Electronic Transactions on Numerical Analysis.
Volume 34, pp. 20-30, 2009.

ISSN 1068-9613.

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UNIQUE SOLVABILITY IN BIVARIATE HERMITE INTERPOLATION*

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Dedicated to Víctor Pereyra on the occasion of his 70th birthday

Abstract. We consider the question of unique solvability in the context of bivariate Hermite interpolation. Starting from arbitrary nodes, we prescribe arbitrary conditions of Hermite type, and find an appropriate interpolation space in which the problem has a unique solution. We show that the coefficient matrix of the associated linear system is a nonsingular submatrix of a generalized Kronecker product of nonsingular matrices corresponding to univariate Hermite interpolation problems. We also consider the case of generalized polynomials, such as Cauchy-Vandermonde systems.

Key words. Hermite interpolation, bivariate interpolation, generalized Kronecker product.

AMS subject classifications. 41A05, 41A63, 65D05

1. Introduction. Bivariate Hermite interpolation has recently been the subject of two related papers by B. Shekhtman [15, 16]. Although the purpose of those papers is to look for spaces for which a certain Hermite interpolation problem is solvable *for any configuration of interpolation points*, the author starts from his assumption of *the lack of unicity for Hermite interpolation in the multivariate case*. This assumption was a main item in one of Shekhtman's sources, a survey paper by R. A. Lorentz [11] that emphasized the existence of *singular* multivariate interpolation problems.

Our aim in this paper is to show that, just as in the univariate case, a bivariate Hermite interpolation problem has a unique solution *if one chooses the appropriate interpolation space* corresponding to the given arbitrary interpolation conditions of Hermite type.

This positive result, in contrast with the results presented in [15, 16], is important if one considers the *applicability* of interpolation, for example in the finite element method [2]. Another relevant application of bivariate interpolation has recently been recalled in [12].

A key idea in our work is a generalization of the use of submatrices of a Kronecker product of matrices. F. Stenger presented this idea in [17], and its importance was recognized very early by G. Galimberti and V. Pereyra [3], who extended some of Stenger's results to the trivariate case. V. Pereyra and G. Scherer gave an early computational treatment of the Kronecker product [14] in which they emphasized the application to multivariate interpolation problems.

More recently, Hack [8], in the context of bivariate Birkhoff interpolation, obtained sufficient conditions for a bivariate problem to be uniquely solvable. The work of Hack improves, also by using tensor-product methods, previous results obtained in [9, 10].

In his work, Hack does not consider the structure of the coefficient matrix of the linear system associated with the interpolation problem. Taking the matrix structure into account, and extending the work of Stenger [17], Gasca and Martínez observe in [5] that the sufficient conditions given by Hack are also necessary for the unique solvability of the Birkhoff interpolation problem.

*Received March 27, 2008. Accepted October 3, 2008. Published online on January 7, 2009. Recommended by José Castillo. This work was supported by Research Grant MTM 2006-03388 from the Spanish Ministerio de Educación y Ciencia

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In addition, the matrix formulation allows a natural generalization to other spaces of univariate functions, such as rational functions with prescribed poles, a situation not addressed by Hack.

The rest of the paper is organized as follows. In Section 2, we describe the polynomial bivariate Hermite interpolation problem. We prove the existence of a unique solution for each given problem, and give an algorithm for computing it, in Section 3. We give examples in Section 4. In Section 5, we extend the approach of Section 2 to the case of generalized polynomials. Finally, in Section 6, we summarize some important features of our approach.

2. The bivariate Hermite interpolation problem. Following [5], let us consider the set of bivariate functions

$$\{F_{ij}(x,y) = \Phi_i(x)\Psi_j(y) \mid i = 0,\ldots,n; j = 0,\ldots,m\},$$

where $\Phi_i(x) = x^i$ and $\Psi_j(y) = y^j$, and the set of interpolation data of Hermite type given by

$$L_{ij}(f) = \frac{\partial^{r_i+t_{ij}} f}{\partial x^{r_i}\partial y^{t_{ij}}}(x_i, y_{ij}), \qquad i = 0,\ldots,n; j = 0,\ldots,m, \tag{2.1}$$

where $x_i \in G$ (respectively $y_{ij} \in H$) are not necessarily different, $r_i \le n$, $t_{ij} \le m$, and G and H are real intervals. *Hermite interpolation data* means that if

$$\frac{\partial^{r_i+t_{ij}} f}{\partial x^{r_i}\partial y^{t_{ij}}}(x_i, y_{ij})$$

is an interpolation datum, then

$$\frac{\partial^{r+t} f}{\partial x^{r}\partial y^{t}}(x_i, y_{ij}),$$

where $r \le r_i$ and $t \le t_{ij}$, are also interpolation data.

We assume, without loss of generality, that the index set $I = \{(i,j)\}$ in the equation (2.1) is ordered so that

1. $$I = \{(i,j) \mid i = 0,\ldots,n; j = 0,\ldots,k(i)\},$$

   with

   $$\begin{aligned} m = k(0) = k(1) = \cdots = k(i_0) > k(i_0+1) \\ = \cdots = k(i_1) > \cdots > k(i_{s-1}+1) = \cdots = k(i_s) \end{aligned} \tag{2.2}$$

   and

   $$i_s = n. \tag{2.3}$$

2. For any $(i,j),\ (h,l) \in I$, one has

   $$i = h \iff x_i = x_h \text{ and } r_i = r_h. \tag{2.4}$$

We can always achieve this order by reordering the abscissas of the points (x_i, y_{ij}).

The *bivariate Hermite interpolation problem* consists of finding a polynomial p in the interpolation space $\Pi(x,y) = \operatorname{span}\{F_{ij} \mid (i,j) \in I\}$ such that

$$L_{ij}(p) = z_{ij} \qquad \forall (i,j) \in I,$$

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where z_{ij} are given real numbers.

It is important to realize that

$$L_{ij}(F_{hk}) = L_i(\Phi_h) \cdot L_j^{(i)}(\Psi_k),$$

where

$$L_i(\Phi_h) = \frac{d^{r_i}\Phi_h}{dx^{r_i}}(x_i), \quad L_j^{(i)}(\Psi_k) = \frac{d^{t_{ij}}\Psi_k}{dy^{t_{ij}}}(y_{ij}).$$

3. Unique solvability. For the interpolation space $\Pi(x,y)$, if we consider the basis

$$\{x^i y^j \mid (i,j) \in I\} = \{1, y, \ldots, y^{k(0)}, x, xy, \ldots, xy^{k(1)}, \ldots, x^{i_0}, x^{i_0}y, \ldots, x^{i_0}y^{k(i_0)}, \\ x^{i_0+1}, x^{i_0+1}y, \ldots, x^{i_0+1}y^{k(i_0+1)}, \ldots, x^{i_s}, x^{i_s}y, \ldots, x^{i_s}y^{k(i_s)}\},$$

with that precise order, and the interpolation points and the interpolation data in the corresponding order, then we can write the interpolation conditions $L_{ij}(p) = z_{ij}$ as a linear system of equations

$$Dp^* = z,$$

where the coefficient matrix D is a submatrix of

$$C = \begin{bmatrix} L_0(\Phi_0)B_0 & L_0(\Phi_1)B_0 & \cdots & L_0(\Phi_n)B_0 \\ L_1(\Phi_0)B_1 & L_1(\Phi_1)B_1 & \cdots & L_1(\Phi_n)B_1 \\ \vdots & \vdots & \ddots & \vdots \\ L_n(\Phi_0)B_n & L_n(\Phi_1)B_n & \cdots & L_n(\Phi_n)B_n \end{bmatrix}, \tag{3.1}$$

with

$$B_i = \left[b_{kl}^i\right]_{k,l=0,\ldots,m} = \left[L_k^{(i)}(\Psi_l)\right]_{k,l=0,\ldots,m}, \quad i = 0, \ldots, n, \tag{3.2}$$

$$z = [z_{00}, \ldots, z_{0k(0)}, z_{10}, \ldots, z_{1k(1)}, \ldots, z_{n0}, \ldots, z_{nk(n)}]^T, \text{ and} \tag{3.3}$$

$$p^* = [p_{00}, \ldots, p_{0k(0)}, p_{10}, \ldots, p_{1k(1)}, \ldots, p_{n0}, \ldots, p_{nk(n)}]^T. \tag{3.4}$$

More precisely, D is the submatrix of C obtained by considering:

- The first $k(i)+1$ rows of the row of blocks corresponding to B_i, $i = 0, \ldots, n$.
- The first $k(i)+1$ columns of the column of blocks corresponding to Φ_i, $i = 0, \ldots, n$.

The nonsingularity of D follows from the following theorem [5], which generalizes Stenger's result on the nonsingularity of a submatrix of the Kronecker product of two matrices [17]. Before stating the theorem, we introduce some notation. We define

$$\gamma_i = \{0, 1, \ldots, k(i)\} \qquad i = 0, \ldots, n, \tag{3.5}$$

$$S_i = \{(m+1)i, (m+1)i+1, \ldots, (m+1)i+k(i)\} \qquad i = 0, \ldots, n, \tag{3.6}$$

$$S = \bigcup_{i=0}^{n} S_i, \text{ and} \tag{3.7}$$

$$\alpha_p = \{0, 1, \ldots, i_p\} \qquad i = 0, \ldots, s. \tag{3.8}$$

With this notation, D is the submatrix of C consisting of the rows S and the columns S of C, which we write as

$$D = C[S|S]. \tag{3.9}$$

In general, if J and T are ordered subsets of the index sets of rows and columns, respectively, of a matrix M, we denote by $M[J|T]$ the matrix consisting of the entries of the rows J and columns T of M.

THEOREM 3.1. *Let $A = [a_{ij}]_{i,j=0,\ldots,n}$ be a matrix of order $n+1$, $B_i = [b^i_{kl}]_{k,l=0,\ldots,m}$, $i = 0,1,\ldots,n$, be $n+1$ matrices of order $m+1$, and*

$$C = \begin{bmatrix} a_{00}B_0 & a_{01}B_0 & \cdots & a_{0n}B_0 \\ a_{10}B_1 & a_{11}B_1 & \cdots & a_{1n}B_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0}B_n & a_{n1}B_n & \cdots & a_{nn}B_n \end{bmatrix}.$$

If (2.2) and (2.3) hold, and $A[\alpha_p|\alpha_p]$, $p = 0,\ldots,s$, and $B_i[\gamma_i|\gamma_i]$, $i = 0,\ldots,n$, are nonsingular, then $C[S|S]$ is nonsingular.

We now describe an algorithm for solving a linear system of the type $Dx = z$ with $D = C[S|S]$. It will be very useful in the proof of Theorem 3.1. This algorithm, which is an extension to our problem of the algorithm given by Stenger in [17], is due to Gasca and Martínez (see [5] and references therein).

The linear system $Dx = z$, where

$$x = [x_{00},\ldots,x_{0k(0)},x_{10},\ldots,x_{1k(1)},\ldots,x_{n0},\ldots,x_{nk(n)}]^T$$

and

$$z = [z_{00},\ldots,z_{0k(0)},z_{10},\ldots,z_{1k(1)},\ldots,z_{n0},\ldots,z_{nk(n)}]^T,$$

can be written in explicit form as

$$\sum_{r=0}^{s}\sum_{l=k(i_{r+1})+1}^{k(i_r)} b^i_{kl}\sum_{j=0}^{i_r} a_{ij}x_{jl} = z_{ik}, \tag{3.10}$$

where $i = 0,\ldots,n$ and $k = 0,\ldots,k(i)$. Setting

$$c_{il} = \sum_{j=0}^{i_r} a_{ij}x_{jl} \qquad l = k(i_{r+1})+1,\ldots,k(i_r);\ r = 0,\ldots,s;\ i = 0,\ldots,n,$$

we write (3.10) in the form

$$\sum_{r=0}^{s}\sum_{l=k(i_{r+1})+1}^{k(i_r)} b^i_{kl}c_{il} = z_{ik},$$

where $i = 0,\ldots,n$ and $k = 0,\ldots,k(i)$.

The expressions above lead to Algorithm 3.2 for solving the system $Dx = z$, which we use to prove Theorem 3.1:

Proof. If the conditions of Theorem 3.1 are satisfied, every linear system solved in Step 2 and Step 3 of Algorithm 3.2 has a unique solution. Therefore, the algorithm gives a solution of the linear system $Dx = z$ for all vectors z, and consequently the square matrix D is nonsingular. □

Taking into account that $\Phi_i(x) = x^i$, $i = 0,\ldots,n$, and $\Psi_j(y) = y^j$, $j = 0,\ldots,m$, it is easy to see that in our case the matrix

$$A = \left[L_i(\Phi_j)\right]_{i,j=0,\ldots,n}$$

ALGORITHM 3.2. Solve $Dx = z$

Define $i_{(-1)} = -1$ and $k(i_{s+1}) = -1$.
for $r = 0, \ldots, s$
Step 1. for $i = i_{r-1} + 1, \ldots, i_r$
Define $z_i = [z_{i0}, \ldots, z_{i,k(i)}]^T$.
if $r > 0$
Define $c_i^{(1)} = [c_{i,k(i_r)+1}, \ldots, c_{i,k(0)}]^T$.
$\widetilde{z}_i = z_i - B_i[\gamma_i \mid k(i_r) + 1, \ldots, k(i_0)]\; c_i^{(1)}$.
else
$\widetilde{z}_i = z_i$.
Step 2. for $i = i_{r-1} + 1, \ldots, i_r$,
Solve $B_i[\gamma_i|\gamma_i]\; c_i^{(2)} = \widetilde{z}_i$, where $c_i^{(2)} = [c_{i0}, \ldots, c_{i,k(i)}]^T$.
Step 3. for $l = k(i_{r+1}) + 1, \ldots, k(i_r)$
Define $c^l = [c_{0l}, \ldots, c_{i_r,l}]^T$.
Solve $A[0, \ldots, i_r \mid 0, \ldots, i_r]\; x^l = c^l$, where $x^l = [x_{0l}, \ldots, x_{i_r,l}]^T$.
Step 4. if $r < s$
for $l = k(i_{r+1}) + 1, \ldots, k(i_r)$
$d^l = [c_{i_r+1,l}, \ldots, c_{nl}]^T = A[i_r + 1, \ldots, n \mid 0, \ldots, i_r]\; x^l$.
end for.

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is a confluent Vandermonde matrix of order $n + 1$, and the matrices

$$B_i = \left[L_k^{(i)}(\Psi_l)\right]_{k,l=0,\ldots,m}, \qquad i = 0, \cdots, n$$

are confluent Vandermonde matrices of order $m+1$. Consequently, all the matrices $A[\alpha_p|\alpha_p]$, $p = 0, \ldots, s$, and $B_i[\gamma_i|\gamma_i]$, $i = 0, \cdots, n$, are also confluent Vandermonde matrices corresponding to univariate Hermite interpolation problems, and therefore they are nonsingular. In this way, by Theorem 3.1, we find that the coefficient matrix D corresponding to the bivariate Hermite interpolation problem is nonsingular; that is, the bivariate Hermite interpolation problem has unique solution. Fast algorithms for solving linear systems whose coefficient matrices are confluent Vandermonde matrices can be found, for example, in [4].

4. Examples. We start this section with a detailed example that illustrates the algorithm described in Section 3.

EXAMPLE 4.1. Let us consider the following interpolation data:

$$f(1,2) = 46, \qquad \frac{\partial f}{\partial x}(1,2) = 70, \qquad \frac{\partial f}{\partial y}(1,2) = 10,$$
$$f(3,4) = 490, \qquad \frac{\partial f}{\partial x}(3,4) = 384, \qquad \frac{\partial f}{\partial y}(3,4) = 24.$$

Our aim is to find the interpolation polynomial corresponding to these data by using the algorithm described in the previous section.

In this case, we have:

$$x_0 = x_2 = 1,\; x_1 = x_3 = 3,\; y_{00} = y_{01} = 2,\; y_{20} = 2,\; y_{10} = y_{11} = 4,\; y_{30} = 4,$$
$$i_0 = 1,\;\; i_1 = 3,\;\; k(0) = k(1) = 1,\;\; k(2) = k(3) = 0,\;\; s = 1,\;\; n = 3,\;\; m = 1.$$

The corresponding interpolation space $\Pi(x, y)$ in which our problem has the unique solution

$$p(x, y) = 1 + 3y + 5x + 7xy + 9x^2 + 11x^3$$

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has the following basis:

$$\{1, y, x, xy, x^2, x^3\}.$$

The matrix corresponding to univariate Hermite interpolation in x with interpolation data $f(1)$, $f(3)$, $f'(1)$, $f'(3)$ is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 6 & 27 \end{bmatrix},$$

the matrix for univariate Hermite interpolation in y with interpolation data $f(2)$, $f'(2)$ is

$$B_0 = B_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

and the matrix for univariate Hermite interpolation in y with interpolation data $f(4)$, $f'(4)$ is

$$B_1 = B_3 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}.$$

Let us show the steps of the algorithm in detail (from $r = 0$ to $r = s = 1$), taking into account that for $r = 0$, step 1 is not necessary, and that for $r = 1$, step 4 is not necessary.

$r = 0$: **Step 2.** We have

$$\tilde{z}_0 = [46, 10]^T \quad \text{and} \quad \tilde{z}_1 = [490, 24]^T.$$

Solving the linear systems

$$B_0[0, 1|0, 1]\, c_0^{(2)} = \tilde{z}_0 \quad \text{and} \quad B_1[0, 1|0, 1]\, c_1^{(2)} = \tilde{z}_1,$$

we obtain

$$c_0^{(2)} = [c_{00}, c_{01}]^T = [26, 10]^T \quad \text{and} \quad c_1^{(2)} = [c_{10}, c_{11}]^T = [394, 24]^T.$$

Step 3. For $l = 1$, we have $c^1 = [c_{01}, c_{11}]^T = [10, 24]^T$. Solving $Ax^1 = c^1$, we obtain $x^1 = [p_{01}, p_{11}]^T = [3, 7]^T$.

We observe that in writing $p = Q_0(x)1 + Q_1(x)y$, where

$$Q_0(x) = p_{00} + p_{10}x + p_{20}x^2 + p_{30}x^3 \quad \text{and} \quad Q_1(x) = p_{01} + p_{11}x,$$

we have already obtained $Q_1(x) = 3 + 7x$.

Step 4. Now, for $l = 1$, we obtain

$$d^1 = [c_{21}, c_{31}] = A[2, 3|0, 1]\, x^1 = [7, 7]^T,$$

i.e. $Q_1'(x_0) = 7$, $Q_1'(x_1) = 7$.

$r = 1$: **Step 1.** For $i = 2$, we have $z_2 = [z_{20}] = [70]$, $c_2^{(1)} = [c_{21}] = [7]$, and so we obtain

$$\tilde{z}_2 = z_2 - B_2[0|1]\,[7] = [56].$$

For $i = 3$, we have $z_3 = [z_{30}] = [384]$, $c_3^{(1)} = [c_{31}] = [7]$, and we obtain

$$\tilde{z}_3 = z_3 - B_3[0|1]\,[7] = [356].$$

Step 2. Since $B_2[0|0] = B_3[0|0] = [1]$, for $i = 2$, we must solve the linear system

$$[1]\ [c_{20}] = \tilde{z}_2 = [56],$$

and for $i = 3$, the linear system

$$[1]\ [c_{30}] = \tilde{z}_3 = [356],$$

and therefore we obtain

$$[c_{20}] = [56] \quad \text{and} \quad [c_{30}] = [356].$$

Step 3. Finally, we must solve the linear system

$$A\, x^0 = [c_{00}, c_{10}, c_{20}, c_{30}]^T = [26, 394, 56, 356]^T,$$

whose solution is

$$x^0 = [p_{00}, p_{10}, p_{20}, p_{30}]^T = [1, 5, 9, 11]^T.$$

All the linear systems involved correspond to univariate Hermite interpolation problems (either in x or in y), and so they have unique solutions.

The following examples are taken from [15] and [16], and all of them consider interpolation data of Hermite type at two arbitrary interpolation nodes (a_1, b_1) and (a_2, b_2) in which only the first partial derivatives of an arbitrary function $f(x, y)$ are involved. In [15] and [16], it is shown that, in some of these examples (Examples 4.2, 4.3, 4.7 and 4.8), when selecting an interpolation space for any configuration of the nodes (a_1, b_1) and (a_2, b_2), the dimension of the interpolation space is necessarily greater than the number of interpolation data, and therefore the interpolation problem does not have a unique solution. In each example, we will show that to select the interpolation space described in Section 3 for which the interpolation problem has unique solution, we have only to distinguish two different configurations of the nodes: the two nodes along the same vertical line, and the two nodes on different vertical lines. In both configurations, the choice of the interpolation basis is completely natural, and the matrix in the system $Dp^* = z$ corresponding to the interpolation problem is nonsingular, that is, the interpolation problem has a unique solution.

EXAMPLE 4.2. The Lagrange case. Let us consider the interpolation data:

$$f(a_1, b_1), \quad f(a_2, b_2).$$

- (a_1, b_1) and (a_2, b_2) are along the same vertical line $(a_1 = a_2)$.
  - The basis of the interpolation space is: $\{1, y\}$
  - $\det(D) = b_2 - b_1$
- (a_1, b_1) and (a_2, b_2) are on two different vertical lines ($a_1 \neq a_2$).
  - The basis of the interpolation space is: $\{1, x\}$
  - $\det(D) = a_2 - a_1$

EXAMPLE 4.3. Let us consider the interpolation data:

$$f(a_1, b_1), \quad \frac{\partial f}{\partial y}(a_1, b_1), \quad f(a_2, b_2).$$

- (a_1, b_1) and (a_2, b_2) are along the same vertical line $(a_1 = a_2)$.
  - The basis of the interpolation space is: $\{1, y, y^2\}$
  - $\det(D) = (b_1 - b_2)^2$

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- (a_1, b_1) and (a_2, b_2) are on two different vertical lines ($a_1 \neq a_2$).
  - The basis of the interpolation space is: $\{1, y, x\}$
  - $\det(D) = a_2 - a_1$

EXAMPLE 4.4. Let us consider the interpolation data:

$$f(a_1, b_1), \;\; \frac{\partial f}{\partial x}(a_1, b_1), \;\; \frac{\partial f}{\partial y}(a_1, b_1), \;\; f(a_2, b_2).$$

- (a_1, b_1) and (a_2, b_2) are along the same vertical line $(a_1 = a_2)$.
  - The basis of the interpolation space is: $\{1, y, y^2, x\}$
  - $\det(D) = (b_2 - b_1)^2$
- (a_1, b_1) and (a_2, b_2) are on two different vertical lines ($a_1 \neq a_2$).
  - The basis of the interpolation space is: $\{1, y, x, x^2\}$
  - $\det(D) = (a_1 - a_2)^2$

EXAMPLE 4.5. Let us consider the interpolation data:

$$f(a_1, b_1), \;\; \frac{\partial f}{\partial y}(a_1, b_1), \;\; f(a_2, b_2), \;\; \frac{\partial f}{\partial x}(a_2, b_2).$$

- (a_1, b_1) and (a_2, b_2) are along the same vertical line $(a_1 = a_2)$.
  - The basis of the interpolation space is: $\{1, y, y^2, x\}$
  - $\det(D) = (b_1 - b_2)^2$
- (a_1, b_1) and (a_2, b_2) are on two different vertical lines ($a_1 \neq a_2$).
  - The basis of the interpolation space is: $\{1, y, x, x^2\}$
  - $\det(D) = (a_1 - a_2)^2$

EXAMPLE 4.6. Let us consider the interpolation data:

$$f(a_1, b_1), \;\; \frac{\partial f}{\partial y}(a_1, b_1), \;\; f(a_2, b_2), \;\; \frac{\partial f}{\partial y}(a_2, b_2).$$

- (a_1, b_1) and (a_2, b_2) are along the same vertical line $(a_1 = a_2)$.
  - The basis of the interpolation space is: $\{1, y, y^2, y^3\}$
  - $\det(D) = (b_2 - b_1)^4$
- (a_1, b_1) and (a_2, b_2) are on two different vertical lines ($a_1 \neq a_2$).
  - The basis of the interpolation space is: $\{1, y, x, xy\}$
  - $\det(D) = (a_1 - a_2)^2$

EXAMPLE 4.7. Let us consider the interpolation data:

$$f(a_1, b_1), \;\; \frac{\partial f}{\partial x}(a_1, b_1), \;\; \frac{\partial f}{\partial y}(a_1, b_1), \;\; f(a_2, b_2), \;\; \frac{\partial f}{\partial x}(a_2, b_2).$$

- (a_1, b_1) and (a_2, b_2) are along the same vertical line $(a_1 = a_2)$.
  - The basis of the interpolation space is: $\{1, y, y^2, x, xy\}$
  - $\det(D) = (b_2 - b_1)^3$
- (a_1, b_1) and (a_2, b_2) are on two different vertical lines ($a_1 \neq a_2$).
  - The basis of the interpolation space is: $\{1, y, x, x^2, x^3\}$
  - $\det(D) = (a_1 - a_2)^4$

EXAMPLE 4.8. Let us consider the interpolation data:

$$f(a_1, b_1), \;\; \frac{\partial f}{\partial x}(a_1, b_1), \;\; \frac{\partial f}{\partial y}(a_1, b_1), \;\; f(a_2, b_2), \;\; \frac{\partial f}{\partial x}(a_2, b_2), \;\; \frac{\partial f}{\partial y}(a_2, b_2).$$

- (a_1, b_1) and (a_2, b_2) are along the same vertical line $(a_1 = a_2)$.

  - The basis of the interpolation space is: $\{1, y, y^2, y^3, x, xy\}$
  - $\det(D) = (b_2 - b_1)^5$
- (a_1, b_1) and (a_2, b_2) are on two different vertical lines ($a_1 \neq a_2$).
  - The basis of the interpolation space is: $\{1, y, x, xy, x^2, x^3\}$
  - $\det(D) = (a_1 - a_2)^5$

The multivariate interpolation problem is involved in the finite element method for solving partial differential equations. The example below will show the existence of a unique interpolant for the well-known Bogner-Fox-Schmit finite element [1], for the case in which four interpolation data are prescribed on each node. We consider a more general situation in which the nodes are not necessarily the four vertices of a rectangle with sides parallel to the coordinate axes, but they are on two different vertical lines.

EXAMPLE 4.9. Let us consider the interpolation data

$$f(x, y), \quad \frac{\partial f}{\partial x}(x, y), \quad \frac{\partial f}{\partial y}(x, y), \quad \frac{\partial^2 f}{\partial x \partial y}(x, y)$$

at the interpolation nodes $(a, c_1), (a, c_2), (b, d_1)$ and (b, d_2), where $a \neq b$. The nodes are on two different vertical lines, with $c_1 \neq c_2$ and $d_1 \neq d_2$, i.e., there are two different nodes along each one of the two vertical lines.

The basis of the interpolation space in which the problem has a unique solution is:

$$\{1, y, y^2, y^3, x, xy, xy^2, xy^3, x^2, x^2y, x^2y^2, x^2y^3, x^3, x^3y, x^3y^2, x^3y^3\}.$$

That is, the interpolation space is in this case the whole space $\Pi_{33}(x, y)$. The determinant of the coefficient matrix of the linear system corresponding to the interpolation problem is

$$(b - a)^{16}(c_2 - c_1)^8(d_2 - d_1)^8 \neq 0.$$

The coefficient matrix of the linear system is the generalized Kronecker product of confluent Vandermonde matrices.

5. The case of generalized bivariate polynomials. Our aim in this section is to show the unique solvability of the bivariate Hermite interpolation problem in the more general situation in which $(\Phi_0, \Phi_1, \ldots, \Phi_n)$ and $(\Psi_0, \Psi_1, \ldots, \Psi_m)$ are extended complete Tchebycheff systems. We recall the definition of an extended complete Tchebycheff system.

DEFINITION 5.1. *Let G be a real interval. We say that $(\Phi_0, \Phi_1, \ldots, \Phi_n)$ is an extended complete Tchebycheff (ECT) system of order n on G if $\Phi_i \in C^n(G)$ $\forall i$, and for $k = 0, \ldots, n$ and for every choice of points $x_0, \ldots, x_n \in G$ not necessarily different, the determinant of the generalized Vandermonde matrix*

$$\left[\frac{d^{\mu(x_j)} \Phi_i}{dx^{\mu(x_j)}}(x_j) \right]_{j,i=0,\ldots,k}$$

is nonzero, where $\mu(x_j)$ is the multiplicity of x_j in the ordered system $(x_0, x_1, \ldots, x_{j-1})$.

REMARK 5.2. $(1, x, x^2, \ldots, x^n)$ and $(1, y, y^2, \ldots, y^m)$ are ECT systems.

Proceeding in the same way as in Section 3, we obtain that the interpolation problem can be written as a linear system $Dp^* = z$ satisfying (3.1)–(3.9).

Since $(\Phi_0, \Phi_1, \ldots, \Phi_n)$ is an ECT system, all the matrices $A[\alpha_p|\alpha_p]$, $p = 0, \ldots, s$, are nonsingular, where $A = \left[L_i(\Phi_j)\right]_{i,j=0,\ldots,n}$. In the same way, since $(\Psi_0, \Psi_1, \ldots, \Psi_m)$ is an ECT system, the matrices $B_i[\gamma_i|\gamma_i]$ are nonsingular, where $B_i = \left[L_k^{(i)}(\Psi_l)\right]_{k,l=0,\ldots,m}$, $i = 0, \cdots, n$.

By using Theorem 3.1, we conclude that the coefficient matrix D of the linear system corresponding to the bivariate interpolation problem is nonsingular, and therefore, the bivariate Hermite interpolation problem has a unique solution.

An important particular case arises when $(\Phi_0, \Phi_1, \ldots, \Phi_n)$ and $(\Psi_0, \Psi_1, \ldots, \Psi_m)$ are systems of generalized polynomials, such as those considered in [6, 13]. In this situation the matrix D is a submatrix of C, where C is the generalized Kronecker product of confluent Cauchy-Vandermonde matrices. We illustrate this situation with the following example.

EXAMPLE 5.3. Let us consider the following two systems of generalized polynomials:

$$(\Phi_0, \Phi_1, \Phi_2, \Phi_3) = \left(1, x, \frac{1}{x+1}, \frac{1}{x+2}\right),$$
$$(\Psi_0, \Psi_1, \Psi_2, \Psi_3) = \left(1, y, \frac{1}{y+3}, \frac{1}{y+4}\right).$$

Our aim is to find an appropriate interpolation space, a subspace of the tensor-product space generated by the bivariate functions

$$\{F_{ij}(x,y) = \Phi_i(x)\Psi_j(y) \mid i = 0, \ldots, 3; j = 0, \ldots, 3\},$$

in which the following Hermite interpolation problem has a unique solution.

We consider the interpolation data

$$f(x,y),\ \frac{\partial f}{\partial x}(x,y),\ \frac{\partial f}{\partial y}(x,y),$$

at the interpolation nodes $(a_1, b_1), (a_1, b_2)$ and (a_2, b_3), where $a_1 \neq a_2$. The nodes are on two different vertical lines, and $b_1 \neq b_2$, i.e., there are two different nodes along one of the vertical lines.

The basis of the interpolation space of dimension 9 in which this problem has unique solution is

$$\left\{1, y, \frac{1}{y+3}, \frac{1}{y+4}, x, xy, \frac{1}{x+1}, \frac{1}{x+1}y, \frac{1}{x+2}\right\}.$$

The determinant of the coefficient matrix of the linear system corresponding to the interpolation problem is

$$\frac{(b_2-b_1)^5(a_2-a_1)^6}{(a_1+1)^4(a_1+2)^2(a_2+1)^3(a_2+2)^2(b_1+3)^2(b_1+4)^2(b_2+3)^2(b_2+4)^2} \neq 0.$$

An analogous case is $(\Phi_0, \Phi_1, \Phi_2, \Phi_3) = (1, x, x^2, x^3)$ and $(\Psi_0, \Psi_1, \Psi_2, \Psi_3) = (1, y, y^2, y^3)$, an example considered in [7]. In this case, the basis of the interpolation space is

$$\{1, y, y^2, y^3, x, xy, x^2, x^2y, x^3\},$$

and the determinant of the coefficient matrix of the linear system corresponding to the interpolation problem is

$$(b_2-b_1)^5(a_2-a_1)^6 \neq 0.$$

Therefore, using the approach presented in this paper, we have found an interpolation space such that the interpolation problem has a unique solution, which cannot be obtained by using the techniques presented in [7].

6. Final remarks. In this section, we briefly state some of the advantageous features of our approach:

1. In the polynomial case, the interpolation basis elements are always monomials.
2. Our approach applies not only to the polynomial case, but also to the more general case in which $(\Phi_0, \Phi_1, \ldots, \Phi_n)$ and $(\Psi_0, \Psi_1, \ldots, \Psi_m)$ are extended complete Tchebycheff systems.
3. The matrix formulation of the bivariate Hermite interpolation problem presented in this paper can easily be extended to multivariate Hermite interpolation problems.
4. Our approach applies to the construction of more general finite element schemes.
5. Using this approach, every new result obtained in the univariate setting can readily be extended to the multivariate case.

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