=1}^{n_{21}} \sum_{k=1}^{m} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l \ne q_r} P_l(\xi_k) y_{j-q_r+l}, \end{aligned} \tag{4.11}$$

where $\delta = 1$ if $s_+ = n_1 + 1$ (implicit method; see Section 2), otherwise $\delta = 0$. The sum over $l \ne q_r$ contains $s + 1 - \delta$ terms.

a) We consider the case $\lambda + \mu x > 0$ for $x \in [\tau_1, \tau_2]$, since the case $\lambda + \mu x < 0$ can be treated similarly. From (4.11) it follows that

$$(1 - \delta\Theta)|y_j| \le 1 + h \sum_{r=1}^{n_{21}} \sum_{k=1}^{m} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l \ne q_r} |P_l(\xi_k)||y_{j-q_r+l}|,$$

where $\Theta := h \sum_{r=1}^{n_{21}} \sum_{k=1}^{m} w_k(\lambda + \mu h(q_r - \xi_k))|P_{q_r}(\xi_k)|$. Now we proceed step by step. For $j < n_2 + 1$, since $y_j = y(t_j) = \varphi(t_j)$, then $|y_j| < \Phi$. For $j = n_2 + 1$, since $\delta\Theta < 1$, we have

$$|y_{n_2+1}| \le \frac{1 + ((2\sigma + 1)\rho - \delta\Theta)\Phi}{1 - \delta\Theta}.$$

For $j = n_2 + 2$, we have

$$(1 - \delta\Theta)|y_{n_2+2}| \le 1 + ((2\sigma+1)\rho - \delta\Theta)\max\left(\Phi, \frac{1 + ((2\sigma+1)\rho - \delta\Theta)\Phi}{1 - \delta\Theta}\right).$$

Now assume that $\Phi < \frac{1}{1-(2\sigma+1)\rho}$. Then we have

$$|y_{n_2+2}| \le \frac{1}{1-\delta\Theta}\left(1 + \frac{(2\sigma+1)\rho - \delta\Theta}{1-\delta\Theta}\right) + \left(\frac{(2\sigma+1)\rho - \delta\Theta}{1-\delta\Theta}\right)^2 \Phi.$$

Since $\frac{(2\sigma+1)-\delta\Theta}{1-\delta\Theta} < 1$,

$$|y_{n_2+2}| \le \Phi + \frac{1}{1-(2\sigma+1)\rho}.$$

Continuing with the same procedure, we can prove

$$|y_j| \le \Phi + \frac{1}{1-(2\sigma+1)\rho}, \qquad j = 1, \dots, N, \tag{4.12}$$

which is equivalent to (4.10), since in this case $|\alpha(h)| + |\beta(h)| = \rho$.

We arrive at the same result when $\Phi \ge \frac{1}{1-(2\sigma+1)\rho}$.

b) By (4.11) we get

$$(1-\delta\Theta)|y_j| \le 1 + h\sum_{r=1}^{n_{21}}\sum_{k=1}^{m} w_k|\lambda + \mu h(q_r - \xi_k)| \sum_{l \ne q_r} |P_l(\xi_k)||y_{j-q_r+l}|, \tag{4.13}$$

where $\Theta := h\sum_{r=1}^{n_{21}}\sum_{k=1}^{m} w_k|\lambda + \mu h(q_r - \xi_k)||P_{q_r}(\xi_k)|$. As in the case a), we proceed step by step. We find that $|y_j| \le \Phi$, $j = 0, \dots, n_2$, and $\delta\Theta < 1$. Therefore, for $j_2 + 1$, we have

$$|y_{n_2+1}| \le \frac{1 + \Phi((2\sigma+1)\left(|\alpha(h)| + |\beta(h)|\right) - \delta\Theta)}{1-\delta\Theta}. \tag{4.14}$$

For $j = n_2 + 2$, from (4.13) and (4.14), we obtain

$$\begin{aligned} |y_{n_2+2}| \le{}& \frac{1}{1-\delta\Theta}\left[1 + \frac{(2\sigma+1)(|\alpha(h)| + |\beta(h)|) - \delta\Theta}{1-\delta\Theta}\right] \\ &+ \left(\frac{(2\sigma+1)(|\alpha(h)| + |\beta(h)|) - \delta\Theta}{1-\delta\Theta}\right)^2 \Phi \\ \le{}& \Phi + \frac{1}{1-(2\sigma+1)(|\alpha(h)| + |\beta(h)|)}, \end{aligned}$$

Here we have assumed that $\Phi \le 1/(1-(2\sigma+1)\left(|\alpha(h)| + |\beta(h)|\right))$. The opposite case can be treated in an analogous way. By proceeding step by step the theorem follows. ◻

In addition to the previous results, we can show that the limit of the numerical solution is the same as for the analytical solution.

THEOREM 4.4. *Let the conditions of Theorem 4.3 be satisfied. Then*

$$\lim_{j\to\infty} y_j = \frac{1}{1-\rho},$$

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with ρ given by (4.2).

Proof. If $l' = \liminf_{j\to\infty} y_j$ and $l'' = \limsup_{j\to\infty} y_j$, then there exist two subsequences $\{y_{j_u}\}_{j_u}$, $\{y_{j_v}\}_{j_v}$ such that $l' = \lim_{u\to\infty} y_{j_u} \le \lim_{v\to\infty} y_{j_v} = l''$, with

$$y_{j_u} = 1 + h\sum_{r=1}^{n_{21}}\sum_{k=1}^{m} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l=s_-}^{s_+} P_l(\xi_k) y_{j_u - q_r + l}, \tag{4.15}$$

$$y_{j_v} = 1 + h\sum_{r=1}^{n_{21}}\sum_{k=1}^{m} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l=s_-}^{s_+} P_l(\xi_k) y_{j_v - q_r + l}. \tag{4.16}$$

a) If we assume that $\lambda + \mu\tau_1 > 0$ and $\lambda + \mu\tau_2 > 0$, then we can split the sum over l in (4.15) into two parts according to the sign of the Lagrange fundamental polynomials,

$$\begin{aligned} y_{j_u} &= 1 + h\sum_{r=1}^{n_{21}}\sum_{k=1}^{m} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l\in L_1} P_l(\xi_k) y_{j_u - q_r + l} \\ &\quad + h\sum_{r=1}^{n_{21}}\sum_{k=1}^{m} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l\in L_2} P_l(\xi_k) y_{j_u - q_r + l}. \end{aligned} \tag{4.17}$$

By computing the limit inferior of both sides of (4.17) and by recalling that $l' = \lim_{u\to\infty} y_{j_u} = \liminf_{u\to\infty} y_{j_u}$, we get

$$l' \ge 1 + \bar\alpha(h) l'' + \bar\beta(h) l', \tag{4.18}$$

where

$$\bar\alpha(h) = h\sum_{r=1}^{n_{21}}\sum_{k=1}^{m} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l\in L_1} P_l(\xi_k),$$

$$\bar\beta(h) = h\sum_{r=1}^{n_{21}}\sum_{k=1}^{m} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l\in L_2} P_l(\xi_k).$$

In the same way, by considering (4.16) and by computing the limit superior, we find that

$$l'' \le 1 + \bar\alpha(h) l' + \bar\beta(h) l'', \tag{4.19}$$

By substracting (4.18) to (4.19), we find that

$$l'' - l' \le (\bar\beta(h) - \bar\alpha(h))(l'' - l').$$

Since $\bar\beta(h) - \bar\alpha(h) \le (2\sigma + 1)|\rho| < 1$, we have $l'' = l'$ and, thus, y_j admits a limiting value $l'' = l' = y^*$, where

$$y^* = \frac{1}{1-\rho}.$$

As a matter of fact, when $j \to +\infty$ in the (4.4), we have

$$y^* = 1 + \rho y^*,$$

by recalling that $\sum_{l=-s_-}^{s_+} P_l(\xi_k) = 1$ and observing that $h \sum_{r=1}^{n_{21}} \sum_{k=1}^{m} w_k(\lambda + \mu h(q_r - \xi_k)) = \rho$. In a similar way, we are able to prove the theorem when $\lambda + \mu\tau_1 < 0$ and $\lambda + \mu\tau_2 < 0$.

b) The equation (4.4) can be written as

$$\begin{aligned} y_j = 1 &+ h \sum_{r,k \in R_1} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l \in L_1} P_l(\xi_k) y_{j-q_r+l} \\ &+ h \sum_{r,k \in R_1} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l \in L_2} P_l(\xi_k) y_{j-q_r+l} \\ &+ h \sum_{r,k \in R_2} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l \in L_1} P_l(\xi_k) y_{j-q_r+l} \\ &+ h \sum_{r,k \in R_2} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l \in L_2} P_l(\xi_k) y_{j-q_r+l}. \end{aligned}$$

Therefore, by computing the limit inferior, we find that

$$l' \geq 1 + \bar{\bar{\alpha}}(h) l'' + \bar{\bar{\beta}}(h) l',$$

where

$$\begin{aligned} \bar{\bar{\alpha}}(h) &= h \sum_{r,k \in R_1} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l \in L_2} P_l(\xi_k) \\ &\quad + h \sum_{r,k \in R_2} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l \in L_1} P_l(\xi_k), \\ \bar{\bar{\beta}}(h) &= h \sum_{r,k \in R_1} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l \in L_1} P_l(\xi_k) \\ &\quad + h \sum_{r,k \in R_2} w_k(\lambda + \mu h(q_r - \xi_k)) \sum_{l \in L_2} P_l(\xi_k), \end{aligned} \tag{4.20}$$

By following the same procedure for y_{j_v}, we obtain

$$l'' \leq 1 + \bar{\bar{\alpha}}(h) l'' + \bar{\bar{\beta}}(h) l',$$

and

$$l'' - l' \leq (\bar{\bar{\beta}}(h) - \bar{\bar{\alpha}}(h))(l'' - l') = (|\bar{\bar{\beta}}(h)| + |\bar{\bar{\alpha}}(h)|)(l'' - l').$$

If $|\bar{\bar{\beta}}(h)| + |\bar{\bar{\alpha}}(h)| \leq (2\sigma + 1)(|\beta(h)| + |\alpha(h)|) < 1$, then $l'' = l' = l$ and

$$\lim_{j \to \infty} y_j = l = y^*. \qquad \square$$

By means of Theorems 4.3 and 4.4, we have shown the following theorem.

THEOREM 4.5. *Assume that one of the following set of conditions holds:*

a) $(\lambda + \mu\tau_1)(\lambda + \mu\tau_2) \geq 0$, $(2\sigma + 1)|\rho| < 1$,
b) $(\lambda + \mu\tau_1)(\lambda + \mu\tau_2) \leq 0$, $(2\sigma + 1)(|\alpha(h)| + |\beta(h)|) < 1$.

Then the method (2.7) is stable with respect to the test equation (4.1).

Here we have used Definition 4.1 of numerical stability.

In order to establish a connection between the behavior of the numerical solution produced by our method and the behavior of the analytical solution, provided that the stepsize h is sufficiently small, we need this final result.

THEOREM 4.6. *Let $\alpha(h)$ and $\beta(h)$ be defined by (4.7. Then*

$$\lim_{h\to 0}\alpha(h)=\alpha \ \textit{ and } \lim_{h\to 0}\beta(h)=\beta,$$

where α and β are defined by (4.3).

Proof. Let us suppose that $\mu>0$. Then $\lambda+\mu x<0$ for $x<-\lambda/\mu$. The opposite case can be treated in the same way. Let $q_{\bar{r}+1}h$ be the rightmost point, such that $\lambda+\mu q_r\le 0$. Hence $R_1=\{(r,k): r>\bar{r},\ k=1,...,m\}\cup\left\{(\bar{r},k): k=\bar{k},...,m\right\}$ and $R_2=\left\{(\bar{r},k): k=1,...,\bar{k}-1\right\}\cup\{(r,k): r=1,...,\bar{r}-1 \text{ e } k=1,...,m\}$ for a suitable $\bar{k}$. It follows that

$$\begin{aligned}\alpha(h)&=h\sum_{r=\bar{r}+1}^{n_{21}}\sum_{k=1}^{m}w_k(\lambda+\mu h(q_r-\xi_k))+h\sum_{k=\bar{k}}^{m}w_k(\lambda+\mu h(q_{\bar{r}}-\xi_k))\\&=\int_{\tau_1}^{q_{\bar{r}+1}h}(\lambda+\mu x)dx+h\sum_{k=\bar{k}}^{m}w_k(\lambda+\mu h(q_{\bar{r}}-\xi_k))\\&=\frac{1}{2\mu}\left[(\lambda+\mu q_{\bar{r}+1}h)^2-(\lambda+\mu\tau_1)^2\right]+h\sum_{k=\bar{k}}^{m}w_k(\lambda+\mu h(q_{\bar{r}}-\xi_k))\\&=\alpha+p(h),\end{aligned}$$

with

$$p(h)=\frac{1}{2\mu}(\lambda+\mu q_{\bar{r}+1}h)^2+h\sum_{k=\bar{k}}^{m}w_k(\lambda+\mu h(q_{\bar{r}}-\xi_k)),$$

and

$$\begin{aligned}\beta(h)&=h\sum_{k=1}^{\bar{k}-1}w_k(\lambda+\mu h(q_{\bar{r}}-\xi_k))+h\sum_{r=1}^{\bar{r}-1}\sum_{k=1}^{m}w_k(\lambda+\mu h(q_r-\xi_k))\\&=h\sum_{k=1}^{\bar{k}-1}w_k(\lambda+\mu h(q_{\bar{r}}-\xi_k))+\int_{q_{\bar{r}}h}^{\tau_2}(\lambda+\mu x)dx\\&=h\sum_{k=1}^{\bar{k}-1}w_k(\lambda+\mu h(q_{\bar{r}}-\xi_k))+\frac{1}{2\mu}\left[(\lambda+\mu\tau_2)^2-(\lambda+\mu q_{\bar{r}}h)^2\right]\\&=\beta-q(h),\end{aligned}$$

with

$$q(h)=\frac{1}{2\mu}(\lambda+\mu q_{\bar{r}}h)^2-h\sum_{k=1}^{\bar{k}-1}w_k(\lambda+\mu h(q_{\bar{r}}-\xi_k)).$$

According to the definition of $\bar{r}$, it follows that $q_{\bar{r}+1}h\to-\lambda/\mu$ and $(\bar{k}-1)\to m$ as $h\to 0$. As a consequence

$$\lim_{h\to 0}p(h)=\frac{1}{2\mu}(\lambda+\mu\left(-\lambda/\mu\right))^2=0$$

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and

$$\lim_{h\to 0} q(h) = \lim_{h\to 0} \frac{1}{2\mu}(\lambda+\mu q_{\bar r} h)^2 - h\sum_{k=1}^{m} w_k(\lambda+\mu h(q_{\bar r}-\xi_k))$$
$$= \lim_{h\to 0} \frac{1}{2\mu}(\lambda+\mu q_{\bar r} h)^2 - \int_{q_{\bar r+1}h}^{q_{\bar r}h}(\lambda+\mu x)dx = \lim_{h\to 0}\frac{1}{2\mu}(\lambda+\mu q_{\bar r+1}h)^2 = 0. \qquad \square$$

REMARK 4.7. Theorem 4.6 shows that, when the stepsize h is sufficiently small, both sufficient conditions of Theorem 4.5 differ from the hypotheses of Theorem 4.2 by the presence of the factor $(2\sigma+1)$, if $\sigma \neq 0$. However, it is always possible to determine the parameters of the method in such a way that the numerical solution mimics the behavior of the analytical solution.

In the following section, we will illustrate how to choose the best parameters of the method.

5. Numerical experiments. Theorems 3.1 and 4.5 establish that the parameter s influences both convergence and the stability of the method (2.7). For a concrete application of the method, one has to consider the following points:

- s should be chosen as $s = 2m-1$ in order to preserve the order of the Gaussian quadrature formula without loss of efficiency;
- For any stable equation of type (1.1) there exists a stable method of order $p \geq 2$;
- The parameter σ, which determines the stability of the method, is equal to zero for $s = 1$ and $s_- = 0$, and increases with s, thus limiting the order of the method. According to Theorem 4.5, as ρ or $\beta-\alpha$ approach one (that is the analytical solution is almost unstable), the maximum order of convergence we may expect is lower. In Table 5.1 we list some practical choices of s and of s_- with respect to the value of ρ (the same holds for $\beta-\alpha$, if h is sufficiently small) and the corresponding order p, which guarantee stability of the method. These values are not the maximal ones, since they come from the sufficient but not necessary conditions for stability of Theorem 4.5.

Now we illustrate the performance of our methods for equations of the type

$$y(t) = f(t) + \int_{t-\tau_2}^{t-\tau_1} (\lambda+\mu(t-\tau))g(y(\tau))d\tau, \quad t\in[\tau_2, T], \tag{5.1}$$

with $\tau_1 = 0.5,\ \tau_2 = 1.0,\ T = 5.0,\ \lambda = 1.0,\ \mu = 1.2.$

We have chosen the following methods of type (2.7):

1. order 2 method, with $s_- = 0,\ s_+ = 1,\ m = 1$ (3.4)–(3.5),

TABLE 5.1
Practical choices of the parameters of the method.

ρ	s	s_-	p
$[0.110, 0.235]$	9	1	10
$[0.235, 0.417]$	7	1	8
$[0.417, 0.615]$	5	1	6
$[0.615, 0.857]$	3	1	4
$[0.857, 1]$	1	0	2

TABLE 5.2
Correct digits for problem (5.1)–(5.2).

h	Gauss $p=2$	Gauss $p=4$	Newton-Cotes $p=2$	Newton-Cotes $p=4$
0.1	3.04	5.62	2.87	5.29
0.05	3.64	6.80	3.46	6.49
0.025	4.24	8.00	4.06	7.69
0.0125	4.84	9.19	4.65	9.03
0.00625	5.45	10.40	5.25	10.29
0.003125	6.05	11.60	5.86	11.47

TABLE 5.3
Correct digits for problem (5.1)–(5.3).

h	$p=2$	$p=4$
0.1	1.82	4.12
0.05	2.16	5.32
0.025	2.70	6.52
0.0125	3.29	7.72
0.00625	3.89	8.93
0.003125	4.49	10.13

2. order 4 method, with $s_-=0$, $s_+=3$, $m=2$,
3. order 6 method, with $s_-=1$, $s_+=4$, $m=3$,
4. order 8 methods, with $s_-=0$, $s_+=7$, or $s_-=1$, $s_+=6$, and $m=4$.

All four methods are explicit. Analogous tests carried out with implicit methods have produced similar results.

We have tested the convergence properties of our methods on the folloowing two problems [7]:

- problem (5.1) with

$$g(y)=1, \quad f(t) \text{ s.t. } y(t)=e^{-t}, \phi(t)=e^{-t}; \tag{5.2}$$

- problem (5.1) with

$$g(y)=(1+y)^2, \quad f(t) \text{ s.t. } y(t)=\sin t, \phi(t)=\sin t; \tag{5.3}$$

In Tables 5.2–5.3 the number of correct digits values (cd) of the solution of (5.1) obtained by the methods 1 and 2, are listed for different values of h. These tables clearly show that our methods produce the desired order according to Theorem 3.1. Moreover, in Table 5.2, we have compared our results with those obtained by the only other numerical approach we know, that is, DQ methods based on Newton-Cotes formulas (in particular the trapezoidal and Simpson 3/8 formulas) [7]: the errors of the two families of methods are very similar, in spite of the error of approximation due to the interpolation technique used for the Gaussian formulas.

In order to test the sharpness of the estimates of the stability parameters of our methods, we have carried out a large number of numerical experiments. Here the most significant ones

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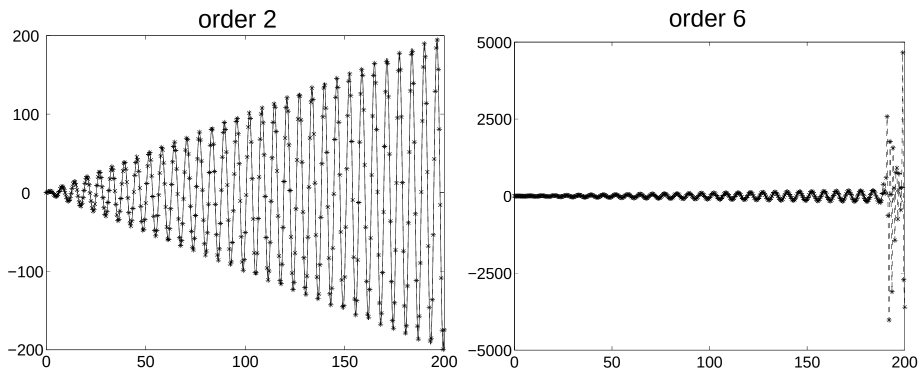

FIG. 5.1. *On the left:* $\lambda = -1$, $\mu = -1.2$, $|\rho| = 0.95$, $(2\sigma+1)|\rho| = 0.95$, $p = 2$, $s_- = 0$, $h = 0.1$. *On the right:* $\lambda = -1$, $\mu = -1.2$, $|\rho| = 0.95$, $(2\sigma+1)|\rho| = 1.54$, $p = 6$, $s_- = 1$, $h = 0.1$.

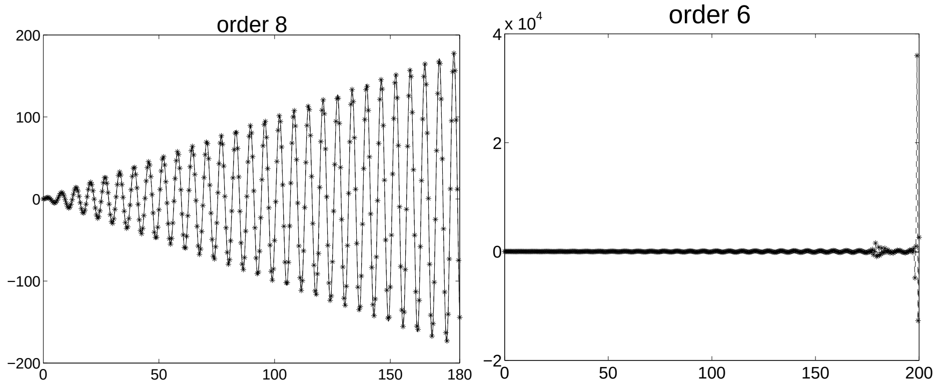

FIG. 5.2. *On the left:* $\lambda = -1$, $\mu = 1$, $|\rho| = 0.25$, $(2\sigma+1)|\rho| = 0.6$, $p = 8$, $s_- = 1$, $h = 0.1$. *On the right:* $\lambda = -1$, $\mu = -1.2$, $|\rho| = 0.95$, $(2\sigma+1)|\rho| = 2.84$, $p = 6$, $s_- = 0$, $h = 0.1$.

are reported. We consider the performances of the method (2.7) when applied to the test problem

$$y(t) = f(t) + \int_{t-\tau_2}^{t-\tau_1} (\lambda + \mu(t-\tau))y(\tau)d\tau, \quad t \in [\tau_2, T], \tag{5.4}$$

with $\tau_1 = 0.5$, $\tau_2 = 1.0$ and f such that $y(t) = t\sin t$. In our examples λ and μ satisfy the hypotheses of Theorem 4.2 for the stability of the analytical solution. Problems of type (5.4) may be assimilated to the test equation (4.1) with the simplified assumption $f(t) \approx f(0)$, as it is usually done in the formulation of test equations; see [6] and references therein.

The numerical solution (*-) and the analytical one (-) are compared in Figures 5.1–5.3, for different values of the parameters λ and μ and for different choices of the parameters of the method. Figures 5.1–5.2 show tests related to the case $(\lambda+\mu\tau_1)(\lambda+\mu\tau_2) > 0$. When the hypothesis $a)$ of Theorem 4.5 is satisfied, the numerical solution behaves like the analytical one (left plots of Figures 5.1 and 5.2), while in the other case instability could arise, both for $s_- = 0$ and for $s_- = 1$ (right plots of Figures 5.1-5.2). Similar results are found when $(\lambda+\mu\tau_1)(\lambda+\mu\tau_2) < 0$. This is illustrated in Figure 5.3. We emphasize that we dispose of high order and stable methods, as shown by test of Figure 5.2, where an order 8 method is

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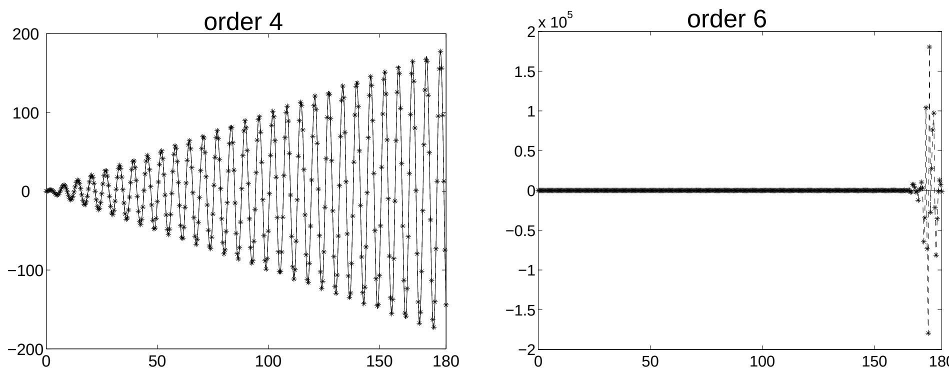

FIG. 5.3. $\lambda = 8$, $\mu = -10$, $\beta - \alpha = 0.65$, $(2\sigma + 1)(\beta(h) - \alpha(h)) = 0.95$, $p = 4$, $s_- = 0$, $h = 0.1$. *On the right:* $\lambda = 8$, $\mu = -10$, $\beta - \alpha = 0.65$, $(2\sigma + 1)(\beta(h) - \alpha(h)) = 1.06$, $p = 6$, $s_- = 1$, $h = 0.1$.

used.

Our tests confirm what we have noticed in Remark 4.7 about the influence of the factor $2\sigma + 1$ on the numerical stability: for example, in the right plot of Figure 5.1, $|\rho| < 1$ but $\sigma = 0.083$, so that $(2\sigma + 1)|\rho| > 1$ and instability occurs. We observe that the estimate on the stability parameter $(2\sigma + 1)|\rho|$ or $(2\sigma + 1)(\beta(h) - \alpha(h))$ is quite sharp, as shown for example by Figure 5.3, where $(2\sigma + 1)(\beta(h) - \alpha(h))$ varies between the values 0.95 and 1.06.

In the example illustrated by the right plot of Figure 5.2, ρ is very close to 1 and so, in principle, we expect an order of accuracy not exceeding 2 for a stable method; see Table 5.1. Therefore, in order to increase the order, we have applied a cubic spline interpolation technique (which is supposed to have order 4). This method proved to be stable; on the other hand also our DQ method of order 4 with the Lagrange interpolating polynomial is stable and guarantees the same accuracy.

6. Concluding remarks. In this paper we constructed Direct Quadrature methods based on Gaussian quadrature formulas for equation (1.1). In order to solve the problem of the evaluation of the solution at points not belonging to the mesh, an interpolation technique has been used. In Theorem 3.1 we have shown that the order of convergence of the method depends both on the order of convergence of the Gaussian formula and on the degree of accuracy of the interpolating polynomial.

In order to complete the study of the method proposed, we analyzed the stability with respect to a class of significant test equations introduced in [7]. We found sufficient conditions for numerical stability. These conditions are such that, if the starting problem satisfies the conditions of Theorem 4.2, then it is easy to determine the parameters of the method that secure the stability of the numerical solution. The numerical experiments clearly confirm the theoretical results and show the sharpness of the estimates of the stability parameters. Finally we note that, in order to increase the order of convergence when ρ is close to one, it may be possible to consider another interpolation technique such as one using cubic splines. This new approach, together with other approximation techniques, requires complete analysis of convergence and numerical stability. We intend to dedicate a future work to this topic.

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